Invariances in Physics and Group Theory

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…in that respect, many physicists, like Monsieur Jourdain, are following Klein's program without knowing…

Outline

Symmetries in Physics and discrete groups : crystallography

Special Relativity and its invariance groups. General Relativity and Gauge theories

Emmy Noether and her theorems (1918)

Invariances in Quantum Mechanics ; Weyl ; Wigner (and von Neumann) ; van der Waerden

Implementations of symmetry in the physical world : three examples . . .

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Implementations of symmetry in the physical world : three examples . . .

a partial and biaised view of Symmetries and Group Theory by a physicist

Early group theory in 19th century physics : crystallography

The classification of space groups in 3d took a good part of the 19th century. Completed by [Schönflies, Fedorov, Barlow] (1891–94)

Dimension d	Space groups
$d = 1$	7
$d = 2$	17
$d = 3$	230

"The most important application of group theory to natural science heretofore has been in this field" [Weyl 1928].

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Breaking of symmetry. Curie principle (1894) : "elements of symmetry of causes must be found in effects ; when some effects reveal some asymmetry, that asymmetry must be found in causes."

"C'est la dissymétrie qui crée le phénomène" . Example : occurrence of piezoelectricity (an electric field created by a mechanical stress) depends on existence of a rank 3 tensor $\gamma_{i,jk} \neq 0$; ruled out by "inversion" in many crystal classes.

Importance of groups and "intergroupes" (=subgroups)...

Special Relativity and Lorentz Group : Lorentz, Poincaré, Einstein. . .

- Lorentz (1892-1904) (also FitzGerald) discovers the L. transformations to 1st order in v^2/c^2 and the contraction of lengths, (by a "coup de pouce" according to Poincaré !). . ., to make the Michelson-Morley experiment consistent with aether, but misses the covariance of Maxwell equations ;
- Poincaré (1905) finds the right Lorentz transformations (to all orders) ; corrects Lorentz mistake and finds the the covariance of Maxwell equations ; sees that L. transfos leave the form $x^2+y^2+z^2-c^2t^2$ invariant and form a group . . . but Lorentz group is not derived from first principles ; (also Poincaré keeps the aether) ;
- Einstein (1905) starting from first principles (c independent of observer ; principle of relativity : physical observations do not depend on inertial frame of observer) *constructs* the Lorentz transformations ; notices as a side remark that they form a group ("wie die sein muss") but does not notice nor comment the fact that they preserve the form $x.x$; proves that the invariance of Maxwell equations ;

– Minkowski (1908) introduces "space-time", the Lorentz group as the invariance group of the metric $x_1^2+x_2^2+x_3^2+x_4^2$, $x_4=ict$, 4-vectors and tensors, the covariant way of writing Maxwell equations etc. Einstein not impressed first, ("überflüssige Gelehrsamkeit" : superfluous erudition !), then realizes the power of tensor methods…

Thus, although Einstein made a real breakthrough in physics and in our view of space and time, thus "propounding a new chronogeometry" [Darrigol], it seems fair to say that group theory played a very minor role in his work and thoughts.

Klein (1910) (as cited by E. Noether) observes : The term "relativity" current in physics is replaceable by "invariance relative to a group".

References : A. Pais, O. Darrigol

General Relativity... and gauge theories

Invariance of equations of gravitational field under general coordinate transformations.

Looking for equations knowing the invariance group...

Einstein, Hilbert (1915)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,.$$

Similar case of gauge theories (abelian case : Weyl (1918) : U(1) gauge invariance of electrodynamics ; non-abelian : Yang–Mills (1954))

Invariances and geometry of space (either real space-time or "internal" space, e.g. color space of QCD) dictate the dynamics (Einstein–Hilbert, Yang–Mills)

Very soon, Einstein, Hilbert, Weyl attempt at merging electromagnetism and GR ; later Kaluza (1921) and Klein (1926) (not the same Klein !)...

Emmy Noether : invariances and conservation laws

Her paper (1918) : 2 theorems on group invariance in variational problems.

Action principle $S = \int \mathcal{L}(x;\, \phi^i(x), \partial\phi^i(x), \cdots) d^d x$ with invariance of S under $x \mapsto x'$, $\phi \mapsto \phi'$. Then

1st theorem : n diml Lie group of invariance of $\mathcal{L}\, d^d x \Rightarrow n$ independent "conserved" currents $j_s^\mu(\vec{x},t) = (j_s^0(\vec{x},t), \vec{j}_s(\vec{x},t))$, i.e. $\partial_\mu j_S^\mu \equiv \partial_t j_S^0 - \mathrm{div}\vec{j}_s = 0$, $s = 1, \cdots, n$, from which by Stokes theorem, n conservation laws follow

$$\frac{d}{dt} Q_s := \frac{d}{dt} \int d^d x j_S^0(\vec{x},t) = \int d^d x (\mathrm{div}\vec{j}_s)(x,t) = 0.$$

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If $\mathcal{L}$ depends only on ϕ and $\partial\phi$,

$\delta x^\mu = X_s^\mu \delta a^s$, $\qquad \delta\phi^i = Z_s^i \delta a^s$, $\qquad s = 1, \cdots n \qquad$ (Einstein convention)

$$\begin{aligned} j_s^\mu &= -\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}(Z_s^i - \partial_\rho\phi^i X_s^\rho) - X_s^\mu \mathcal{L} \\ \partial_\mu j_s^\mu\, \delta a^s &= \sum_i \Psi_i\, \delta\phi^i \ \text{ where }\ \Psi_i := \frac{\delta\mathcal{L}}{\delta\phi^i} := \frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i} \\ &= 0 \ \text{ by Euler}-\text{Lagrange equations}\ . \end{aligned}$$

Noether 1st theorem : invariances and conservation laws
– early precursors : Lagrange (1811), Hamilton (1834), Jacobi (1837),
(cons. laws in mechanics : energy, momentum, angular momentum)
– late precursors : Schütz, Hamel (1904), Herglotz (1911), ...
– her legacy : after initial applause by Klein, Hilbert, ..., comes a long freeze, (quantum mechanics makes no use of Lagrangian formalism) ;
late 40's–early 50's, development of covariant QFT, revival of Lagrangian formalism and Noether theorem becomes important in QFT.
Role of *Ward identities*. If $\partial_\mu j^\mu = 0$,

$$\partial_\mu \langle T j^\mu(x)\phi_1(y_1)\cdots\phi_n(y_n)\rangle = \sum_j \delta(x^0 - y_j^0)\langle T\phi_1 \cdots [j_s^0, \phi_j]\cdots\rangle \,.$$

2nd theorem : for an “infinite dimensional group” of invariance (like diffeomorphisms in G.R., or gauge transformations in gauge theories), invariance within a variational principle $\Rightarrow$ constraints between the $\Psi_i = \delta\mathcal{L}/\delta\phi^i \ldots$. For example, Bianchi identities....

(Also in Noether’s paper, converse statements, cons. laws $\Rightarrow$ invariance ; Also E. Noether clarifies the issue of energy conservation in G.R., a most confusing issue at the time...)

References : H. Kastrup (1987), Y. Kosmann-Schwarzbach (2006)

Invariances in Quantum Mechanics, Weyl, Wigner, van der Waerden…

Symmetries in Q.M. and group theory : a new paradigm.

Wigner theorem $\Rightarrow$ representation theory enters Physics

Transformations in QM are implemented on state vectors $\Psi \in \mathcal{H}$ and "observables" A (self-adjoint oprs on $\mathcal{H}$) by $\Psi \to U\Psi$, $A \to UAU^{-1}$ with U unitary or anti-unitary, and unique *up to a phase*.

$$g, g' \in G \qquad U(g)U(g') = U(g.g')e^{i\omega(g,g')}$$

$\Rightarrow U(g)$ gives a projective (up to a phase) representation of G.

Invariances : group action commutes with dynamics (Hamiltonian), i.e.

$$[H, U(g)] = 0\,.$$

Diagonalize a maximal commuting subgroup of G :

Invariances $\Rightarrow$ conserved quantities, "good quantum numbers" .

Example : quantization of angular momentum (and "spin") : conservation of $\vec{J}^2$ and J_z $\longrightarrow$ representations of SO(3) or SU(2)

First applied to rotation group and symmetric group (Fermi-Dirac or Bose-Einstein statistics) with innumerable applications to atomic and molecular physics (electronic orbitals and the structure of atoms and molecules ; Zeeman effect ; selection rules in transitions etc etc) ; then to condensed matter and to particle physics (scattering theory).

J. von Neumann and E. Wigner (1928) : atomic wave functions with spinning electrons
H. Weyl *Gruppentheorie und Quantenmechanik* (1928)
E. Wigner *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (1931)
B. van der Waerden *Die gruppentheoretische Methode in der Quantenmechanik* (1932)

Resistance of some . . . “the group pest”. . .

Conversely : inferring a symmetry group from the existence of conserved quantities : the case of “flavor groups” in particle physics :
Heisenberg (1932) : neutron and proton form a dim-2 representation of a new SU(2) of “isospin” ; later, more instances : pions (π^+,π^0,π^-) (1947), kaons (K^+,K^0), Δ resonance $(\Delta^{++},\Delta^+,\Delta^0,\Delta^-)$, etc, form other reprns. This is a symmetry of *hadrons* (strongly interacting particles), broken by electromagnetic interactions.

Gell-Mann–Ne’eman (1961) : same game with the newly discovered “strange” particles : flavor SU(3) group
Argument : there are *two* independent conserved quantities (isospin and *hypercharge* or *strangeness*), hence group must be of rank 2. There are “octets” (8-dim representations), which points to SU(3).
This is an *approximate* symmetry of strong interactions.

More symmetries in contemporary particle physics…
(Weak interactions : current algebra, Cabibbo angle, …, gauge theory)

Nature and the many implementations of symmetries in the quantum world

– space-time or “internal” symmetries

– global or local

implemented as

– exact (QED, QCD)

– broken explicitly (SU(2) by electromagnetism, SU(3) approximate)

– broken spontaneously (ferromagnets), Goldstone phenomenon if continuous symmetry : appearance of massless excitations/particles

– broken spontaneously with gauge symmetry (electro-weak sector of Standard Model, the Higgs mechanism. . .)

– anomalously (breaking of a classical symmetry by quantum effects)

– quantum symmetries, “quantum groups”

– with supersymmetry (susy)

– . . .

Example 1 : "Linear/non linear sigma models" : Klein's most direct heirs ?
Take a field $\phi \in \mathbb{R}^n$ or $\in S^{n-1}$. Write a Lagrangian in the form

$$\mathcal{L} = \frac{1}{2}(\partial\phi, \partial\phi) - V((\phi, \phi))$$

Invariance group $\mathrm{O}(n)$. May be generalized to a field $\phi \in \mathcal{M}$, a Riemannian manifold. Case of symmetric spaces $G/K \ldots$
First set up by Gell-Mann and Lévy (1960) to describe the physics of pion particles $\pi^{\pm}, \pi^0$ and a hypothetical σ. Original model has O(4) symmetry.

Questions

– how is the symmetry realized, exact, broken, spontaneously broken ?
– how is the symmetry preserved by renormalization ? (use Noether currents and Ward identities)
– physical consequences (pion as a Goldstone particle, or an almost G. particle (low mass) ?) ; generation of mass ? ; scale or conformal invariance ? etc

Sigma models extensively used on all kinds of manifolds and groups in particle physics, in stat. mech. and solid state physics ("effective theories" for various phenomena, membranes, excitations, "order parameters", …), in string theory (low energy limit), generalized geometry à la Hitchin, etc etc.

Also non compact and/or supersymmetric sigma models : a currently fairly active subject…

Example 2 : the Standard Model of particle physics : symmetry group SU(3)$\times$ SU(2)$\times$U(1), three gauge groups realized in an utterly different way…

SU(3) color (gauge) symmetry of QCD : exact (crucial for quark confinement) ; SU(2)$\times$ U(1) : weak isospin and hypercharge : spont. broken to exact U(1), gauge symmetry of electrodynamics ($\Rightarrow$ Higgs boson).

On top of that, SU(2)$\subset$ SU(3)… flavor symmetries (broken).
Possibly, in some extension of the SM, supersymmetry ? (broken).

Absence of anomalies in the Standard Model, crucial for consistency of the theory, points to a remarkable matching between numbers of families of leptons and quarks :
currently 3 families $(e,\nu_e),\ (\mu,\nu_\mu),\ (\tau,\nu_\tau) \longleftrightarrow$ (u,d), (c,s), (t,b).

Example 3 : Quantum integrable systems and Quantum Groups

Consider the spin 1/2 Heisenberg XXZ quantum chain

$$H = \sum_{i=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z + \text{boundary terms}$$

acting in $(\mathbb{C}^2)^{\otimes N}$, a quantum integrable system [Heisenberg, Lieb & Sutherland, Yang and Yang, Gaudin, Baxter, Faddeev et al. . .]

For $\Delta = 1$, (and no bdy term), SU(2) invariance

For $\Delta \neq 1$, $|\Delta| < 1$, deformed symmetry $U_q sl(2)$ (quantum SU(2)), $q = e^{i\alpha}$, $\Delta = \cos\alpha$ [Pasquier–Saleur 1989]

For $\Delta = \frac{1}{2}$, amazing connections with problems of combinatorics, the quantum Knizhnik–Zamolodchikov equation, Razumov–Stroganov (ex)-conjecture etc etc.

Recent progress –correlation functions, etc– made possible by representation theoretic considerations [Jimbo–Miwa]. . .

Role of symmetry :

to *predict* (more invariants $\longrightarrow$ less independence)

to *protect* in the quantization (and renormalization) : example, gauge theories

Conversely, no symmetry, no protection, possible “mixings”. Example : mixing generations of quarks or leptons.

Conclusions

Symmetries play an *essential* role in modern physics ;

Study of symmetry groups, representation theory : part of the education of a modern physicist ;

Symmetry group and its breaking, residual group of symmetry ;

Many possible implementations of symmetries in (quantum) physics ;

At some stages, resistance of some . . .

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. . .but the mathematical beauty of Physics will prevail !

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