Logarithmic Minimal Models and their Integrable Lattice Realization

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- What is a LCFT?
- Loop model; the planar vc the linear Temperley-Lieb algebras
- Yang-Baxter integrable boundary conditions
- Commuting Double-Row Transfer Matrices
- Spectra on a strip and what they teach us

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Logarithmic CFT

Usually, in CFT's, power law singularities of correlation functions. In LCFT's, appearance of logarithms! How is that possible? Presence of reducible but indecomposable representations (of Virasoro or some other conformal algebra)

A bit of history (partial !)

1987? Knizhnik: *logarithms in CFT's*

1991 Saleur, Rozanski & Saleur :

Logarithms and indecomposable representations in WZW on supergroup

1993 Gurarie: *Logarithmic operators in CFT*

1996– Flohr, Rohsiepe, Gaberdiel and Kausch, Feigin et al. :

Algebraic approach: Indecomposable representations, Null vectors, Fusion rules,. . .

2000 Kogan, Wheater, Kawai: *Boundary logarithmic CFT*

1992– Many people: *LCFT's with fermions*

1992– Many people: *applications to polymers, percolation, disordered systems*

Many papers but still many unanswered questions . . .

What is a LCFT?

• Occurrence of **logarithms**

An example [Gurarie 93]

Take $c = -2$ theory, primary field $\mu := \phi_{12}$ $(h = -1/8)$. It is "degenerate" at level 2, thus the conformal block $F(z) := \langle \mu(\infty)\mu(z)\mu(1)\mu(0)\rangle$ satisfies a 2nd order ODE,

$$\left(z(1-z)\frac{d^2}{dz} + (1-2z)\frac{d}{dz} - \frac{1}{4}\right) F(z) = 0$$

Two independent solutions are

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) \quad \text{and} \quad \log z \; F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) + \text{reg.}$$

i.e. the OPE $\mu(z)\mu(0) = z^{1/4}(\log z \; \Phi + \Phi_1)$.

Further analysis reveals a Jordan cell structure of L_0: $L_0\Phi|0\rangle = 0$; $L_0\Phi_1|0\rangle = \Phi|0\rangle$, etc.

• Presence of **distinct fields with the same dimension** (or with dimensions differing by integers).

• Occurrence of **reducible but indecomposable representations** of the Virasoro algebra.

• Logarithmic partner t of the energy-momentum T [Gurarie & Ludwig]

Minimal LCFT's [many people]

These theories are non-unitary and non-rational.

- Central charges and conformal weights:

$$c = 1 - \frac{6(p-p')^2}{pp'}, \qquad p < p', \qquad p, p' = 1, 2, 3, \ldots \text{ coprime}$$

$$h_{r,s} = \frac{(p'r - ps)^2 - (p-p')^2}{4pp'}, \qquad r, s = 1, 2, 3, \ldots$$

- Extended Kac Tables of the "principal series" $p = m, p' = m+1$: In light blue, the first m columns. In deep blue, the ordinary Kac table ($r \le m-1, \; s \le m$).

Critical Dense Polymers

$(p, p') = (1, 2), \; c = -2$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
6	3	1	0	0	1	$\cdots$
$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
3	1	0	0	1	3	$\cdots$
$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
1	0	0	1	3	6	$\cdots$
$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
0	0	1	3	6	10	$\cdots$
$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
0	1	3	6	10	15	$\cdots$

Critical Percolation

$(p, p') = (2, 3), \; c = 0$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$

- Extended Kac table (cont'd)

Logarithmic Ising

$(p, p') = (3, 4), \; c = 1/2$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$\frac{225}{16}$	$\frac{161}{16}$	$\frac{323}{48}$	$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\cdots$
11	$\frac{15}{2}$	$\frac{14}{3}$	$\frac{5}{2}$	1	$\frac{1}{6}$	$\cdots$
$\frac{133}{16}$	$\frac{85}{16}$	$\frac{143}{48}$	$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\cdots$
6	$\frac{7}{2}$	$\frac{5}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$	$\cdots$
$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\cdots$
$\frac{5}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{5}{3}$	$\cdots$
$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\frac{5}{16}$	$\frac{21}{16}$	$\frac{143}{48}$	$\cdots$
$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{5}{2}$	$\frac{14}{3}$	$\cdots$
$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\frac{33}{16}$	$\frac{65}{16}$	$\frac{323}{48}$	$\cdots$
0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{2}$	6	$\frac{55}{6}$	$\cdots$

Logarithmic Tricritical Ising

$(p, p') = (4, 5), \; c = 7/10$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$\frac{153}{10}$	$\frac{899}{80}$	$\frac{39}{5}$	$\frac{399}{80}$	$\frac{14}{5}$	$\frac{99}{80}$	$\cdots$
12	$\frac{135}{16}$	$\frac{11}{2}$	$\frac{51}{16}$	$\frac{3}{2}$	$\frac{7}{16}$	$\cdots$
$\frac{91}{10}$	$\frac{483}{80}$	$\frac{18}{5}$	$\frac{143}{80}$	$\frac{3}{5}$	$\frac{3}{80}$	$\cdots$
$\frac{33}{5}$	$\frac{323}{80}$	$\frac{21}{10}$	$\frac{63}{80}$	$\frac{1}{10}$	$\frac{3}{80}$	$\cdots$
$\frac{9}{2}$	$\frac{39}{16}$	1	$\frac{3}{16}$	0	$\frac{7}{16}$	$\cdots$
$\frac{14}{5}$	$\frac{99}{80}$	$\frac{3}{10}$	$-\frac{1}{80}$	$\frac{3}{10}$	$\frac{99}{80}$	$\cdots$
$\frac{3}{2}$	$\frac{7}{16}$	0	$\frac{3}{16}$	1	$\frac{39}{16}$	$\cdots$
$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$	$\frac{63}{80}$	$\frac{21}{10}$	$\frac{323}{80}$	$\cdots$
$\frac{1}{10}$	$\frac{3}{80}$	$\frac{3}{5}$	$\frac{143}{80}$	$\frac{18}{5}$	$\frac{483}{80}$	$\cdots$
0	$\frac{7}{16}$	$\frac{3}{2}$	$\frac{51}{16}$	$\frac{11}{2}$	$\frac{135}{16}$	$\cdots$

- The highest weight representations labelled by the Kac table are fully reducible

  $\Longrightarrow$ irreducible...

Do these representations close under fusion?

Lattice Approach

How to realize such a LCFT on the lattice?

For six-vertex, RSOS models, . . . (local bosonic degrees of freedom)

local degrees of freedom	$\Rightarrow$	symmetric transfer matrices	$\Rightarrow$	diagonalizable transfer matrices	$\Rightarrow$	no indecomposable representations	$\Rightarrow$	not logarithmic theory

Paradigm Shift: New ingredients needed. . .

Fermionic degrees of freedom, or non local ones

bosonic logarithmic theory	$\Rightarrow$	nonlocal degrees of freedom

Non locality: connectivity of lines (loop models) or of clusters (Potts)

Overview of lattice $\mathcal{LM}(p,p')$

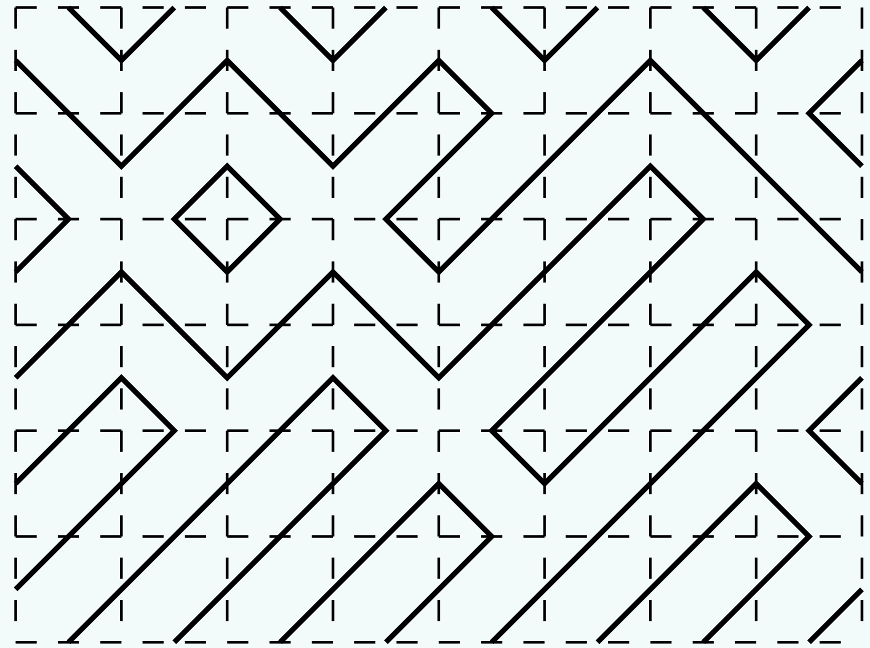
Planar Algebra
(Temperley-Lieb Algebra)

↓ YBE

Nonlocal Statistical Mechanics
(Integrable Link Models)

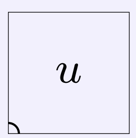
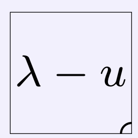
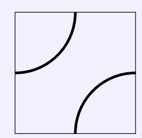
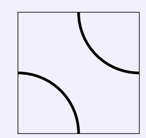
↓ continuum limit ↑ lattice realization

Logarithmic CFTs
(Logarithmic Minimal Models)



Nonlocal Degrees of Freedom = Connectivities

- **Face Operators:**

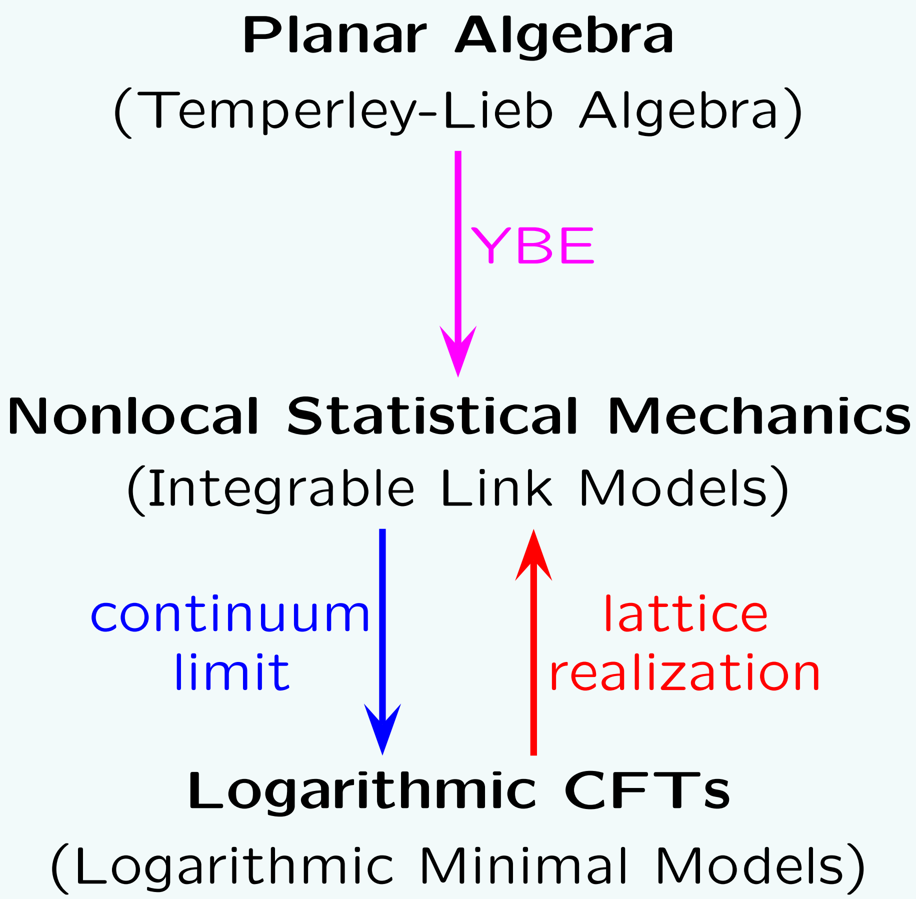
$X(u) =$  $=$  $= \rho\Big(\sin(\lambda-u)$  $+ \sin u$  $\Big)$

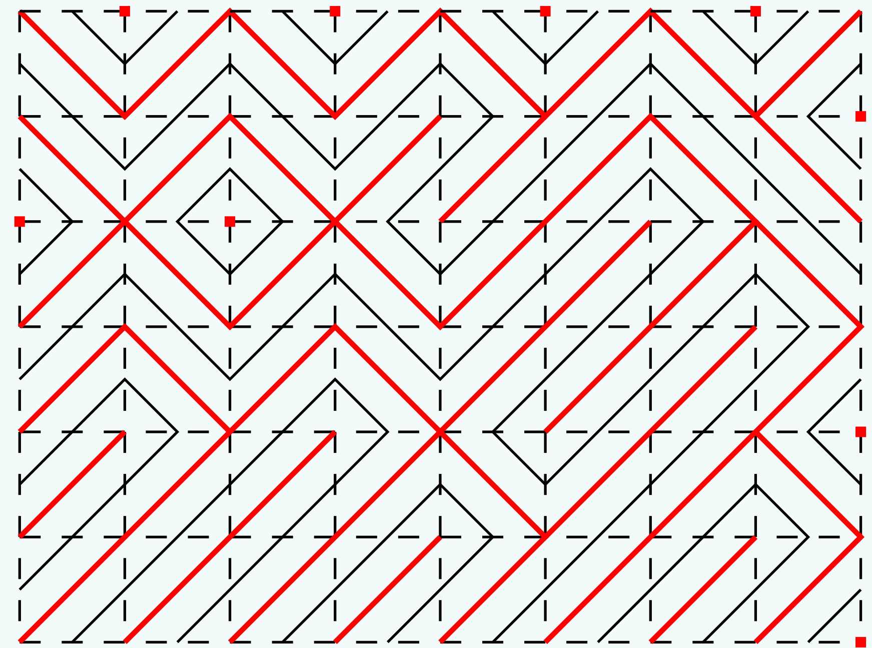
u = spectral parameter, $\quad \lambda = \dfrac{(p'-p)\pi}{p'}$ = crossing parameter, $\quad p < p'$ coprime

- **Critical Percolation:** $(p,p') = (2,3), \qquad \rho = 1$

$$\lambda = \frac{\pi}{3}, \qquad u = \frac{\lambda}{2} = \frac{\pi}{6}, \qquad \text{Probability} = \sin(\lambda-u) = \sin u = \frac{1}{2}$$

Overview of lattice $\mathcal{LM}(p,p')$

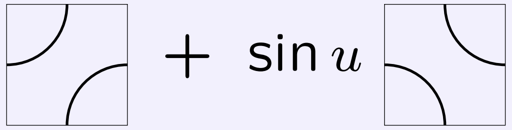




Nonlocal Degrees of Freedom $=$ Connectivities

- **Face Operators:**

$$X(u) \;=\; \boxed{u} \;=\; \boxed{\lambda - u} \;=\; \rho\Big(\sin(\lambda-u)\;\; + \;\; \sin u \;\Big)$$



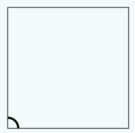
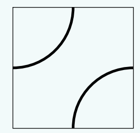
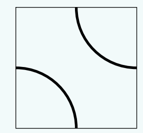
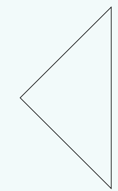
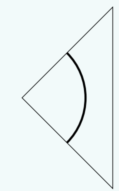
$u \;=\;$ spectral parameter, $\qquad \lambda \;=\; \dfrac{(p'-p)\pi}{p'} \;=\;$ crossing parameter, $\qquad p<p'$ coprime

- **Critical Percolation:** $\qquad (p,p')=(2,3), \qquad \rho=1$

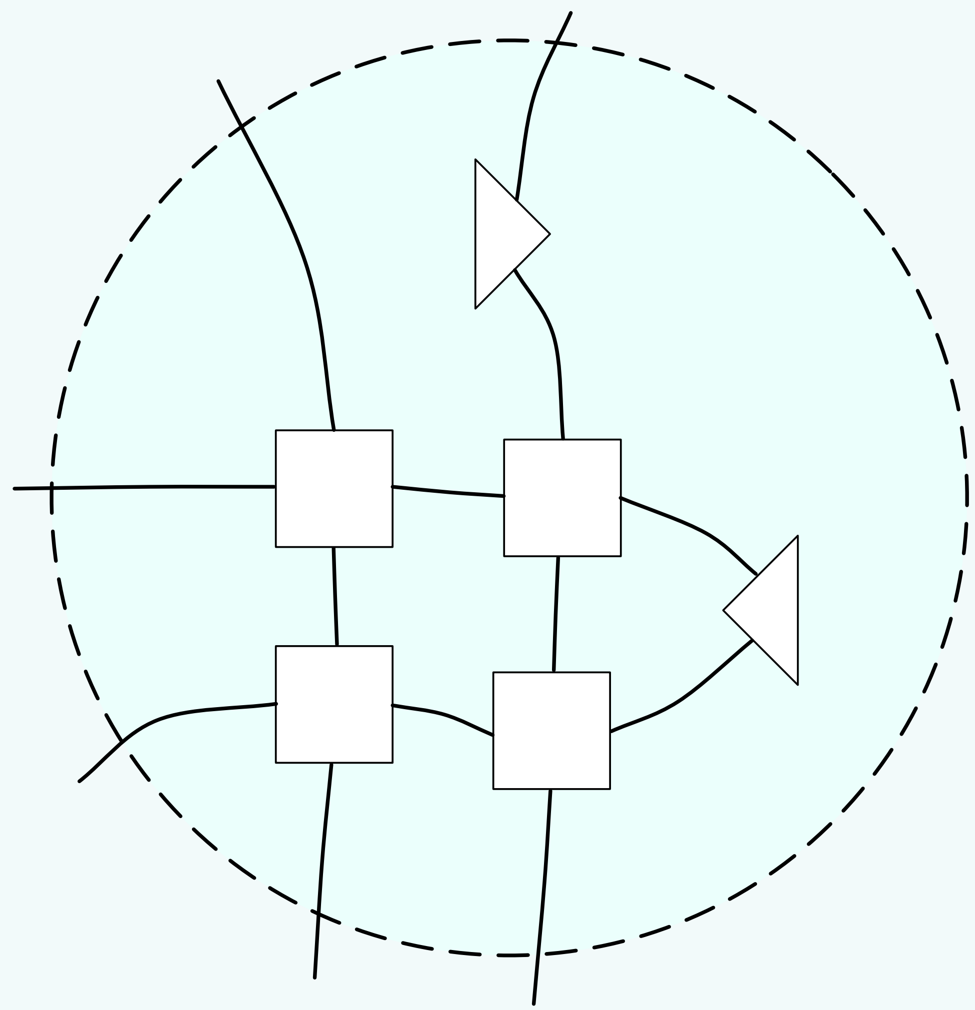
$$\lambda \;=\; \frac{\pi}{3}, \qquad u \;=\; \frac{\lambda}{2} \;=\; \frac{\pi}{6}, \qquad \text{Probability} \;=\; \sin(\lambda-u) \;=\; \sin u \;=\; \frac{1}{2}$$

Planar Temperley-Lieb Algebra (Jones 1999)

- Algebra of tangles, generated by composition of elementary 2-tangles (2-boxes or oriented monoids) and elementary 1-tangles (1-triangles)

 = linear comb. of  and ,  = 

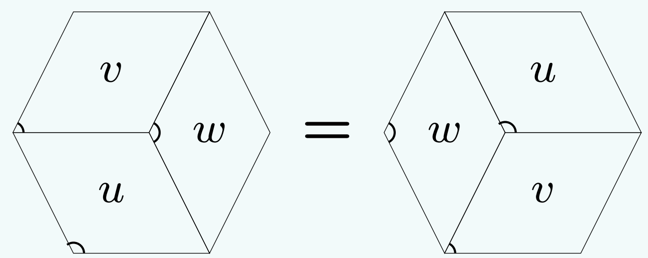
- Example: a 3-tangle: Any 3 consecutive strings can be taken as “in-states”, the other 3 are then “out-states”. As a planar operator, the 3-tangle can act in “6 different directions”.



- Two N-tangles are equal if they have the same connectivities with the same weights.

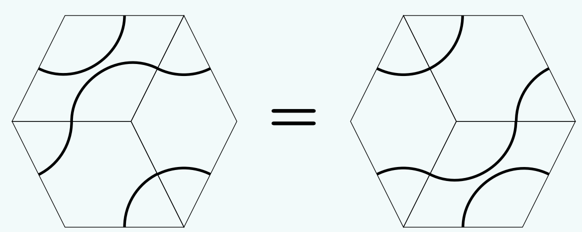
Integrability I: Yang-Baxter Equation

- The Yang-Baxter Equations (YBE) express the equality of two planar 3-tangles

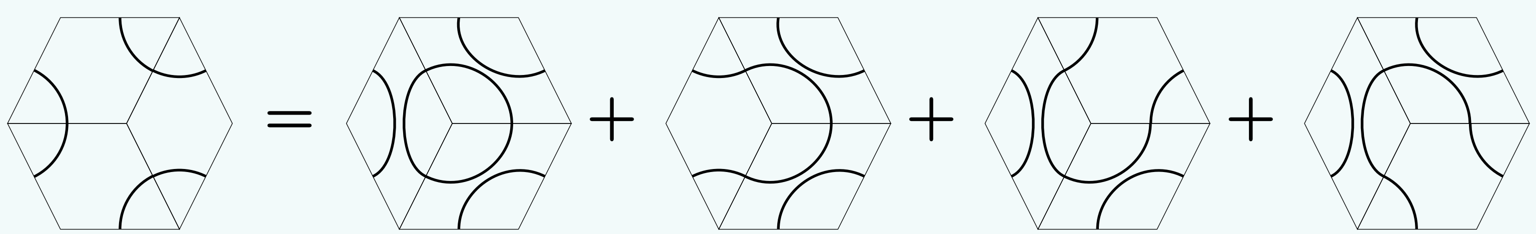


$w = v - u$

- The five possible connectivities of the external nodes give the diagrammatic equations



$\times\ 3$ (120° rotations)



$\times\ 2$ (180° rotations)

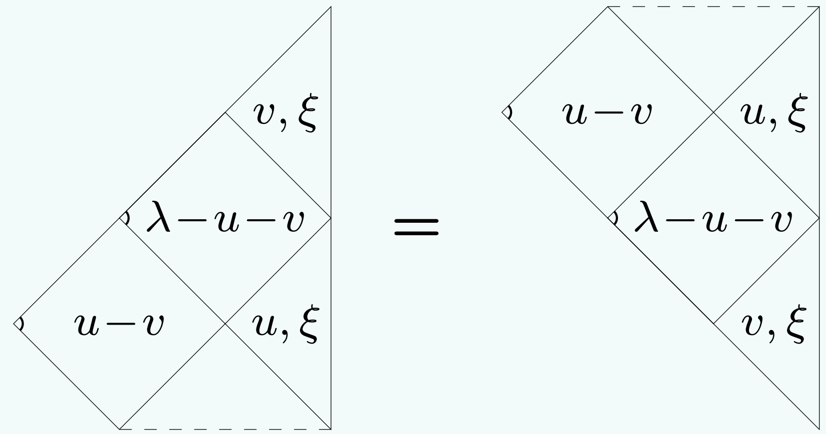
- The first equation is trivial. The second equation follows from the identity

$$
\begin{aligned}
s_1(-u)s_0(v)s_1(-w) \;=\;& \beta\, s_0(u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_1(-w) \\
&+ s_1(-u)s_1(-v)s_0(w) + s_0(u)s_0(v)s_0(w)
\end{aligned}
$$

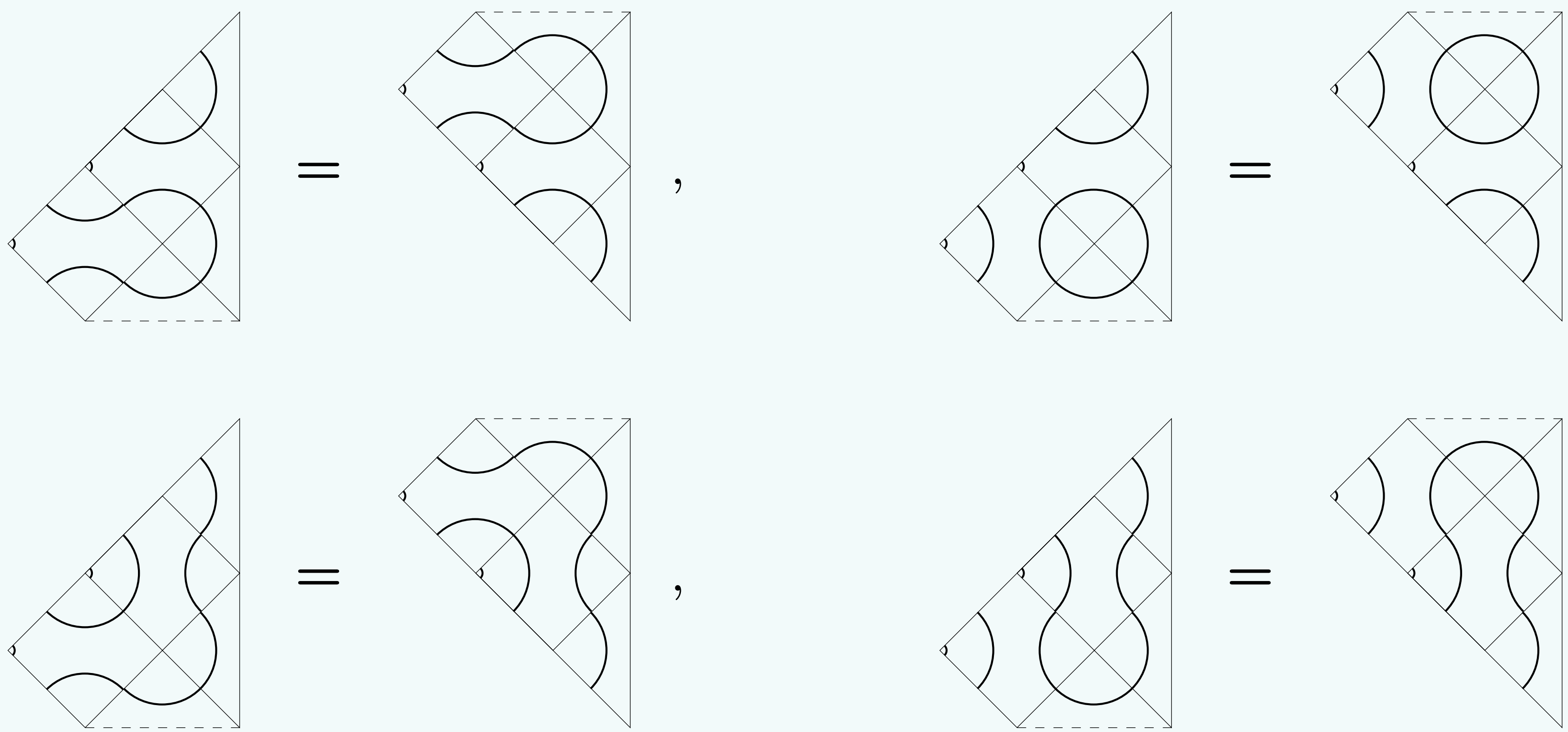
$$
s_r(u) = \frac{\sin(u + r\lambda)}{\sin\lambda}, \qquad \beta = 2\cos\lambda = \text{loop fugacity}
$$

Integrability II: Boundary Yang-Baxter Equation

- The Boundary Yang-Baxter Equation (BYBE) is the equality of boundary 2-tangles

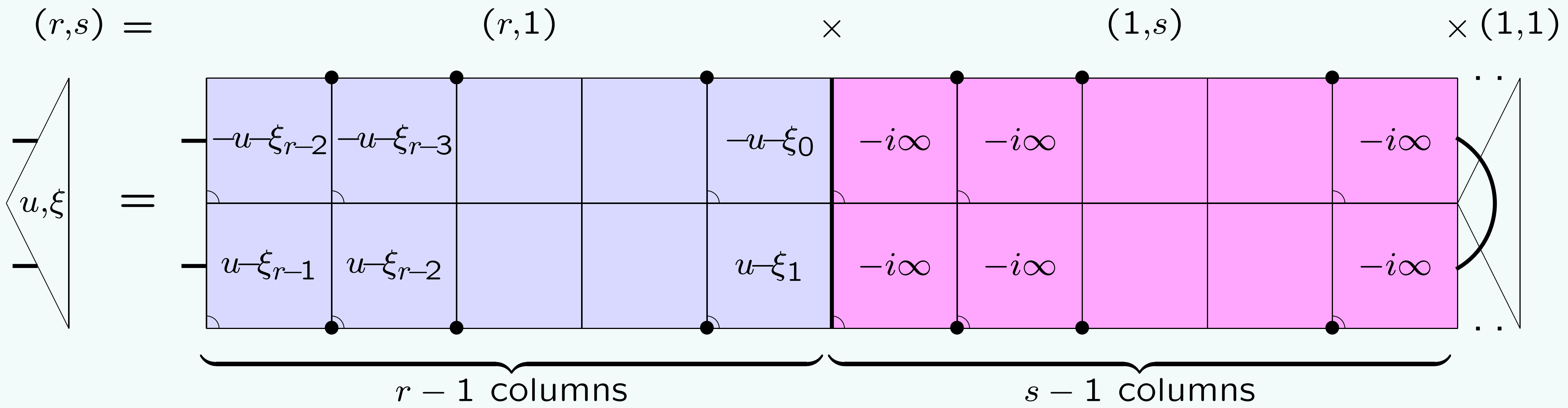


- For the elementary 1-triangle, this follows from four identities among the weights



General (r, s) Boundary Conditions

- The $(r, s) = (r, 1) \times (1, s)$ BYBE solution (1-triangle) is built as the fusion product of $(r, 1)$ and $(1, s)$ integrable seams acting on the vacuum $(1, 1)$ 1-triangle:



- The column inhomogeneities are: $\xi_k = \xi + k\lambda$

- The solid dots indicate that a fusion projector is applied along the bottom and/or top of the integrable seams. This projector projects out any configuration with closed half arches. It is built from the face operators $X(k\lambda)$ with $k \in \mathbb{Z}$.

- The $r + s - 2$ columns are considered part of the right boundary. Left boundary solutions are constructed similarly.

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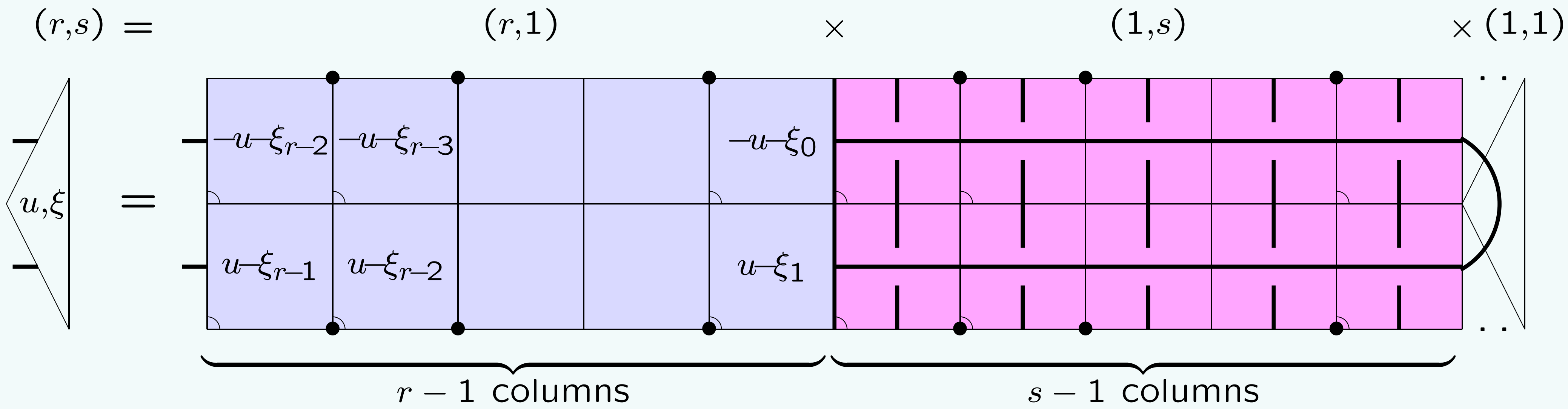

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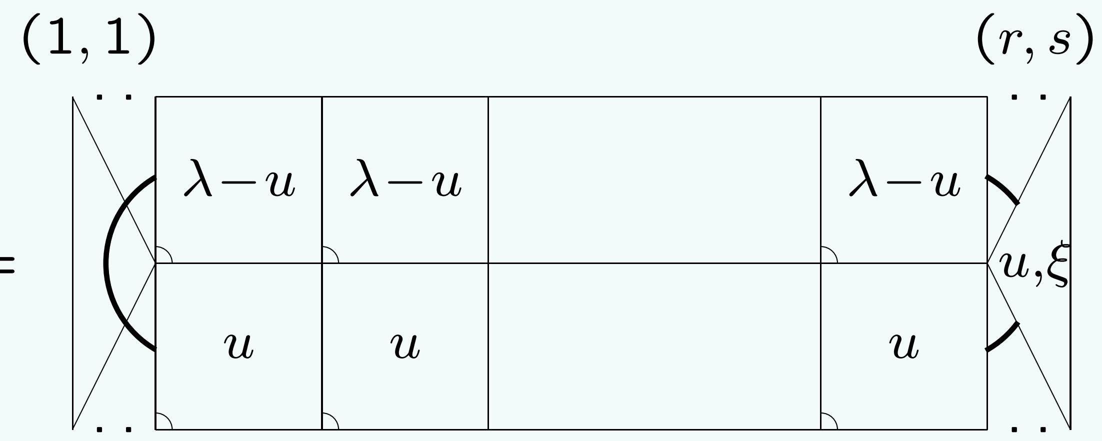
- The $s - 1$ rightmost strings pass through the boundary tangle.

- The $r + s - 2$ columns are considered part of the right boundary. Left boundary solutions are constructed similarly.

- These bounadry weights satisfy the BYBE's.

Double-Row Transfer Matrices

- Consider a strip with the vacuum $(1,1)$ boundary on the left and the (r,s) boundary on the right. The N column double-row transfer "matrix" is the N-tangle

$\boldsymbol{D}(u) \;=$ 

- Using local relations in the planar Temperley-Lieb (TL) algebra it can be shown that, for any (r,s), these commute

$$\boldsymbol{D}(u)\boldsymbol{D}(v) = \boldsymbol{D}(v)\boldsymbol{D}(u)$$

- Multiplication is vertical concatenation of diagrams, equality is the equality of N-tangles.

- Crossing symmetry can also be shown in the planar TL algebra

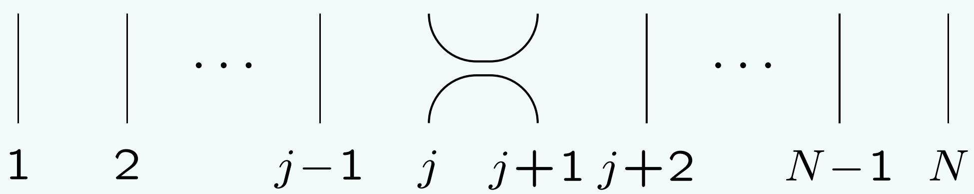
$$\boldsymbol{D}(u) = \boldsymbol{D}(\lambda - u)$$

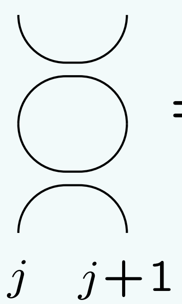
Linear Temperley-Lieb Algebra

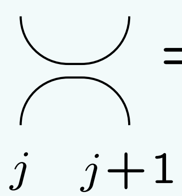
- The linear TL algebra is obtained by fixing the in- and out-states, that is a distinguished direction of transfer, in the planar TL algebra. It is generated by $e_1, \ldots, e_{N-1}$ and the identity I acting on N strings

$$\begin{cases} e_j^2 = \beta\, e_j, & \\ e_j\, e_k\, e_j = e_j, & |j-k| = 1, \qquad j,k = 1,2,\ldots,N-1; \quad \beta = 2\cos\lambda \\ e_j\, e_k = e_k\, e_j, & |j-k| > 1 \end{cases}$$

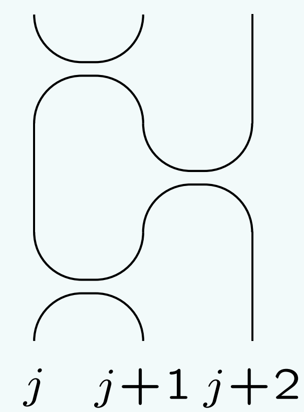
- Using the dense loop representation the TL generators e_j are represented graphically by

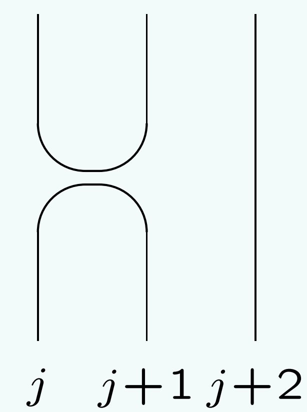
$e_j =$ 

$e_j^2 =$ 

$= \beta$ 

$= \beta\, e_j,$

$e_j e_{j+1} e_j =$ 

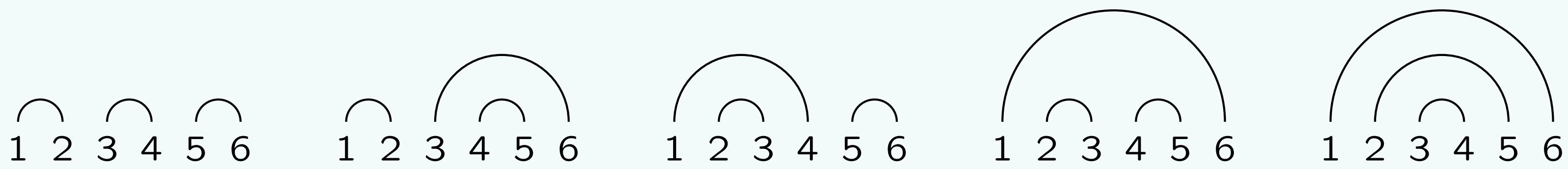
$=$ 

$= e_j$

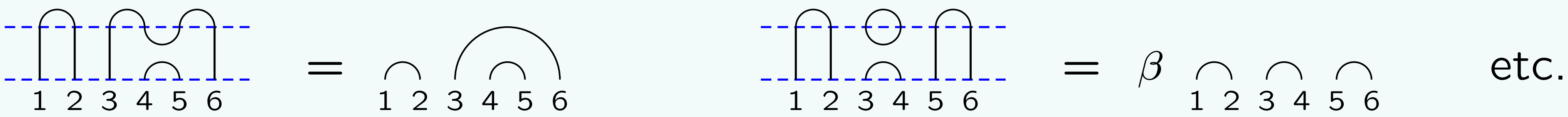
- For $\beta \neq 0$, $\beta^{-1} X_j(\lambda) = \beta^{-1} e_j$ and $\beta^{-1} X_j(-\lambda) = I - \beta^{-1} e_j$ are orthogonal projectors.

Link Diagrams

- In a "fixed time direction", the planar N-tangles act on a vector space $\mathcal{V}_N$ of *planar link diagrams*. For $N = 6$, there is a basis of 5 link diagrams:



- The first link diagram is the "vacuum state". The excited states are generated by the action of the TL generators by concatenation *from below*



- The action of the TL generators on the states is inherently nonlocal. It leads to non-symmetric matrices with entries $0, 1, \beta$ that represent the TL generators. For $N = 6$, the action of e_1 and e_2 on $\mathcal{V}_6$ is

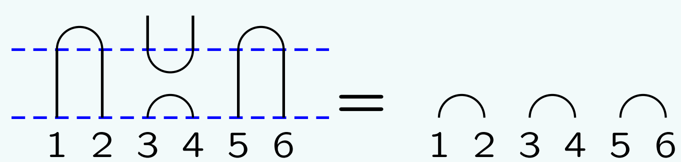
$$e_1 = \begin{pmatrix} \beta & 0 & 1 & 0 & 1 \\ 0 & \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \beta & 0 & 0 \\ 0 & 1 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Defects

- More generally, for $s > 1$, the vector space of states $\mathcal{V}_N^{(s)}$ contains $\ell = s - 1$ *defects*. For $N = 4$ and $\ell = 2$, there are 3 link diagrams

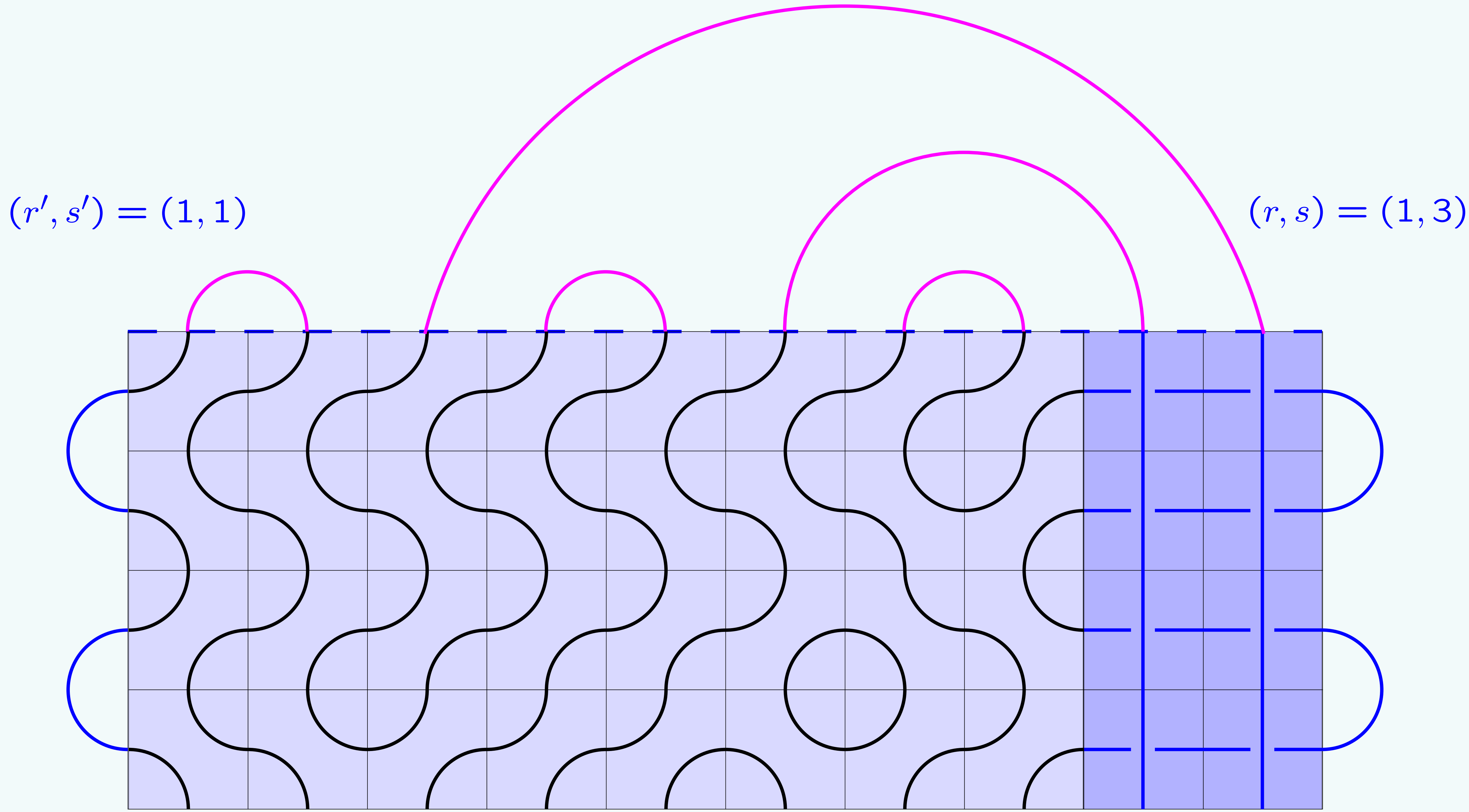
- The $s-1$ rightmost strings of an s-type boundary condition join to these defects, and so, the action of an s-type boundary condition is to close these defects on the right so that they propagate along the right boundary.

- Defects can be annihilated in pairs but not created under the action of TL



etc.

Connectivity Configuration on a Strip



Black	$=$	Connectivities
Blue	$=$	Boundary Condition
Purple	$=$	Link State

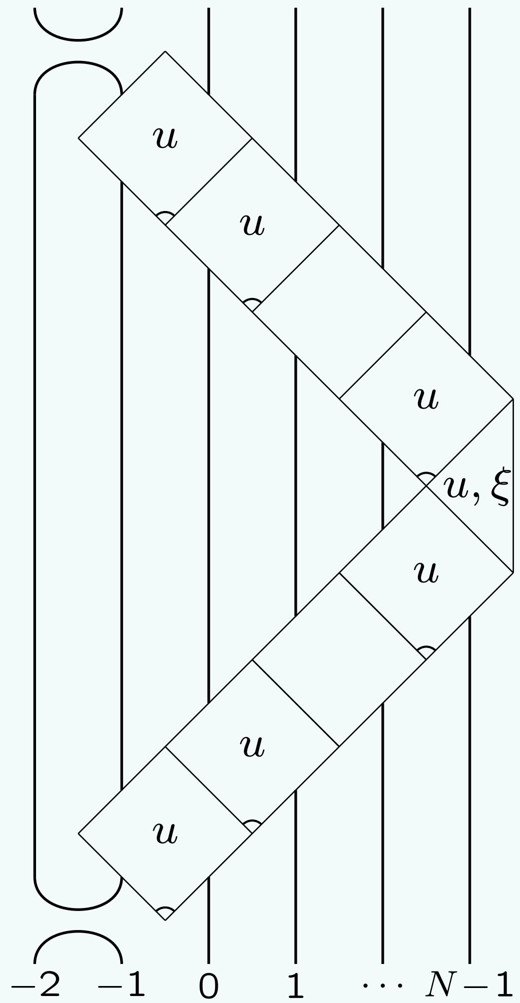
$\ell = s - 1 = 2 = \#$ of Defects

$D(u)$ in Linear TL Algebra

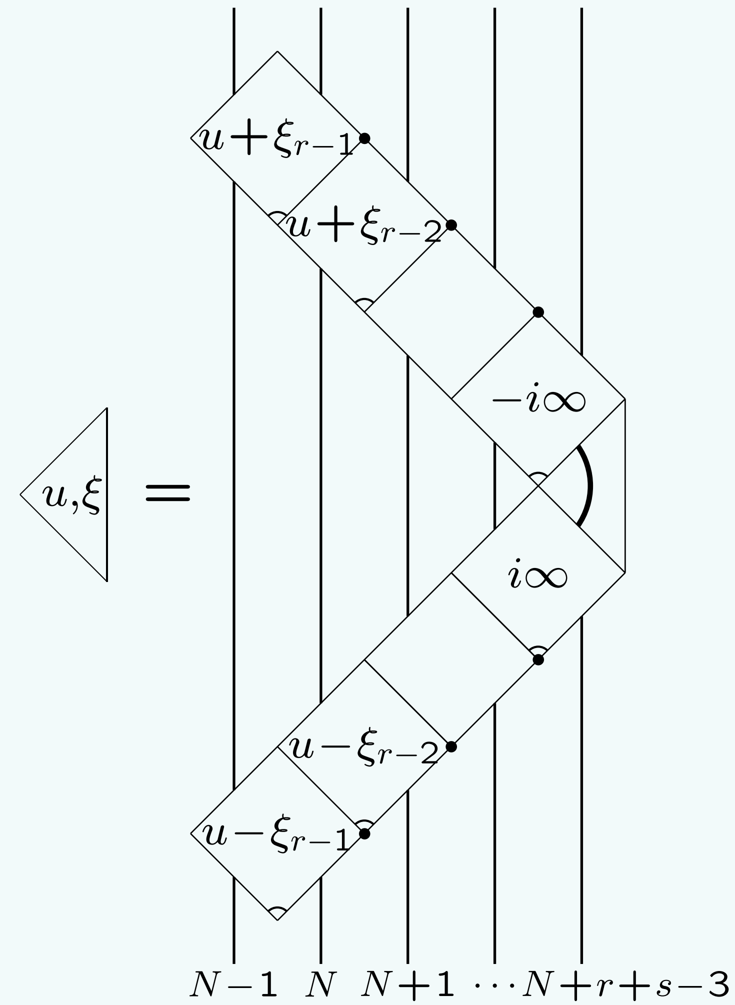
- Assuming $\beta \neq 0$, we can write $\boldsymbol{D}(u)$ in the linear TL algebra as

$$\boldsymbol{D}(u) = \beta^{-1} e_{-1} \Big(\prod_{j=0}^{N-1} X_j(u) \Big) K_N^{(r,s)}(u,\xi) \Big(\prod_{j=N-1}^{0} X_j(u) \Big) \beta^{-1} e_{-1}$$

$\boldsymbol{D}(u) =$



$K_N^{(r,s)}(u,\xi) =$



- The TL generators e_{-1} enforce closure on the left and $K_j^{(1,1)} = I$.
- $\boldsymbol{D}(u)$ acts on the vector space $\mathcal{V}_{N+r+s-2}$ from below.
- Although the matrices $\boldsymbol{D}(u)$ are not normal, it seems the "one-boundary matrices" are diagonalizable. In contrast, the "two-boundary matrices" are not in general diagonalizable.

Hamiltonian Limits

- For small u

$$\boldsymbol{D}(u) = \boldsymbol{D}(0)\, e^{2u\mathcal{H}/\sin\lambda + O(u^2)}, \qquad \boldsymbol{D}(0) = e_{-1}$$

- The Hamiltonian $\mathcal{H}$ is defined by the logarithmic derivative

$$\boxed{\mathcal{H} = \tfrac{1}{2}\sin\lambda\,\frac{d}{du}\log\boldsymbol{D}(u)\Big|_{u=0}}$$

- Explicit evaluation yields the family of integrable Hamiltonians

$$\boxed{\mathcal{H}^{(r,s)} = \sum_{j=1}^{N-1} e_j - \frac{s_{r-1}(0)}{s_0(\xi)s_r(\xi)}\,P^r_{N+1}e_N P^r_{N+1}}$$

where P^r_j is a fusion projector. These act on the vector space $\mathcal{V}^{(s)}_{N+r-1}$ of link diagrams with $s-1$ defects.

- The Hamiltonian $\mathcal{H}^{(1,1)}$ is the usual $U_q(s\ell(2))$ invariant Hamiltonian.

- Finite size corrections $\frac{N}{\pi v_s}\Big(\mathcal{H}^{(r,s)} - (N f_{bulk} + f_{bdy})I\Big) \;\to\; L_0 - \frac{c}{24}$, $v_s = \frac{\pi\sin\lambda}{\lambda}$, $f_{bulk}(\lambda)$, $f^{(r)}_{bdy}(\lambda,\xi)$ known functions.

Continuum limit : Logarithmic Minimal CFTs

- It is asserted that the continuum scaling limit of these lattice models yields logarithmic minimal CFTs $\mathcal{LM}(m, m+1)$.

- Central charges and conformal weights:

$$c = 1 - \frac{6(p-p')^2}{pp'}, \qquad p < p', \qquad p, p' = 1, 2, 3, \ldots \text{ coprime}$$

$$h_{r,s} = \frac{(p'r - ps)^2 - (p-p')^2}{4pp'}, \qquad r, s = 1, 2, 3, \ldots$$

- Consider the characters

$$\chi_{r,s}(q) = q^{-c/24} \frac{q^{h_{r,s}} - q^{h_{r,-s}}}{\prod_{n=1}^{\infty}(1-q^n)} = q^{-c/24} \frac{q^{h_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

Although the representations (r, s) are not irreducible, we find that these are the appropriate characters to describe the conformal spectra of the logarithmic minimal models on a strip, with the boundary conditions $(1, 1)$ on the left and (r, s) on the right.

Some Numerical Partition Functions

- Bethe Ansatz ($r = 1$) and direct numerical extrapolation from $N \le 16$:

- **Critical Percolation**

$$\boldsymbol{D}(\tfrac{\lambda}{2})\begin{cases} (r,s)=(1,1): & Z_{(1,1)}(q)=q^{-c/24}(1+q^2+q^3+2q^4+2q^5+\cdots), & c=.0000000(1) \\ (r,s)=(1,2): & Z_{(1,2)}(q)=q^{-c/24+h}(1+q+q^2+2q^3+3q^4+\cdots), & h=.0000000(1) \\ (r,s)=(2,1): & Z_{(2,1)}(q)=q^{-c/24+h}(1+q+q^2+2q^3+\cdots), & h=.624(2) \quad \tfrac{5}{8} \end{cases}$$

$$\mathcal{H}\begin{cases} (r,s)=(1,2): & Z_{(1,2)}(q)=q^{-c/24+h}(1+q+q^2+2q^3+3q^4+\cdots), & h=.00000000(1) \\ (r,s)=(1,3): & Z_{(1,3)}(q)=q^{-c/24+h}(1+q+2q^2+2q^3+\cdots), & h=.33333333(1) \quad \tfrac{1}{3} \\ (r,s)=(1,4): & Z_{(1,4)}(q)=q^{-c/24+h}(1+q+2q^2+\cdots), & h=1.000000(3) \quad 1 \end{cases}$$

- **Logarithmic Ising Model**

$$\boldsymbol{D}(\tfrac{\lambda}{2})\begin{cases} (r,s)=(1,1): & Z_{(1,1)}(q)=q^{-c/24}(1+q^2+q^3+2q^4+2q^5+\cdots), & c=.49999999(3) \quad \tfrac{1}{2} \\ (r,s)=(1,2): & Z_{(1,2)}(q)=q^{-c/24+h}(1+q+q^2+2q^3+3q^4+\cdots), & h=.062499999(2) \quad \tfrac{1}{16} \\ (r,s)=(1,3): & Z_{(1,3)}(q)=q^{-c/24+h}(1+q+2q^2+2q^3+\cdots), & h=.49999999(7) \quad \tfrac{1}{2} \\ (r,s)=(1,4): & Z_{(1,4)}(q)=q^{-c/24+h}(1+q+2q^2+\cdots), & h=1.3125(1) \quad \tfrac{21}{16} \\ (r,s)=(2,1): & Z_{(2,1)}(q)=q^{-c/24+h}(1+q+q^2+2q^3+\cdots), & h=.4999(2) \quad \tfrac{1}{2} \end{cases}$$

$$\mathcal{H}\begin{cases} (r,s)=(1,2): & Z_{(1,2)}(q)=q^{-c/24+h}(1+q+q^2+2q^3+3q^4+\cdots), & h=.062499999(2)) \quad \tfrac{1}{16} \\ (r,s)=(1,3): & Z_{(1,3)}(q)=q^{-c/24+h}(1+q+2q^2+2q^3+\cdots), & h=.5000000(1) \quad \tfrac{1}{2} \end{cases}$$

Indecomposable Representations

- The fusion product $(r',s')\times(r,s)$ of certain non-trivial boundaries on the left and right of the strip leads to indecomposable representations of Virasoro.

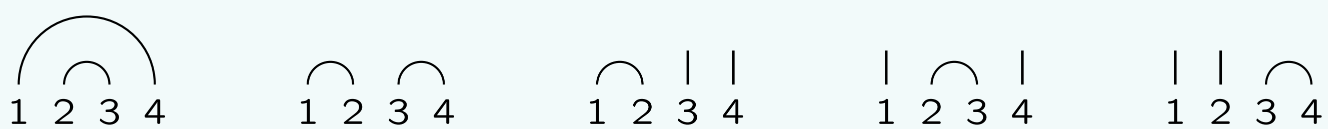
Example: For *Critical Dense Polymers*, the $(1,2)_{h=-1/8}\times(1,2)$ fusion yields an indecomposable representation. Specifically, for $N=4$, the finitized partition function is

$$Z^{(N)}_{(1,2)|(1,2)}(q)=\underbrace{\chi^{(N)}_{(1,1)}(q)}_{0\text{ defects}}+\underbrace{\chi^{(N)}_{(1,3)}(q)}_{2\text{ defects}}=q^{-c/24}[(1+q^2)+(1+q+q^2)]=q^{-c/24}(2+q+2q^2)$$

- The Hamiltonian

$$\mathcal{H}\;=\left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0\\ 2 & 0 & 1 & 0 & 1\\ \hline 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 \end{array}\right)+\sqrt{2}\,I$$

acts on the five states with $\ell=0$ or $\ell=2$ defects



- The Jordan canonical form for $\mathcal{H}$ has rank 2 Jordan cells

$$\mathcal{H}\sim\left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0\\ 0 & \sqrt{8} & 0 & 0 & 1\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{2} & 0\\ 0 & 0 & 0 & 0 & \sqrt{8} \end{array}\right)\sim\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 0 & \sqrt{8} & 1\\ 0 & 0 & 0 & 0 & \sqrt{8} \end{pmatrix}\sim\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}=L_0^{(4)}$$

- The eigenvalues of $\mathcal{H}$ approach the integer energies indicated in $L_0^{(4)}$ as $N\to\infty$.

A Conjectured Jordan form for L_0

- For $N \leq 10$, every eigenvalue of the $(1,1)$ block has an exactly equal eigenvalue in the $(1,3)$ block. Together they form a rank 2 Jordan cell.

- We conjecture the exact form in the limit $N \to \infty$. Symbolically,

$$L_0 = \begin{pmatrix} \text{Diag}(0,2,3,4,4,\ldots) & \boldsymbol{J} \\ \mathbf{0} & \text{Diag}(0,1,2,2,3,3,4,4,4,4,\ldots) \end{pmatrix}$$

Here it is understood that each term q^E occurs on the matrix diagonal and a rank 2 Jordan cell is formed between every term in $\chi_{(1,1)}(q)$ and its corresponding partner in $\chi_{(1,3)}(q)$. Symbolically again,

$$(1,2) \times_f (1,2) = (1,1) \text{``} \oplus_J \text{''} (1,3) \qquad \text{[Gaberdiel \& Kausch]}$$

- This generalizes to some other products $(1,s_1) \times_f (1,s_2)$, for higher values of m, provided $\exists s', s'' \;: h_{1s'} - h_{1s''} \in \mathbb{Z}$

- The set of all representations (including indecomposables) is supposed to be *quasi-rational*: this means that there are a countable number of representations but the fusion of any two involves only a finite number of representations. How to test that on the lattice?

Concluding Remarks

- A family of integrable link models on the lattice has been proposed. These are inherently nonlocal and it is asserted that, in the continuum scaling limit, these are associated with logarithmic CFTs. Because of obvious parallels, we call these *logarithmic minimal models*.

- The lattice realization of minimal models offer a laboratory for studying their properties. Yang-Baxter integrability brings new tools to bear on the problem, such as functional equations, Bethe ansatz, T-systems, and Thermodynamic Bethe Ansatz (TBA).

- Further directions for study : effects of non-contractible loops on the cylinder and torus, systematic study of the fusion algebras, ...