Vingt Ans Après

J.–B. Zuber

This will **not** be a portrait of John as a Musketeer. . .

This will be a review of an outstanding 5-year period 1984-1988$\pm\epsilon$ in John's career

As a tribute of admiration and gratitude to

John,

who has influenced so deeply so many of us. . .

A selection from SPIRES [Google Scholar]

42) Conformal Invariance And Universality In Finite Size Scaling.
John L. Cardy, J. Phys. A17 (1984) L385-L387. [261]

41) Conformal Invariance and Surface Critical Behavior.
John L. Cardy, Nucl.Phys.B240:514-532,1984 [459]

45) Conformal Invariance And The Yang-Lee Edge Singularity In Two-Dimensions.
John L. Cardy, Phys.Rev.Lett.54:1354-1356,1985. [126]

46) Conformal Invariance, the Central Charge, and Universal Finite Size Amplitudes at Criticality.
H.W.J. Bloete, John L. Cardy, M.P. Nightingale,
Phys.Rev.Lett.56:742-745,1986. [644]

47) Operator Content of Two-Dimensional Conformally Invariant Theories.
John L. Cardy, Nucl.Phys.B270:186-204,1986. [1011]

48) Effect of Boundary Conditions on the Operator Content of Two-Dimensional Conformally Invariant Theories.
John L. Cardy, Nucl.Phys.B275:200-218,1986. [> 200?]

63) Boundary Conditions, Fusion Rules and the Verlinde Formula.
John L. Cardy, Nucl.Phys.B324:581,1989. [689]

71) Bulk and boundary operators in conformal field theory.
John L. Cardy and David C. Lewellen, Phys. Lett.B259:274-278,1991. [183]

+ many other papers on other aspects of CFT's, perturbation away from criticality, random systems, universal quantities, finite size corrections, $d > 2$, etc.

+ many other review articles and courses, including **Les Houches 1988**

J. Phys. A: Math. Gen. **17** (1984) L385–L387. Printed in Great Britain

LETTER TO THE EDITOR

Conformal invariance and universality in finite-size scaling

John L Cardy
Department of Physics, University of California, Santa Barbara, California 93106, USA

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Abstract. The universal relation between critical exponents and the amplitude of the correlation length divergence as a function of finite size at the critical point of two-dimensional systems is shown to be a consequence of conformal invariance. Both periodic and free boundary conditions are considered.

In studies of the finite-size scaling behaviour of two-dimensional systems on infinitely long strips of finite width, a remarkable universality has been observed. The theory of finite-size scaling (Barber 1983) predicts that the inverse correlation length $\varkappa_n$ (measured in lattice spacings), for a strip of width n lattice spacings, should behave, at the critical point of the infinite system, as

$$\varkappa_n \sim A/n \tag{1}$$

where the amplitude A is universal in the usual sense (Privman and Fisher 1984). However, exact and numerical calculations (Luck 1982, Derrida and de Seze 1982, Nightingale and Blöte 1983, Privman and Fisher 1984) also suggest that the ratio A/x, where x is the scaling dimension of the operator concerned, is equal to 2π for several different operators in a wide variety of isotropic two-dimensional models.

In this letter, we point out that this result is a simple consequence of conformal covariance of the correlation functions, which is believed to hold in the continuum limit at the critical point (Polyakov 1970, 1974). In two dimensions, conformal invariance implies that the correlation functions of a local scalar operator $\varphi(z)$ satisfy

$$\langle\varphi(z_1)\varphi(z_2)\rangle = |w'(z_1)|^x|w'(z_2)|^x\langle\varphi(w(z_1))\varphi(w(z_2))\rangle \tag{2}$$

where $z \to w(z)$ is an arbitrary conformal transformation, and x is the scaling dimension of φ. Now consider the particular transformation

$$w = \ln z \tag{3}$$

which maps the whole z-plane into the strip $|\mathrm{Im}\, w| \leqslant \pi$, with periodic boundary conditions. Using the result that in the infinite plane

$$\langle\varphi(z_1)\varphi(z_2)\rangle \sim |z_1 - z_2|^{-2x} \tag{4}$$

and writing $z_j = \exp(y_j + \mathrm{i}\theta_j)$, ($|\theta_j| \leqslant \pi$), it follows from (2) that

$$\langle\varphi(y_1+\mathrm{i}\theta_1)\varphi(y_2+\mathrm{i}\theta_2)\rangle_S \sim \exp\{-x\ln[\exp(y_1-y_2)+\exp(y_2-y_1)-2\cos(\theta_1-\theta_2)]\} \tag{5}$$

where the correlation function on the left-hand side is evaluated in the strip geometry.

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Conformal Invariance And Universality In Finite Size Scaling.

Conformal invariance useful not only for local infinitesimal transformations, but also mappings from one domain to another.

On a cylinder (aka strip of width L with pbc), $w = u + iv$, for large $|u_1 - u_2|$

$$\langle\varphi(w_1)\varphi(w_2)\rangle_{\text{cyl}} \sim \exp -2\pi x \frac{|u_1 - u_2|}{L}$$

where $x =$ conformal dimension of scaling field φ.

This results simply from mapping $z = e^{2\pi w/L}$ of cylinder to plane and $\langle\varphi(z_1)\varphi(z_2)\rangle = |z_1 - z_2|^{-2x}$

Indeed for $\varphi(z)$ a primary (spinless) field,

$$\begin{aligned}
\langle\varphi(w_1)\varphi(w_2)\rangle_{\text{cyl}} &= \left|\frac{dz_1}{dw_1}\right|^x \left|\frac{dz_2}{dw_2}\right|^x \langle\varphi(z_1)\varphi(z_2)\rangle \\
&= \left(\frac{2\pi}{L}\right)^{2x} \frac{|z_1 z_2|^x}{|z_1 - z_2|^{2x}} \\
&= \left(\frac{2\pi}{L}\right)^{2x} |2\sinh(\pi(w_1 - w_2)/L)|^{-2x} \\
&\sim \exp\left(-\frac{2\pi x}{L}|u_1 - u_2|\right) \qquad \text{QED}
\end{aligned}$$

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CONFORMAL INVARIANCE AND SURFACE CRITICAL BEHAVIOR

John L CARDY*

Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA

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Conformal invariance constrains the form of correlation functions near a free surface In two dimensions, for a wide class of models, it completely determines the correlation functions at the critical point, and yields the exact values of the surface critical exponents They are related to the bulk exponents in a non-trivial way For the Q-state Potts model ($0 \leqslant Q \leqslant 4$) we find $\eta_{\parallel} = 2/(3\nu - 1)$, and for the O($N$) model ($-2 \leqslant N \leqslant 2$), $\eta_{\parallel} = (2\nu - 1)/(4\nu - 1)$

1. Introduction

It has been recognized for some time that a statistical system at a critical point is not only scale invariant, but also conformally invariant [1,2]. This additional symmetry imposes constraints on the form of the bulk correlation functions In two dimensions, conformal invariance is much stronger. More recently, Belavin, Polyakov and Zamolodchikov [3], Dotsenko [4], and Friedan, Qiu and Shenker [5] have shown that conformal invariance completely determines the critical exponents and bulk correlation functions of a wide class of two-dimensional theories at criticality

In this paper we apply these ideas to surface critical behavior [6]. For definiteness, consider a semi-infinite d-dimensional system bounded by a $(d-1)$-dimensional plane surface. We are interested in the ordinary and special transitions, at which the bulk and the surface order simultaneously in the absence of symmetry-breaking fields. It is known from ε-expansion methods, as well as other exact and approximate calculations, that correlation functions near the surface decay with critical exponents different from their bulk values, and that their functional form is more complicated than in the bulk

The results of the present paper fall into two classes. For general dimension d, we show that conformal invariance constrains the form of the general two-point functions at criticality, to within an unknown function of a single scaling variable The functional forms of the correlations between surface and bulk quantities are, however, completely determined by conformal invariance.

* Permanent address Department of Physics, University of California, Santa Barbara CA 93106, USA

Conformal Invariance and Surface Critical Behavior

Surface critical behaviour in arbitrary d : constraints from conformal invariance.

General form of the two-point function in semi-half space.

In $d = 2$, general setting of conformal invariance in the presence of a boundary, the foundation of future "BCFT", (with later applications to many situations in stat. mech, cond. mat. and string theory, branes . . .).

Consider upper half-plane. Conformal transf. must respect that geometry. Instead of two independent $T(z)$ and $\bar{T}(\bar{z})$ and two commuting copies of Virasoro algebra, $T(z)$ and $\bar{T}(\bar{z})$ are analytic continuations of one another, and there is only one copy of Virasoro.

Two-point function in half-plane : same equation as four-point function in bulk.

Numerous applications to various critical models, Ising, Potts, O(n), explicit two-point function, surface critical exponents, . . .

Non unitary cft's may also be interesting !

VOLUME 54, NUMBER 13 PHYSICAL REVIEW LETTERS 1 APRIL 1985

Conformal Invariance and the Yang-Lee Edge Singularity in Two Dimensions

John L. Cardy
Department of Physics, University of California, Santa Barbara, California 93106
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It is shown that very general features of the critical theory of the Yang-Lee edge singularity in two dimensions completely determine the way in which the theory realizes conformal invariance. This leads to the value $\sigma = -\frac{1}{6}$ for the edge exponent, and makes possible the calculation of the correlation functions.

PACS numbers: 05.50.+q, 05.70.Jk

Recently there has been considerable progress in exploiting the principle of conformal invariance of two-dimensional systems at the critical point to obtain information about critical exponents and correlation functions.[1,2] Unfortunately this principle by itself is not sufficiently restrictive, and other criteria, such as full unitarity of the theory,[2] have been invoked. While such criteria are necessary for a sensible quantum field theory, many interesting critical points do not correspond to unitary theories. Another difficulty of this approach is in the identification of a given realization of conformal symmetry with a particular universality class. So far, this has been accomplished only by matching the predicted exponents, or the conformal anomaly c, with those values already known by other means.

The Yang-Lee edge singularity[3,4] is perhaps the simplest nonunitary critical point. In addition, as will be discussed below, it also corresponds to the simplest universality class. It will turn out that these properties are sufficient to determine a simple way in which conformal invariance can be realized in this model in two dimensions. This determines the critical exponents and the correlation functions.

The Yang-Lee edge singularity[3,4] occurs in an Ising model above its critical temperature in a nonzero, purely imaginary magnetic field ih. For h larger than some critical value $h_c(T)$ the partition function acquires zeros, which become dense on the line $\mathrm{Re}h > h_c$ in the thermodynamic limit. The density of these zeros behaves near h_c like[5] $(h-h_c)^\sigma$. Fisher[6] showed how the point $h=h_c$ can be regarded as a conventional critical point. In high dimensions it corresponds to the infrared behavior of the field theory of a single scalar field $\phi(\mathbf{r})$ with an action

$$A = \int d^dr\,[\tfrac{1}{2}(\nabla\phi)^2 + i(h-h_c)\phi + \tfrac{1}{3}ig\phi^3]. \qquad (1)$$

The imaginary coupling makes the theory nonunitary. The critical point is where the renormalized coefficients of ϕ and ϕ^2 vanish. In $6-\epsilon$ dimensions there are apparently two relevant fields, coupling to ϕ and ϕ^2. However, correlations of ϕ^2 are related to those of ϕ by the equation of motion, so in fact ϕ^2 is a redundant operator.[7] The two-point function $\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle$ behaves like $|\mathbf{r}_1-\mathbf{r}_2|^{-2x}$ at the critical point, where $2x = d-2+\eta$ is related to σ by Fisher's relation[6] $\sigma = (d-2+\eta)/(d+2-\eta)$. The simplicity of this universality class lies in its lack of any internal symmetry, and in the existence of only one independent (relevant) exponent.

In order to characterize the theory in two dimensions, the following properties [valid to all orders in the $(6-d)$ expansion[8]] will be assumed: (a) there is only one[9] relevant operator ϕ; (b) the three-point function $\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\phi(\mathbf{r}_3)\rangle$ is nonzero.

Belavin, Polyakov, and Zamolodchikov[1] have shown that there is a large class of field theories which realize conformal symmetry in a simple way. The allowed scaling dimensions of scalar operators in these theories are given by the Kac formula[10]

$$x_{p,q} = \tfrac{1}{2}[(p\alpha_+ + q\alpha_-)^2 - (\alpha_+ + \alpha_-)^2], \qquad (2)$$

where p,q are positive integers, $\alpha_\pm = \alpha_0 \pm (1+\alpha_0^2)^{1/2}$, and α_0 is related to the conformal anomaly c of the theory by $c = 1-24\alpha_0^2$. In such theories, the correlation functions satisfy linear differential equations, and there are restrictions on which three-point functions may be nonzero: $\langle\phi_{p_1,q_1}\phi_{p_2,q_2}\phi_{p_3,q_3}\rangle$ is zero unless (i) $p_1+p_2+p_3=1$ (mod2), (ii) (p_1-1), (p_2-1), and (p_3-1) satisfy the triangle inequalities $(p_1-1)+(p_2-1) \leq (p_3-1)$, etc. Similar conditions must be satisfied by the q_i. For theories where α_+/α_- is a rational number, these conditions imply that there are only a finite number of basic operators in the theory. Such cases appear to be connected to integrable models in a way which is not yet understood.[11]

Let us assume that the critical theory corresponding to the Yang-Lee edge in two dimensions is in the class considered by Belavin, Polyakov, and Zamolodchikov, and use conditions (a) and (b) as constraints on the possible realizations. Since ϕ is to be relevant, it must have a scaling dimension $x<2$. This restricts it to lie in the strip

$$\alpha_- - \alpha_+ < p\alpha_+ + q\alpha_- < \alpha_+ - \alpha_-. \qquad (3)$$

The conditions (i) and (ii) imply that the correlation functions $\langle\phi_{p,q}\phi_{p,q}\phi_{p',q'}\rangle$ may be nonzero if $p'=p\pm1$

Lee-Yang cft has $c = -22/5$,

it is the "(2,5) minimal model"

Conformal Invariance, the Central Charge, and Universal Finite-Size Amplitudes at Criticality

H. W. J. Blöte
Laboratorium voor Technische Natuurkunde, 2600 GA Delft, The Netherlands

John L. Cardy
Department of Physics, University of California, Santa Barbara, California 93106

and

M. P. Nightingale
Department of Physics, University of Rhode Island, Kingston, Rhode Island 02881

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We show that for conformally invariant two-dimensional systems, the amplitude of the finite-size corrections to the free energy of an infinitely long strip of width L at criticality is linearly related to the conformal anomaly number c, for various boundary conditions. The result is confirmed by renormalization-group arguments and numerical calculations. It is also related to the magnitude of the Casimir effect in an interacting one-dimensional field theory, and to the low-temperature specific heat in quantum chains.

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The principle of conformal invariance at a critical point has been shown to be remarkably powerful, especially in two dimensions.[1,2] Universality classes appear to be characterized by a single dimensionless number c, the conformal anomaly or the value of the central charge of the Virasoro algebra.[3] It was shown by Friedan, Qiu, and Shenker[2] that unitarity constrains those values of c less than unity to be quantized. For such theories, the critical exponents are given by the Kac formula,[4] and the correlation functions are determined.[1,5] For various models, c has been determined indirectly by use of exact information on exponents and correlation functions obtained by other means.[1,2,5] In this Letter we give a simple means of determining c.

The free energy (measured in units of k_BT) per unit length of an infinitely long strip of width L at criticality has the finite-size scaling form $F=fL+f^\times+\Delta/L+\dots$, where f is the bulk free energy per unit area, and $\frac{1}{2}f^\times$ is the surface free energy, which vanishes in the case of periodic boundary conditions. It has been argued, from the assumption that L^{-1} is a scaling field which does not require the introduction of a metric factor,[6,7] that Δ is universal. We find that

$$\Delta=\begin{cases}-\pi c/6, & \text{periodic boundary conditions,} \quad (1)\\ -\pi c/24, & \text{free or fixed boundary conditions,} \quad (2)\end{cases}$$

where, in the last case, the order parameter is fixed to the *same* value on either side of the strip.

These results have several other interesting physical interpretations. Since F corresponds to the ground-state energy of a (1+1)-dimensional quantum field theory in a finite volume, Eq. (2) also gives the magnitude of the Casimir effect[8] in such a theory. The partition function of a classical system of finite width with periodic boundary conditions may also be interpreted as the Feynman path integral for an infinitely long quantum chain at finite temperature $T\propto L^{-1}$. In that case Eq. (1) gives the leading $T\rightarrow 0$ correction to the free energy, from which may be deduced the specific heat C. In fact the conformal result applies only if the two-dimensional classical system is rotationally invariant at large distances. This is equivalent to the requirement that the dispersion relation for gapless excitations of the quantum chain is of the form $\omega\sim vk$ with $v=1$. The case $v\neq 1$ can be accommodated by a suitable rescaling of time versus length for the quantum chain. The result is $C\sim \pi c k_B^2 T/3\hbar v$. This is confirmed by exact results for the spin-$\frac{1}{2}$ XXZ chain[9] ($c=1$) and for the anisotropic spin-$\frac{1}{2}$ XY model in a critical transverse field[10] ($c=\frac{1}{2}$). In three dimensions, the analog of Δ is the interaction energy (in units of k_BT) per unit area of two plates immersed in a critical system.[11] Universality in this case was verified by Monte Carlo techniques.[12] The same constant also plays a role in determining the thickness of gravity-thinned, critical wetting layers.[13] Two-dimensional analogs of these systems, which would allow an experimental determination of c, are conceivable.

A system at a critical point is governed by a reduced fixed-point Hamiltonian[14] $\mathcal{H}^*$. Under a coarse graining in which lengths are rescaled uniformly, the form of the Hamiltonian is invariant. For short-range interactions, the Hamiltonian remains at the fixed point also under conformal transformations, which corre-

Conformal Invariance, the Central Charge, and Universal Finite Size Amplitudes at Criticality.

The interpretation of the central charge as a Casimir effect, i.e. as a finite size correction to the (free) energy.

(Simultaneously, same observation by I. Affleck)

Fundamental for stat. phys. interpretation of central charge c and its identification from numerical or from analytical computations on finite width strips.

Transformation of $T(z)$ involves schwarzian derivative $\tilde{T}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z, w\}$, $\left(\{z, w\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2\right)$, hence for $z = e^{2\pi w/L}$,

$$T_{\text{cyl}}(w) = \left(\frac{2\pi}{L}\right)^2 \left(z^2 T(z) - \frac{c}{24}\right)$$

hence $L_{-1}^{\text{cyl}} = \frac{2\pi}{L}(L_0 - \frac{c}{24})$ and the Hamiltonian = generator of ("time") translations along the cylinder
$H = L_{-1}^{\text{cyl}} + \bar{L}_{-1}^{\text{cyl}} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$

In a unitary theory, lowest eigenvalue of L_0 is zero, whence "ground state energy" is $E_0 = -\frac{\pi c}{6L}$. Alternatively finite size correction to free energy per unit length

$$\lim_{T\to\infty} \frac{\log Z(L, T)}{T} = fL + \frac{c\pi}{6L}$$

In a non unitary theory, $c_{\text{eff}} = c - 24 h_{\text{min}}$

[Itzykson-Saleur-Z '86]

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OPERATOR CONTENT OF TWO-DIMENSIONAL CONFORMALLY INVARIANT THEORIES

John L. CARDY

Department of Physics, University of California, Santa Barbara CA 93106, USA

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It is shown how conformal invariance relates many numerically accessible properties of the transfer matrix of a critical system in a finite-width infinitely long strip to bulk universal quantities. Conversely, general properties of the transfer matrix imply constraints on the allowed operator content of the theory. We show that unitary theories with a finite number of primary operators must have a conformal anomaly number $c < 1$, and therefore must fall into the classification of Friedan, Qiu and Shenker. For such theories, we derive sum rules which constrain the numbers of operators with given scaling dimensions.

1. Introduction

The fact that a statistical system with short range interactions at a critical point should be conformally invariant has many interesting consequences, particularly in two dimensions [1]. A simple example is the mapping of the plane into a finite-width strip, from which the correlation functions [2] and other quantities accessible to numerical calculation may be determined. They are related to properties of the transfer matrix along the strip, which we shall denote by $e^{-a\hat{H}}$, where a is the lattice spacing. In the continuum limit, $\hat{H}$ may be thought of as the hamiltonian operator of a quantum field theory in (1 + 1) dimensions.

Two particularly useful results of this mapping, which have already been discussed elsewhere [3, 2], relate to the eigenvalues E_n of $\hat{H}$: for a strip whose width $l \to \infty$, with periodic boundary conditions,

$$E_0 \sim fl - \frac{\pi c}{6l}, \qquad (1.1)$$

$$E_n - E_0 \sim \frac{2\pi x_n}{l}. \qquad (1.2)$$

Eq. (1.1) relates the finite size correction to the lowest eigenvalue E_0 (the ground state energy) to the value of the conformal anomaly number c, which plays a central

Operator Content of Two-Dimensional Conformally Invariant Theories.

Three important results

1. all universal quantities defining a cft −the central charge c, the conformal weights $(h_i, \bar{h}_i)$, the structure constants of OPE C_{ijk}− appear in and may be read off from properties of the transfer matrix (on a cylinder) and the finite size corrections to correlation functions.

2. A unitary theory with a finite number of primary fields ⇒ $c < 1$

3. On a parallelogram with pbc, consistency due to symmetry puts strong constraints on operator content.

4. Summary and further remarks

In the first part of this paper, we have completed the program begun in ref. [2]. We have shown how all important universal properties of conformally invariant two-dimensional theories, including critical exponents and operator product expansion coefficients, may be related to numerically accessible properties of the transfer matrix of a finite width strip. The value of these results will lie in the investigation of new models, rather than in reproducing already known results. The multicritical points in the models obtained by Andrews, Baxter and Forrester [16] whose exponents do not [11] appear to fit the Kac formula are of the first type.

Second, we showed that unitary models with a finite number of primary operators (in the narrow sense defined by Belavin, Polyakov and Zamolodchikov [4]) have $c < 1$. This result partially fills a gap in the line of reasoning which picks out those models in the Friedan, Qiu and Shenker [5] classification as being special. For these models, we showed how the character formulas of Rocha-Caridi [9] give the partition function in an arbitrarily shaped parallelogram, once the number of operators with given scaling dimensions are known. Exploiting the symmetry of the parallelogram, we then derived sum rules which must be satisfied by these numbers. It is remarkable how the scaling dimensions allowed in the models in the Friedan, Qiu and Shenker [5] classification enable this symmetry to be realized. An arbitrary list of scaling dimensions would not have this property. This is another argument pointing to the special role of degenerate theories. We note that the symmetry of the parallelogram, which corresponds to the invariance of $Z(\delta)$ under the modular group, has recently been exploited to limit the possible gauge groups in heterotic string theories [22].

Finally, we obtained all solutions of the sum rules for $m = 3, 4, 5$, and showed that only the models which have been previously identified (Ising, tricritical Ising, 3-state Potts, and generic tetracritical point,) are in fact allowed. We gave for the first time a complete list of primary operators for these models. Solution of the sum rules for larger values of m will require greater effort or sophistication. However, it would appear that the number of solutions should grow with m. This points to the existence of as yet unexplored models, even with $c > 1$. However, it is important to realize that existence of a solution to the sum rules does not imply existence of a corresponding model, since the sum rules are only a necessary condition for the model to be consistent.

The sum rules form a more severe constraint on a theory than closure of the operator product expansion and crossing symmetry, which in some cases does determine the operator product expansion coefficients [17]. For example, in the case $m = 3$, the operator product expansion closes with the operators $\mathbb{1}$ and ε, the energy density. However, the sum rules show that the magnetization σ must be included to get a consistent theory. Once the solution of the sum rules is obtained, the expression (3.16) gives the shape dependence of the free energy at criticality in an arbitrary

1. all universal quantities defining a cft −the central charge c, the conformal weights $(h_i, \bar{h}_i)$, the structure constants of OPE C_{ijk}− appear in and may be read off from properties of the transfer matrix (on a cylinder) and the finite size corrections to correlation functions.

continuation of previous argument on connection between spectrum of Hamiltonian/transfer matrix and $L_0 + \bar{L}_0 - \frac{c}{12}$

2. In a unitary theory with a finite number of primary fields $\Rightarrow$ $c < 1$

Compute the partition function on a rectangle $L \times T$ with doubly pbc, aspect ratio $\delta = T/L$,

$$
\begin{aligned}
Z(L,T) &= \mathrm{tr}\, e^{-TH} = \mathrm{tr}\, e^{\frac{-2\pi T}{L}(L_0+\bar{L}_0-\frac{c}{12})} \\
&= e^{\frac{2\pi c}{12}\delta} \sum_{(h,\bar{h})} \sum_{\substack{\text{descendents} \\ \text{at level } N,\bar{N}}} e^{-2\pi\delta(h+N+\bar{h}+\bar{N})} \\
&\leq e^{\frac{\pi c}{6}\delta} \sum_{(h,\bar{h})} \frac{e^{-2\pi\delta(h+\bar{h})}}{\prod_n (1-q^n)^2} = e^{\frac{\pi c}{6}\delta} \sum_{(h,\bar{h})} \frac{e^{-2\pi\delta(h+\bar{h})}\delta}{\prod_n (1-\tilde{q}^n)^2} e^{-\pi(\delta-\delta^{-1})/6}
\end{aligned}
$$

using Poisson summation formula to transform product $\prod_n (1-q^n)$ under $q = e^{-2\pi\delta} \to \tilde{q} = e^{-2\pi/\delta}$.

Hence in the limit $\delta \to 0$, $q \to 1$, $\tilde{\delta} \to \infty$, $\tilde{q} \to 0$

$$
\mathcal{N}\delta \exp\left(\frac{\pi}{6\delta}\right) \geq Z(L,T) = Z(T,L) \sim \exp\left(\frac{\pi c}{6\delta}\right)
$$

If $\mathcal{N} = \#$ of primary fields is finite, this is consistent only if $c < 1$. QED

The conclusion also holds for non unitary theories [Altschuler, '89]

3. On a parallelogram with pbc, consistency due to symmetry puts strong constraints on operator content.

- Partition function on *torus* of modular ratio $\tau = i\delta$

$$Z(\tau) = \mathrm{tr}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{c}{24}} = \sum_{h,\bar{h}} \mathcal{N}_{h,\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q})$$

  with $q = e^{2\pi i \tau}$ and **character** $\chi_h(q) = \mathrm{tr}\, q^{L_0 - \frac{c}{24}}$ (counting function of states in the h conformal tower.)

- In a (unitary) $c < 1$ theory, $c = 1 - 6/m(m+1)$, $m = 3, 4, \cdots$ and $h, \bar{h}$ take their values in the Kac table [Friedan, Qiu, Shenker]

$$h_{pq} = \frac{(p(m+1) - qm)^2 - 1}{4m(m+1)} \qquad 1 \le q \le p \le m-1$$

  Characters are explicitly known [Rocha-Caridi, Feigin–Fuks] and transform among themselves under the action of $\tau \mapsto -1/\tau$ (Poisson transformation)

$$\chi_h(\tau) = S_{hh'}\, \chi_{h'}(-1/\tau)$$

  (and also of $\tau \mapsto \tau + 1$);

- Consistency, i.e. **modular invariance** of partition fn,

  yields constraints, "sum rules" $\mathcal{N} = S\mathcal{N}$,

$$\sum_{p,q;\bar{p},\bar{q}} \mathcal{N}(p,q;\bar{p},\bar{q})(-1)^{(p+q)(p'+q')+(\bar{p}+\bar{q})(\bar{p}'+\bar{q}')} \times \sin\frac{\pi p p'}{m} \sin\frac{\pi q q'}{m+1} \sin\frac{\pi \bar{p}\bar{p}'}{m} \sin\frac{\pi \bar{q}\bar{q}'}{m+1} = \tfrac{1}{8} m(m+1) \mathcal{N}(p',q';\bar{p}',\bar{q}'). \qquad (3.22)$$

S is unitary, hence "diagonal solution" $\mathcal{N}_{h\bar{h}} = \delta_{h\bar{h}}$ always modular invariant **but** $\exists$ also other solutions, for ex., 3-state (tri)critical Potts model

. . . and more [Itzykson–Z '86]

A fundamental paper in several respects

– Gives a physical interpretation to the mathematical fact that characters have modular properties (i.e. form a finite diml reprn of modular group).
– Opens the route to a classification of cft's.
Classification of $c < 1$ completed soon after [Cappelli–Itzykson–Z; Gepner–Qiu; CIZ, Kato '87]
with some insight from lattice theories [Pasquier].
– Uncovers hidden sectors of theory; for ex. Ising : 1, ϵ, but also σ forced upon us by modular invariance.

Earlier observation of modular properties of partition function [Fisher-Ferdinand, Thorn] or of spectrum [Nahm]
Parallel work by Gepner and Witten, Gepner...

Paper also remarkable by the "economy of means"

4. Summary and further remarks

In the first part of this paper, we have completed the program begun in ref. [2]. We have shown how all important universal properties of conformally invariant two-dimensional theories, including critical exponents and operator product expansion coefficients, may be related to numerically accessible properties of the transfer matrix of a finite width strip. The value of these results will lie in the investigation of new models, rather than in reproducing already known results. The multicritical points in the models obtained by Andrews, Baxter and Forrester [16] whose exponents do not [11] appear to fit the Kac formula are of the first type.

Second, we showed that unitary models with a finite number of primary operators (in the narrow sense defined by Belavin, Polyakov and Zamolodchikov [4]) have $c < 1$. This result partially fills a gap in the line of reasoning which picks out those models in the Friedan, Qiu and Shenker [5] classification as being special. For these models, we showed how the character formulas of Rocha-Caridi [9] give the partition function in an arbitrarily shaped parallelogram, once the number of operators with given scaling dimensions are known. Exploiting the symmetry of the parallelogram, we then derived sum rules which must be satisfied by these numbers. It is remarkable how the scaling dimensions allowed in the models in the Friedan, Qiu and Shenker [5] classification enable this symmetry to be realized. An arbitrary list of scaling dimensions would not have this property. This is another argument pointing to the special role of degenerate theories. We note that the symmetry of the parallelogram, which corresponds to the invariance of $Z(\delta)$ under the modular group, has recently been exploited to limit the possible gauge groups in heterotic string theories [22].

Finally, we obtained all solutions of the sum rules for $m = 3, 4, 5$, and showed that only the models which have been previously identified (Ising, tricritical Ising, 3-state Potts, and generic tetracritical point,) are in fact allowed. We gave for the first time a complete list of primary operators for these models. Solution of the sum rules for larger values of m will require greater effort or sophistication. However, it would appear that the number of solutions should grow with m. This points to the existence of as yet unexplored models, even with $c > 1$. However, it is important to realize that existence of a solution to the sum rules does not imply existence of a corresponding model, since the sum rules are only a necessary condition for the model to be consistent.

The sum rules form a more severe constraint on a theory than closure of the operator product expansion and crossing symmetry, which in some cases does determine the operator product expansion coefficients [17]. For example, in the case $m = 3$, the operator product expansion closes with the operators $\mathbb{1}$ and ε, the energy density. However, the sum rules show that the magnetization σ must be included to get a consistent theory. Once the solution of the sum rules is obtained, the expression (3.16) gives the shape dependence of the free energy at criticality in an arbitrary

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EFFECT OF BOUNDARY CONDITIONS ON THE OPERATOR CONTENT OF TWO-DIMENSIONAL CONFORMALLY INVARIANT THEORIES

John L. CARDY
Department of Physics, University of California, Santa Barbara CA 93106, USA

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The operator content of unitary conformally invariant theories with $c < 1$ is further analysed by deriving the spectrum of the transfer matrix for finite width strips and a variety of boundary conditions: antiperiodic, cyclic, twisted, free, fixed and a mixture of the last two. Complete results are obtained for the Ising model and for the three-state Potts model, as illustrations of the method. They demonstrate how the internal symmetries of these theories are tied in with their conformal properties.

1. Introduction

Recently there has been considerable progress in using the principle of conformal invariance [1] to classify possible universality classes of critical behavior in two dimensions. Friedan, Qiu and Shenker [2] showed that for unitary models with conformal anomaly number c less than unity, c must be quantized according to

$$c = 1 - \frac{6}{m(m+1)}, \qquad \text{for } m = 3,4,\dots, \tag{1.1}$$

and that the allowed values of the scaling dimensions of the primary scaling operators are $(h_{p,q}, h_{\bar{p},\bar{q}})$, where

$$h_{p,q} = \frac{(p(m+1) - qm)^2 - 1}{4m(m+1)}, \tag{1.2}$$

with $1 \leqslant q \leqslant p \leqslant m-1$. For a given m, different universality classes correspond to different subsets of the allowed operators. In a recent paper [3], hereinafter referred

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Modular transformations also useful in connecting or constraining other types of boundary conditions

• Periodic, Antiperiodic, Cyclic or Twisted (for a $\mathbb{Z}_N$ theory like Potts)
Consider again partition function on rectangle of aspect ratio δ and P or A b.c. in the two directions

$$
\begin{aligned}
Z_{PP}(\delta) &= \sum_{h,\bar{h}} \mathcal{N}^{P}_{h\bar{h}}\, \chi_h(q)\chi_{\bar{h}}(q) \\
Z_{PA}(\delta) &= \sum_{h,\bar{h}} \underbrace{\epsilon_{h\bar{h}}}_{\mathbb{Z}_2\ \text{charge}}\ \mathcal{N}^{P}_{h\bar{h}}\, \chi_h(q)\chi_{\bar{h}}(q) \\
Z_{AP}(\delta) &= Z_{PA}(1/\delta) = \sum_{h,\bar{h}} \mathcal{N}^{A}_{h\bar{h}}\, \chi_h(q)\chi_{\bar{h}}(q)
\end{aligned}
$$

thus knowledge of charges in periodic sector + modular transform gives us operator content in antiperiodic sector ! Same argument for T, C b.c. Applications to Ising, Potts, etc

• Free or fixed

Z_{FP} is now a linear combination of characters, see below, but again consistency constraints from modular transformation. . .

Parallel work on Z_{AP} etc and their (sub)modular properties [Z '86].

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BOUNDARY CONDITIONS, FUSION RULES AND THE VERLINDE FORMULA

John L. CARDY

Department of Physics, University of California, Santa Barbara, CA 93106, USA

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Boundary operators in conformal field theory are considered as arising from the juxtaposition of different types of boundary conditions. From this point of view, the operator content of the theory in an annulus may be related to the fusion rules. By considering the partition function in such a geometry, we give a simple derivation of the Verlinde formula.

1. Introduction

Recently there has been considerable progress made in understanding the problem of classifying conformal theories, following the observation of E. Verlinde [1] that the fusion rules of the underlying algebra are related by formula

$$\sum_i S_i^j N_{kl}^i = S_k^j S_l^j / S_0^j \tag{1}$$

to the elements S_i^j of the matrix which represents the modular transformation $\tau \to -1/\tau$ acting on the Virasoro characters.

A conformal field theory defined on a manifold without boundaries has as its underlying symmetry two algebras $\mathscr{A}$ and $\bar{\mathscr{A}}$ which act respectively on the holomorphic (z) and antiholomorphic ($\bar{z}$) dependences of the physical fields of the theory. In a rational conformal field theory, the irreducible representations of these algebras are constructed by acting on a highest weight vector with all possible lowering operators, and then projecting out null states. The fusion rule coefficients N_{jk}^i of the algebra $\mathscr{A}$ give the number of distinct ways that the representation i occurs in the "fusion" of two fields transforming according to the representations j, k respectively. This process of fusion corresponds to considering only the holomorphic, or only the antiholomorphic part of the operator product expansion of two physical operators. The decoupling of the null states after this fusion process then gives strong constraints on the N_{jk}^i, first analysed for the case of the Virasoro algebra by Belavin et al. [2].

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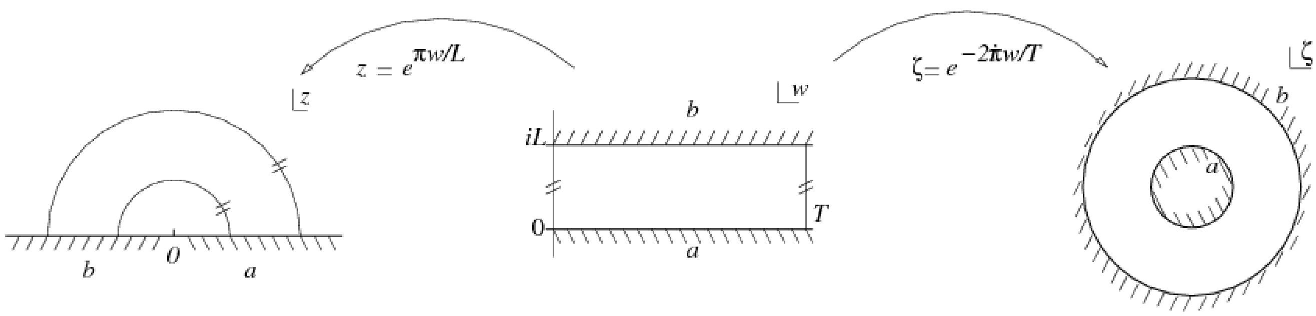
One more fundamental paper ! The key concepts of Boundary CFT : boundary condition changing operators, Cardy (boundary) states and fields, Cardy (consistency) equation, . . .
\+ a physicist's proof of Verlinde fusion formula

In the upper half plane, insertion of **boundary cond. changing operator** Ψ^h_{ab} on real axis creates mixed b.c.

Consider again the partition function, now in an annulus



Boundary states $|a\rangle$ must satisfy $(L_n - \bar{L}_{-n})|a\rangle = 0$: superpositions of "Ishibashi states" $|h\rangle\rangle$, in one-to-one correspondence with $h = \bar{h}$ primary *bulk* operators, (in a diagonal theory, but also in general [Watts '97])

$$Z_{ba}(\delta) = \sum_h n^h_{ba}\chi_h(q) \qquad q = e^{-\pi T/L} \qquad \tilde{q} = e^{-4\pi L/T}$$

$$= \langle b|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|a\rangle = \cdots = \sum_h \langle b|h\rangle\rangle\langle\langle h|a\rangle\chi_(\tilde{q})$$

whence $n^h_{ba} = \sum_{h'} S_{hh'}\langle b|h'\rangle\rangle\langle\langle h'|a\rangle$ **Cardy's equation** .
In diagonal minimal theories, $|a\rangle\rangle \leftrightarrow \tilde{h}$, conformal weight of a primary field, and $\langle\langle h'|a\rangle = S_{\tilde{h}h'}/(S_{0h'})^{1/2}$ whence

$$n^h_{ba} = n^h{}_{\tilde{h}_1\tilde{h}_2} = \sum_{h'} \frac{S_{hh'}S_{\tilde{h}_1h'}S_{\tilde{h}_2h'}}{S_{0h'}}$$

which is Verlinde formula for the fusion coefficients $N^h_{\tilde{h}_1\tilde{h}_2}$!

A paper with some antecedents [John's previous paper, Saleur and Bauer], but a conceptual breakthrough, and a long filiation...

By John himself, from percolation ("Cardy (crossing) formula"), ... to SLE.
Applications from cond. mat. physics, [Affleck, Ludwig, Oshikawa, Saleur, ...]
to string theory : [Pradisi–Sagnotti–Stanev, Recknagel–Schomerus, Fuchs–Schweigert, Runkel,...]

[Di Francesco–Z '89, Behrend–Pearce–Petkova–Z '98] Cardy's equation explains why graphs are the good way to encode the spectrum of cft's...

$$n_{ba}^{i} = \sum_{j} \frac{S_{ij}\psi_b^{j*}\psi_a^{j}}{S_{0j}}$$

are the adjacency matrices of graphs and form a representation of the fusion algebra $n_i\, n_j = N_{ij}^{k} n_k$.
In fact classification of boundary states reduced to classification of "nimreps" (non negative integer valued representations) of fusion algebra.

Further generalization to defects lines... [Petkova–Z 2000, Fröhlich–Fuchs–Runkel–Schweigert, ... Runkel, Bachas–Brunner]

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Bulk and boundary operators in conformal field theory

John L. Cardy [1] and David C. Lewellen
Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

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In conformal field theory on a manifold with a boundary, there is a short-distance expansion expressing local bulk operators in terms of boundary operators at an adjacent boundary. We show how the coefficients of such an expansion are given solely by data appearing in the bulk theory on the sphere and torus. In particular, the coefficients of the identity operator, which fix the one-point functions, are determined by the elements of the matrix S which implements modular transformations on the torus. The other coefficients are related, in addition, to the elements of the matrices implementing duality transformations on the conformal blocks of the four-point functions on the sphere. Some examples are given.

Conformal field theory hypothesizes the existence of a complete set of observable densities called scaling operators, whose correlation functions transform in a simple homogeneous fashion under scale transformations. A given local scaling operator $\phi(r)$ is assigned a scaling dimension x_ϕ. At least for theories in which all scaling operators are non-negative, this implies that if ϕ is a rotational scalar, it may be normal-

expansion [1] is necessarily of the form

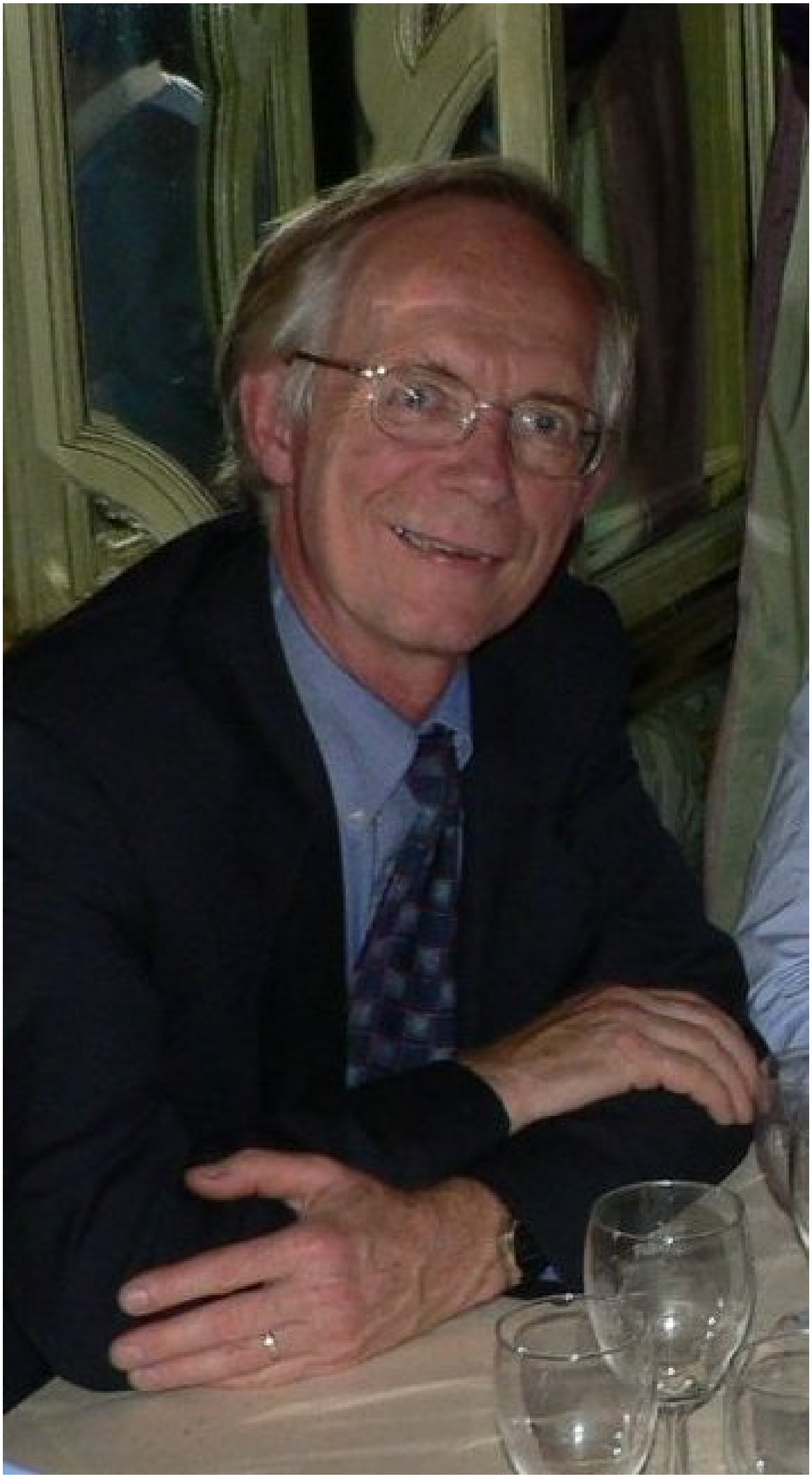
$$\phi(x, y) \sim \sum_i (2y)^{\Delta_i - x_\phi} C^a_{\phi\psi_i} \psi_i(x) , \qquad (1)$$

where the $\psi_i(x)$ are *boundary operators* with boundary scaling dimensions Δ_i, normalized so that on the boundary of the half-plane $\langle \psi_i(x)\psi_i(0)\rangle_a = x^{-2\Delta_i}$.

BCFT requests more than list of boundary states and spectrum.

$\langle a|\Phi(z)|a\rangle$, expansion of bulk fields Φ on boundary fields Ψ, . . .

Again a filiation of papers, "bulk-boundary algebra", fusion of boundary fields and their structure constants, etc. [Lewellen '92, Runkel '98-99, . . .], connections with integrable lattice models, . . .



Congratulations, John,

and best wishes

for many new ascents

to new summits !

⋆