

Introduction to random matrices

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Summary

1. General features.
Why RMT ?
What is a RMT ?
The classical ensembles
Large N , perturbative and topological expansion
2. Computational techniques
Coulomb gas picture and saddle point approximation
Orthogonal polynomials
Loop equations
3. “Angular matrix integrals”
Application to multi-matrix models
4. Universality ...

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1 Introduction and basic definitions

These lectures are devoted to a simple introduction to Random Matrix Theory (RMT).

Let us start with a definition which is a tautology. A RMT is a theory in which the random variables are matrices (of large size).

1.1 Why RMT ?

A short overview of the many occurrences of RMT in mathematics and physics. Originally introduced by Wishart (1928) in multivariate statistics (covariance matrices). Then starting in the 50's with Wigner, many occurrences of RMT in Physics ...

1. i) Wigner and the spectrum of energy levels in large nuclei.
ii) Electromagnetic response of irregular metallic grains.
Transport properties in disordered quantum systems.
iii) Classically chaotic quantum systems with few degrees of freedom.
Rydberg levels of hydrogen atoms in a strong magnetic field, etc, etc.
2. i) 't Hooft large N limit of $U(N)$ QCD, planar diagrams;
ii) From the counting of maps and triangulations to statistics of discretized surfaces to 2D quantum gravity. The double scaling limit.
iii) Random matrices and statistical mechanics models on "random lattices". KPZ formula.
iv) Random matrices and integrable hierarchies (KdV, KP, Toda, ...) and tau-functions.
v) Random matrices and growth models
3. i) Spectrum of Dirac operator, chiral effective actions
ii) String theory: Matrix models, quantum gravity, topological strings and supersymmetric gauge theories

... and in Mathematics:

1. Random matrices, random processes, random operators
2. Topology of the moduli space of curves and matrix integrals
3. Random matrices and "free probabilities".
4. Zeros of the Riemann ζ function and distributions of eigenvalues
5. Random matrices and combinatorics. The longest increasing subsequence of a random permutation.
Counting of various types of maps, of foldings, of colorings, of knots and links, ...

etc etc etc

1.2 Reviews

I list here some review articles and/or books on the subject of RMT.

[Porter] C.E. Porter *Statistical Theories of Spectra: Fluctuations*, Academic Pr, 1965. Contains reprints of all the “historic” papers.

[Mehta2] M.L. Mehta *Random Matrices* 2nd edition, Academic Pr. 1991.

[Mehta3] M.L. Mehta *Random Matrices* 3d edition, Elsevier 2004.

[DFGZJ] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, *2D Gravity and Random Matrices*, *Phys. Rep.* **254** (1995) 1–133.

[Be97] C.W.J. Beenakker, *Random-matrix theory of quantum transport*, *Rev.Mod.Phys.* **69**, (1997), 731, [cond-mat/9612179](#)

[GMGW98] T. Guhr, A. Müller-Groeling and H. A. Weidenmüller, *Random Matrix Theories in Quantum Physics: Common Concepts* *Phys. Reports* **299** (1998) 189-428, [cond-mat/9707301](#).

[FSV03] P.J. Forrester, N.C. Snaith and J.J.M. Verbaarschot, *J. Phys A* **36** (2003) *Special Issue: Random Matrix Theory*, R1-R10 ([cond-mat/0303207](#)) and 2859-3645.

[Ey05] B. Eynard, Saclay lecture notes 2001, thèse d’habilitation 2005.

[LH04] Proceedings of Les Houches School, *Applications of Random Matrices in Physics*, June 6-25, 2004 (Eds. E. Brézin, V. Kazakov, D. Serban, P. Wiegmann, and A. Zabrodin),

[ABDF] G. Akemann, J. Baik, and Ph. Di Francesco, *The Oxford Handbook of Random Matrix Theory*, Oxford Univ. Press 2011.

[Forr] P. J. Forrester, *Log-Gases and Random Matrices*, Princeton Univ. Press 2010.

1.3 The basic sets of RMT

A RMT is defined by the choice of

- a matrix ensemble (dictated by the physics of the problem, its symmetries, etc)
- a probability measure on that ensemble, *i.e.* an integration measure times a weight.

For example, suppose our random matrix describes the Hamiltonian of some quantum, finite dimensional system. It must then be a Hermitian operator *i.e.* (in some basis) a Hermitian matrix $H = H^\dagger$. On the other hand, if we are interested in the scattering of a quantum in a random medium, we shall use a scattering matrix \mathcal{S} which is unitary (conservation of probabilities), $\mathcal{S}\mathcal{S}^\dagger = I$, etc.

In each situation, that random matrix may be subject to additional requirements, due to the symmetries of the problem. For instance, according to a discussion of Wigner and Dyson, the Hamiltonian may have various types of symmetry, depending on the symmetries (time invariance, rotation invariance, half-integer total angular momentum) of the system. One finds that one has to consider three classes of Hermitian matrices, in which the entries take values in \mathbb{R} , \mathbb{C} or \mathbb{H} , the field of quaternions. It is conventional and convenient to refer to them by the integer $\beta = 1, 2, 4$ respectively. Note that β counts the number of real parameters of the off-diagonal entries H_{ij} .

Each of these three sets of matrices is invariant under, respectively, the real orthogonal group $O(N)$, the unitary group $U(N)$ and the “unitary symplectic group” $USp(N)$. The latter is made of unitary matrices that are real quaternionic, *i.e.* $\bar{S}^T = S^\dagger = S^{-1}$ (see Appendix B for more on quaternions and the symplectic group).

$$H = H^\dagger = H^* = H^T \mapsto OHO^T \quad (\beta = 1) . \quad (O)$$

$$H = H^\dagger \mapsto UHU^\dagger \quad (\beta = 2) . \quad (U)$$

$$H = H^R \mapsto SHS^\dagger \quad S^\dagger = S^R = S^{-1} \quad (\beta = 4) . \quad (S)$$

It is thus natural to take an integration measure that is invariant under that group

$$DH = \prod_i dH_{ii} \prod_{i < j} dH_{ij} \quad (1.1)$$

$$DH = \prod_i dH_{ii} \prod_{i < j} d\Re H_{ij} d\Im H_{ij} \quad (1.2)$$

$$DH = \prod_i dH_{ii}^{(0)} \prod_{i < j} \prod_{\alpha=0}^3 dH_{ij}^{(\alpha)} \quad (1.3)$$

If furthermore one assumes a Gaussian distribution

$$P(H)DH = \text{const.} e^{-\frac{1}{2}\beta \text{tr} H^2} DH \quad (1.4)$$

this defines respectively the Orthogonal, Unitary, and Symplectic Gaussian Ensembles of random matrices (GOE, GUE and GSE).

The Gaussian distribution have the peculiarity that the algebraically independent variables are also independent in the probabilistic sense, and even more, are *independent, identically distributed* (i.i.d.) variables¹. This follows from $\text{tr} H^2 = \sum_i H_{ii}^2 + 2 \sum_{i < j} \sum_{\alpha=0}^{\beta-1} (H_{ij}^{(\alpha)})^2$. More generally, one considers random matrices with upper diagonal matrix elements H_{ij} , $i < j$ that are i.i.d. centered random variables, while the diagonal elements H_{ii} are also i.i.d. centered, with possibly an independent law. The lower diagonal H_{ij} , $i > j$, are then obtained by the (real, complex or quaternionic) conjugation $H_{ij} = H_{ji}^*$. Those are called *Wigner matrices* ([Wigner 1957]).

But there is in general no reason (but the search of simplicity) to restrict oneself to Gaussian distributions or to matrix elements that are i.i.d. It is natural to generalize the Gaussian to the form

$$P(H)DH = e^{-N \text{tr} V(H)} DH , \quad (1.5)$$

with $V(H)$ typically a polynomial, and thus to a “potential” $\text{tr} V(H)$ whose trademark is to involve a single trace, hence no correlation between eigenvalues.

In other contexts, one may be led to introduce other classes and ensembles of matrices. *Wishart matrices* make frequent appearances in problems of statistics and statistical physics (random processes). Given a rectangular $M \times N$ matrix X (with real or complex coefficients), one considers the $N \times N$ matrix $W = X^\dagger X$. The Wishart ensemble is the set of matrices W when the X have a Gaussian law

$$P(X)DX = \text{const} e^{-\frac{1}{2}\beta \text{tr} X^\dagger X} \prod_{i,j} dX_{ij} , \quad (1.6)$$

and one could once again consider a more general potential $\text{tr} V(X^\dagger X)$.

There may be other reasons to select a particular ensemble of random matrices and a probability density on it. As we shall see, the large N limit of Hermitian matrix integrals is naturally associated with oriented Riemann surfaces of low genus, while symmetric matrices would lead to non-orientable surfaces etc. And the “action” (i.e. the logarithm of the Boltzmann weight) is dictated by the physics one want to study on such a surface.

1.4 Probability density function of the eigenvalues

Consider the $\beta = 1, 2, 4$ ensembles, of respectively real symmetric, Hermitian, or self-dual quaternionic real. Each such matrix may be diagonalized by a real orthogonal, unitary or unitary symplectic matrix, respectively,

$$M = \begin{cases} O \Lambda O^T & O \in \text{O}(N) & \beta = 1 \\ U \Lambda U^\dagger & U \in \text{U}(N) & \beta = 2 \\ S \Lambda S^R & S \in \text{USp}(N) & \beta = 4 \end{cases} \quad (1.7)$$

¹up to a factor 2 in the variance of off-diagonal vs diagonal elements...

with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)^2$. If the probability density of this set of matrices is invariant under the transformations that diagonalize the matrices, it is useful to recast it in terms of the eigenvalues. The first step is to write the integration measure in terms of the eigenvalues. This is a little exercise that one has to carry out for each set of RM. For example, for the three sets above of Symmetric, Hermitian and (real self-dual) Quaternionic matrices, one finds (see Appendix C) that

$$DM = \text{const.} |\Delta(\lambda)|^\beta \prod_{i=1}^N d\lambda_i DX . \quad (1.8)$$

where

$$\Delta(\lambda) := \prod_{j>i} (\lambda_j - \lambda_i) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \vdots & & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix} = \left(\det \lambda_j^{i-1} \right)_{1 \leq i, j \leq N} \quad (1.9)$$

is the Vandermonde determinant, and DX is the Haar measure on the compact group $O(N)$, $U(N)$ or $USp(N)$.

If we now choose a weight as in (1.5) and change notation $V \mapsto \beta V$, the probability distribution (or density) function (PDF) on eigenvalues for the three classical sets reads

$$P(\lambda_1, \dots, \lambda_N) \prod_{i=1}^N d\lambda_i = \text{const.} \exp -\beta \left(N \sum_{i=1}^N V(\lambda_i) - \sum_{i>j} \log |\lambda_i - \lambda_j| \right) \prod_{i=1}^N d\lambda_i \quad (1.10)$$

It exhibits a fundamental feature. Because of the Jacobian given by the Vandermonde determinant, eigenvalues cannot coincide. This is referred to as *repulsion of eigenvalues* of RM's. Alternatively (1.10) may be regarded as describing a classical statistical mechanical model of N charges submitted to a potential energy $NV(\lambda)$ and to a logarithmic 2D Coulomb repulsion, at inverse temperature β . There are two competing effects : eigenvalues tend to sit at the bottom of the potential, but this is prevented by their Coulomb repulsion. This ‘‘Coulomb gas picture’’ is extremely useful to help us develop some intuition on RMT and on the repulsion between levels or eigenvalues. We shall return to it below.

A similar exercise may be carried out for other sets of matrices, for example the unitary matrices of $U(N)$. They may be diagonalized by unitary transformations $U = V \text{diag}(e^{i\alpha_j}) V^\dagger$ and one finds for the PDF of the α 's

$$P(\alpha_1, \dots, \alpha_N) \prod_{i=1}^N d\alpha_i = \text{const.} |\Delta(e^{i\alpha_j})|^2 \prod_{i=1}^N d\alpha_i. \quad (1.11)$$

Wishart matrices, $W = X^\dagger X$, with X a $M \times N$ matrix and $M \geq N$, are positive definite: the eigenvalues of W are positive and will still be denoted λ_i . One may again compute their PDF

$$P(\lambda_1, \dots, \lambda_N) = \text{const} e^{-\beta N \sum_{i=1}^N V(\lambda_i)} \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(M-N+1)-1} |\Delta(\lambda)|^\beta . \quad (1.12)$$

In particular the Gaussian case $V(W) = \text{tr} W = \text{tr} X^\dagger X$ is sometimes referred to as the *Laguerre ensemble*.

Remark. In all the previous cases, a constant has been left unspecified. This constant may be computed (see [Mehta]) but it is irrelevant in most considerations, as it is absorbed into the normalization of $P(\lambda)$ and does not appear in computations of expectation values.

²In the quaternionic/symplectic case, this is understood as a diagonal of N 2×2 unit matrices times λ_j .

1.5 The quantities of interest.

For invariant measures that only depend on the eigenvalues $\lambda_1, \dots, \lambda_N$ of the random matrix M , it is natural to study the statistical properties of eigenvalues.³ One may be interested in global or in local statistics of the eigenvalues λ_i . Global quantities involve a large number ($O(N)$) of eigenvalues, like for example the expectation values of functions of the λ 's

$$\begin{aligned} \langle F \rangle &= \int P(\lambda_1, \dots, \lambda_N) F(\lambda_1, \lambda_2, \dots, \lambda_N) \prod_{i=1}^N d\lambda_i \\ &= \frac{1}{Z} \int |\Delta(\lambda)|^\beta e^{-\beta \sum_i V(\lambda_i)} F(\lambda_1, \lambda_2, \dots, \lambda_N) \prod_{i=1}^N d\lambda_i \end{aligned} \quad (1.13)$$

where Z is the normalization factor

$$Z = \int |\Delta(\lambda)|^\beta e^{-\beta \sum_i V(\lambda_i)} \prod_{i=1}^N d\lambda_i. \quad (1.14)$$

In the language of statistical physics, Z is the partition function of the system. And the large size limit (large N limit) just corresponds to the thermodynamic limit of that system. Just like in statistical mechanics, its definition may require the rescaling of some of the parameters (parameters of the potential, ratio M/N in the Wishart matrices, etc) and of the eigenvalues.

In particular we want to determine the density of eigenvalues

$$\rho(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i) \right\rangle \quad (1.15)$$

(normalized to 1, $\int d\lambda \rho(\lambda) = 1$). In terms of the joint probability density $P(\lambda_1, \lambda_2, \dots, \lambda_N)$, a symmetric function of its arguments, this is just $\rho(\lambda) = \int d\lambda_2 \dots d\lambda_N P(\lambda, \lambda_2, \dots, \lambda_N)$. More generally, defining $\mathcal{O}(\lambda) := \frac{1}{N} \sum_j \delta(\lambda - \lambda_j)$, the joint probability density of n eigenvalues (when the $N - n$ other are not observed) is

$$\begin{aligned} \rho_n(\lambda_1, \lambda_2, \dots, \lambda_n) &= \int d\lambda_{n+1} \dots d\lambda_N P(\lambda_1, \lambda_2, \dots, \lambda_N) \\ O_n(\lambda_1, \lambda_2, \dots, \lambda_n) &= \langle \mathcal{O}(\lambda_1) \dots \mathcal{O}(\lambda_n) \rangle = \frac{N!}{(N-n)!} \rho_n(\lambda_1, \lambda_2, \dots, \lambda_n) \end{aligned} \quad (1.16)$$

(we assume $\lambda_1, \dots, \lambda_n$ all different, to eliminate “contact terms”)⁴ from which all the correlation functions of eigenvalues follow. For example

$$\left\langle \prod_{i=1}^n \frac{1}{N} \text{tr} M^{k_i} \right\rangle = \int \prod_{i=1}^n (d\lambda_i \lambda_i^{k_i}) O_n(\lambda_1, \lambda_2, \dots, \lambda_n)$$

We may also wish to know the *connected* correlation functions, (or cumulants), $O_{2c}(\lambda_1, \lambda_2) = O_2(\lambda_1, \lambda_2) - O_1(\lambda_1)O_1(\lambda_2)$, etc ... As we'll see later, connected correlation functions are $O(1/N^2)$ w.r.t. the original

³The statistics of eigenvectors may also be of physical relevance. For example, the distribution of the scalar product $x = |\langle \phi_0 | \psi \rangle|$ of an eigenvector ψ of M with a fixed vector ϕ_0 yields the distribution of widths of levels, in a RMT description of nuclear levels, see [Porter]. For a matrix of the GOE, it may be shown to follow the χ^2 -law with one degree of freedom $f(x) = Cx^{-\frac{1}{2}}e^{-\frac{1}{2}x}$.

⁴The notation O_n is unconventional. Many authors unfortunately call it ρ_n , others R_n . As I have used R_n elsewhere, I have adopted O_n here.

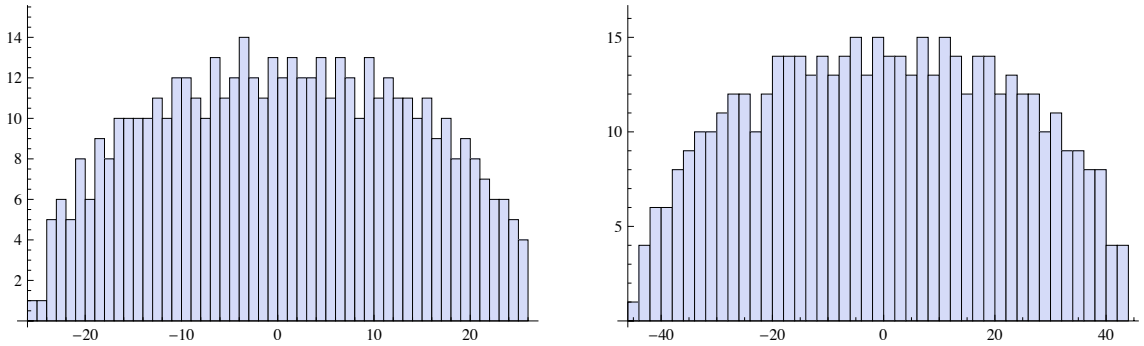


Figure 1: Histogram of the eigenvalues of a typical real symmetric 500×500 Wigner matrix, with i.i.d. elements obeying a uniform distribution over $[-1, 1]$ (left) or a normal centered distribution of variance 1 (right).

correlation functions. For example for a quantity A which is given by a “linear statistics” $\langle A^p \rangle = \text{tr} A^p(M)/N$, the variance is $\text{Var}(A) = \langle A^2 \rangle - \langle A \rangle^2 = \int O_{2c}(\lambda, \lambda') A(\lambda) A(\lambda') d\lambda d\lambda'$, of the fluctuations of conductance in quantum wires.

We shall also encounter the (trace of the) *resolvent*, aka Green function, if one thinks of M as a Hamiltonian, or as the *Stieltjes transform* of the density ρ

$$G(x) = \left\langle \frac{1}{N} \text{tr} \frac{1}{x - M} \right\rangle = \int d\lambda \frac{1}{x - \lambda} \rho(\lambda) . \quad (1.17)$$

It may also be regarded as the generating function of moments

$$G(x) = \sum_{p=0}^{\infty} x^{-p-1} \left\langle \frac{1}{N} \text{tr} M^p \right\rangle . \quad (1.18)$$

If eigenvalues are supported on a real segment (or collection of real segments) S , the discontinuity of $G(x)$ across this support is

$$G(x + i\epsilon) - G(x - i\epsilon) = -2i\pi\rho(x), \quad x \in S . \quad (1.19)$$

Here the Cauchy identity $(x \pm i\epsilon - a)^{-1} = \text{P.P.}(x - a)^{-1} \mp i\pi\delta(x - a)$, has been used, with P.P. the principal part and ϵ is a positive infinitesimally small real.

Other interesting and much studied quantities describe *local* properties of the distribution of the eigenvalues, i.e. on a scale which is $o(N)$. For example the probability $E_\beta(n, s)$ that an interval of length s contains exactly n eigenvalues, or the distribution of the spacings between neighboring eigenvalues: for a given ensemble, $p(s)ds$ is the probability that such a spacing lies in the interval $[s, s + ds]$. By a standard probabilistic argument one may prove that $p_\beta(s) = \partial^2 E_\beta(0, s)/\partial s^2$. See [Mehta 2, p 83ff and A 8].

1.6 Universality properties

Why is RMT ubiquitous? The key to understand why RMT appears in so many different contexts lies presumably in its universality properties. Indeed one may observe/conjecture and in some cases prove universality of its eigenvalue statistics under slight changes of the probability distribution, possibly including breaking of the symmetry that led to the definition of the ensemble.

For example, the distribution of spacings is the same in the Gaussian and Circular ensembles of matrices of a given type (Real symmetric, Complex Hermitian etc), or in the non Gaussian case, (with caveat, however... (several cuts)), see [Mehta 2, chap 10 and A 1]. According to the standard lore, the

“unfolded” spacing density is universal⁵. For the Gaussian ensembles, with $\beta = 1, 2$ or 4 , it is exactly computable, see below. On the other hand, it is very well approximated by the celebrated Wigner “surmise”

$$p_\beta(s) = a_\beta s^\beta e^{-b_\beta s^2}. \quad (1.20)$$

This is not an exact formula, but it turns out to approximate extremely well the exact result. The factor s^β in (1.20) is the origin of the vanishing of this spacing density, hence of the “repulsion of levels”. Based on a former work by von Neumann and himself (1929), Wigner (1957) gave a simple heuristic argument relating the power β to the number of conditions to impose to enforce a level crossing.

Consider a 2×2 matrix $A = (a_{ij})$ for simplicity. Depending whether we are considering a real symmetric, complex Hermitian, or real self-dual quaternionic matrix, the number of real parameters varies: two on the diagonal, and resp. $\beta = 1, 2, 4$, for the off-diagonal one. In this $\beta + 2$ -dimensional space, the condition that the matrix has a doubly degenerate eigenvalue, i.e. that the discriminant of the characteristic equation vanishes implies $\beta + 1$ conditions, *v.i.z.* $(a_{11} - a_{22})^2 + 4|a_{12}|^2 = 0$, i.e. describes a line. Let us compute the “phase space”: if s is the distance away from this line and ℓ the distance along the line, locally, the volume element is proportional to $s^\beta ds$ (times $d\ell$), and so is the probability of crossing of the two eigenvalues.

We return to this important concept of universality at the end of these lectures.

1.7 Feynman diagrams. Large N limits. ’t Hooft topological expansion.

In all the applications reviewed above, one considers matrices of large but finite size. It may be interesting to look at the limits of $N \rightarrow \infty$. By the plural on “limits”, I emphasize that there are many such limits. What will be referred to most of the time as large N (or ’t Hooft) limit in the following, is a limit consistent with a computation of the integrals in terms of Feynman diagrams. It leads to an ordering of these Feynman diagrams in terms of their topological character. But there are other ways of taking the limit $N \rightarrow \infty$, while letting some of the parameters of the theory approach a “critical” value. Such limits are referred to as “double scaling limit(s)”. Here we focus on the standard ’t Hooft limit.

Before we proceed, let me give a “crash course on Feynman diagrams”. Suppose we want to compute the (formal) power series expansion of the integral

$$Z(g) = \int_{\mathbb{R}^n} d^n x \exp\left(-\frac{1}{2} \sum x_\alpha A_{\alpha\beta} x_\beta + g x^4\right) = \sum_{p=0}^{\infty} \frac{g^p}{p!} \int_{\mathbb{R}^n} d^n x (x^4)^p e^{-\frac{1}{2} Q(x)} \quad (1.21)$$

Here x is a vector of \mathbb{R}^n , and $Q(x) := \sum x_\alpha A_{\alpha\beta} x_\beta$ is a positive definite quadratic form, and x^4 stands for $x^4 = \sum_{\alpha,\beta} x_\alpha^2 x_\beta^2$. In order that the integral makes sense, the sign of the “coupling constant” g should originally be chosen negative. One first notes that the following Gaussian integral

$$Z_0(b) = \int_{\mathbb{R}^n} d^n x \exp\left(-\frac{1}{2} \sum x_\alpha A_{\alpha\beta} x_\beta + \sum x_\alpha b_\alpha\right) = \left(\frac{(2\pi)^n}{\det A}\right)^{\frac{1}{2}} \exp\left(\frac{1}{2} b_\alpha A_{\alpha\beta}^{-1} b_\beta\right)$$

is a generating function of all integrals of monomials of x in the Gaussian integral

$$\begin{aligned} \int_{\mathbb{R}^n} dx x_{\alpha_1} \cdots x_{\alpha_{2r}} \exp\left(-\frac{1}{2} Q(x)\right) &= \prod_i \frac{\partial}{\partial j_{\alpha_i}} Z_0(b) \Big|_{b=0} \\ &= Z_0(0) \sum_{\substack{\text{all distinct} \\ \text{pairings } \mathcal{P}}} A_{\alpha_{\mathcal{P}(1)} \alpha_{\mathcal{P}(2)}}^{-1} \cdots A_{\alpha_{\mathcal{P}(2r-1)} \alpha_{\mathcal{P}(2r)}}^{-1}. \end{aligned} \quad (1.22)$$

Note that with each pairing between x_α and x_β is associated a matrix element $A_{\alpha\beta}^{-1} = \langle x_\alpha x_\beta \rangle$, called “propagator” in the physics literature and represented in a graphical way as $\alpha \text{---} \beta$. The resulting diagrammatic representation of the fourth moment is depicted on figure 2.

⁵The unfolding consists in a change of variable that makes the density uniform: $\lambda \mapsto x(\lambda)$ such that $\frac{dx}{d\lambda} = \rho(\lambda)$, see [Mehta]

$$\begin{aligned}
\langle x_\alpha x_\beta x_\gamma x_\delta \rangle &= \left\langle \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ | & | & | & | \\ \alpha & \beta & \gamma & \delta \end{array} \right\rangle \\
&= \begin{array}{c} \bullet \text{---} \bullet \\ \alpha \quad \beta \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \gamma \quad \delta \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \alpha \quad \beta \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \gamma \quad \delta \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \alpha \quad \beta \end{array} \begin{array}{c} \bullet \text{---} \bullet \\ \gamma \quad \delta \end{array} \\
&= A_{\alpha\beta}^{-1} A_{\gamma\delta}^{-1} + A_{\alpha\delta}^{-1} A_{\beta\gamma}^{-1} + A_{\alpha\gamma}^{-1} A_{\beta\delta}^{-1}
\end{aligned}$$

Figure 2: The three Feynman diagrams for the calculation of $\langle x_\alpha x_\beta x_\gamma x_\delta \rangle$

$$\begin{aligned}
\langle x^4 \rangle &= \langle x_\alpha x_\alpha x_\beta x_\beta \rangle = \left\langle \begin{array}{c} \times \\ \bullet \end{array} \right\rangle = \left\langle \begin{array}{cc} \alpha & \beta \\ \times & \times \\ \alpha & \beta \end{array} \right\rangle \\
&= \begin{array}{c} \bullet \text{---} \bullet \\ \times \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \times \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \times \end{array}
\end{aligned}$$

Figure 3: The three Feynman diagrams for the calculation of $\langle x^4 \rangle$

One then returns to (1.21) and uses the same representation: each $x^4 = x_\alpha x_\alpha x_\beta x_\beta$ is represented by a little cross and one is instructed to sum over all possible distinct pairings between the $4p$ lines that emerge from them. Each pairing carries a propagator $A_{\alpha\beta}^{-1}$, and one has finally to sum over all indices α_j . The series expansion in powers of g of the ratio $Z(g)/Z(0)$ is thus obtained by summing over all Feynman diagrams obtained with these rules. An example of the calculation of $Z(g)/Z(0)$ at order g is illustrated on fig. 3 and leads to

$$Z(g) = \left(\frac{\pi^n}{\det Q} \right)^{\frac{1}{2}} \left(1 + g((\text{tr} A^{-1})^2 + 2\text{tr} A^{-2}) + \dots \right)$$

A last useful remark is that, according to a well known combinatorial argument, the “free energy” $F(g) = \log Z(g)/Z(0)$ has a similar diagrammatic expansion, but restricted to *connected* Feynman diagrams.

Let us now apply this technique to matrix integrals and recall the famous result due to G. ’t Hooft [1]: the connected Feynman diagrams contributing to the free energy (or vacuum energy) in the large N limit of a matrix field theory have a topological characterization. We consider for definiteness the following one-Hermitian-matrix integral

$$Z = \int DM e^{-N \text{tr} (\frac{1}{2} M^2 - \frac{g}{4} M^4)}, \tag{1.23}$$

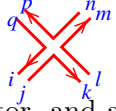
with the integration measure

$$DM = \prod_{i=1}^N dM_{ii} \prod_{i < j} d\Re M_{ij} d\Im M_{ij} \tag{1.24}$$

but the argument is of wider validity.

By a simple generalization of the previous discussion, the “propagator” is given by the inverse of the quadratic term in (2.1), $\langle M_{ij} M_{kl} \rangle_0 = \frac{1}{N} \delta_{il} \delta_{kj}$. According to ’t Hooft’s idea, we introduce a double

line notation to take care of the two indices of the matrix M . (The resulting diagrams are also called “fat graphs” in the mathematical literature). Then the Feynman rules look as follows: the propagator is represented as $\langle M_{ij} M_{kl} \rangle_0 = \overset{i}{\leftarrow} \overset{l}{\rightarrow} \frac{1}{N} \delta_{il} \delta_{kl}$, while

the vertex  $= gN \delta_{jk} \delta_{lm} \delta_{np} \delta_{qi}$. A factor N is thus attached to each vertex, a factor N^{-1} to each propagator, and a factor N is also generated whenever an index loop is closed. The total power of N reads

$$\text{power of } N = \#\text{vertices} - \#\text{propagators} + \#\text{faces}. \quad (1.25)$$

For a *connected* Feynman diagram, imagine we paste a disk (or some piece of surface homeomorphic to a disk) along each index loop: this builds a connected surface, which is compact (no boundaries) and orientable⁶. Then the previous combination $V - E + F$ of the numbers of vertices, edges (propagators) and faces of the diagram is just the Euler characteristics χ of the compact surface, which is a topological invariant of that surface. For a connected orientable surface with no boundaries, $\chi = 2 - 2h$ in terms of its genus (or number of **handles**) h . Thus

$$\text{power of } N \Big|_{\text{conn. diagram}} = V - E + F = \chi = 2 - 2h; \quad (1.26)$$

Note that more generally, for a diagram with c connected parts, the power of N is

$$\text{power of } N \Big|_{\substack{\text{diagram with} \\ c \text{ conn. parts}}} = \sum_{i=1}^c (2 - 2h_i) \leq 2c. \quad (1.27)$$

Thus the large N expansion of $\mathcal{F} = \log Z$ is interpreted as a topological expansion in terms of connected Feynman diagrams drawn on surfaces of increasing genus h :

$$\log Z = \sum_{h=0}^{\infty} N^{2(1-h)} \mathcal{F}^{(h)}(g) \quad (1.28)$$

$$\mathcal{F}^{(h)}(g) = \sum_{\substack{\text{connected Feynman graphs } \Gamma \\ \text{of genus } h \text{ and with } n \text{ vertices}}} \frac{g^n}{\text{Symm } \Gamma} \quad (1.29)$$

(with $\text{Symm } \Gamma$ the “symmetry factor of graph Γ .”) This observation is of great importance in several contexts. The sum over Feynman diagrams with vertices of coordinence k , drawn on a Riemann surface of genus h , amounts, by looking at the dual picture, to a sum over discretizations of that surface by k -gons. (See figure 4). The large N limit of matrix theories thus opens the way to the study of statistical mechanical models on random geometry, to new approaches to quantum gravity, to combinatorics of planar objects and to other mathematical problems, etc. See M. Marino’s lectures for an illustration.

2 Computational techniques

To be specific, consider a one-Hermitian matrix model, where the matrix $M \in H(N)$, the set of $N \times N$ Hermitian matrices, $M = M^\dagger$. We restrict to situations with $U(N)$ invariance: the weight is invariant under $M \rightarrow U M U^\dagger$, $U \in U(N)$, the group of unitary matrices.

Consider for definiteness the following integral

$$Z = \int dM e^{-N \text{tr} V(M)}, \quad (2.1)$$

⁶ Why is this surface orientable? Show that an integral over real symmetric matrices would also produce non-orientable surfaces.

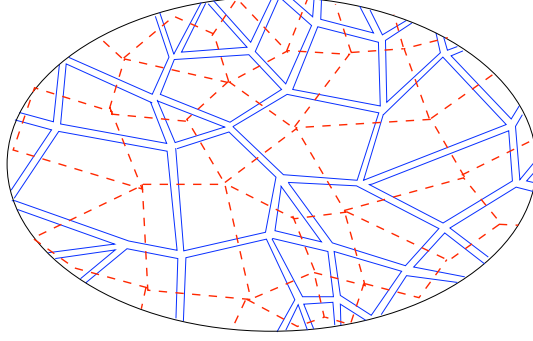


Figure 4: A quadrangulation of a genus 0 surface, or the dual fat graph with 4-vertices

where $V(M)$ is a polynomial of degree $d + 1$. Two particularly interesting cases are those with cubic or quartic potentials $V_3(M) = (\frac{1}{2}M^2 + \frac{g}{3}M^3)$ and $V_4(M) = (\frac{1}{2}M^2 + \frac{g}{4}M^4)$. We might also have logarithmic terms in V . Note that we are excluding the possible occurrence of multi-traces in the potential, for example $(\text{tr}M^2)^2$.

The integration measure is as explained above

$$dM = \prod_{i=1}^N dM_{ii} \prod_{i<j} d\Re M_{ij} d\Im M_{ij} , \quad (2.2)$$

(up to an arbitrary constant, which we choose equal to 1 for convenience). The integrand depends only on the eigenvalues $\lambda_1, \dots, \lambda_N$ of M . It is thus suggested to rewrite the integral as an integral over the λ 's. Paying due attention to the Jacobian, see above and Appendix C, one finds

$$Z = \int \prod_{i=1}^N d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-N \sum_{i=1}^N V(\lambda_i)} , \quad (2.3)$$

(again up to an overall normalization which is of no interest to us).

More generally we shall be interested in correlation function of M ,

$$\langle F \rangle = Z^{-1} \int DM F(M) e^{-NV(M)} , \quad (2.4)$$

with F invariant under $M \rightarrow U M U^\dagger$. For example, $F(M) = \frac{1}{N} \text{tr} M^k \frac{1}{N} \text{tr} M^\ell$, or any polynomial in the $\text{tr} M^k$. Such F are expressible in terms of the eigenvalues λ_i , $F(M) = F(\lambda)$ (with a little abuse of notations).

We shall treat these λ integrals in three different ways.

2.1 Saddle point approximation [2]

In the following, it is essential that the potential V contains a single trace. Rewrite then (2.3) as

$$Z = \int \prod_{i=1}^N d\lambda_i \exp(-N^2 S(\lambda)) \quad (2.5)$$

where the ‘‘action’’ $S(\lambda)$ reads

$$S(\lambda) = \frac{1}{N} \sum_{i=1}^N V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j|. \quad (2.6)$$

As the notation suggests, in the large N limit, $N^2 S$ is of order N^2 if we assume that all λ are of order 1, an assumption to be justified *a posteriori* by its consistency. It is thus legitimate to use the saddle point method, i.e. to look for the stationary point of that action, i.e. the solution $\{\tilde{\lambda}_i\}$ of

$$\frac{2}{N} \sum_{j \neq i} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j} = V'(\tilde{\lambda}_i). \quad (2.7)$$

If we find such a configuration $\tilde{\lambda}$, assumed to be unique⁷, the large N limit of Z will read

$$Z \approx e^{-N^2 S(\tilde{\lambda})}. \quad (2.8)$$

More generally any invariant correlation function of the form (2.4) and of order N^0 , is also computable by the saddle method

$$\langle F \rangle \approx F(\tilde{\lambda}) \quad (2.9)$$

and it follows that, in the large N limit, correlation functions of products of invariant functions factorize

$$\langle FG \rangle \approx F(\tilde{\lambda})G(\tilde{\lambda}) = \langle F \rangle \langle G \rangle \quad (2.10)$$

or in other words, the ‘‘connected correlation function’’ $\langle FG \rangle_c := \langle FG \rangle - \langle F \rangle \langle G \rangle$ is of lower order in N (typically $O(N^{-2})$). This observation is of wider validity than the saddle point method and also follows from the Feynman diagram expansion: in (1.27) above, we noticed that c -connected diagrams have a leading behaviour N^{2c} , and hence the leading contribution to $\langle FG \rangle$ comes from disconnected diagrams and equals $\langle F \rangle \langle G \rangle$.

To solve the problem, namely to find a distribution of eigenvalues satisfying (2.7), one considers the resolvent

$$G(x) = \frac{1}{N} \left\langle \text{tr} \frac{1}{x - M} \right\rangle = \frac{1}{N} \left\langle \sum_{i=1}^N \frac{1}{x - \lambda_i} \right\rangle. \quad (2.11)$$

According to the above argument, its saddle point value reads

$$G(x) = \frac{1}{N} \sum_{i=1}^N \frac{1}{x - \tilde{\lambda}_i}.$$

Computing its square leads to

$$\begin{aligned} G^2(x) &= \frac{1}{N^2} \sum_{i,j=1,\dots,N} \frac{1}{(x - \tilde{\lambda}_i)(x - \tilde{\lambda}_j)} \\ &= \frac{1}{N^2} \sum_{i=1}^N \frac{1}{(x - \tilde{\lambda}_i)^2} + \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \frac{1}{(x - \tilde{\lambda}_i)(x - \tilde{\lambda}_j)} \\ &= -\frac{1}{N} G'(x) + \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j} \left(\frac{1}{x - \tilde{\lambda}_i} - \frac{1}{x - \tilde{\lambda}_j} \right) \\ &= -\frac{1}{N} G'(x) + \frac{2}{N^2} \sum_{i=1}^N \frac{1}{x - \tilde{\lambda}_i} \sum_{j \neq i} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j} = -\frac{1}{N} G'(x) + \frac{1}{N} \sum_{i=1}^N \frac{V'(\tilde{\lambda}_i)}{x - \tilde{\lambda}_i} \\ &= -\frac{1}{N} G'(x) + \frac{1}{N} \sum_{i=1}^N \frac{V'(\tilde{\lambda}_i) - V'(x) + V'(x)}{x - \tilde{\lambda}_i} \\ &= -\frac{1}{N} G'(x) + V'(x)G(x) - \frac{1}{N} \sum_{i=1}^N \frac{V'(x) - V'(\tilde{\lambda}_i)}{x - \tilde{\lambda}_i} \end{aligned} \quad (2.12)$$

⁷See [3] for a discussion of that point. I’m grateful to A. Hardy for drawing my attention to that reference.

where use has been made of (2.7) on the fourth line.

Note that $P(x) := \frac{1}{N} \sum_i \frac{V'(x) - V'(\tilde{\lambda}_i)}{x - \tilde{\lambda}_i}$ is a *polynomial* in x of degree $d - 1$ ⁸. For the Gaussian potential $V(\lambda) = \frac{1}{2}\lambda^2$, $P = 1$, while for $V(\lambda) = \frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4$, $P(x) = 1 + g(x^2 + x \frac{1}{N} \sum_i \tilde{\lambda}_i + \frac{1}{N} \sum_i \tilde{\lambda}_i^2)$, which depends on the first and second moments of the $\tilde{\lambda}_i$'s. We finally get

$$G^2(x) - V'(x)G(x) + \frac{1}{N}G'(x) + P(x) = 0. \quad (2.13)$$

For N large, it is safe to neglect the $1/N$ term. We thus get a quadratic equation for $G(x)$, with a (not yet completely determined) polynomial P , with solution

$$G(x) = \frac{1}{2} \left(V'(x) - \sqrt{V'(x)^2 - 4P(x)} \right) \quad (2.14)$$

where the minus sign in front of the square root is dictated by the requirement that for large $|x|$, $G(x) \sim 1/x$. Show that (2.14) does satisfy $G(x) \sim 1/x$ for large $|x|$

In that same large N limit, we imagine the $\tilde{\lambda}$'s to form a continuous distribution with density $\rho(\tilde{\lambda})$ on a real support \mathcal{S} ⁹ and thus $\int_{\mathcal{S}} d\lambda \rho(\lambda) = 1$. Then $G(x)$ reads

$$G(x) = \int_{\mathcal{S}} \frac{d\mu \rho(\mu)}{x - \mu}, \quad (2.15)$$

and $G(x)$ is thus an analytic function of x in the complex plane cut along the real support \mathcal{S} , and behaving as $1/x$ for large $|x|$. As in (1.19) one may extract ρ from the discontinuity of G across that cut

$$G(x + i\epsilon) - G(x - i\epsilon) = -2i\pi\rho(x)$$

(with ϵ infinitesimal > 0). Also (2.7) yields

$$2\text{P.P.} \int_{\mathcal{S}} \frac{d\mu \rho(\mu)}{\lambda - \mu} = V'(\lambda) \quad \text{if } \lambda \in \mathcal{S}. \quad (2.16)$$

(P.P.= principal part), and along the cut, G reads

$$G(x \pm i\epsilon) = \frac{1}{2}V'(\lambda) \mp i\pi\rho(x) \quad x \in \mathcal{S}. \quad (2.17)$$

For a purely Gaussian potential $V(\lambda) = \frac{1}{2}\lambda^2$, $V'(\lambda) = \lambda$, $P = 1$ and it follows from (2.14, 2.17) that $\rho(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2}$ on the segment $\lambda \in [-2, 2]$: this is the celebrated “semi-circle law” of Wigner¹⁰. For more general potentials, it is reasonable to assume first¹¹ \mathcal{S} to be still a finite segment $[-2a', 2a'']$

The solution (2.14) thus reduces to

$$G(x) = \frac{1}{2}V'(x) - Q(x)\sqrt{(x + 2a')(x - 2a'')} \quad (2.18)$$

where the coefficients of the polynomial $Q(x)$ and a' , a'' are determined by the condition that $G(x) \sim 1/x$ for large $|x|$. Q is of degree $d - 1$, and one verifies that the number of unknowns equals that of conditions, namely $d + 2$. The solution is unique (under the one-cut assumption).

⁸I recall that V is of degree $d + 1$

⁹That is $\tilde{\lambda}_i \mapsto \lambda(u := i/N)$, $du/d\lambda = \rho(\lambda)$, $1/N \sum_i f(\tilde{\lambda}_i) \mapsto \int_0^1 du f(\lambda(u)) = \int_{\mathcal{S}} d\lambda \rho(\lambda) f(\lambda)$.

¹⁰Caveat: if one returns to the original normalization of Gaussian matrices with a weight $\exp -\frac{1}{2}\text{tr}M^2$ with no factor N , this means that the semi-circle law gives a spectrum on $[-2\sqrt{N}, 2\sqrt{N}]$.

¹¹We shall return to this point at the end of this section.

Example For the quartic potential $V(\lambda) = \frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4$, we have by symmetry $a' = a'' =: a$ and

$$G(x) = \frac{1}{2}(x + gx^3) - \left(\frac{1}{2} + \frac{g}{2}x^2 + ga^2\right)\sqrt{x^2 - 4a^2} \quad (2.19)$$

with a^2 the solution of

$$3ga^4 + a^2 - 1 = 0 \quad (2.20)$$

which goes to 1 as $g \rightarrow 0$ (a limit where we recover Wigner's semi-circle law). From (2.19) we extract

$$\rho(\lambda) = \frac{1}{\pi} \left(\frac{1}{2} + \frac{g}{2}\lambda^2 + ga^2\right)\sqrt{4a^2 - \lambda^2} \quad (2.21)$$

and we may compute all invariant quantities like the free energy or the moments

$$G_{2p} := \left\langle \frac{1}{N} \text{tr} M^{2p} \right\rangle = \int d\lambda \lambda^{2p} \rho(\lambda) .$$

For example $G_2 = (4 - a^2)a^2/3$, $G_4 = (3 - a^2)a^4$, etc. All these functions of a^2 are singular as functions of g at the point where the two roots of (2.20) coalesce

$$g_c = -\frac{1}{12} . \quad (2.22)$$

For example the genus 0 free energy

$$\begin{aligned} \mathcal{F}^{(0)}(g) &:= \lim_{N \rightarrow \infty} (1/N^2) \log \left(\frac{Z(g)}{Z(0)} \right) = \frac{1}{2} \log a^2 - \frac{1}{24}(a^2 - 1)(9 - a^2) \\ &= \sum (-3g)^n \frac{(2n-1)!}{n!(n+2)!} \end{aligned} \quad (2.23)$$

has a power-law singularity

$$\mathcal{F}^{(0)}(g) \underset{g \rightarrow g_c}{\approx} |g - g_c|^{5/2} \quad (2.24)$$

which reflects on its series expansion

$$\mathcal{F}^{(0)}(g) = \sum_{n=0}^{\infty} f_n g^n \quad , \quad f_n \underset{n \rightarrow \infty}{\approx} \text{const} |g_c|^{-n} n^{-7/2} . \quad (2.25)$$

Several comments are in order at this point.

1. In contrast with the severe divergences of ordinary perturbation theories (with typical n -th order growing as $n!$), here we get a *convergent* g -expansion, and $f_n \sim |g_c|^n$. The result has been derived in the explicit case of the quartic potential but is quite general. The large N limit of matrix integrals, or for that matter, of any such large N field theory, is expected to have much better behaved perturbative expansions than ordinary QFT integrals. Notice that even the badly defined matrix integral with a *cubic* potential acquires a decent meaning through a convergent expansion in the large N limit. We conclude that interchanging the perturbative g expansion and the large N limit is not innocent!
2. Within a large class of potentials, one finds the same singular behavior (2.24), with the exponent $5/2$: this ‘‘critical’’ exponent is *universal*, while the radius of convergence g_c is non-universal. Likewise the density of eigenvalues is a non universal function, as is apparent on expression (2.21) or on Fig. 5, but its vanishing behavior like a square root at the edge $|\lambda| \sim 2a$ is universal, for $|g| < g_c$. The situation is strongly reminiscent of what happens in critical phenomena.

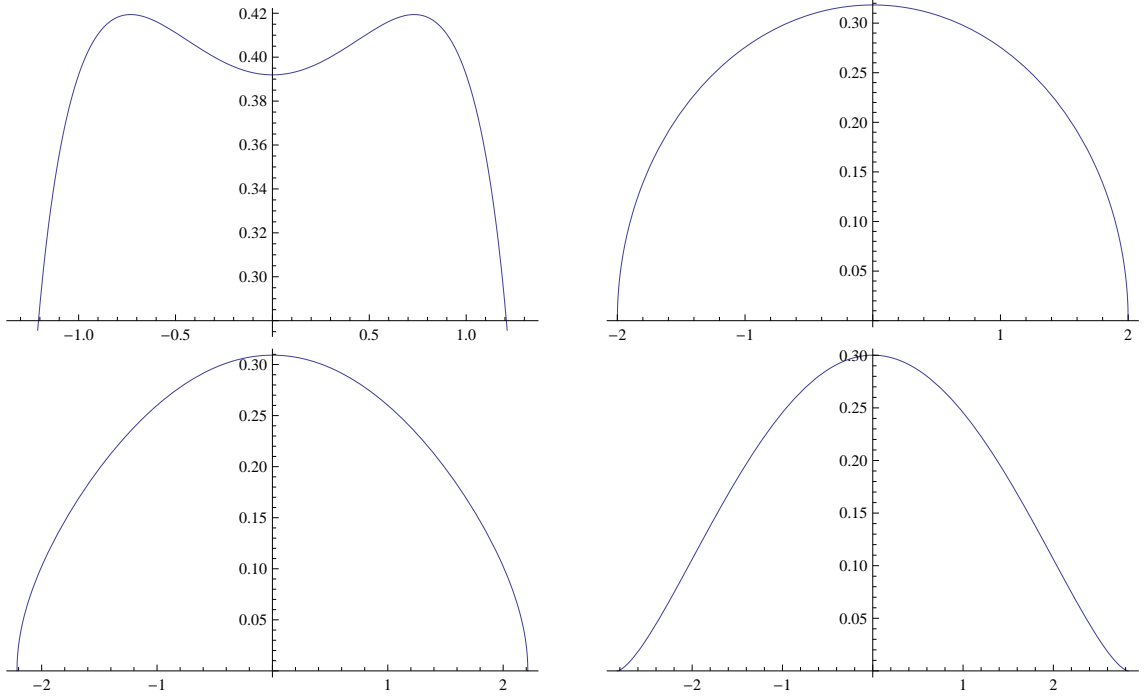


Figure 5: Density of eigenvalues for the quartic potential; from left to right, and from top to bottom, $g = 1$, $g = 0$, $g = -1/20$ and $g = g_c = -1/12$.

2.2 Orthogonal polynomials

The saddle point method is fine, the resulting density of eigenvalues gives a most helpful intuition of what's going on, but the method has its limitations: in particular it doesn't allow to compute the subdominant terms in the large N limit (in the “higher genus” free energy, or in (connected) correlation functions). We now turn to a more powerful and more systematic procedure, based on orthogonal polynomials. For a review, see [4]. We illustrate it again on the same integral (2.3).

Introduce the polynomials $P_n(\lambda)$ of degree n in λ , normalized by $P_n(\lambda) = \lambda^n + \dots$ (they are “monic”), and orthogonal for the weight appearing in (2.3)

$$\int d\lambda P_n(\lambda) P_m(\lambda) e^{-NV(\lambda)} = h_n \delta_{nm} . \quad (2.26)$$

The coefficients h_n are yet to be determined.

There exist closed formulae expressing these orthogonal polynomials in a determinant form, see Appendix D.

Remember from (1.9) that the Jacobian $\Delta(\lambda)$ may be written as a Vandermonde determinant $\Delta(\lambda) = \det \left(\lambda_j^{i-1} \right)_{1 \leq i, j \leq N}$ and may thus be rewritten in terms of the *monic* polynomials P_n as

$$\Delta(\lambda) = \det (P_{i-1}(\lambda_j))_{1 \leq i, j \leq N} = \sum_{\sigma \in \mathcal{S}_N} \epsilon_\sigma \prod_{i=1}^N P_{\sigma(i)-1}(\lambda_i) \quad (2.27)$$

by linear combinations of its rows. Thus (2.3) reads

$$\begin{aligned}
Z &= \int \Delta(\lambda)^2 \prod_{i=1}^N e^{-NV(\lambda_i)} d\lambda_i \\
&= \sum_{\sigma, \tau \in \mathcal{S}_N} \epsilon_\sigma \epsilon_\tau \prod_{i=1}^N \int P_{\sigma(i)-1}(\lambda_i) P_{\tau(i)-1}(\lambda_i) e^{-NV(\lambda_i)} d\lambda_i \\
&= \sum_{\sigma, \tau \in \mathcal{S}_N} \delta_{\sigma\tau} \prod_{j=0}^{N-1} h_j \\
&= N! \prod_{j=0}^{N-1} h_j
\end{aligned} \tag{2.28}$$

Evaluating the genus 0 free energy $\mathcal{F}^{(0)} := \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z$ thus amounts to computing

$$\mathcal{F}^{(0)} = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{n=0}^{N-1} \log h_n, \tag{2.29}$$

(but one may also extract the subdominant terms as $N \rightarrow \infty$, hence the higher genus contributions to the free energy)

• *The recursion formulae on the P_n .* The orthogonal polynomials satisfy as usual a 3-term recursion formula

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda). \tag{2.30}$$

Homework: Check that the decomposition of $\lambda P_n(\lambda)$ on the P_k 's involves only $k = n, n \pm 1$. Check that, for an even potential $V(\lambda)$ like $V(\lambda) = \frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4$, the term $k = n$ is absent, hence $S_n = 0$.

Computing in two different ways $\int d\lambda e^{-V(\lambda)} \lambda P_n(\lambda) P_{n-1}(\lambda) = R_n h_{n-1} = h_n$ yields the relation

$$R_n = \frac{h_n}{h_{n-1}}. \tag{2.31}$$

(In the Gaussian case, $R_n = n/N$.) The expression (2.28) of the partition function leads therefore to

$$Z = N! h_0^N \prod_{i=1}^{N-1} R_i^{N-i}. \tag{2.32}$$

On the other hand, there is a non-linear recursion formula obtained by looking at $\int d\lambda e^{-V(\lambda)} \lambda P'_n(\lambda) P_n(\lambda)$, using repeatedly the orthogonality property and integrating by parts

$$nh_n = \int d\lambda e^{-NV(\lambda)} \lambda P'_n(\lambda) P_n(\lambda) \tag{2.33}$$

$$= \int d\lambda e^{-NV(\lambda)} P'_n(\lambda) (P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda)) \tag{2.34}$$

$$= R_n \int d\lambda e^{-NV(\lambda)} P'_n(\lambda) P_{n-1}(\lambda) \tag{2.35}$$

$$= NR_n \int d\lambda e^{-NV(\lambda)} V'(\lambda) P_n(\lambda) P_{n-1}(\lambda). \tag{2.36}$$

By repeated use of (2.30), for any polynomial V' of degree d , $V'(\lambda) P_{n-1}(\lambda)$ decomposes on $P_k(\lambda)$ with $n-1-d \leq k \leq n-1+d$, and we project the result on P_n . For example, in the case of $V'(\lambda) = \lambda + g\lambda^3$,

$$V'(\lambda) P_{n-1}(\lambda) = (1 + g(R_{n+1} + R_n + R_{n-1})) P_n(\lambda) + \dots \tag{2.37}$$

where the dots stand for terms orthogonal to P_n . Hence (2.33) gives the desired relation

$$\frac{n}{N} = R_n (1 + g(R_{n+1} + R_n + R_{n-1})) . \quad (2.38)$$

• *Free fermion picture. Results for the correlation functions*

One may rephrase these relations in terms of the orthonormalized ψ_n

$$\psi_n(\lambda) = \frac{P_n(\lambda)e^{-\frac{1}{2}NV(\lambda)}}{\sqrt{h_n}} , \quad (2.39)$$

regarded as the wave functions of one particle states. Out of them, we construct the N -particle state of wave function

$$\Psi(\lambda_1, \dots, \lambda_N) = \det [\psi_{i-1}(\lambda_j)]_{1 \leq i, j \leq N} \quad (2.40)$$

which is a Slater determinant, a wave function of N independent fermions. In terms of Ψ , the partition function or any expectation value reads

$$\left(\prod_0^{N-1} h_j \right)^{-1} Z = \int d\lambda_1 \cdots d\lambda_N |\Psi(\lambda_1, \dots, \lambda_N)|^2 = \langle \Psi | \Psi \rangle = N! \quad (2.41)$$

$$\left\langle \frac{1}{N} \text{tr} A(M) \right\rangle = \frac{\langle \Psi | A(\lambda) | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad (2.42)$$

The correlation functions may be written in a compact form in terms of the ψ

$$\rho(\lambda_1, \dots, \lambda_N) = \frac{1}{N!} (\det \psi_j(\lambda_i))^2 \quad (2.43)$$

or

$$\rho(\lambda_1, \dots, \lambda_N) = \frac{1}{N!} \det K(\lambda_i, \lambda_j)_{1 \leq i, j \leq N} \quad (2.44)$$

where the kernel K reads

$$K(\lambda, \mu) = \sum_{i=0}^{N-1} \psi_i(\lambda) \psi_i(\mu) = C \frac{\psi_N(\lambda) \psi_{N-1}(\mu) - \psi_{N-1}(\lambda) \psi_N(\mu)}{\lambda - \mu} . \quad (2.45)$$

Passing from (2.43) to (2.44) involves the Gram identity, and the last reexpression uses Darboux-Christoffel identity (see Appendix D). The kernel K satisfies

$$\int K(\lambda, \lambda) d\lambda = N \quad \int K(\lambda, \mu) K(\mu, \nu) d\mu = K(\lambda, \nu) . \quad (2.46)$$

By repeated use of Dyson-Mehta theorem, $\int d\lambda_n \det_n K = \frac{N-n+1}{N} \det_{n-1} K$, (see Appendix D), and one may integrate over $\lambda_{n+1}, \dots, \lambda_N$ to get

$$O_n(\lambda_1, \dots, \lambda_n) = \frac{N!}{(N-n)!} \rho_n(\lambda_1, \dots, \lambda_n) = \det K(\lambda_i, \lambda_j)_{1 \leq i, j \leq n} . \quad (2.47)$$

For example, the 1-point and 2-point correlation functions read $\rho(\lambda) = O_1(\lambda) = K(\lambda, \lambda)$, $O_2(\lambda, \mu) = K(\lambda, \lambda)K(\mu, \mu) - K^2(\lambda, \mu)$, $O_{2c}(\lambda, \mu) = -K^2(\lambda, \mu)$, see figure 6.

• *Large N limit*

So far our discussion was conducted for N finite. We now take the limit $N \rightarrow \infty$.

If we are just interested in the limit of Z or of its logarithm, the free energy \mathcal{F} , it is sufficient to control the large n behavior of the h_n .

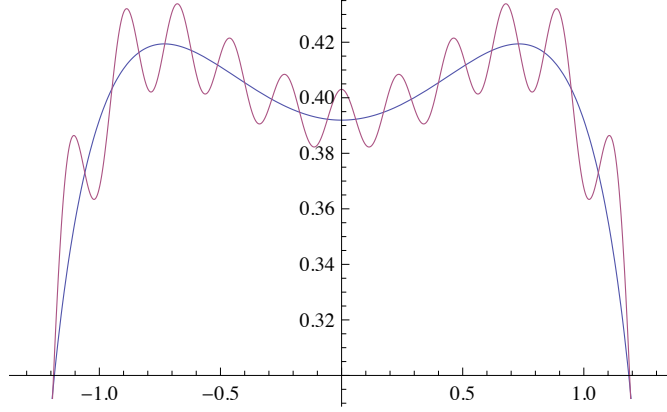


Figure 6: The density of eigenvalues ρ for the potential $V(x) = \frac{1}{2}x^2 + \frac{g}{4}x^4$ for $g = 1$: limit curve and as computed from $K(\lambda, \lambda) = \frac{1}{N} \sum_{n=0}^{N-1} \psi(\lambda)^2$ for $N = 11$.

The result of (2.32) yields for N large

$$\frac{1}{N^2} \log \frac{Z}{Z(g=0)} \approx \frac{1}{N} \sum_{i=1}^{N-1} \left(1 - \frac{i}{N}\right) \left(\log R_i - \log \frac{i}{N}\right) \approx \int_0^1 dy (1-y) \log \frac{R(y)}{y} \quad (2.48)$$

(with the additive constant fixed by comparison with the Gaussian case). Here we are assuming that R_n admits a limit depending only on $y = n/N$ for n and N large. In that limit, we identify to $R(y)$ all R_k appearing in the recursion relation following from (2.33). This recursion relation is thus a quadratic equation for $R(y)$. Solving it and plugging it into (2.48) yields $\mathcal{F}^{(0)} = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z$. For example in the quartic case, $R(y)$ satisfies

$$y = R(y)(1 + 3gR(y))$$

whence

$$R_{\text{quartic}}(y) = \frac{\sqrt{1 + 12gy} - 1}{6g} \quad (2.49)$$

and the Gaussian case ($g = 0$) used to normalize our formulae has $R_{\text{Gaussian}}(y) = y$, whence

$$\mathcal{F}_{\text{quartic}}^{(0)} = \int_0^1 dy (1-y) \log \left(\frac{\sqrt{1 + 12gy} - 1}{6gy} \right), \quad (2.50)$$

an expression which may be shown to lead to (2.23).

But the orthogonal polynomials allow us to go beyond the large N limit and to compute the subdominant terms, for example of the free energy, i.e. the higher genus contributions $\mathcal{F}^{(h)}(g)$. One finds that the leading g -singularity of any $\mathcal{F}^{(h)}(g)$ is the same as for the leading term $\mathcal{F}^{(0)}(g)$, but with a different exponent

$$\mathcal{F}^{(h)}(g) \underset{g \rightarrow g_c}{\approx} |g - g_c|^{5/2(1-h)}. \quad (2.51)$$

The fact that all subdominant terms have a g -singularity at the same point is in agreement with our discussion of the end of previous section, on the interchange of the g -expansion and the large N limit. . . The behavior (2.51) opens the route to the *double scaling limit*, in which $N \rightarrow \infty$ while $g \rightarrow g_c$ in such a way that $N^2|g - g_c|^{5/2}$ is fixed, with a remarkable relationship with 2D quantum gravity and Liouville theory, see M. Marino's lectures and [DFGZJ].

Let us look now at the large N limit of correlation functions, in particular of the kernel K of (2.45). It depends on the study of the asymptotic behavior of the orthogonal polynomials P_n or of their orthonormalized version ψ_n . Various techniques of increasing sophistication have been used for this purpose: saddle point approximation [Brézin-Zee, Eynard], WKB and Riemann-Hilbert problem [Bleher-Its, Baik-Deift-Johansson...]. It is convenient to use the determinant representation of the P_n . But the problem is made difficult by the oscillations of P_{N-1} and P_N , see fig. 6.

For the two-point function of the one-matrix model with an even potential $V(\lambda)$, and assuming one cut $[-2a, 2a]$, if one writes $\lambda = 2a \cos \varphi$, $f(\lambda) = 2a \sin \varphi$

$$\psi_n(\lambda) \sim e^{\frac{1}{2}NV(\lambda)} \frac{1}{\sqrt{f(\lambda)}} \cos(N\zeta(\lambda) + (N-n)\varphi(\lambda) + \frac{1}{2}\varphi(\lambda) + \text{const.}) \quad (2.52)$$

where $\zeta(\lambda)$ is the primitive of $-\pi\rho(\lambda)$ vanishing at the lower bound of the cut [Brézin-Zee 93, Eynard 94].

From this expression one derives the behavior of the two-point correlation function in different regimes. At short separations, $|\lambda - \mu| \sim O(N^{-1}) \ll 1$, or at finite separation, but after smoothing, one finds universal functions.

2.3 Loop equation

Now that we have recognized the central role played by the resolvent $G(x) = \langle \frac{1}{N} \text{tr} \frac{1}{x-M} \rangle$, let us try to compute it directly. The loop equation (alias equation of motion) expresses the invariance of the integral defining $G(x)$ under infinitesimal changes of the integration variable M , or alternatively, that the integral of a total derivative vanishes

$$\int dM \frac{d}{dM_{kl}} \left[\left(\frac{1}{x-M} \right)_{ij} e^{-N \text{tr} V(M)} \right] = 0 \quad (2.53)$$

with, as before, $V(M)$ a polynomial of M . One finds

$$\int dM \left[\left(\frac{1}{x-M} \right)_{ik} \left(\frac{1}{x-M} \right)_{lj} - N \left(\frac{1}{x-M} \right)_{ij} (V'(M))_{lk} \right] e^{-N \text{tr} V(M)} \quad (2.54)$$

After summation over $i = k$ and $j = l$ and division by $N^2 Z$, one finds

$$\left\langle \left(\frac{1}{N} \text{tr} \frac{1}{x-M} \right)^2 \right\rangle - \left\langle \left(\frac{1}{N} \text{tr} \frac{V'(M)}{x-M} \right) \right\rangle = 0. \quad (2.55)$$

We finally write

$$\left\langle \left(\frac{1}{N} \text{tr} \frac{V'(M)}{x-M} \right) \right\rangle = V'(x)G(x) + \left\langle \left(\frac{1}{N} \text{tr} \frac{V'(M) - V'(x)}{x-M} \right) \right\rangle \quad (2.56)$$

$$= V'(x)G(x) - P(x) \quad (2.57)$$

with $P(x)$ a polynomial in x , of degree $d-1$ if V is of degree $d+1$, whose coefficients are given in terms of moments of M . For example, for our favorite quartic potential, $P(x) = 1 + g(x^2 + \langle \frac{1}{N} \text{tr} M^2 \rangle)$. The loop equation thus reads in general

$$\left\langle \left(\frac{1}{N} \text{tr} \frac{1}{x-M} \right)^2 \right\rangle - V'(x)G(x) + P(x) = 0. \quad (2.58)$$

Exercise Perform the change of integration variable $M \mapsto M' = M + \epsilon M^{k+1}$ in the integral (2.1). Compute carefully the Jacobian and show that one gets

$$\sum_{\ell=0}^{k-1} \left\langle \frac{1}{N} \text{tr} M^\ell \frac{1}{N} \text{tr} M^{k-\ell} \right\rangle = \left\langle \frac{1}{N} \text{tr} (V'(M) M^{k+1}) \right\rangle. \quad (2.59)$$

Multiplying by z^{-k-2} and summing over k from 0 to ∞ , check that one recovers (2.55). Suppose we write $V(M) = \sum_{k=0}^{\infty} t_k M^k$. Now, (2.59) may also be regarded as expressing that $L_k Z = 0$, $k \geq -1$ with

$$L_k := \sum_{\substack{m, \ell \geq 0 \\ m + \ell = k}} \frac{\partial^2}{\partial t_m \partial t_\ell} + \sum_{m \geq 0} m t_m \frac{\partial}{\partial t_{m+k}}$$

while $\frac{\partial}{\partial t_0} Z = -N^2 Z$. Check that the L_n , $n \geq -1$, satisfy the commutation relations of the Virasoro (or Witt) algebra with $c = 0$. This tells us that Z is a ‘‘highest weight’’ of that algebra with the weight 0.

So far, everything holds true for arbitrary N . In the large N limit, however, the expectation value of invariant traces factorizes

$$\langle \text{tr} A \text{tr} B \rangle \approx \langle \text{tr} A \rangle \langle \text{tr} B \rangle \quad (2.60)$$

as noticed above in (2.10), so that (2.55) simplifies into

$$G^2(x) - V'(x)G(x) + P(x) = 0. \quad (2.61)$$

Thus we recover the same equation as (2.13), with the same polynomial P depending on $d-1$ yet unknown moments $\langle \frac{1}{N} \text{tr} M^p \rangle$, $p = 1, \dots, d-1$. This leads to a general form

$$G(x) = \frac{1}{2} \left(V'(x) - \sqrt{V'(x)^2 - 4P(x)} \right) \quad (2.62)$$

as already noticed in (2.14) above. The missing coefficients are finally determined uniquely by the condition that $G(x)$ has a single cut with a discontinuity across it of a definite sign.

Imposing that the degree $2d$ polynomial $V'^2 - 4P$ has $d-1$ double zeros and reads $Q^2(x + 2a')(x - 2a'')$ gives $d-1$ relations which determine these moments, and hence the edges a', a'' of the cut. It remains to prove that the discontinuity is positive on the cut, i.e. that Q has the requested sign along it.

The loop equation thus offers an alternative to the saddle point equation. In some more complicated matrix models, where the saddle point equation is hard to solve and the orthogonal polynomials inapplicable, or the reduction to an eigenvalue problem impossible, it may be the only way to the solution. (Example: Chain of coupled matrices.) Notice, however, that the fact that loop equations close on a finite set is a non trivial property, not to be taken as granted in general.

2.4 Several cuts. A more global approach

To discuss in a systematic way the possibility of a more complicated structure, with several cuts rather than one, it is profitable to rephrase the previous discussion in a compact form, using an algebraic geometric formalism. Algebraic geometry made a timid appearance with the simple branch points in the x plane of the resolvent, and the curve $y^2 = V'(x)^2 - 4P(x) = Q^2(x)(x + 2a')(x - 2a'')$. In general, having more cuts, i.e. more branch points, leads to a more general complex curve, a hyperelliptic one for one-matrix models. It turns out that the data (holomorphic differential forms, Bergmann kernel, ...) that appear in that description are also of prime importance for the computation of the partition function and of correlation functions at arbitrary genus. [Eynard et al 05]...

3 More matrices

3.1 Angular matrix integrals

Suppose we have to deal with the following two-matrix integral

$$Z_2 = \int DADB \exp -N \text{tr}(V(A) + W(B) - cAB) \quad (3.1)$$

where the two (say, Hermitian) matrices A and B are subject to a potential V , resp. W and are coupled to one another by the term $c \text{tr} AB$. In the absence of that coupling, the integrand would be invariant under independent transformations $A \rightarrow UAU^\dagger$ and $B \rightarrow U'BU'^\dagger$ and it would be natural to reformulate it as an

integral over the eigenvalues a_i of A and b_j of B . In the presence of that coupling, there is only invariance under such transformations with $U = U'$. If we write $A = U_A \text{diag}(a_i)U_A^\dagger$ and $B = U_B \text{diag}(b_j)U_B^\dagger$, $\text{tr} AB = \text{tr} \text{diag}(a)U_A^\dagger U_B \text{diag}(b)U_B^\dagger U_A$, we want to integrate over the relative $U = U_A^\dagger U_B$

$$I(a., b.) = \int DU \exp\left(\frac{1}{t} \text{tr} \text{diag}(a)U \text{diag}(b)U^\dagger\right), \quad (3.2)$$

with $t^{-1} = cN$, and then

$$Z_2 = \text{const.} \int \prod_{i=1}^N da_i db_i \Delta^2(a) \Delta^2(b) I(a, b) e^{-N \sum_i V(a_i) + W(b_i)}, \quad (3.3)$$

where as before, Δ is the Vandermonde determinant (recall (1.9)). It turns out that the integral (3.2) may be computed exactly and enjoys remarkable properties, making it worth a separate discussion.

Assume that all the eigenvalues a_i and b_j of A and B , respectively, are distinct. One finds [5, 6]

$$\begin{aligned} I(a., b.; t) &= \int DU \exp\left(\frac{1}{t} \text{tr} AUBU^\dagger\right) = C' \frac{\det e^{\frac{1}{t} a_i b_j}}{\Delta(a)\Delta(b)} \\ C' &= \left(\prod_{p=1}^{N-1} p! \right) t^{N(N-1)/2}. \end{aligned} \quad (3.4)$$

Note that both the numerator and the denominator of the r.h.s. are completely antisymmetric functions of the a 's and of the b 's independently, making the expression $I(a., b.; t)$ a completely *symmetric* function of the a 's and of the b 's, and that the limit where some eigenvalues coalesce is well defined.

This expression (3.4) may be obtained by different routes.

3.1.1 Heat equation [6]

For two hermitian $N \times N$ matrices A and B , let us consider

$$K(A, B; t) = \left(\frac{1}{2\pi t} \right)^{N^2/2} \exp -\frac{1}{2t} \text{tr}(A - B)^2. \quad (3.5)$$

$K(A, B; t)$ satisfies the heat equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta_A \right) K(A, B; t) = 0, \quad (3.6)$$

where Δ_A is the Laplacian over A , together with the boundary condition that for $t \rightarrow 0$, $K(A, B; t) \rightarrow \delta(A - B)$. Applied on a function $\psi_0(B)$, it gives $\psi(A; t) = \int DB K(A, B; t) \psi_0(B)$, the solution of the heat equation,

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta_A \right) \psi(A; t) = 0 \quad \text{with the b.c. } \psi(A; 0) = \psi_0(A). \quad (3.7)$$

The heat kernel $K(A, B; t)$ is invariant under the simultaneous adjoint action on A and B by the same unitary matrix U . If we diagonalise $A = U_A \text{diag}(a)U_A^\dagger$ and $B = U_B \text{diag}(b)U_B^\dagger$, $K(A, B; t) = K(\text{diag}(a), U \text{diag}(b)U^\dagger; t)$ where $U = U_A^\dagger U_B$. Upon integration

$$\begin{aligned} \tilde{K}(A, B; t) &:= \int DU K(A, UBU^\dagger; t) = \int DU K(\text{diag}(a), UB \text{diag}(b)U^\dagger; t) \\ &= \left(\frac{1}{2\pi t} \right)^{N^2/2} e^{-\frac{1}{2t} \text{tr}(A^2 + B^2)} I(a., b.; t) \end{aligned} \quad (3.8)$$

is again a solution of the heat equation (3.6), but depends only on the eigenvalues a_i of A (and b_i of B). We still denote it $\tilde{K}(a., b.; t)$. Using the explicit form of the Laplacian in terms of the eigenvalues a_i , (see (C.10)), \tilde{K} satisfies

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \sum_k \left(\frac{\partial}{\partial a_k} \right)^2 \right) \Delta(a) \tilde{K}(a., b.; t) = 0 . \quad (3.9)$$

The product $\Delta(a)\Delta(b)\tilde{K}(a., b.; t)$ is an antisymmetric function of the a 's and of the b 's and is a solution of the heat equation with the flat Laplacian. To see the boundary conditions it satisfies for $t \rightarrow 0$, apply \tilde{K} as in (3.7) to a symmetric function $\psi_0(b.)$ of the eigenvalues: $\psi(a.; t) = C \int d^N b \Delta^2(b) \tilde{K}(a., b.; t) \psi_0(b.)$ is¹² a symmetric function of the eigenvalues $a.$ of A and \tilde{K} satisfies the boundary condition $C \Delta^2(b.) \tilde{K}(a., b.; t) \rightarrow \frac{1}{N!} \sum_P \delta(a_i - b_{Pi})$, or equivalently

$$C \Delta(a) \Delta(b) \tilde{K}(a., b.; t) \rightarrow \frac{1}{N!} \sum_{P \in \mathfrak{S}_N} \epsilon_P \prod_i \delta(a_i - b_{Pi}) .$$

The product $\Delta(a)\Delta(b)\tilde{K}(a., b.; t)$ is thus given by an alternate sum of products of elementary solutions to the heat equation

$$\begin{aligned} C \Delta(a) \Delta(b) \tilde{K}(a., b.; t) &= \frac{1}{N!} \sum_{P \in \mathfrak{S}_N} \epsilon_P \prod_i \exp -\frac{1}{2t} (a_i - b_{Pi})^2 \\ &= \frac{1}{N!} \left(\frac{1}{2\pi t} \right)^{N/2} \det \left[\exp -\frac{1}{2t} (a_i - b_j)^2 \right] . \end{aligned} \quad (3.10)$$

In physical terms it is the Green function of N independent free fermions, and (3.10) is their Slater determinant. The result (3.10) is consistent with (3.2).

3.1.2 Character expansion [6]

We shall just sketch the successive steps of that method. The exponential in (3.2) is expanded as

$$I(a., b.; t) = \sum_{n=0}^{\infty} \frac{(1/t)^n}{n!} \int DU (\text{tr} AUBU^\dagger)^n . \quad (3.11)$$

The power $(\text{tr} AUBU^\dagger)^n$ may be expressed as a linear combination of characters χ_λ of irreducible representations of the linear group $\text{GL}(N)$ labelled by Young diagrams λ through the Frobenius formula:

$$\text{tr}^n X = \sum_{|\lambda|=n} \hat{d}_\lambda \chi_\lambda(X) , \quad (3.12)$$

where

$$\chi_\lambda(X) = \frac{\Delta_\lambda(x)}{\Delta(x)} , \quad (3.13)$$

$\Delta_\lambda(x) = \det(x_i^{\lambda_j + j - 1})$ is a generalized Vandermonde determinant and $\hat{d}_\lambda = \hat{\chi}_\lambda([1^n])$ is the dimension of the λ -representation of the symmetric group \mathcal{S}_n . Integration over the unitary group then yields

$$\int DU \chi_\lambda(AUBU^\dagger) = \frac{\chi_\lambda(A) \chi_\lambda(B)}{\chi_\lambda(I)} = \frac{\chi_\lambda(A) \chi_\lambda(B)}{d_\lambda} \quad (3.14)$$

and a well known formula [7] gives

$$\frac{\hat{d}_\lambda}{d_\lambda} = n! \prod_{p=1}^N \frac{(p-1)!}{(\lambda_p + p - 1)!} . \quad (3.15)$$

¹² C is the constant computed in (C.9)

Using (3.13) and putting everything together we find

$$\Delta(a) \Delta(b) I(a., b.; t) = \left(\prod_{p=1}^{N-1} p! \right) \sum_{n=0}^{\infty} t^{-n} \sum_{\substack{\lambda \\ |\lambda|=n}} \frac{1}{\prod_p (\lambda_p + p - 1)!} \Delta_\lambda(a) \Delta_\lambda(b) \quad (3.16)$$

$$= \left(\prod_{p=1}^{N-1} p! \right) \sum_{0 \leq \ell_1 < \dots < \ell_N} \prod_{q=1}^N \frac{t^{-\ell_q + q - 1}}{(\ell_q)!} \det(a_i^{\ell_j}) \det(b_i^{\ell_j}) \quad (3.17)$$

Eq. (3.16) is interesting in its own sake, while an extension of Binet–Cauchy theorem¹³ enables one to resum (3.17) into

$$\Delta(a) \Delta(b) I(a., b.; t) = \left(\prod_{p=1}^{N-1} p! \right) t^{N(N-1)/2} \det \left(e^{\frac{1}{t} a_i b_j} \right) \quad (3.19)$$

which is precisely (3.2).

3.1.3 Duistermaat-Heckman theorem [9]

Let us compute the integral (3.2) by the stationary phase method, with no very good justification at this stage [8]. The stationary points U_0 of the “action” $\text{Tr} AUBU^\dagger$ satisfy $\text{Tr} \delta U U_0^\dagger [U_0 B U_0^\dagger, A] = 0$ for arbitrary antihermitian $\delta U U_0^\dagger$, hence $[U_0 B U_0^\dagger, A] = 0$. For diagonal matrices A and B with distinct eigenvalues, this implies that $U_0 B U_0^\dagger$ is diagonal and therefore that the saddle points U_0 are permutation matrices.

Gaussian fluctuations around the stationary point $U_0 = P$ may be computed by writing $U = e^X P$, X antihermitian, and by integrating over X after expanding the action to second order. Summing over all stationary points thus gives the “one-loop approximation” to the integral (3.2):

$$\begin{aligned} I(a., b.; t) &= C' \sum_{P \in \mathcal{S}_N} e^{\frac{1}{t} \sum_{i=1}^N a_i b_{P_i}} \int \prod_{i < j} d^2 X_{ij} e^{-\frac{1}{t} \sum_{i < j} |X_{ij}|^2 (a_i - a_j)(b_{P_i} - b_{P_j})} \\ &= C' \sum_{P \in \mathcal{S}_N} e^{\frac{1}{t} \sum_{i=1}^N a_i b_{P_i}} \frac{\left(\frac{\pi t}{N}\right)^{N(N-1)/2}}{\prod_{i < j} (a_i - a_j)(b_{P_i} - b_{P_j})} \\ &= C' (t\pi)^{N(N-1)/2} \frac{1}{\Delta(a)\Delta(b)} \sum_{P \in \mathcal{S}_N} \epsilon_P e^{\frac{1}{t} \sum_{i=1}^N a_i b_{P_i}} \\ &= C' (t\pi)^{N(N-1)/2} \frac{\det \left(e^{\frac{1}{t} a_i b_j} \right)}{\Delta(a)\Delta(b)} \end{aligned} \quad (3.20)$$

which reproduces the previous result up to a constant C' . The latter may be determined, for example by considering the $t \rightarrow 0$ limit, and the result reproduces (3.2). Thus the stationary phase approximation of the original integral (3.2) or (3.8) turns out to give the exact result! This empirical fact turned out to be a particular case of a general situation analysed later by Duistermaat and Heckman [9]: roughly speaking, if a classical system has only periodic trajectories with the same period, the stationary phase (or saddle point) method is exact.

For more details on this angular matrix integral, see [10].

¹³Recall [Balantekin A. B., 2000, Phys. Rev. D 62 085017, hep-th/0007161] that if $f(x) = \sum_{\ell \geq 0} f_\ell x^\ell$,

$$\begin{aligned} \sum_{0 \leq \ell_1 < \ell_2 < \dots < \ell_N} f_{\ell_1} \dots f_{\ell_N} \det a_i^{\ell_j} \det b_i^{\ell_j} &= \frac{1}{N!} \sum_{\ell_i \geq 0} \sum_{P, P'} \epsilon_P \epsilon_{P'} f_{\ell_1} \dots f_{\ell_N} \prod_i a_{P_i}^{\ell_i} b_{P'_i}^{\ell_i} \\ &= \frac{1}{N!} \sum_{P, P'} \epsilon_P \epsilon_{P'} \prod_i \sum_{\ell_i} f_{\ell_i} (a_{P, P'_i} b_i)^{\ell_i} \\ &= \det f(a_i b_j) \end{aligned} \quad (3.18)$$

3.2 Extensions, remarks, and applications of (3.4)

Formula (3.4) admits several generalizations. In particular, for any compact group G , and for A and B belonging to the Lie algebra of G , Harish Chandra [5] has given a simple and explicit expression of the corresponding integral. It is a good exercise to derive the result by application of the stationary phase method, which is exact according to Duistermaat-Heckman theorem. In particular if $G = O(n)$ and A and B are $n \times n$ skew-symmetric matrices, $\int dO \exp \operatorname{tr} OAO^T B$ may be computed exactly in a closed form. See for example [11] for explicit formulae. In contrast, the same integral with A, B symmetric matrices has a much more elusive form. . .

The explicit knowledge of the “angular” integral (3.4) allows to reduce 2-matrix integrals like (3.3) to integrals over eigenvalues. Suitable generalizations of the methods of sect. 2, in particular the introduction of appropriate orthogonal polynomials (Mehta), then enable one to compute Z_2 . This also generalizes to integrals over *chains* of matrices.

Physical applications are numerous, from the study of the Ising model on a random graph [12] to models of 2-d quantum gravity coupled to “matter fields”, relations with integrable hierarchies, etc, see [?].

4 Universality

(still very sketchy. . .)

Universality is to be understood in the same sense as in critical phenomena, namely insensitivity of some (statistical) property to small variations of the probability distribution within the same RM ensemble.

Which quantities are universal ? Roughly speaking, “local” statistics are, whereas global statistics are not, although . . .

4.1 Global density

Within a given ensemble, for instance the Hermitian matrices with $\beta = 2$, the density $\rho(\lambda)$ of eigenvalues depends on the potential $V(\lambda)$ and *is not* universal. To wit, the shape of ρ for the quartic potential $\frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4$ at different values of the coupling g , see Fig 5.

On the other hand, *all* Wigner matrices (with the same mean and variance) have the same ρ , irrespective of the law of their i.i.d. matrix elements.

Theorem (Wigner 1955; Anderson, Guionnet 2009). For a Wigner matrix with off-diagonal elements M_{ij} i.i.d. of mean 0 and variance 1 and with diagonal elements of finite variance, there is weak convergence a.s. towards the semi-circle law

$$\left\langle \frac{1}{N} \sum_i f(\lambda_i) \right\rangle \rightarrow \int_{-2}^2 \frac{1}{2\pi} \sqrt{4 - \lambda^2} f(\lambda) d\lambda$$

for any f of polynomial growth.

Local behavior of ρ

One finds a universal behavior near end points or singular points of the spectrum:

at a “soft” edge, for a spectrum of support $[b, a]$, $\lambda \sim 2a$, $\rho(\lambda) \sim (2a - \lambda)^{\frac{1}{2}}$;

at a “hard” edge, for example for positive definite matrices, $\lambda > 0$, then for $\lambda \sim 0$, $\rho(\lambda) \sim \frac{1}{\sqrt{\lambda}}$;

at the merging point of two segments of the (real) support,

when the vanishing at the soft edge is non-generic, $g = g_c$, $\rho(\lambda) \sim (2a - \lambda)^{\frac{3}{2}}$, etc.

4.2 Largest eigenvalue

For a RMT with real spectrum on $[b, a]$, $P(\lambda_{\max} \leq t) \underset{N \rightarrow \infty}{\approx} F(N^{2/3}(t - a))$, supposed to be a universal function in each class of RMT. For GUE, GOE, GSE, $F_\beta(t)$ is known explicitly [Tracy-Widom]. Relation with Painlevé functions, large deviations, etc.

4.3 Correlation functions

Given the joint probability density of the N eigenvalues, $P(\lambda_1, \dots, \lambda_N)$, (a symmetric function of its arguments), one defines the correlators of n of them $O_n(\lambda_1, \dots, \lambda_n) = \frac{N!}{(N-n)!} \int P(\lambda_1, \dots, \lambda_N) d\lambda_{n+1} \dots d\lambda_N$. It is given by the determinant of the kernel K , see above sect 2.2

$$O_n(\lambda_1, \dots, \lambda_n) = \left(\det K_N(\lambda_i, \lambda_j) \right)_{1 \leq i, j \leq n}.$$

Universality of the correlation functions at short distances

In the bulk, i.e. near a λ^* far from the edges of the spectrum etc, such that $\rho(\lambda^*) = r \neq 0$, take $\lambda = \lambda^* + \frac{x}{rN}$, $\mu = \lambda^* + \frac{y}{rN}$. Then one may prove that

$$\lim_{N \rightarrow \infty} \frac{1}{rN} K_N(\lambda, \mu) = K^{\sin}(x, y) := \frac{\sin \pi(x - y)}{\pi(x - y)}$$

For example the two-point function $O_2(\lambda, \mu) = \det \begin{pmatrix} K_N(\lambda, \lambda) & K_N(\lambda, \mu) \\ K_N(\mu, \lambda) & K_N(\mu, \mu) \end{pmatrix} = K_N(\lambda, \lambda)K_N(\mu, \mu) - K_N(\lambda, \mu)K_N(\mu, \lambda)$ $O_1(\lambda) = K_N(\lambda, \lambda) = N\rho(\lambda) = Nr$.

★

A Dyson discussion

In Quantum Mechanics [13], time reversal is an antiunitary transformation of state vectors $\psi \rightarrow \psi^T$, i.e. \exists unitary K s.th. $\psi^T(t') = K\psi^*(t)$ with $t' = -t$; time reversal of an operator A is $A^T := KA^TK^{-1}$ (A^T is the transpose of A); in particular for the Hamiltonian, $H^T = KH^TK^{-1}$ and $i\frac{\partial}{\partial t}\psi(t) = H\psi(t) \Leftrightarrow i\frac{\partial}{\partial t'}\psi^T(t') = H^T\psi^T(t')$. As time reversal is involutive (square one), $(\psi^T)^T = e^{i\alpha}\psi$, $KK^* = e^{i\alpha}I$, hence (upon conjugation) $e^{2i\alpha} = 1$. Whether this phase $e^{i\alpha} = \pm 1$ depends on the total angular momentum of the system. Independently of its conservation, for a given system, J changes sign under time reversal $J^T = KJ^*K^{-1} = -J$ and J_z takes a value which is either integer or half-integer. As is well known (see any book of Quantum Mechanics !), the phase $e^{i\alpha}$ depends on that value, and equals $+1$ for integer spin, and -1 for half-integer spin.

Assume there is time reversal invariance, i.e. $H = H^T$. If $KK^* = I$, or equivalently $K = K^T$, $H^T = KH^TK^{-1} = H$, let us show that, up a unitary transformation, H may be chosen real symmetric $H = H^\dagger = H^T = H^*$.

By a change of basis, $\psi \rightarrow U\psi$, $K \rightarrow UKU^T$. Given $K = K^T$, write it as $K = A + iB$, $A = A^T = A^*$, $B = B^T = B^*$. Unitarity of K implies that A and B commute, hence are simultaneously diagonalizable by a real orthogonal change of basis $A = O \text{diag}(a.)O^T$, $B = O \text{diag}(B.)O^T$, $a., b.$ real, $K = O \text{diag}(a. + ib.)O^T$, $a_i^2 + b_i^2 = 1$ and $\text{diag}(a. + ib.) = \exp \text{diag}(i\beta.)$. We then choose $U = O \exp \text{diag}(\frac{i}{2}\beta.)$, $U^T = \exp \text{diag}(\frac{i}{2}\beta.)O^T$ and $K = UU^T$, $UU^\dagger = I$. Thus, up to the U change of basis, $K = I$, and $H = H^T$ qed.

In general, if we have rotation invariance, (and angular momentum is integer or half-integer), pick a basis where J_x and J_z are real, J_y imaginary. Then choose $K = \exp i\pi J_y$. This satisfies $KK^T =$

$\exp 2i\pi J_y = \pm I$ depending on total angular momentum, as it should. By rotation invariance, J and H commute, hence H and K commute, hence $H^T = H^T$ and time invariance means $H = H^T$, with again symmetric matrices.

In cases J is half integer and there is no rotation symmetry, $KK^* = -I \Leftrightarrow K = -K^T$ and K is unitary antisymmetric. It may be brought to a block diagonal form $\text{diag} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. If $2N$ denotes the size of that matrix, the permitted unitary transformations (preserving that form) are those of the symplectic group $\text{USp}(N)$ ¹⁴. Written as a $N \times N$ matrix of quaternions, the Hamiltonian H is changed under time reversal into its (quaternionic) conjugate transposed, i.e.

$$H_{ij} = h_{ij}^{(0)} e_0 + \sum_{\alpha=1}^3 h_{ij}^{(\alpha)} e_\alpha \mapsto H_{ij}^R := e_2(h_{ji}^{(0)} e_0 + \sum_{\alpha=1}^3 h_{ji}^{(\alpha)} e_\alpha^T)(-e_2) = h_{ji}^{(0)} e_0 - \sum_{\alpha=1}^3 h_{ji}^{(\alpha)} e_\alpha = \bar{H}_{ji} ,$$

while Hermiticity means that $H_{ij} = h_{ji}^{(0)*} e_0 - \sum_{\alpha=1}^3 h_{ji}^{(\alpha)*} e_\alpha$, so that time invariance means that H is “real selfdual”, $H_{ij} = h_{ij}^{(0)} e_0 + \sum_{\alpha=1}^3 h_{ij}^{(\alpha)} e_\alpha$ with $h^{(0)}$, $h^{(\alpha)}$ real, $H = H^R$. By a symplectic transformation, it may be brought to a diagonal of 2×2 blocks multiple of the identity, exposing the characteristic multiplicity two of the levels of such an Hamiltonian (Kramers degeneracy).

Thus, to summarize, if the system is invariant under time reversal,

- with either an integer total angular momentum, or a conserved total angular momentum, the appropriate set of (Hamiltonian) matrices is symmetric, up to orthogonal ones. We are dealing with the “Orthogonal ensemble” (of random matrices)
- if angular momentum is not conserved and total angular momentum is half-integer, the Hamiltonian must be taken in the space of real self-dual quaternionic matrices. This is the so-called “Symplectic ensemble” of random matrices, because the remaining permitted transformations are the matrices of $\text{USp}(N)$.

Finally in case there is no time reversal invariance, the Hamiltonian is just some Hermitian matrix, up to unitary transformations, (“Unitary ensemble” of random matrices).

B Quaternions. Symplectic group

The quaternion algebra is generated by 4 elements 1, e_i , $i = 1, 2, 3$,

$$q = q^{(0)}1 + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3 \quad q^{(\cdot)} \in \mathbb{C} \quad (\text{B.1})$$

with multiplication $e_i^2 = e_1e_2e_3 = -1$, hence

$$e_1e_2 = -e_2e_1 = e_3$$

and cyclic permutations. One may represent the generators e_i by Pauli matrices $e_i = -i\sigma_i$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The quaternionic conjugate of q is

$$\bar{q} = q^{(0)}1 - q^{(1)}e_1 - q^{(2)}e_2 - q^{(3)}e_3 . \quad (\text{B.2})$$

¹⁴see Appendix B for a reminder about quaternions and the symplectic group

A quaternion q is *real* iff $q^{(\mu)} \in \mathbb{R}$. For a non vanishing real quaternion q , $\|q\|^2 := q\bar{q} = \sum_{\alpha=0}^3 (q^{(\alpha)})^2 > 0$ and the inverse is $q^{-1} = \frac{\bar{q}}{\|q\|^2}$. The real quaternions thus form a field, and also a 4-dimensional space over \mathbb{R} .

Quaternionic matrices

Let us consider $N \times N$ matrices Q whose elements are quaternions $Q_{ij} = (q_{ij})$. The *dual* Q^R of a quaternionic matrix is the matrix

$$(Q^R)_{ij} = \bar{q}_{ji} \quad (\text{B.3})$$

A quaternionic matrix is thus called *real self-dual* if

$$Q^R = Q = (q_{ij}) = (\bar{q}_{ji}) \quad , \quad (\text{B.4})$$

with its elements q_{ij} real quaternions.

If one thinks of the basic quaternions e_i as Pauli matrices, a $N \times N$ quaternion matrix may also be regarded as a $2N \times 2N$ matrix.

Symplectic group $\text{USp}(N)$

The (compact) symplectic group $\text{USp}(N)$ is generated by real quaternionic matrices satisfying $S^R = S^{-1}$.

The set of *real self-dual quaternion matrices* $Q = Q^R$ is invariant under $Q \rightarrow S^R Q S$, for any symplectic matrix S .

C Measures on sets of matrices, Jacobians and all that.

Consider the general setting of a Riemannian manifold \mathcal{M} . If a metric g has been introduced on \mathcal{M} , i.e. for a certain choice of coordinates (x^i) , a tensor g_{ij} , the square of the length element reads

$$ds^2 = g_{ij} dx^i dx^j \quad . \quad (\text{C.1})$$

Under a coordinate change, g_{ij} transforms as a covariant tensor

$$x^i \mapsto \tilde{x}^i \Rightarrow g_{ij} \mapsto \tilde{g}_{ij} = \left(\frac{\partial x^k}{\partial \tilde{x}^i} \right) \left(\frac{\partial x^l}{\partial \tilde{x}^j} \right) g_{kl} \quad (\text{C.2})$$

so as to preserve ds^2 . One may then define

$$g = \det(g_{ij}) \quad (\text{C.3})$$

(which is not invariant under a coordinate change) and

$$g^{ij} = (g^{-1})_{ij} \quad (\text{C.4})$$

which transforms as a contravariant tensor, i.e. with the inverse power of the Jacobian, viz

$$g^{ij} \mapsto \tilde{g}^{ij} = \left(\frac{\partial \tilde{x}^i}{\partial x^k} \right) \left(\frac{\partial \tilde{x}^j}{\partial x^l} \right) g^{kl} \quad . \quad (\text{C.5})$$

One may then define an *integration measure*, invariant under coordinate changes

$$d\mu(x) = \sqrt{g} \prod_i dx^i \quad (\text{C.6})$$

$d\mu(x) = d\mu(\tilde{x})$ and a Laplacian Δ

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} g^{ij} \sqrt{g} \frac{\partial}{\partial x^j} \quad (\text{C.7})$$

such that

$$\int d\mu(x) \frac{\partial f(x)}{\partial x^i} g^{ij} \frac{\partial g(x)}{\partial x^j} \equiv \int d\mu(x) f(x) (-\Delta) g(x) . \quad (\text{C.8})$$

is invariant under coordinate changes, for any pair of square integrable functions on the manifold.

Let us apply this to Hermitian, Symmetric and Quaternionic matrices.

If M is Hermitian, $M = U\Lambda U^\dagger$, Λ diagonal, U unitary, we have $dM = U(d\Lambda + [dX, \Lambda])U^\dagger$, where $dX := U^\dagger dU$ is antihermitian. Then $\text{Tr}(dM)^2 = \sum_i d\lambda_i^2 + 2 \sum_{i<j} |dX_{ij}|^2 |\lambda_i - \lambda_j|^2$ defines the metric tensor $g_{\alpha\beta}$ in the coordinates $\xi^\alpha = (\lambda_i, \Re X_{ij}, \Im X_{ij})$. This determines first the measure $DM = \sqrt{\det g} \prod d\xi^\alpha = 2^{N(N-1)/2} \prod_i dM_{ii} \prod_{i<j} d\Re M_{ij} d\Im M_{ij} = 2^{N(N-1)/2} \Delta^2(\lambda) \prod d\lambda_i \prod d^2 X_{ij} = C \Delta^2(\lambda) \prod_i d\lambda_i DU$. The constant C is fixed by computing in two different ways the integral of a $U(N)$ invariant function of M , for example a Gaussian

$$1 = \int DM \frac{e^{-\frac{1}{2} \text{Tr} M^2}}{(2\pi)^{N^2/2}} = \frac{C}{(2\pi)^{N(N-1)/2}} \int \prod_{i=1}^N \frac{d\lambda_i}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2} \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 .$$

The latter integral equals $\prod_{p=1}^N p!$ thus

$$C = \frac{(2\pi)^{N(N-1)/2}}{\prod_{p=1}^N p!} , \quad DM = \frac{(2\pi)^{N(N-1)/2}}{\prod_{p=1}^N p!} \Delta^2(\lambda) \prod_{i=1}^N d\lambda_i DU . \quad (\text{C.9})$$

From the metric above, one also computes the Laplacian

$$\begin{aligned} \Delta_M &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^\alpha} g^{\alpha\beta} \sqrt{g} \frac{\partial}{\partial \xi^\beta} = \frac{1}{\prod_{i<j} (\lambda_i - \lambda_j)^2} \sum_k \frac{\partial}{\partial \lambda_k} \prod_{i<j} (\lambda_i - \lambda_j)^2 \frac{\partial}{\partial \lambda_k} + \Delta_X \\ &= \Delta(\lambda)^{-1} \sum_k \left(\frac{\partial}{\partial \lambda_k} \right)^2 \Delta(\lambda) + \Delta_X , \end{aligned} \quad (\text{C.10})$$

where the last equality results from the vanishing of $\sum_k [\partial_{\lambda_k}, [\partial_{\lambda_k}, \Delta(\lambda)]] = 0$, as this completely antisymmetric function of the λ 's is a polynomial of degree $N(N-1)/2 - 2$.

The calculation for symmetric (resp. quaternionic real self-dual) matrices proceeds along the same lines, with again $dM = U d\Lambda U^\dagger + [dX, M]$ with U orthogonal real (resp. symplectic), dX in the corresponding Lie algebra, so that $DM = |\Delta(\lambda)|^\delta \prod_i d\lambda_i DU$, with $\delta = 1$, resp 4, the dimension (over \mathbb{R}) of the Lie algebra.

D Orthogonal polynomials

Consider Hermitian matrices M of size N , with a weight $\exp -N \text{tr} V(M)$, V a polynomial. The orthogonal polynomials for the measure $d\mu(\lambda) = d\lambda \exp -NV(\lambda)$ may be expressed as

$$P_n(\lambda) = \langle \det(\lambda - M) \rangle_n \quad (\text{D.1})$$

which is computed on **matrices of size n** [Szegö].

$$\int d\mu(\lambda) P_n(\lambda) P_m(\lambda) = \frac{Z_{n+1}}{(n+1)Z_n} \delta_{n,m} . \quad (\text{D.2})$$

Proof : Let $P_n(\lambda) = Z_n^{-1} \int \prod_{i=1}^n d\mu(\lambda_i) (\lambda - \lambda_i) \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2$, compute its scalar product with λ^k . One may write

$$\begin{aligned}
\int d\mu(\lambda) P_n(\lambda) \lambda^k &= \int d\mu(\lambda_{n+1}) P_n(\lambda_{n+1}) \lambda_{n+1}^k \\
&= \int \prod_{i=1}^{n+1} \Delta_{n+1}(\lambda_1, \dots, \lambda_{n+1}) \Delta_n(\lambda_1, \dots, \lambda_n) \lambda_{n+1}^k \\
&= \frac{1}{n+1} \int \prod_{i=1}^{n+1} \Delta_{n+1}(\lambda_1, \dots, \lambda_{n+1}) \sum_{s=1}^{n+1} (-1)^{n+1-s} \Delta_{n+1}(\lambda_1, \dots, \widehat{\lambda}_s, \dots, \lambda_{n+1}) \lambda_s^k \\
&= \frac{1}{n+1} \int \prod_{i=1}^{n+1} \Delta_{n+1}(\lambda_1, \dots, \lambda_{n+1}) \begin{vmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} & \lambda_1^k \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} & \lambda_2^k \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \lambda_{n+1} & \dots & \lambda_{n+1}^{n-1} & \lambda_{n+1}^k \end{vmatrix}
\end{aligned} \tag{D.3}$$

where the sign $(-1)^{n+1-s}$ is the signature of the cyclic permutation which maps $(s+1, \dots, n+1, s)$ onto $(s, s+1, \dots, n+1)$. It is clear that the latter expression vanishes for $k = 0, \dots, n-1$ hence the P_n are indeed orthogonal, while for $k = n$ the same computation leads to

$$Z_n \int d\mu(\lambda) P_n(\lambda) \lambda^n = \frac{Z_{n+1}}{n+1} .$$

Christoffel–Darboux formula

We have seen that multiplication of $P_n(\lambda)$ by λ gives a simple sum of three terms, see eqn (2.30). On the orthonormalized functions ψ_n of (2.39)

$$\lambda \psi_n(\lambda) = \sum_{m=n-1}^{n+1} Q_{nm} \psi_m(\lambda) \tag{D.4}$$

and $Q_{nm} = \int d\lambda \psi_n(\lambda) \psi_m(\lambda) \lambda = Q_{mn}$ is symmetric. Then form

$$\begin{aligned}
(\lambda - \mu) \sum_{n=0}^{N-1} \psi_n(\lambda) \psi_n(\mu) &= \sum_{n=0}^{N-1} \sum_{m=0}^N Q_{nm} \psi_m(\lambda) \psi_n(\mu) - \sum_{n=0}^{N-1} \sum_{m=0}^N Q_{nm} \psi_n(\lambda) \psi_m(\mu) \\
&= Q_{N-1 N} (\psi_N(\lambda) \psi_{N-1}(\mu) - \psi_{N-1}(\lambda) \psi_N(\mu))
\end{aligned} \tag{D.5}$$

which after division by $\lambda - \mu$ yields the desired formula, with some constant $Q_{N-1 N}$ [Mehta3].

Dyson–Mehta theorem

The kernel $K(\lambda, \mu) = \frac{1}{N} \sum_{i=0}^{N-1} \psi_i(\lambda) \psi_i(\mu)$ satisfies

$$\int K(\lambda, \lambda) d\lambda = 1 \quad \int K(\lambda, \mu) K(\mu, \nu) d\mu = \frac{1}{N} K(\lambda, \nu) . \tag{D.6}$$

If we denote for short $\det_n K = \det K(\lambda_i, \lambda_j) |_{1 \leq i, j \leq n}$, we have Dyson–Mehta theorem,

$$\int d\lambda_n \det_n K = \frac{N - n + 1}{N} \det_{n-1} K , \tag{D.7}$$

which is proved by a simple expansion over the permutations of \mathcal{S}_n and integration using (D.6).

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