ON THE COUNTING
OF COLORED TANGLES

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ABSTRACT

The connection between matrix integrals and links is used to define matrix models which count alternating tangles in which each closed loop is weighted with a factor n, i.e. may be regarded as decorated with n possible colors. For $n = 2$, the corresponding matrix integral is that recently solved in the study of the random lattice six-vertex model. The generating function of alternating 2-color tangles is provided in terms of elliptic functions, expanded to 16-th order (16 crossings) and its asymptotic behavior is given.

Keywords: Alternating knots; Tangles; Matrix integrals

1. Introduction

The problem of counting topologically distinct knots remains a challenging one: see [1] for a review and for a report on recent advances. It was recently noticed [2] that combinatorial methods developed in quantum field theory, namely Feynman diagrams applied to matrix integrals, may provide a new way to count knots. In particular, the counting of *alternating* tangles, which had been achieved in [3], was reproduced. This counting, however, is not capable of discriminating between objects with different numbers of connected components: in the usual terminology, it gives the number of links rather than that of knots. In the present article, we

reexamine this question and show how the introduction of a number n of possible "colors" for the knotted loops would solve this question, and how this may be formulated in terms of a matrix integral. For $n = 2$, this integral is equivalent to one recently studied in detail and computed in the framework of the random lattice 6-vertex model [4–5]. We thus carry out the explicit counting of alternating 2-color tangles: their generating function is the solution of coupled equations involving elliptic functions. We are able to give the 13 first terms of its expansion and its asymptotic behavior. We conclude with more conjectural considerations on the number of (2-color alternating) links.

2. A Matrix Model for Colored Links

We want to consider a model which describes alternating links in which each of the intertwined loops can have n different colors. This can be achieved via a large $N \times N$ matrix integral [2]. As a first attempt, let us consider the following integral

$$Z^{(N)}(n, g) = \int \prod_{a=1}^{n} \mathrm{d}M_a \, \mathrm{e}^{N \operatorname{tr} \left(-\frac{1}{2} \sum_{a=1}^{n} M_a^2 + \frac{g}{4} \sum_{a,b=1}^{n} M_a M_b M_a M_b \right)} \tag{2.1}$$

over $N \times N$ hermitean matrices, and the corresponding "free energy"

$$F(n, g) = \lim_{N \to \infty} \frac{\log Z^{(N)}(n, g)}{N^2} . \tag{2.2}$$

Such an integral has a g-series expansion ("perturbative expansion") which admits a graphical representation in terms of Feynman diagrams. In the large N limit, only planar diagrams survive: see for example [6] for a review. The integral (2.1) has an $O(n)$-invariance where the M_a form the vector representation of $O(n)$. The Feynman diagram expansion of $F(n, g)$ generates planar diagrams with four-legged vertices and colored edges such that colors cross each other at each vertex (Fig. 1).

To each planar diagram we can associate an alternating link diagram (see e.g. [7], page 21) by following colored loops as they cross other loops and choosing alternatingly under- and over-crossings. This can be carried out in a consistent way throughout the whole diagram.
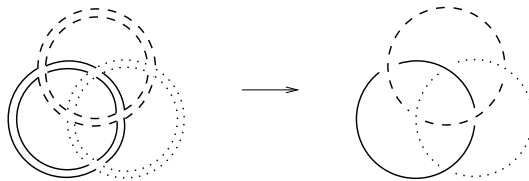

Fig. 1: A planar Feynman diagram of (2.2) and the corresponding alternating link diagram.

We have thus generated alternating link diagrams with n colors. Diagrams with different numbers of connected components can now be distinguished by the n dependence of their weight: indeed, to a diagram with k connected components is associated a factor n^k. We can therefore write

$$F(n,g) = \sum_{k=1}^{\infty} F_k(g)n^k \qquad (2.3)$$

where $F_k(g)$ is the sum over alternating link diagrams with exactly k intertwined loops. Note that if one can define $F(n,g)$ for non-integer n and in particular in a neighborhood of 0, one has access to the individual contributions $F_k(g)$ since they form the small n expansion of $F(n,g)$. As usual in $O(n)$ vector models, one can perform (at least formally) an analytic continuation in n by a Hubbard–Stratonovitch transformation. Unfortunately such a transformation breaks planarity of the diagrams and is not suitable for our purposes. However, another Hubbard–Stratonovitch transformation exists for the particular values $n = \pm 2$ (besides $n = 1$, of course) which preserves planarity, as will be explained later.

We can also define correlation functions in the model. There is only one 2-point function,

$$G(g) = \lim_{N\to\infty} \left\langle \frac{1}{N}\mathrm{tr}M_a^2 \right\rangle \qquad (2.4)$$

where a is fixed, $1 \le a \le n$. Here the brackets $\langle \cdot \rangle$ refer to the normalized average with the exponential weight of (2.1). Due to $O(n)$-invariance, there are *two* independent 4-point functions. We consider connected 4-point functions only; we choose

$$\Gamma_1(g) = \lim_{N\to\infty} \left\langle \frac{1}{N}\mathrm{tr}(M_a M_b)^2 \right\rangle_{\mathrm{c}} \qquad (2.5a)$$

$$\Gamma_2(g) = \lim_{N\to\infty} \left\langle \frac{1}{N}\mathrm{tr}(M_a^2 M_b^2) \right\rangle_{\mathrm{c}} \qquad (2.5b)$$

where a and b are fixed and distinct. They have the following interpretation: $\Gamma_i(g)$ is the generating function of the number of colored alternating tangle diagrams of type i; a diagram with four external legs is of type 1 if the external strings come in and out in diagonally opposite corners; it is of type 2 if the external strings come in and out in the same upper/lower half-plane; see Fig. 2. Note that any diagram with four external legs is either of type 1, or of type 2, or the image of a type 2 diagram under a $\pi/2$ rotation.

We must now proceed as in [2]. Our goal is 1) to count only diagrams which are prime and reduced, and 2) to count as a single contribution diagrams which correspond to the same link. First it is necessary to take care of non-prime and non-reduced diagrams. This is achieved by introducing an extra parameter t in the action:

$$Z^{(N)}(n,t,g) = \int \prod_{a=1}^{n} \mathrm{d}M_a\, \mathrm{e}^{N\,\mathrm{tr}\left(-\frac{t}{2}\sum_{a=1}^{n} M_a^2 + \frac{g}{4}\sum_{a,b=1}^{n} M_a M_b M_a M_b\right)} , \qquad (2.6)$$
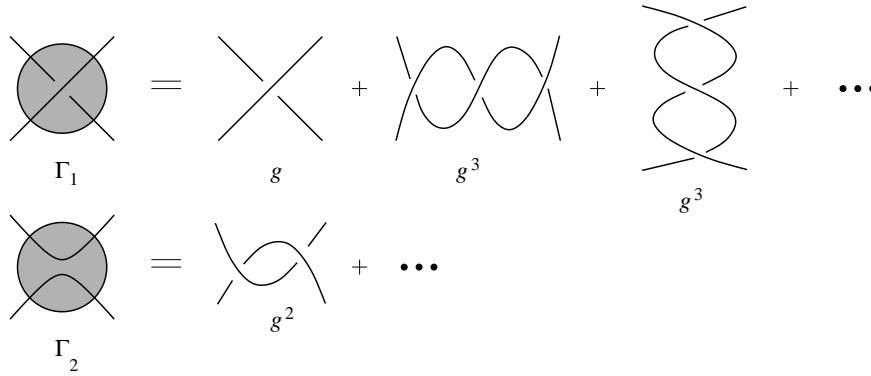
Fig. 2: The two types of tangle diagrams.

and choosing t as a function of g in such a way that

$$G(t(g), g) = 1 . \tag{2.7}$$

We have the obvious scaling property $G(t, g) = \frac{1}{t} G(1, g/t^2)$, which means that given the two-point function of the original model $G(1, g) \equiv G(g)$, $t(g)$ is the solution of the equation:

$$t(g) = G(g/t^2(g)) . \tag{2.8}$$

We have similar scaling properties for the higher correlation functions; in particular, $\Gamma_i(t, g) = \frac{1}{t^2} \Gamma_i(1, g/t^2)$.

Fig. 3: The flype of a tangle

The next step is to remove the over-counting of links due to the fact that several diagrams can correspond to a single link. According to the Tait flyping conjecture (proven in [8]), two reduced alternating diagrams are equivalent if and only if they are related by a sequence of flypes (Fig. 3). In order to take into account the flyping equivalence, we must now consider, as advocated in [9], the most general $O(n)$-invariant model with quartic interaction

$$Z^{(N)}(n, t, g_1, g_2) \tag{2.9}$$
$$= \int \prod_{a=1}^{n} dM_a\, e^{N \operatorname{tr} \left(-\frac{t}{2} \sum_{a=1}^{n} M_a^2 + \frac{g_1}{4} \sum_{a,b=1}^{n} (M_a M_b)^2 + \frac{g_2}{2} \sum_{a,b=1}^{n} M_a^2 M_b^2 \right)} .$$

There is a new type of vertex which allows loops to "avoid" each other. The appearance of two types of vertices can be understood as follows: the unwanted

(i.e. overcounted) tangle diagrams due to the flyping equivalence can be of either type 1 or type 2, and we must introduce "counterterms" of both types to cancel them. To pursue the analogy with renormalization, we can rephrase this by saying that it is only the "renormalized" coupling constants which must be of the form $(g, 0)$, but in order to reach this point we must consider more general "bare" coupling constants (g_1, g_2).

We impose again the condition

$$G(t(g_1, g_2), g_1, g_2) = 1 \ . \tag{2.10}$$

From now on t will be assumed to be fixed by (2.10), and we shall be concerned with the correlation functions $\Gamma_i \equiv \Gamma_i(t(g_1, g_2), g_1, g_2)$, which are the generating functions of diagrams of type i with four external legs and two types of vertices weighted by g_1 and g_2 respectively.

As in [2], we make use of the concepts of two-particle irreducibility. Following the language of field theory, we recall that a four-legged diagram is *two-particle-irreducible* (2PI) if cutting any two distinct propagators leaves it connected; otherwise it is *two-particle-reducible* (2PR). Also, we shall consider skeleton diagrams, which must be "dressed" to recover ordinary diagrams. This will be implicit in the following and we refer the reader to [2] for details.

Let us define D_i (resp. H_i, V_i) to be the generating functions of 2PI (resp. 2PI in the horizontal, vertical channel) four-legged diagrams of type i. Note that $H_1 = V_1$, but $H_2 \neq V_2$, since the defining property of diagrams of type 2 is not invariant by rotation of $\pi/2$.

By decomposing a general diagram Γ_1 according to the number of times it is reducible in the *horizontal* channel (Fig. 4), we find the following formulae:

$$\Gamma_1 = \frac{1}{2} \left(\frac{H_2 + H_1}{1 - (H_2 + H_1)} - \frac{H_2 - H_1}{1 - (H_2 - H_1)} \right) \tag{2.11a}$$

$$\Gamma_2 = \frac{1}{2} \left(\frac{H_2 + H_1}{1 - (H_2 + H_1)} + \frac{H_2 - H_1}{1 - (H_2 - H_1)} \right) \ . \tag{2.11b}$$
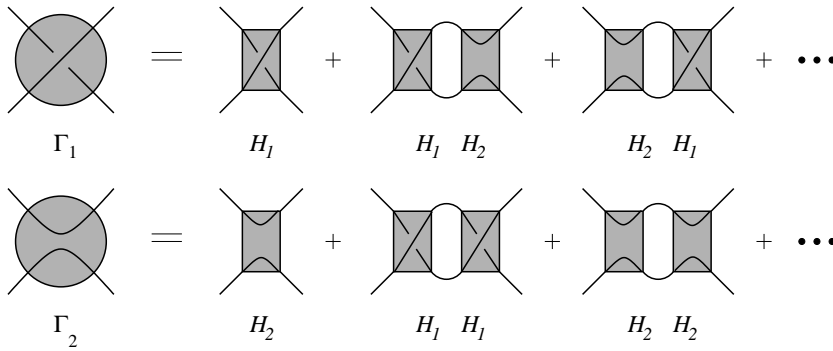

Fig. 4: Decomposition of a tangle diagram in the horizontal channel.

This can be simplified by introducing the combinations $\Gamma_\pm = \Gamma_2 \pm \Gamma_1$ and similarly for $H_\pm$. We find:

$$\Gamma_\pm = \frac{H_\pm}{1 - H_\pm} \; . \qquad (2.11')$$

Note that these equations are independent of n.

It is not as obvious how to decompose diagrams of type 2 using the *vertical* channel. It is simpler to proceed as follows: define Γ_0 to be generating function of diagrams with four external legs such that the color of the two left outgoing strings is free, whereas the color of the two right outgoing strings is fixed and equal; and similarly H_0, V_0, D_0. We have the formulae

$$\Gamma_0 = (n + 1)\Gamma_2 + \Gamma_1 \qquad (2.12a)$$

$$H_0 = H_2 + nV_2 + H_1 \; . \qquad (2.12b)$$

We can now proceed to decompose Γ_0 in the horizontal channel. It is easy to convince oneself that the simple formula

$$\Gamma_0 = \frac{H_0}{1 - H_0} \qquad (2.13)$$

holds.

The three equations (2.11) and (2.13) determine $H_1 = V_1$, H_2 and V_2 as functions of Γ_1 and Γ_2.[1] More explicitly, one should invert them to

$$H_{0,\pm} = \frac{\Gamma_{0,\pm}}{1 + \Gamma_{0,\pm}} \qquad (2.14)$$

and take appropriate linear combinations. Furthermore, noting that any diagram with four external legs and four-legged vertices must be irreducible in one of the two channels, we have

$$D_i = H_i + V_i - \Gamma_i \; , \qquad (2.15)$$

which means that we have managed to express the D_i in terms of the Γ_i.

The flype equivalence does not affect 2PI skeleton diagrams; this means in practive that the expression of $D_i' = D_i - g_i$ as a function of the Γ_i is left unchanged by the removal of the overcounting of flype equivalent diagrams. Note that the $D_i[\Gamma_1, \Gamma_2]$ have a simple expression given by Eqs. (2.14)–(2.15), but the $g_i[\Gamma_1, \Gamma_2]$ are non-trivial functions which can only be obtained by actually solving the matrix model (2.9) (or some equivalent procedure). For example, perturbatively we find: (Fig. 5)

$$D_1' = n\Gamma_1^5 + 8\Gamma_1^4\Gamma_2 + (4n + 4)\Gamma_1^3\Gamma_2^2 + 24\Gamma_1^2\Gamma_2^3 + 16\Gamma_1^6\Gamma_2 + O(g^9) \qquad (2.16a)$$

$$D_2' = n\Gamma_1^4\Gamma_2 + 2\Gamma_1^7 + 16\Gamma_1^3\Gamma_2^2 + (20 + 8n)\Gamma_1^2\Gamma_2^3 + (14 + 6n)\Gamma_1^6\Gamma_2 + 3\Gamma_1^8 + O(g^9) \qquad (2.16b)$$

where the order of truncation is dictated by the rule:[2] $\Gamma_1 \sim g$, $\Gamma_2 \sim g^2$.

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[1] If one sets $n = 0$, two equations become identical and one has to include the derivative of (2.13) with respect to n: $\Gamma_2 = V_2/(1 - (H_1 + H_2))^2$.

[2] This rule can be justified as follows: at leading order, the bare coupling constants can
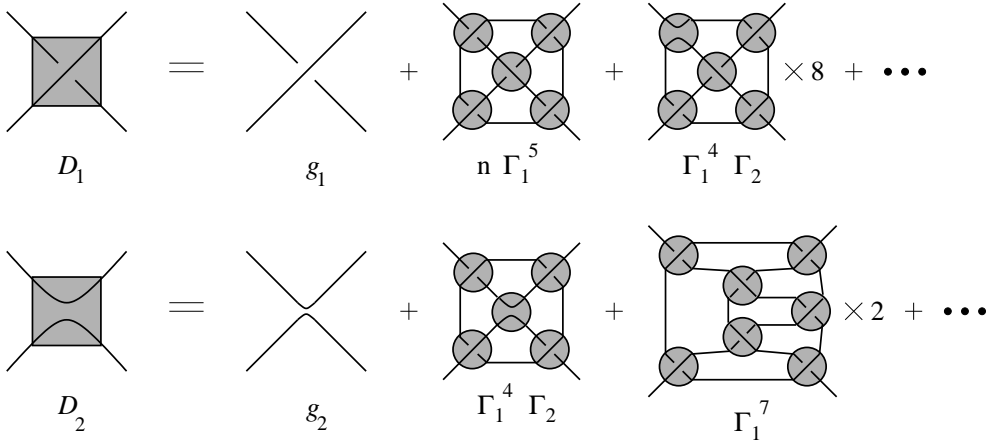
Fig. 5: First terms in the perturbative expansion of D'_1 and D'_2.

The data $D'_i[\Gamma_1, \Gamma_2]$ is all we need from the generalized matrix model (2.9), and we can now come back to the problem of tangle diagrams, with a single coupling constant g weighting a simple crossing.

We want to redo the decompositions of a general diagram in the horizontal channel, but this time taking into account the flyping equivalence. This forces us to distinguish between simple crossings and other diagrams, that is to use the primed objects defined by $\tilde{\Gamma}'_1 = \tilde{\Gamma}_1 - g$, $\tilde{H}'_1 = \tilde{H}_1 - g$, $\tilde{V}'_1 = \tilde{V}_1 - g$. Here, as in [2], the tilde on one of the symbols Γ, H or V denotes generating functions of flype equivalence classes of 2PI skeleton diagrams. We find:

$$\tilde{\Gamma}'_1 = g\tilde{\Gamma}_2 + \frac{1}{2}\left(\frac{\tilde{H}_2 + \tilde{H}'_1}{1 - (\tilde{H}_2 + \tilde{H}'_1)} - \frac{\tilde{H}_2 - \tilde{H}'_1}{1 - (\tilde{H}_2 - \tilde{H}'_1)}\right) \tag{2.17a}$$

$$\tilde{\Gamma}'_2 = g\tilde{\Gamma}_1 + \frac{1}{2}\left(\frac{\tilde{H}_2 + \tilde{H}'_1}{1 - (\tilde{H}_2 + \tilde{H}'_1)} + \frac{\tilde{H}_2 - \tilde{H}'_1}{1 - (\tilde{H}_2 - \tilde{H}'_1)}\right) \ . \tag{2.17b}$$

Introducing $\tilde{H}'_\pm = \tilde{H}'_2 \pm \tilde{H}_1$, we rewrite this

$$(1 \mp g)\tilde{\Gamma}_\pm = \pm g + \frac{\tilde{H}'_\pm}{1 - \tilde{H}'_\pm} \ . \tag{2.18}$$

Similarly, if $\tilde{H}'_0 = \tilde{H}_0 - g$, we find

$$(1 - g)\tilde{\Gamma}_0 = g + \frac{\tilde{H}'_0}{1 - \tilde{H}'_0} \ . \tag{2.19}$$

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be replaced with their renormalized values: $g_1 = g$, $g_2 = 0$. One then finds $\Gamma_1 = g + O(g^2)$, $\Gamma_2 = g^2 + O(g^3)$.

We invert these relations to

$$\tilde{H}'_{\pm} = \frac{(1 \mp g)\tilde{\Gamma}_{\mp} \mp g}{1 + (1 \mp g)\tilde{\Gamma}_{\mp} \mp g} \tag{2.20a}$$

$$\tilde{H}'_0 = \frac{(1 - g)\tilde{\Gamma}_0 - g}{1 + (1 - g)\tilde{\Gamma}_0 - g} \tag{2.20b}$$

and express the 2PI functions

$$\tilde{D}'_1 = \tilde{H}'_1 + \tilde{V}'_1 - \tilde{\Gamma}'_1 = \tilde{H}'_+ - \tilde{H}'_- - \tilde{\Gamma}_1 - g \tag{2.21a}$$

$$\tilde{D}'_2 = \tilde{H}'_2 + \tilde{V}'_2 - \tilde{\Gamma}'_2 = \frac{1}{2}(\tilde{H}'_+ + \tilde{H}'_-) + \frac{1}{n}(\tilde{H}'_0 - \tilde{H}'_+) - \tilde{\Gamma}_2 \ . \tag{2.21b}$$

Finally, the generating functions $\tilde{\Gamma}_i(g)$ are given by the implicit equations

$$\tilde{D}'_i(g) = D'_i[\tilde{\Gamma}_1(g), \tilde{\Gamma}_2(g)] \tag{2.22}$$

where $D'_i[\Gamma_1, \Gamma_2]$ comes from the solution of the matrix model, and $\tilde{D}'_i(g)$ is given by Eqs. (2.20)–(2.21) with $\tilde{\Gamma}_i = \tilde{\Gamma}_i(g)$.

We can solve the implicit equation (2.22) perturbatively using (2.16); the result is summarized in Table 1.

Table 1. Table of the number of prime alternating tangles up to 8 crossings. The power of n indicates the number of closed loops, i.e. the number of connected components besides the two outgoing strings.

	g	g^2	g^3	g^4	g^5	g^6	g^7	g^8
$\tilde{\Gamma}_1$	1	2	2	$6 + 3n$	$30 + 2n$	$62 + 40n + 2n^2$	$382 + 106n + 2n^2$	
$\tilde{\Gamma}_2$		1	1	$3 + n$	$9 + n$	$21 + 11n + n^2$	$101 + 32n + n^2$	$346 + 153n + 24n^2 + n^3$

These results are compatible with the exact solution at $n = 1$ [3], and they will give us a non-trivial check of our new solution at $n = 2$.

3. The $n = 2$ Case: a Model of Oriented Links

Let us now carry out explicitly the procedure outlined in the previous section, for a value of n corresponding to a solved matrix model. The case $n = 1$ has already been considered in [2]; let us now set $n = 2$. The partition function

$$Z = \tag{3.1}$$
$$\int dM_1 dM_2 \, e^{N \, \text{tr} \left(-\frac{t}{2}(M_1^2 + M_2^2) + \frac{g_1 + 2g_2}{4}(M_1^4 + M_2^4) + \frac{g_1}{2}(M_1 M_2)^2 + g_2 M_1^2 M_2^2 \right)}$$

is conveniently rewritten in terms of a complex matrix $X = \sqrt{\frac{t}{2}}\,(M_1 + iM_2)$:

$$Z = \int dX dX^\dagger \, e^{N \, \text{tr} \left(-XX^\dagger + b_0 X^2 X^{\dagger 2} + \frac{1}{2} c_0 (XX^\dagger)^2 \right)} \tag{3.2}$$
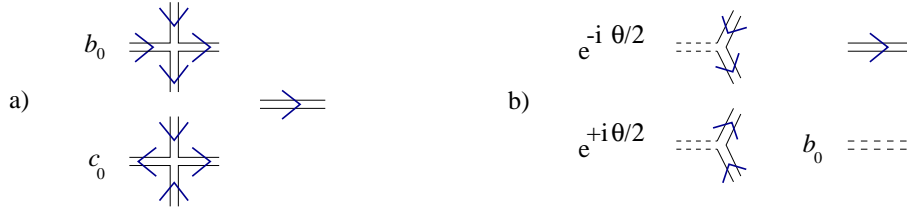
Fig. 6: Feynman rules corresponding to a) (3.2) and b) (3.3).

where we have absorbed t in the coupling constants: $b_0 = b/t^2$, $c_0 = c/t^2$, with $b = g_1 + g_2$ and $c = 2g_2$. It is clear that the partition function has been left unchanged by the transformation, but the individual diagrams are different. The new Feynman rules are depicted on Fig. 6 a). If we simply set $b = g$, $c = 0$ this model describes oriented links, each closed loop having now two possible orientations instead of two colors.

We recognize in (3.2) the partition function of the *six-vertex model* on random dynamical lattices, which has recently been solved [4,5]. In the 6 vertex formulation it is particularly clear that there is a Hubbard–Stratonovitch transformation which preserves planarity;[3] introducing the notation $c = 2b\cos\theta$, with $\theta \in [0, \pi[$ in the regime of interest to us, we have

$$Z = \int \mathrm{d}A\mathrm{d}X\mathrm{d}X^\dagger \mathrm{e}^{N\mathrm{tr}\left(-XX^\dagger - \frac{1}{2b_0}A^2 + A(XX^\dagger \mathrm{e}^{i\theta/2} + X^\dagger X \mathrm{e}^{-i\theta/2})\right)} \tag{3.3}$$

where A is a hermitean matrix. (Here and in the following integral (3.4), we omit an overall constant factor, irrelevant in the computation of correlation functions.) We have rewritten our model as a model of oriented loops which are *not* the intersecting loops we started from, since they can *only* avoid each other, see Fig. 6 b). This may seem unnatural, but is what allows to solve exactly the model, since we can now integrate over the original matrix X, shift the matrix A:

$$Z = \int \mathrm{d}A \det{}^{-1}\left(\mathrm{e}^{i\theta/2} \otimes A + A \otimes \mathrm{e}^{-i\theta/2}\right) \mathrm{e}^{-N\frac{1}{2b_0}\mathrm{tr}\left(A - \frac{1}{2\cos(\theta/2)}\right)^2} \tag{3.4}$$

and do a saddle point analysis of the eigenvalues of A. This is the basis of the analytic solutions [4,5] of the model.

It is natural to redefine the two independent 4-point functions to be

$$\Gamma_b = \left\langle \frac{1}{N}\mathrm{tr}(X^2 X^{\dagger 2}) \right\rangle_c \tag{3.5a}$$

$$\Gamma_c = \left\langle \frac{1}{N}\mathrm{tr}(XX^\dagger)^2 \right\rangle_c \tag{3.5b}$$

i.e. they are characterized by the position of the ingoing/outgoing arrows on the external legs. They are related to the 4-point functions defined earlier by: $\Gamma_b = \Gamma_1 + \Gamma_2$, $\Gamma_c = 2\Gamma_2$, as is clear diagrammatically.

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[3] The same remark applies to the corresponding fermionic model, i.e. $n = -2$.

Let us summarize the combinatorial relations of the previous section in the case $n = 2$. First, the relations which determine the corresponding 2PI correlation functions D_b and D_c can be obtained either by direct diagrammatic arguments or by setting $n = 2$ in the general formulae (2.11)–(2.13). We find:

$$\Gamma_b = \frac{H_b}{1 - H_b} \tag{3.6a}$$

$$\Gamma_c \pm \Gamma_b = \frac{V_c \pm V_b}{1 - (V_c \pm V_b)} \tag{3.6b}$$

$$D'_b = H_b + V_b - \Gamma_b - b \tag{3.6c}$$

$$D'_c = H_c + V_c - \Gamma_c - c \ . \tag{3.6d}$$

Similarly, the modified relations which take into account the flyping equivalence read: (we have reintroduced the renormalized coupling constant g corresponding to a simple crossing, and $\tilde{H}'_b = \tilde{H}_b - g$, etc)

$$\tilde{\Gamma}_b = g(1 + \tilde{\Gamma}_b) + \frac{\tilde{H}'_b}{1 - \tilde{H}'_b} \tag{3.7a}$$

$$(1 \mp g)(\tilde{\Gamma}_c \pm \tilde{\Gamma}_b) = \pm g + \frac{\tilde{V}_c \pm \tilde{V}'_b}{1 - (\tilde{V}_c \pm \tilde{V}'_b)} \tag{3.7b}$$

$$\tilde{D}'_b = \tilde{H}_b + \tilde{V}_b - \tilde{\Gamma}_b - g \tag{3.7c}$$

$$\tilde{D}'_c = \tilde{H}_c + \tilde{V}_c - \tilde{\Gamma}_c \ . \tag{3.7d}$$

We shall now use the solution of the matrix model (3.2) [4,5] to show how to extract the functions $\tilde{\Gamma}$ perturbatively at an arbitrary order, and to find their large order behavior.

3.1 Perturbative expansion of the off-critical solution.

In [5], the large N saddle point density of eigenvalues of A is explicitly constructed in terms of elliptic functions. More precisely if we define the resolvent

$$W(a) = \lim_{N \to \infty} \left\langle \frac{1}{N} \operatorname{tr} \frac{a}{a - A} \right\rangle \tag{3.8}$$

where the average is with respect to the measure in (3.4), and

$$J(a) = i \left[W\left(i\, a\, e^{-i\theta/2}\right) - W\left(-i\, a\, e^{i\theta/2}\right) \right] + \frac{1}{2 \sin \theta \, b_0} a^2 - \frac{1}{4 \cos^2(\theta/2)\, b_0} a \tag{3.9}$$

then we have the following expression for J using an elliptic parametrization u:

$$J = A + B \frac{1}{\operatorname{sn}^2(u - u_\infty)} \tag{3.10b}$$

$$a = a_0 \frac{H(u_\infty + u)}{H(u_\infty - u)} \tag{3.10a}$$

where H is the Jacobi theta function and A, B, u_∞ a_0 are constants which depend on b_0 and θ.

If we consider the small b, c perturbative expansion of the correlation functions of the model, we are in the region where the elliptic nome q is close to zero, and the elliptic functions can be expressed to a given order in q in terms of trigonometric functions. We can therefore write every quantity as a power series in q with coefficients dependent on θ. From the $1/a$ term in the $a \to \infty$ expansion of $J(a)$ we can extract $W_1 = \lim_{N \to \infty} \langle \frac{1}{N} \mathrm{tr} A \rangle$, and from there the two-point function using the formula

$$G = \frac{1}{b_0} \left(\frac{1}{2 \cos(\theta/2)} - W_1 \right) . \qquad (3.11)$$

We can then go back to the free energy by integrating once (at fixed θ)

$$F = \int^{b_0} \mathrm{d}b_0 \frac{G-1}{2b_0} \qquad (3.12)$$

and differentiate again to get the 4-point functions:

$$H = \frac{\partial}{\partial \theta} F_{|b_0 \text{ fixed}} . \qquad (3.13)$$

The rescaling which allows to set $G = 1$ simply amounts to writing that the coupling constant b is

$$b = b_0 \, G^2 . \qquad (3.14)$$

Then we have

$$\Gamma_b = \frac{(G-1)/2 + H \cot \theta}{b} - 1 \qquad (3.15a)$$

$$\Gamma_c = -\frac{H}{b \sin \theta} - 2 . \qquad (3.15b)$$

We must apply formulae (3.6) to evaluate D'_b and D'_c. Finally, we must consider q and θ as power series in the renormalized coupling constant g and solve perturbatively in g the equations (3.7).

This can be programmed on a computer using for example Mathematica$^{\text{TM}}$. At the first non-trivial orders we find:

$$b_0 = q^2 - 6(1 + 2 \cos \theta)q^4 + O(q^6)$$

$$G = 1 + 2(1 + 2 \cos \theta)q^2 + O(q^4)$$

$$\Gamma_b = q^2 - q^4 + O(q^6)$$

$$\Gamma_c = 2 \cos \theta \, q^2 + 2(1 - \cos \theta)q^4 + O(q^6)$$

$$D'_b = (6 + 12 \cos \theta + 4 \cos(2\theta) + 4 \cos(3\theta))q^{10} + O(q^{12})$$

$$D'_c = (8 + 24 \cos \theta + 8 \cos(2\theta) + 10 \cos(3\theta) + 2 \cos(5\theta))q^{10} + O(q^{12}) .$$

The next step is to solve perturbatively the Eqs. (3.7). We can theoretically proceed to an arbitrary order in q and therefore in g. In Table 2 we show the results obtained after 20 minutes of CPU time on a Sun work-station.

Table 2. Table of the number of oriented prime alternating tangles up to 16 crossings, and related quantities. $\Delta\theta \equiv \theta - \pi/2$.

	g	g^2	g^3	g^4	g^5	g^6	g^7	g^8	g^9	g^{10}	g^{11}
q^2	1	2	7	29	137	679	3515	18677	101463	560062	3132639
$\Delta\theta$		1	1	4	13	$\frac{319}{6}$	$\frac{437}{2}$	$\frac{1941}{2}$	$\frac{13424}{3}$	$\frac{858263}{40}$	$\frac{844871}{8}$
b	1		-1	-3	-9	-27	-103	-411	-1838	-8484	-41000
c			-2	-2	-6	-18	-74	-314	-1420	-6696	-32592
g_1	1			-2	-6	-18	-66	-254	-1128	-5136	-24704
g_2			-1	-1	-3	-9	-37	-157	-710	-3348	-16296
$\tilde{\Gamma}_b$	1	1	3	7	23	81	319	1358	6132	28916	140852
$\tilde{\Gamma}_c$		2	2	10	22	94	338	1512	6700	31944	155200
$\tilde{\Gamma}_1$	1		2	2	12	34	150	602	2782	12944	63252
$\tilde{\Gamma}_2$		1	1	5	11	47	169	756	3350	15972	77600

	g^{12}	g^{13}	g^{14}	g^{15}	g^{16}
q^2	17708417	100998567	580364663	3356435363	19519575163
$\Delta\theta$	$\frac{6386963}{12}$	$\frac{21859105}{8}$	$\frac{1596578121}{112}$	$\frac{18091381093}{240}$	$\cdots$
b	-202822	-1027954	-5300828	-27768068	-147347688
c	-162628	-828244	-4290608	-22553488	-120042224
g_1	-121508	-613832	-3155524	-16491324	-87326576
g_2	-81314	-414122	-2145304	-11276744	-60021112
$\tilde{\Gamma}_b$	704020	3592394	18648750	98217650	523734140
$\tilde{\Gamma}_c$	778168	3972088	20646268	108809588	580654216
$\tilde{\Gamma}_1$	314936	1606350	8325616	43812856	233407032
$\tilde{\Gamma}_2$	389084	1986044	10323134	54404794	290327108

As a check, note that $g_1 = (1-\cos\theta)b = g+O(g^4)$ and $g_2 = \cos\theta\, b = -g^3+O(g^4)$ which is consistent with the fact that at order 3 there is exactly one diagram of type 2 (up to a $\pi/2$ rotation) which is overcounted (see Fig. 7). Also, these data are compatible with those of Table 1.
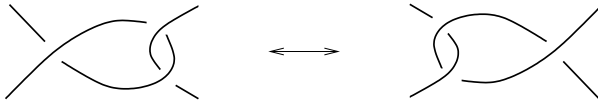

Fig. 7: First flype-equivalent tangle diagrams.

3.2 Solution at criticality and large order behavior of the correlation functions.

In [4], the model (3.2) is analyzed in the critical regime, that is when one encounters the first singularity of the free energy (or of the correlation functions) by analytic continuation from the gaussian model. This singularity – or more precisely, the line of singularities in the (b_0, c_0) plane – is what determines the large order behavior of the correlation functions, and ultimately of the functions $\tilde{\Gamma}$.

We shall now summarize some of the relevant results from [4]. If the parameter θ is fixed and the coupling constant b_0 is increased, for any value of $\theta \in [0, \pi[$ there will be a critical value $b_0^\star(\theta)$ for which the free energy becomes singular:

$$b_0^\star(\theta) = \frac{1}{32} \frac{\tan(\theta/2)}{\theta/2} \frac{1}{\cos^2(\theta/2)} \ . \tag{3.16}$$

This forms a line of critical points, describing what is known in physics as the line of $c = 1$ conformal field theories – i.e. a free boson compactified on circles of varying radius – coupled to gravity, see for example [10] for a review and references. In [4], all correlation functions of the form $\lim_{N\to\infty} \left\langle \frac{1}{N}\text{tr}A^s \right\rangle$ are calculated on the critical line. It is easy to see that the explicit expressions of $W_1 = \lim_{N\to\infty} \left\langle \frac{1}{N}\text{tr}A \right\rangle$ and $W_2 = \lim_{N\to\infty} \left\langle \frac{1}{N}\text{tr}A^2 \right\rangle$ allow to extract the 2-point function and the free energy (still on the critical line); one finds

$$G^\star(\theta) = 8\frac{\theta/2}{\tan(\theta/2)} - \frac{2}{3}(\pi^2 - \theta^2) \tag{3.17a}$$

$$F^\star(\theta) = -4\frac{\theta/2}{\tan(\theta/2)} + \frac{1}{6}(\pi^2 - \theta^2) \ . \tag{3.17b}$$

By definition $F^\star(\theta) = F(b_0^\star(\theta)), \theta)$, which means that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}F^\star(\theta) = \frac{1}{2b_0^\star(\theta)}G^\star(\theta)\frac{\mathrm{d}b_0^\star}{\mathrm{d}\theta} + \frac{\partial}{\partial\theta}F(b_0^\star(\theta), \theta) \ . \tag{3.18}$$

This gives us access to $H = \frac{\partial}{\partial\theta}F_{|b_0 \text{ fixed}}$ at criticality. One finds:

$$H^\star(\theta) = -2\theta + \frac{1}{2}(\pi^2 - \theta^2)\tan(\theta/2) - \frac{1}{3}\pi^2\frac{1}{\theta} + \frac{1}{6}(\pi^2 - \theta^2)\cot(\theta/2) \ . \tag{3.19}$$

One must then use again the formulae (3.14)–(3.15) and (3.6) to find the values of Γ_b, Γ_c, D_b', D_c' on the critical line as a function of θ, and finally solve (3.7) for g and θ. This is a set of 2 complicated coupled equations,[4] and we can only solve them numerically. We find the following numerical values:

$$\theta_c = 1.60780446\ldots \tag{3.20a}$$

$$1/g_c = 6.28329764\ldots \tag{3.20b}$$

The value (3.20b) is non-universal, and only this explicit calculation could give us access to it. It is related to the leading behavior of the coefficients of the power series $\tilde{\Gamma}_b(g)$ and $\tilde{\Gamma}_c(g)$, i.e. of the numbers of oriented prime alternating tangles. On the other hand, the subleading behavior is universal; it is characteristic of $c = 1$ conformal field theories coupled to gravity, which exhibit zero "string susceptibility"

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[4] In fact, one of the two equations is quadratic in g, so that remains only one (transcendental) equation in θ.

with logarithmic corrections [4,5]. Therefore we can state that if $\tilde{\Gamma}_b(g) = \sum \gamma_p g^p$, then

$$\gamma_p \overset{p \to \infty}{\sim} \text{const } g_c^{-p} p^{-2} (\log p)^{-1} \tag{3.21}$$

and similarly for $\tilde{\Gamma}_c$. The constant $1/g_c = 6.28329764\ldots$ is slightly less than the corresponding constant $16/(\pi(\pi - 4)^2) = 6.91167\ldots$ [9] for oriented alternating link or tangle diagrams (without taking into account the flype equivalence); and should also be compared to the numbers obtained for $n = 1$: $(101 + \sqrt{21001})/40 = 6.147930\ldots$ and $27/4 = 6.75$ with and without the flype equivalence, respectively. The fact that the $n = 1$ and $n = 2$ results are fairly close shows in particular that the entropy generated by tangles with large numbers of connected components is small.

As a final note, we can make some slightly conjectural statements on the number of oriented prime alternating *links*. It is clear that if we close a tangle by pasting a simple crossing to its four external legs, we obtain a link; if the tangle had k closed loops, then the number of connected components of the resulting link is simply $k + 2$ for a tangle of type 1 and $k + 1$ for a tangle of type 2. Inversely, from a link with p crossings one can produce $4p$ tangles by removing one of its vertices and fixing the circular permutation of the four legs. Even though we cannot use this fact to correctly count links (one cannot make it a one-to-one correspondence), by assuming that most links have low symmetry, we have the approximate relation $f_p \approx \gamma_{p-1}/p$ satisfied by the number f_p of links with p crossings. Therefore we conjecture that

$$f_p \overset{p \to \infty}{\sim} \text{const } g_c^{-p} p^{-3} (\log p)^{-2} \ . \tag{3.22}$$

Acknowledgements

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