A short trip to cumuland

An hommage to the memory of Ivan Todorov

Jean-Bernard Zuber (LPTHE, Sorbonne Université)

Sofia, 30 May 2026

Recollections of Ivan T. Todorov

Starting in the mid 80's

A frequent visitor of our lab in Saclay, an old friend of Claude Itzykson (1938–1995).

1 ELECTROPRODUCTION STRUCTURE FUNCTIONS 2823

Because of causality, the limit in (A17) is the light-cone limit $x^2=4\sigma\lambda-\mathbf{x}^2\to 0$. It indeed follows from (A14) and (A17) that

$$\pi\hat{W}(z,\lambda)\sim\delta(z)\tfrac{1}{2}\epsilon(\lambda)\hat{f}(\lambda) \tag{A23}$$

for $z\sim 0$. We therefore have determined the leading singularity of $\hat{W}$ on the light cone and, by (A20), the large-λ behavior of its coefficient. The behavior of W near the light cone can also be determined from the configuration-space form

$$\pi\hat{W}(x^2,x_0)=-\tfrac{1}{2}i\int dadb\,\sigma(a,b)\,\exp(-ibx_0)\Delta(x;a+b^2) \tag{A24}$$

of (A1) and the behavior (3.58) of Δ. These give

$$\pi\hat{W}(z,\lambda)\sim\delta(z)L(\lambda) \tag{A25}$$

for $z\sim 0$, where

$$L(\lambda)=-[i\epsilon(\lambda)/4\pi]\int dadb\,\sigma(a,b)\,\exp(-ib\lambda). \tag{A26}$$

Equations (A6) and (A9) give

$$L(\lambda)\underset{\lambda\to\infty}{\longrightarrow}-2^{-\alpha-1}igw\epsilon(\lambda)\,|\,\lambda\,|^{\alpha}. \tag{A27}$$

In view of (A22), Eqs. (A25) and (A27) are in perfect agreement with (A23) and (A20).

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Three-Dimensional Formulation of the Relativistic Two-Body Problem and Infinite-Component Wave Equations

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(Received 14 November 1969)

A relativistic quasipotential equation is derived from the conventional Hamiltonian formalism and old-fashioned "noncovariant" off-energy-shell perturbation theory in a similar way to that by which the four-dimensional Bethe-Salpeter equation is obtained from the off-mass-shell Feynman rules. The three-dimensional equation for the (off-energy-shell) scattering amplitude appears as a straightforward generalization of the nonrelativistic Lippmann-Schwinger equation. The corresponding homogeneous equation for the bound-state wave function and the normalization condition for its solutions are derived from the equation for the complete four-point Green's function. In order to obtain a solvable model, we consider a simplified version of the quasipotential equation which still reproduces correctly the on-shell scattering amplitude and is consistent with the elastic unitarity condition. It involves a "local" approximation to the potential $V(p-q)$ which defines the kernel of our integral equation (the integration being carried over a two-sheeted hyperboloid in the energy-momentum space). It is shown that for the scalar Coulomb potential $V(p-q)=\alpha/(p-q)^2$, our model equation is equivalent to a simple infinite-component wave equation of the type considered by Nambu, Barut, and Fronsdal. The energy eigenvalues for the bound-state problem are calculated explicitly in this case and are found to be $O(4)$ degenerate (just as in the nonrelativistic Coulomb problem and in Wick and Cutkosky's treatment of the Bethe-Salpeter equation in the same approximation).

I. INTRODUCTION

THE purpose of this paper is to show the relationship between a modification of the quasipotential approach to the relativistic two-body problem developed in Refs. 1–3 and the infinite-component wave equations for the "relativistic hydrogen atom" of the type considered in Refs. 4–6.

A three-dimensional relativistic quasipotential equation for the two-particle scattering amplitude and for the bound-state wave function was first proposed by Logunov and Tavkhelidze.[7] It was derived in the framework of the Bethe-Salpeter equation using the nonuniqueness of the off-shell extrapolation of the scattering amplitude.

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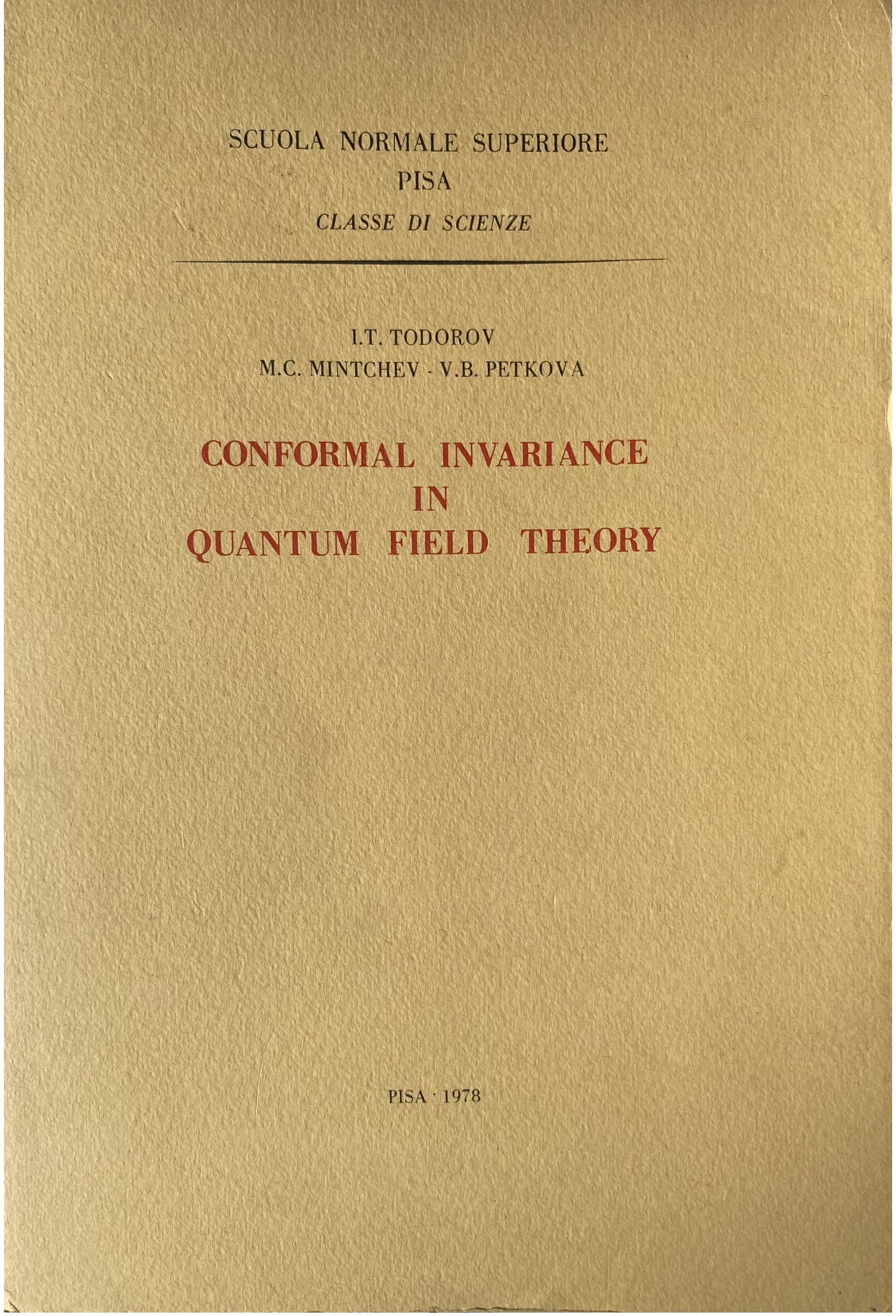
High reputation of
(field and group) theorist
and conformal field theorist..



SCUOLA NORMALE SUPERIORE
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I.T. TODOROV
M.C. MINTCHEV - V.B. PETKOVA

**CONFORMAL INVARIANCE
IN
QUANTUM FIELD THEORY**

PISA · 1978

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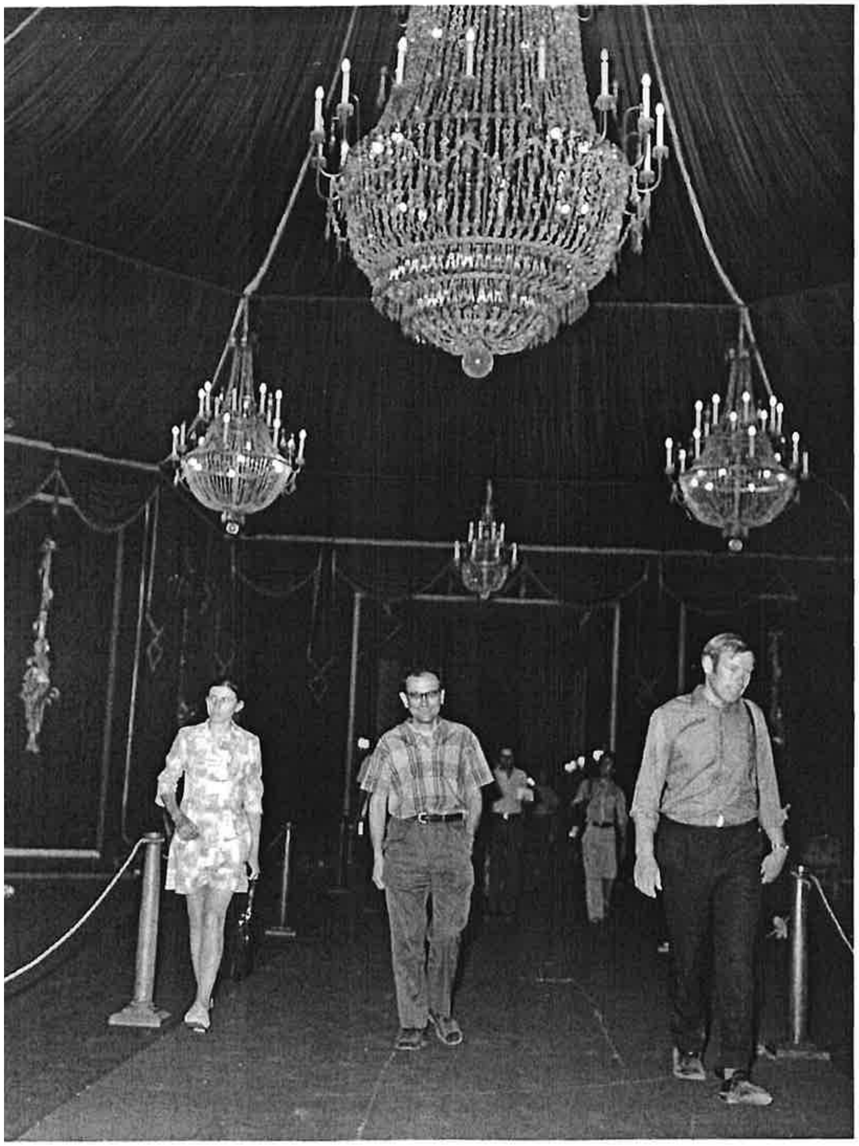

Who ? Where ?

High reputation of
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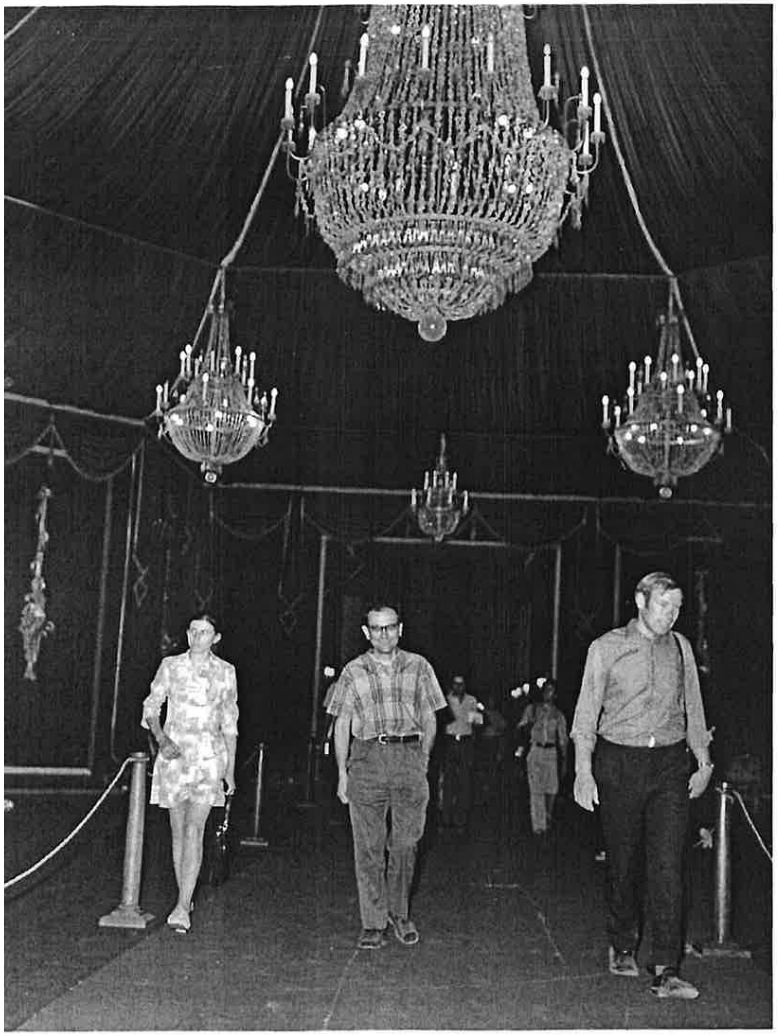
and conformal field theorist..



Tehran 1972

with Valentina Petkova and Werner Rühl

Many encounters in various places,
Varna (1987),
Sofia (1989),
Durham (1991),
Vienna (1999),
CERN (2000), Trieste (200?),

Saclay, Orsay,
Bures-sur-Yvette, ...

Sofia October 1989

Durham 1991

A man of indefatigable curiosity . . .

A man of culture . . .

The father of a school of mathematical physics here in Bulgaria. . .

Respect and admiration . . .
and gratitude . . .

A short trip to cumuland

with Sylvain Lacroix

`math-ph/2508.21483`

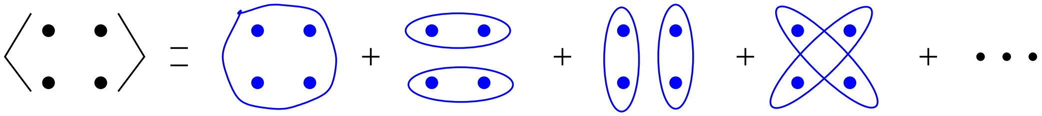
I. Classical cumulants

In probability, statistics, statistical physics. . .
X random variable, $m_n(X) = \langle X^n \rangle$ its moments;
if several such r.v. X_i, mixed moments $m_n(X_{i_1}, \cdots, X_{i_n}\rangle$.
Decompose moments following *partitions* of the set $\{1, \cdots, n\}$

$$m_n(X_{i_1}, \cdots, X_{i_n}\rangle = \sum_{\pi \in P[n]} \prod_{B \in \pi} c_{|B|}(X_{i_a}, a \in B)$$

Thus

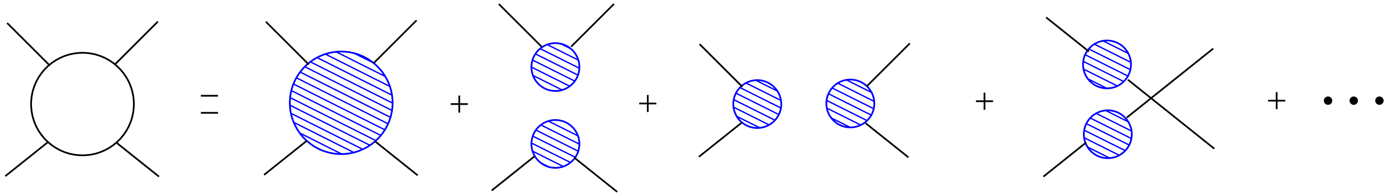
$$\begin{aligned} m_1(X) &= c_1(X) \\ m_2(X) &= c_2(X) + c_1^2(X) \qquad\qquad m_2(X,Y) = c_2(X,Y) + c_1(X)c_1(Y) \\ &\vdots \\ m_4(X) &= c_4(X) + 3\,c_2^2(X) + \underbrace{\cdots\cdots\cdots\cdots}_{0 \text{ if } c_1(X)=0} \end{aligned}$$



This defines recursively the cumulants c_n as (signed) homogeneous polynomials of the moments m_k's.

Thus $c_2(X,Y) = \langle XY \rangle - \langle X \rangle \langle Y \rangle$ is the (co-)variance, etc.

In QFT or particle physics, write the 4-point function (or the scattering amplitude) as



“Exponential” generating functions

$$Z_X(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} m_n(X) = \langle e^{Xt} \rangle$$

$$W_X(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} c_n(X)$$

$$Z_X(t) = e^{W_X(t)}$$

Fundamental property, in connection with independence.
X, Y independent variables: for all (polynomials) f and g,

$$\langle f(X) g(Y) \rangle = \langle f(X) \rangle \; \langle g(Y) \rangle \tag{1}$$

X, Y independent $\Longrightarrow$ $c_n(X+Y) = c_n(X) + c_n(Y)$

$$e^{W_{X+Y}(t)} = \langle e^{t(X+Y)} \rangle = \langle e^{tX} \rangle \langle e^{tY} \rangle = e^{W_X(t)} e^{W_Y(t)} = e^{W_X(t) + W_Y(t)}$$

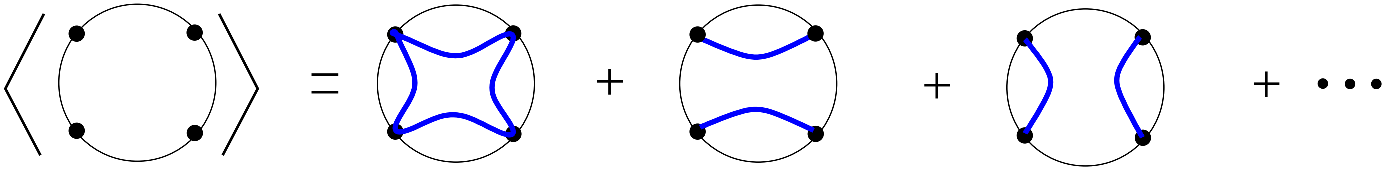
X, Y independent $\Longleftrightarrow$ all <u>mixed</u> cumulants vanish [Rota]

II. Free cumulants

Consider now *non commuting* random variables $X_1, X_2, \cdots$
again $m_n(X_1, \cdots, X_n) = \langle X_1 \cdots X_n \rangle$
Change the rule: replace the set $P[n]$ of all partitions by the set NC$[n]$ of *non-crossing* partitions [Kreweras '72]

$$m_n(X_{i_1}, \cdots, X_{i_n}\rangle = \sum_{\pi \in NC[n]} \prod_{B \in \pi} \kappa_{|B|}(X_{i_a}, a \in B)$$

Thus now $m_4(X) = \kappa_4(X) + 2\, \kappa_2^2(X) + \cdots$



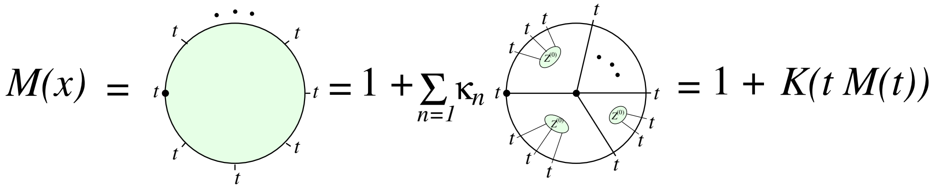
This defines recursively the free cumulants κ_n as (signed) homogeneous polynomials of the moments.

Ordinary generating functions

$$M_X(t) \;=\; 1 + \sum_{n\geq 1} t^n m_n(X)$$

$$K_X(t) \;=\; \sum_{n\geq 1} t^n \kappa_n(X)$$

$$M_X(t) \;=\; 1 + K_X(t\, M_X(t))$$



[Brézin–Itzykson–Parisi-Z '78, Cvitanovic '81]

(Voiculescu's ('86) equivalent functional relation: "R-transform" of the resolvent of X)

Now, independence ⟶ **free independence** (or "freeness") [Voiculescu '86]

Def. X and Y are free if $\langle f_1(X) g_1(Y) f_2(X) \cdots g_r(Y)\rangle = 0$ whenever $\langle f_i(x)\rangle = \langle g_i(Y)\rangle = 0$ for all $i = 1, \cdots, r$

X, Y are free $\Longrightarrow$ Their non-crossing (or "free") cumulants are additive

$$K_{X+Y}(t) = K_X(t) + K_Y(t)$$ [Speicher '93]

X, Y are free $\Longleftrightarrow$ all mixed **free** cumulants vanish [Speicher '97]

Ordinary probability	Free probability
$m_n = \sum_{\pi \in P[n]} \prod_{B \in \pi} c_{\lvert B\rvert}$	$m_n = \sum_{\pi \in NC[n]} \prod_{B \in \pi} \kappa_{\lvert B\rvert}$
Exp. g.f. $Z_X(t) = e^{W_X(t)}$	Ordinary g.f. $M_X(t) = 1 + K_X(tM_X(t))$
X, Y independent	X, Y free
$\Longrightarrow c_n(X+Y) = c_n(X) + c_n(Y)$	$\Longrightarrow \kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y)$
$\Longleftrightarrow$ all mixed cumulants vanish	$\Longleftrightarrow$ all mixed free cumulants vanish

III. Freeness and Random Matrices

Large Random Matrices : natural setting of Free Probability

[Voiculescu '91]

Sequences of $N \times N$ matrices A_N, assume convergence $m_n(A_N) = \frac{1}{N}\mathrm{Tr} A_N^n \to m_n(A)$ ("limiting distribution").

In particular, take two such sequences of Hermitian matrices A_N and B_N with (at least) B_N with a $\mathrm{U}(N)$-invariant distribution.

Theorem Then A_N and B_N are *asymptotically free* (as $N \to \infty$).

[Voiculescu '91]

In particular, A_N, B_N deterministic (and with limiting distribution), A_N and $U_N B_N U_N^\dagger$, with U_N Haar-uniform in $\mathrm{U}(N)$, are free.

Thus $\lim_{N\to\infty}\left[\int_{\mathrm{U}(N)} DU\, \kappa_n(A_N + U_N B_N U_N^\dagger) - \kappa_n(A_N) - \kappa_n(B_N)\right] = 0$.

Thus $\lim_{N\to\infty}\left[\int_{\mathsf{U}(N)} DU\,\kappa_n(A_N + U_N B_N U_N^\dagger) - \kappa_n(A_N) - \kappa_n(B_N)\right] = 0$

Question Can one find *at finite* N, invariant functions $K_n(A)$ such that

$$\int_{\mathsf{U}(N)} DU\,K_n(A_N + U_N B_N U_N^\dagger) - K_n(A_N) - K_n(B_N) = 0 \qquad (*)$$

holds true, and $K_n(A_N) - \kappa(A_N) = O(N^{-2})$?

The answer is **yes** and the solution is surprisingly simple. For $n \le N$

$K_n(A) =$ coefficient of $\frac{Nt^n}{n}\mathrm{Tr}C^n$ in $Z(A,C;t) := \int_{\mathsf{U}(N)} DU\,e^{Nt\mathrm{Tr}AUCU^\dagger}$

More generally, for a partition $\alpha = (\alpha_1, \cdots, \alpha_\ell) = [n^{\widehat{\alpha}_n} \ldots 1^{\widehat{\alpha}_1}]$ of n $(n \le N)$, $n = \alpha_1 + \cdots + \alpha_\ell$, define

$$K_\alpha^{(N)}(A) := \frac{\prod_{k=1}^n k^{\widehat{\alpha}_k}\,\widehat{\alpha}_k!}{N^\ell}\left[t^n\,\mathrm{Tr}(C^{\alpha_1})\cdots\mathrm{Tr}(C^{\alpha_\ell})\right]Z(A,C\,;t)\,.$$

Proof of (*)

$$\int DV\,Z(A+VBV^\dagger, C; t) = \int DU\,DV\,e^{Nt\mathrm{Tr}(AUCU^\dagger + BV^\dagger UCU^\dagger V)} = \int DU\,DV\,e^{Nt\mathrm{Tr}(AUCU^\dagger + BV^\dagger CV)} = Z(A,C;t)Z(B,C;t)$$

and identify the "simple trace terms" $\mathrm{Tr}C^n$ in the two sides:

$$1 + \sum_n \frac{Nt^n}{n}\mathrm{Tr}C^n \int DV\,K_n(A + VBV) + \cdots = (1 + \sum_n \frac{Nt^n}{n}\mathrm{Tr}C^n K_n(A) + \cdots)(1 + \sum_n \frac{Nt^n}{n}\mathrm{Tr}C^n K_n(B) + \cdots)\,.$$

$\square$

The functions $K_n(A)$ and $K_\alpha(A)$ (symmetric functions of the eigenvalues) enjoy all kinds of nice properties, anticipating those of free cumulants:

$\lim_{N\to\infty} K_n = \kappa_n$ (whence the name "precursors of free cumulants")

That the "simple trace terms", or the "low rank" case, in the large N limit of $Z(A,C;t)$, are given by free cumulants has been known for long [Itzykson–Z 1980, Marinari–Parisi–Rittort 1994, P. Zinn-Justin 1999, 2002, Guionnet–Maïda '05, Collins–Śniady '02,'07, Tanaka '08]

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For example

$$\begin{aligned}
K_1 &= m_1 = \kappa_1 \\
K_2 &= \frac{N^2}{N^2-1}(m_2 - m_1^2) = \frac{N^2}{N^2-1}\kappa_2 \\
K_3 &= \frac{N^4}{(N^2-1)(N^2-4)}(m_3 - 3m_2m_1 + 2m_1^3) = \frac{N^4}{(N^2-1)(N^2-4)}\kappa_3\,, \\
K_4 &= \frac{N^4}{(N^2-1)(N^2-4)(N^2-9)}\Big((N^2+1)(m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4) + 5(m_2 - m_1^2)^2 \\
&= \frac{N^4}{(N^2-1)(N^2-4)(N^2-9)}\Big((N^2+1)\kappa_4 + 5\kappa_2^2\Big)
\end{aligned}$$

etc.

The functions $K_n(A)$ and $K_\alpha(A)$ (symmetric functions of the eigenvalues) enjoy all kinds of nice properties, anticipating those of free cumulants:

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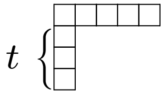
$K_n(A_N) - \kappa(A_N) = O(N^{-2})$, a computable $\frac{1}{N^2}$-expansion, in terms of "monotone Hurwitz numbers" [Goulden–Gay-Paquet–Novak, 2011-17]

K_n explicitly known in terms of Schur functions

$$K_n(A) = \frac{1}{N}\sum_{t=0}^{n-1}\frac{(-1)^t}{C_{\lambda_{n,t}}(N)}\, s_{\lambda_{n,t}}(A)\,, \qquad \underbrace{C_{\lambda_{n,t}}}_{\text{“content”}}(N) = N^{-n}\frac{(N+n-t-1)!}{(N-t-1)!}\,.$$

$\lambda_{n,t}$: "hook diagrams" with n boxes



G.f. of the K_n: $\int DU\, e^{N\,\mathrm{Tr} AU^\dagger}\ \frac{1}{N}\mathrm{Tr}\frac{1}{1-tU} = 1 + \sum_{n=1} t^n K_n(A)$

(which implies a novel form of the g.f. of free cumulants

$\lim_{N\to\infty} \int DU e^{N\,\mathrm{Tr} AU^\dagger}\ \frac{1}{N}\mathrm{Tr}\frac{1}{1-tU} = 1 + \sum_{n=1} t^n \kappa_n(A)$)

Define $A^U := UAU^\dagger$. Then $K_n(A) = N^{n-1} \int DU\, A^U_{i_1 i_2} A^U_{i_2 i_3} \cdots A^U_{i_n i_1}$.

Known at large N: [Collins et al '06, Maillard et al '19, Bernard–Hruza '24]

and more generally, for $\sigma \in S_n$, $[\sigma]$ its cycle decomposition

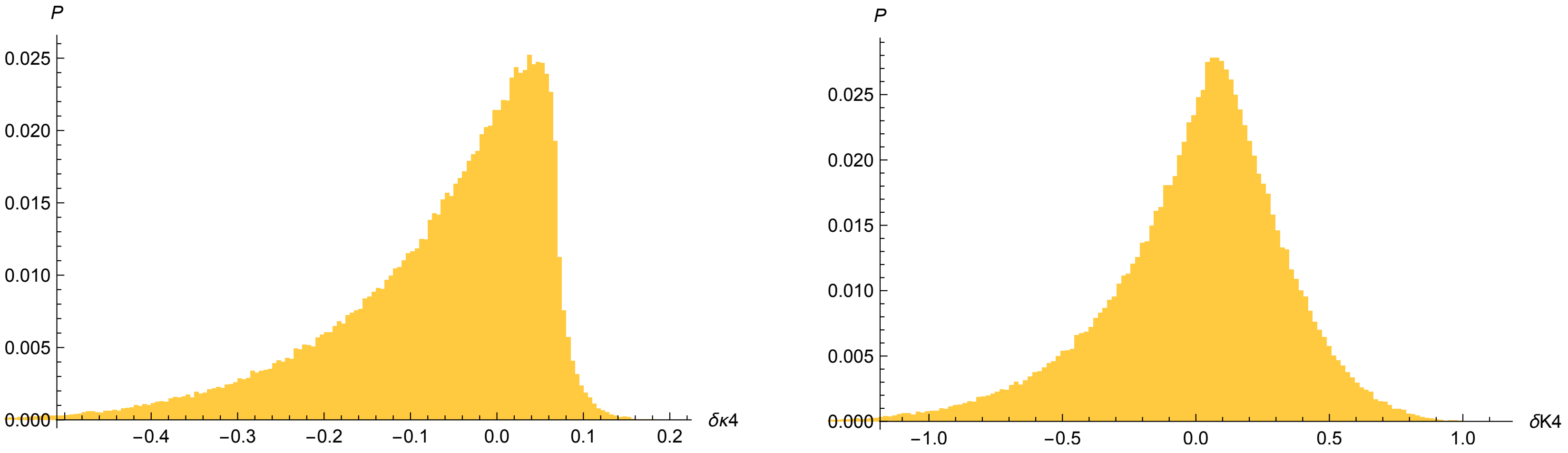
$$K_{[\sigma]}(A) = N^{n-\ell([\sigma])} \int_{\mathrm{U}(N)} DU\, A^U_{1\,\sigma(1)} A^U_{2\,\sigma(2)} \cdots A^U_{n\,\sigma(n)}\,.$$

"Wick property". Take A a random Gaussian Hermitian matrix ("GUE(N)"). Then $\mathbb{E}_{A\sim \mathrm{GUE}(N,\sigma)}[K_\alpha(A)] = \delta_{\alpha,[2^n]}\,\sigma^{2n}$.

Approach to additivity.

Let $\delta K_n(A,B;U) := K_n(A+UBU^\dagger) - K_n(A) - K_n(B)$, where U is taken randomly and uniformly following the Haar measure of $\mathrm{U}(N)$.
$\mathbb{E}_U(\delta K_n(A,B;U)) = 0$. Higher (classical) cumulants ?
For large N, $\mathrm{var}(\delta K_n) = \mathrm{O}(N^{-2})$ and the higher (classical) cumulants of δK_n are more and more suppressed: $c_{2k}(\delta K_n) = \mathrm{O}(N^{-2k})$, $c_{2k+1}(\delta K_n) = \mathrm{O}(N^{-2k-2})$ (Narrow Gaussian).

Application: (probabilistic) Horn problem: distribution of eigenvalues of the sum of two (Hermitian) matrices A and B of given spectrum, *i.e.*,sum of *orbits* of A and B. At finite N, non trivial distribution with a convex support.
For large N, spectrum of $A+B$ must localise on the “free convolution” of the spectra of A and B.
At finite N, $\delta K_n(A,B)$ closer to Gaussian variable than $\delta \kappa_n(A,B)$.



A numerical experiment with 4×4 matrices A and B of regularly spaced spectrum on $[-1, 1]$ and a sample of 200,000 Haar-distributed matrices U, showing the histograms of $\delta\kappa_4(U)$ (left), $\delta K_4(U)$ (right). The improvement of the latter two with respect to the former is manifest: smaller skewness, distribution closer to Gaussian, etc.

Are there other applications of these *finite N precursors* of free cumulants ?

Recently many appearances of free cumulants in various **physical** contexts

– large time behaviour of diffusion with a random diffusion coefficient

[Guéneau–Majumdar–Schehr, '25]

– "quantum exclusion processes" [D. Bernard–Hruza, '24-25]

– "ETH" (Eigenstate Thermalization Hypothesis) [Pappalardi–Foini–Kurchan, '22]

$$\vdots$$

Algebraic aspects

A (new) coproduct on the algebra $\mathcal{A}^{(N)}$ of symmetric functions

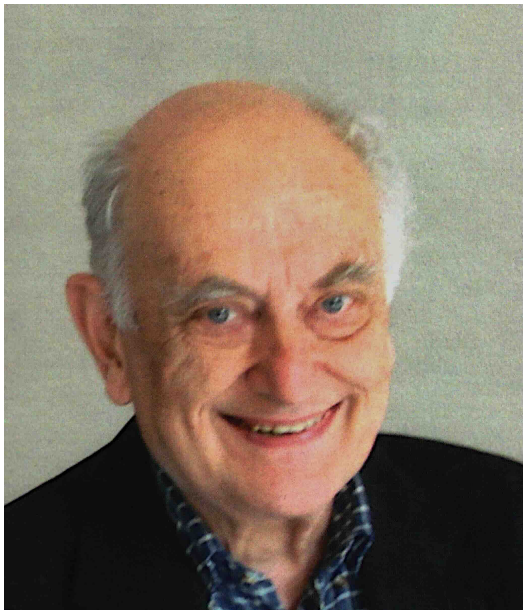
$$\mathcal{A}^{(N)} \to \mathcal{A}^{(N)} \otimes \mathcal{A}^{(N)} \qquad : \qquad \Delta f(A,B) = \int_{\mathsf{U}(N)} DU\, f(A + UBU^\dagger)\,,$$

for which the K_n are the primitive elements

$$\Delta K_n = K_n \otimes 1 + 1 \otimes K_n$$

$$\vdots$$

more in `math-ph/2508.21483`



Thank you !