# “Angular” matrix integrals

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## Matrix integrals

$$\mathcal{Z}_G = \int_G D\Omega \, \exp N\beta \, \Re e \left( \mathrm{tr} \left( \Omega J \right) \right) \tag{1}$$

$$Z^{(G)} = \int_G D\Omega \, \exp N\beta \, \Re e \left( \mathrm{tr} \left( A\Omega B\Omega^\dagger \right) \right) \tag{2}$$

over a compact group $G$, are frequently encountered in physics (and in maths) : “Bessel matrix functions” or “angular matrix integrals”.

$G = \mathrm{O}(N), \mathrm{U}(N), \mathrm{Sp}(N)$, with respectively $\beta = 1, 2, 4$.

Invariance under $J \mapsto \Omega_1 J \Omega_2$ and $A \mapsto \Omega_1 A \Omega_1^\dagger$, $B \mapsto \Omega_2 B \Omega_2^\dagger$, resp.

$\Rightarrow \mathcal{Z}_G$ expressible as a sum of $\prod_i \mathrm{tr} \, (JJ^\dagger)^{p_i}$ and $Z^{(G)}$ as a sum of $\prod_i \mathrm{tr} \, A^{p_i} \prod_j \mathrm{tr} \, B^{q_j}$

Matrix integrals

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over a compact group $G$, are frequently encountered in physics (and in maths) : "Bessel matrix functions". Mostly studied for $G = \mathrm{U}(N)$ ($\beta = 2$).

What happens for other groups, e.g. $G = \mathrm{O}(N)$ ($\beta = 1$), $\mathrm{Sp}(N)$ ($\beta = 4$)?

- If $A$ and $B$ are both real skew-symmetric (i.e. in the Lie algebra of $G = \mathrm{O}(N)$), $Z$ is known exactly from the work of Harish-Chandra '57. Also correlation functions are known [Eynard *et al*].

- If $A$ and $B$ are both real symmetric, much more complicated and elusive, [Brézin & Hikami '02-06, Bergère & Eynard 08].

- if they are neither, ...?

- Expect simplification as $N \to \infty$ [Weingarten '78]. Universality of (1), (2).

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## Outline of this talk

- Review of (2) in the Harish-Chandra case (*A* and *B* in the Lie algebra)
- Correlation functions
- The integral (2) in the symmetric case
- The large *N* limit

## 1. The Harish-Chandra integral

For $A$ and $B$ in the *Lie algebra* $\mathfrak{g}$ of $G$, in fact in a *Cartan* algebra

$$Z^{(G)} = \int_G D\Omega \exp N\beta \,\mathrm{tr}\,(A\Omega B\Omega^\dagger) = \text{const.} \sum_{w\in\mathcal{W}} \frac{\exp N\beta \,\mathrm{tr}\, AB^w}{\Delta_G(A)\Delta_G(B^w)} \qquad (3)$$

$\Delta_G(A) := \prod_{\alpha>0}\langle\alpha, A\rangle$, a product over the positive roots, $\mathcal{W}$ the Weyl group.

More concretely, for $G = \mathrm{U}(N)$, take $A = \mathrm{diag}\,(a_i)$, $B = \mathrm{diag}\,(b_i)$

$$Z^{(\mathrm{U})} = \text{const.} \frac{\det e^{\beta N a_i b_j}}{\prod_{i<j}(a_i - a_j)(b_i - b_j)}$$ [Itzykson-Z '80]

and for $G = \mathrm{O}(N)$, take $A$ and $B$ both skew-symmetric, block-diagonal form

$A = \mathrm{diag} \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}_{i=1,\cdots,m}$, $B$ likewise

$$Z^{(\mathrm{O})} = \text{const.} \frac{\det(2\cosh 2N a_i b_j)}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}$$

for $\mathrm{O}(N)$, $N = 2m$, with $\Delta_{\mathrm{O}}(a) = \prod_{1\le i<j\le m}(a_i^2 - a_j^2)$.

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$$Z^{(\mathrm{O})} = \text{const.} \frac{\det(2\sinh 2N a_i b_j)}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}$$

for $\mathrm{O}(N)$, $N = 2m+1$ with $\Delta_{\mathrm{O}}(a) = \prod_{i=1}^m a_i \prod_{1\le i<j\le m}(a_i^2 - a_j^2)$.

## Proofs of this H-C formula

– *Heat kernel*

$Z' = t^{-\frac{1}{2}\dim G} \int_G D\Omega\, e^{-\frac{1}{2t} N\beta\, \mathrm{tr}\,(A-\Omega B\Omega^\dagger)^2}$ satisfies $(N\beta\frac{\partial}{\partial t} - \frac{1}{2}\Delta_A)Z' = 0$ and boundary cond $Z' \underset{t\to 0}{\longrightarrow} \text{const} \int_G d\Omega\, \delta(A - \Omega B\Omega^\dagger)$. Rewrite in "radial coordinates" $a_i$ using the expression of the Laplacian

$$\Delta_A = \Delta_G^{-2}(A) \sum_i \partial_i \Delta_G^2(A) \partial_i + \text{angular part} = \Delta_G^{-1}(A) \sum_i \partial_i^2 \Delta_G(A) + \text{ang.}$$

Thus $Z'' := \Delta_G(A)\Delta_G(B)Z'$ solution of $(N\beta\partial_t - \frac{1}{2}\sum_i \partial_i^2)Z'' = 0$ and is an alternate sum over the Weyl group of $\exp -\frac{1}{2t}N\beta(a_i - b_i^w)^2$. QED

– *Character expansion* . . .

– *Exact semi-classical expression* [Duistermaat-Heckman theorem]

Stationary points of $\mathrm{tr}\,(A\Omega B\Omega^\dagger)$ w.r.t $\Omega$ satisfy $[A, \Omega B\Omega^\dagger] = 0$ and are for *generic* $A$ and $B$ in $\mathfrak{g}$, (distinct eigenvalues), in 1-to-1 correspondence with the elements of $\mathcal{W}$, whence the numerator of the H-C formula. Then the Gaussian fluctuations around each of these stationary points yield the denominator of the H-C formula.

## Correlation functions

What about the associated “correlation functions” of invariant traces

$$\int \mathrm{D}\Omega\, \mathrm{e}^{-\mathrm{tr}\, A\Omega B\Omega^\dagger} \prod \mathrm{tr}\left(A^{p_1}\Omega B^{q_1}\Omega^\dagger A^{p_2}\cdots\right) \qquad ?$$

(still invariant under $A \to \Omega_1 A \Omega_1^\dagger$, $B \to \Omega_2 B \Omega_2^\dagger$)

Is there still some localization property?

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Is there still some localization property?

$$\int D\Omega\, e^{-\mathrm{tr}\, A\Omega B\Omega^\dagger} F(A, \Omega B\Omega^\dagger) = c_n \sum_{w\in W} \frac{e^{-\mathrm{tr}\, AB^w}}{\Delta(A)\Delta(B^w)} \int_{\mathfrak{n}_+=[\mathfrak{b},\mathfrak{b}]} DT\, e^{-\mathrm{tr}\, TT^\dagger} F(A+T, B^w+T^\dagger)$$

with $\mathfrak{b}$ the Borel subalgebra (upper triangular matrices), $\mathfrak{n}_+$ its “derived ideal” (generated by positive roots) (strictly upper triangular matrices), whence a *finite* number of correction terms to the semi-classical approximation.

[Eynard–Prats Ferrer ’04, P F–E–Di Francesco–Z ’06 , Bertola–P F ’08] generalizing or making more explicit previous expressions [Morozov ’92, Shatashvili ’93].

## 2. The integral (2) in the symmetric case

$$Z^{(G)} = \int_G D\Omega \exp N\beta \,\mathrm{tr}\,(A\Omega B\Omega^\dagger)$$

for $A = A^\dagger$ and $B = B^\dagger$.

For $G = \mathrm{U}(N)$, $A$ and $B$ hermitian rather than *anti*hermitian, no difference, HCIZ formula works.

For $G = \mathrm{O}(N)$, $A$ and $B$ real symmetric, ??

For $G = \mathrm{Sp}(N)$, $A$ and $B$ quaternionic self-dual. ??

A case much studied in the recent years [Guhr–Kohler ’00, Brézin–Hikami ’02–06, Bergère–Eynard ’08, Collins–Guionnet–Maurel-Segala ’08]

Many nice features

– finite (semi-classical) expansion and “τ-expansion” for β an *even* integer

$$Z^{(G)} = \sum_{\sigma\in\mathfrak{S}_N} \frac{e^{N\beta a_i b_{\sigma(j)}}}{\Delta(a)^\beta \Delta(b^\sigma)^\beta} \mathcal{P}_{\beta,N}(A,B^\sigma) \quad (5)$$

with $\mathcal{P}_{\beta,N}(A,B)$ a *polynomial* of degree $\beta/2$ in each variable $\tau_{ij} := (a_i - a_j)(b_i - b_j)$,

hence $Z^{(G)} = \sum_{\sigma\in\mathfrak{S}_N} \frac{e^{N\beta a_i b_{\sigma(j)}}}{\Delta(a)^{\beta/2}\Delta(b^\sigma)^{\beta/2}} P_{\beta,N}(\frac{1}{\tau^\sigma})$.

– Differential equation [Bergère-Eynard ’08]: take $A$ and $B$ diagonal

Let $K = \{(K)_{ij}\}$ be the *matrix* differential operator

$K_{ii} = \frac{\partial}{\partial a_i} + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{a_i - a_j}$ and for $i \neq j$, $K_{ij} = -\frac{\beta}{2}\frac{1}{a_i - a_j}$ and let

$M_{ij} := Z\langle|\Omega_{ij}|^2\rangle$. Then $Z = \sum_i M_{ij} = \sum_j M_{ij}$ and

$$\sum_j K_{ij} M_{jk} = N\beta M_{ik} b_k \qquad \text{no summation over } k$$

$\sum_k M_{ik} = \sum_i M_{ik} = Z$ and $\sum_j K_{ij} M_{jk} = (N\beta) M_{ik} b_k$. Can iterate that equation to get

$$\sum_j K^p_{ij} M_{jk} = M_{ik} (N\beta)^p b^p_k$$

and summing over $i$ and $k$

$$\underbrace{(\sum_{ij} K^p_{ij})}_{\text{a differential operator of order } p} \quad Z = (N\beta)^p \operatorname{tr} B^p Z \; . \tag{6}$$

**Two remarks**

1. *This solves the following problem :*

Define the differential operator $D_p(\partial/\partial A)$ by
$D_p(\partial/\partial A) e^{N \operatorname{tr} AB} = N^p \operatorname{tr} B^p \, e^{N \operatorname{tr} AB}$
If $D_p$ acts on *invariant functions* $F(A) = F(\Omega A \Omega^\dagger)$, how to write it in terms

of $\partial/\partial a_i$ ? For $G = \mathrm{U}(N)$,

$$D_p(\partial/\partial A) = \mathrm{tr}\left(\frac{\partial}{\partial A}\right)^p := \sum_{i_1,\cdots i_p} \frac{\partial}{\partial A_{i_1 i_2}} \frac{\partial}{\partial A_{i_2 i_3}} \cdots \frac{\partial}{\partial A_{i_p i_1}}$$

and

$$D_p = \frac{1}{\Delta(a)} \sum_i \left(\frac{\partial}{\partial a_i}\right)^p \Delta(a) \; .$$

$\Delta(a) = \prod_{i<j}(a_i - a_j)$ (a non trivial calculation !) [Itzykson–Z '80].
In general, "radial" expression of $D_p$ is given by $D_p = \sum_{i,j}(K^p)_{ij}$

2. *Connection with Calogero*

Note that by construction the $D_p := \sum_{ij} K^p_{ij}$ commute.

Consider $H_p := \Delta(a)^{\beta/2} D_p \Delta(a)^{-\beta/2}$. $H_2 = \sum_i \partial_i^2 + \frac{\beta}{2}\left(1 - \frac{\beta}{2}\right)\sum_{i\neq j}\frac{1}{(a_i - a_j)^2}$ is the Calogero Hamiltonian, and the $H_p$ are the higher conserved quantities.

## 3. Large N limit

Expect things to simplify as $N \to \infty$ [Weingarten '78]. Look at the "free energies" :

$$W_G(J.J^\dagger) = \lim_{N\to\infty} \frac{1}{N^2} \log \mathcal{Z}_G$$

and

$$F_G(A,B) = \lim_{N\to\infty} \frac{1}{N^2} \log Z^{(G)}$$

Then $W(X)$ and $F(A,B)$ are, *up to an overall factor*, independent of $G = \mathrm{O}(N),\ \mathrm{U}(N)$ !

(Not true at finite $N$ !)

More precisely,

$$W_{\mathrm{O}}(J.J^{\dagger}) = \frac{1}{2} W_{\mathrm{U}}(J.J^{\dagger}) \tag{7}$$

and

$$F_{\mathrm{O}}(A,B) = \frac{1}{2} F_{\mathrm{U}}(A,B) \tag{8}$$

Intuitively, counting of # of degrees of freedom : $\beta N^2/2$ real parameters in $\mathrm{O}(N)$, $\mathrm{U}(N)$.

Actual proof relies either on inspection of explicit formulae ("Harish-Chandra case"), or on the use of differential equations satisfied by $\mathcal{Z}$, resp. $Z$, which simplify in the $N \to \infty$ limit.

For $\mathcal{Z}_{\mathrm{O}} = \int_{\mathrm{O}(N)} \mathrm{D}O \exp N \mathrm{tr}\,(J.O)$, follow the steps of [Brézin-Gross '80]: the trivial identity $\sum_j \frac{\partial^2 \mathcal{Z}_{\mathrm{O}}}{\partial J_{ij} \partial J_{kj}} = N^2 \delta_{ik} \mathcal{Z}_{\mathrm{O}}$ is reexpressed in terms of the eigenvalues $\lambda_i$ of the real symmetric matrix $J.J^t$:

$$4\lambda_i \frac{\partial^2 \mathcal{Z}_{\mathrm{O}}}{\partial \lambda_i^2} + \sum_{j \neq i} \frac{2\lambda_j}{\lambda_j - \lambda_i} \left( \frac{\partial \mathcal{Z}_{\mathrm{O}}}{\partial \lambda_j} - \frac{\partial \mathcal{Z}_{\mathrm{O}}}{\partial \lambda_i} \right) + 2N \frac{\partial \mathcal{Z}_{\mathrm{O}}}{\partial \lambda_i} = N^2 \mathcal{Z}_{\mathrm{O}} \,.$$

Writing as above $\mathcal{Z}_{\mathrm{O}} = \mathrm{e}^{N^2 W_{\mathrm{O}}}$ and dropping subdominant terms in the large $N$ limit, with $W_{\mathrm{O}}$ and $W_i := N \partial W_{\mathrm{O}} / \partial \lambda_i$ of order 1, we get

$$4\lambda_i W_i^2 + 2W_i + \frac{1}{N} \sum_{j \neq i} \frac{2\lambda_i}{\lambda_j - \lambda_i} (W_j - W_i) = 0 \qquad (9)$$

which is precisely the equation satisfied by $\frac{1}{2} W_{\mathrm{U}}$ in [B-G]. This, together with appropriate boundary conditions, suffices to complete the proof of (7).

An explicit expression of $W_{\mathrm{U}}$ is known [O' Brien-Z '84]

$$W_{\mathrm{U}}(J.J^{\dagger}) = \sum_{n=1}^{\infty} \sum_{\alpha \vdash n} W_{\alpha} \frac{\mathrm{tr}_{\alpha} J.J^{\dagger}}{\prod_p (\alpha_p!\, p^{\alpha_p})}$$

$$W_{\alpha} = (-1)^n \frac{(2n + \sum \alpha_p - 3)!}{(2n)!} \prod_{p=1}^{n} \left( \frac{-(2p)!}{p!(p-1)!} \right)^{\alpha_p} ,$$

where $\alpha \vdash n$ denotes a partition of $n = \alpha_1.1 + \alpha_2.2 + \cdots + \alpha_n.n$ and

$$\mathrm{tr}_{\alpha}(X) := \prod_{p=1}^{n} \frac{1}{N} (\mathrm{tr}\, X^p)^{\alpha_p} .$$

For $Z^{(\mathrm{O})} = \int_{\mathrm{O}(N)} \mathrm{D}O \exp N \mathrm{tr}\,(AOBO^t)$, take $A$ and $B$ both skew-symmetric, or both symmetric.

• $A$ and $B$ both skew-symmetric [Harish-Chandra]

block-diagonal form $A = \mathrm{diag}\left(\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}_{i=1,\cdots,m}\right)$, $B$ likewise, recall

$$Z^{(\mathrm{O})} = \mathrm{const.} \frac{\det(2\cosh 2Na_ib_j)}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}$$

(for $\mathrm{O}(N = 2m)$), with $\Delta_{\mathrm{O}}(a) = \prod_{1\le i<j\le m}(a_i^2 - a_j^2)$.

Regard $A$ as $N \times N$ anti-Hermitian, eigenvalues $A_j = \pm i a_j$, $B$ likewise. Easy to check that as $N \to \infty$,

$$Z^{(\mathrm{U})}(A,B) = \frac{\det\left(e^{2NA_iB_j}\right)}{\Delta(A)\Delta(B)} \sim \left(\frac{(\det(e^{2Na_ib_j})_{1\le i,j\le m}}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}\right)^2 = (Z^{(\mathrm{O})}(A,B))^2$$

- $A$ and $B$ both symmetric

Can take them in diagonal form $A = \operatorname{diag} a_i$, $B = \operatorname{diag} b_i$

Then Bergère-Eynard equation $D_p Z = (N\beta)^p \operatorname{tr} B^p Z$ (6), in the large $N$ limit, yields

$$\sum_i \left( \frac{N}{\beta} \frac{\partial F^{(G)}}{\partial a_i} + \frac{1}{2N} \sum_{j \neq i} \frac{1}{a_i - a_j} \right)^p = \operatorname{tr} B^p \tag{10}$$

Hence $F^{(\mathrm{O})}$ $(\beta = 1)$ satisfies same set of equations as $\frac{1}{2} F^{(\mathrm{U})}$ $(\beta = 2)$, QED.

Another proof [Guionnet-Zeitouni '02] (for $A$ and $B$ symmetric) as a by-product of the construction of $F_G$ as the unique solution of Matytsin '94 flow: $\beta$ dependence is explicit. (also [Collins–Guionnet–Maurel-Segala '08])

**A Conjecture** $F^{(\mathrm{O})}(A,B) = \frac{1}{2} F^{(\mathrm{U})}(A,B)$ extends to $A$ and $B$ generic (neither symmetric, nor skew-symmetric). Some evidence from power expansion.

**Origin of this universality?** Diagrammatics ? Relation between $\mathcal{Z} = \int_G d\Omega \, \exp N\beta \, \Re e \, (\mathrm{tr}\,(\Omega J))$ and $Z = \int_G d\Omega \, \exp N\beta \, \Re e \, (\mathrm{tr}\,(A\Omega B\Omega^\dagger))$ ?

In the case of $\mathrm{U}(N)$, yes [P. Zinn-Justin–Zuber '03, Collins '03].
For $\mathrm{O}(N)$ ?? no such simple relation ...

*Heuristic argument*: consider the two-matrix integrals

$$Z_{2RS} = \int_{\substack{\text{real}\\ \text{symmetric}}} dA\,dB\, e^{-N\mathrm{tr}\,(V(A)+W(B)-AB)} \sim e^{-N^2 F_{2RS}} \tag{11}$$

$$Z_{2CH} = \int_{\substack{\text{complex}\\ \text{hermitian}}} dA\,dB\, e^{-\mathbf{2}N\mathrm{tr}\,(V(A)+W(B)-AB)} \sim e^{-N^2 F_{2CH}}\,, \tag{12}$$

over real symm., resp complex Hermitian, matrices. It is “well known” that for large $N$, same perturbative expansion of $F$, up to rescalings and a global factor 2 i.e.

$$F_{2CH} = 2F_{2RS} \tag{13}$$

On the other hand, if we diagonalize the matrices $A = \mathrm{diag}\,(a_i)$, $B = \mathrm{diag}\,(b_i)$, we see that (11-12) reduce to

$$Z_{2RS} = \int d\mathbf{a}\,d\mathbf{b}\,|\Delta(a)\Delta(b)|\, e^{-N\sum_i(V(a_i)+W(b_i))} \int DO\, e^{N\mathrm{tr}\,AOBO^t} \tag{14}$$

$$Z_{2CH} = \int d\mathbf{a}\,d\mathbf{b}\,(\Delta(a)\Delta(b))^2 e^{-2N\sum_i(V(a_i)+W(b_i))} \int DU\, e^{2N\mathrm{tr}\,AUBU^t}\,. \tag{15}$$

If integrals (14-15) dominated as $N \to \infty$ by a saddle point configuration, scaling $F_{\mathrm{O}}(A,B) = \frac{1}{2} F_{\mathrm{U}}(A,B)$ of angular part *consistent* with the scaling (13) of the integral.

**Particular case** where $A$ is of finite *rank* $r$. Then in the expansion of $F = \sum_{p,q} \prod(\frac{1}{N}\mathrm{tr}\, A^{p_i}) \prod(\frac{1}{N}\mathrm{tr}\, B^{q_j})$, terms with a single trace of $A$ dominate.

In the U$(N)$ case (and $N \longrightarrow \infty$) ([IZ '80])

$$F^{(\mathrm{U})} \sim \sum_{p=1}^{\infty} \frac{1}{p} (\frac{1}{N}\mathrm{tr}\, A^p) \psi_p(B)$$

where $\psi_p(B) = p$-th "non-crossing cumulant" of $B$ ([Brézin–Itzykson–Parisi–Z '78, Speicher '94]).

**Application :** $\star$ The $\beta$ universality of $F_G$ first pointed out in that finite rank case [Marinari, Parisi, Ritort, '94], Spin glass Hamiltonian with $n$ replicas of $N$ Ising spins

$$\mathcal{H} = \sum_{i,j=1}^{N} \underbrace{\sum_{a=1}^{n} \sigma_i^a \sigma_j^a}_{\Omega_{ij}} O_{ij} \qquad \Omega \text{ of rank } \leq n$$

with a coupling $O_{ij}$, a real, orthogonal, symmetric matrix with an equal number of $\pm 1$ eigenvalues, $O = V^t.D.V$.

Have to compute $Z = \int_{\mathrm{O}(N)} dV \exp \beta \mathrm{tr}\, DV\Omega V^t$.

Now according to Marinari, Parisi, Ritort, pretend you integrate over the unitary group,
compute $\sum \frac{1}{p} \mathrm{tr}\, \Omega^p \psi_p(D) =: \mathrm{tr}\, G(\Omega)$

and (with some insight ...) the correct formula is $\frac{1}{2} G(2\Omega)$ ! ...

Proved later by Collins, Collins and Sniady, Guionnet & Maida

## Conclusion and Open issues

– More explicit formulae for $Z$, $F$
– A priori argument for universality, graphical argument ?
– Relations with integrability: D-H localization, finite semi-classical expansions, Calogero, . . .