An Introduction to Conformal Field Theory

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The aim of these lectures is to present an introduction at a fairly elementary level to recent developments in two dimensional field theory, namely in conformal field theory. We shall see the importance of new structures related to infinite dimensional algebras: current algebras and Virasoro algebra. These topics will find physically relevant applications in the lectures by Shankar and Ian Affleck.

1st Lecture

Infinite dimensional algebras

Let us start by introducing some basic notions related to finite and infinite dimensional Lie algebras.

As an example of a finite-dimensional simple Lie group, describing the internal global symmetry of a field theory in D-dimensional spacetime, let us take the orthogonal group $O(N)$. A multiplet of fields $\Phi_\alpha(x)$ ($\alpha = 1, ..., N$) is assumed to form a N-dimensional fundamental representation of the group $O(N)$. The infinitesimal transformation of fields is given by

$$\delta^a \Phi_\alpha(x) = i T^a_{\alpha\beta} \Phi_\beta(x) \ , \tag{1}$$

where T^a are the generators of the infinitesimal transformations, so that $\exp(i\, T^a \delta\varepsilon^a)$ belongs to $O(N)$. They span the Lie algebra associated with the symmetry group, completely defined by the structure constants f^{abc}

$$[T^a, T^b] = i f^{abc} T^c \ . \tag{2}$$

The generators of the group $O(N)$ are taken as (hermitian) antisymmetric matrices,

$$(T^a)^\dagger = T^a = -(T^a)^t \ , \tag{3}$$

satisfying also the normalisation condition

$$\mathrm{tr}\, T^a T^b = \delta_{ab} \ . \tag{4}$$

In a quantum theory, the transformation law (1) for the field operator $\hat\Phi$ is generated by the conserved charge operator $\hat Q^a$

$$\delta^a \hat\Phi = i[\hat Q^a, \hat\Phi] \ . \tag{5}$$

Here and in the following, the hat above a field intends to stress its operator nature. It will be dropped whenever it is unambiguous. The following algebra of charges holds,

$$[\hat{Q}^a, \hat{Q}^b] = if^{abc}\hat{Q}^c \ . \tag{6}$$

In a local field theory, the charges resulting from global symmetries are given by

$$\hat{Q}^a = \int d^{D-1}x \ \hat{J}^a_0(\vec{x}, t) \ , \tag{7}$$

where $\hat{J}^a_0$ are time components of the Noether currents. They are conserved $\frac{d}{dt}\hat{Q}^a = 0$ if the currents satisfy

$$\partial^\mu \hat{J}^a_\mu = 0 \ . \tag{8}$$

Then, we can look at the equal time commutation relations between the time components of the currents,

$$[\hat{J}^a_0(\vec{x}, t), \hat{J}^b_0(\vec{y}, t)] = if^{abc}\hat{J}^c_0(\vec{x}, t)\delta(\vec{x} - \vec{y}) + \dots \ , \tag{9}$$

and, if possible, at analogous relations for space components of the currents. The first term on the right hand side of (9) follows from the structure of the symmetry algebra $O(N)$. The dots stand here for possible extra terms, that cannot be deduced from the sole properties of the $O(N)$ charge algebra. They are the so-called Schwinger terms.

If these extra terms are under control and the algebra (9) closes on the terms $\hat{J}_0$ plus a finite number of other terms, we see that these current components form an infinite dimensional algebra. The structure of this algebra is particularly simple in two spacetime dimensions.

Free Euclidean fermions

Let us consider a simple model of free massless fermions in two–dimensional Euclidean spacetime. Euclidean coordinates are denoted by $x^\mu = (x^1, x^2)$. It is convenient to adopt complex coordinates,

$$z = x^1 + ix^2 \ , \bar{z} = x^1 - ix^2 \ , \tag{10}$$

The line element is given by,

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 = g_{zz}(dz)^2 + 2g_{z\bar{z}}dzd\bar{z} + g_{\bar{z}\bar{z}}(d\bar{z})^2 \ . \qquad (11)$$

The flat metric $g_{\mu\nu} = \delta_{\mu\nu}$ corresponds to off-diagonal $z, \bar{z}$ components

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \ , \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} \ . \qquad (12)$$

The complex contravariant components are respectively

$$g^{zz} = g^{\bar{z}\bar{z}} = 0 \ , \quad g^{z\bar{z}} = g^{\bar{z}z} = 2 \ , \qquad (13)$$

and thus the complex indices are raised and lowered according to

$$V_z = \frac{1}{2}V^{\bar{z}} \ , \quad V^z = 2V_{\bar{z}} \ . \qquad (14)$$

It is easy to find relations between real and complex tensor components. For example, we can relate respective components of the gradient operator,

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \ , \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \ . \qquad (15)$$

We will further abbreviate ∂_z by ∂ and $\partial_{\bar{z}}$ by $\bar{\partial}$. The volume element reads

$$`d^2z' \equiv d^2x = dx^1 \wedge dx^2 = \frac{d\bar{z} \wedge dz}{2i} \ . \qquad (16)$$

Dirac or Majorana fermions in two dimensions are two-component objects,

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \ . \qquad (17)$$

The bar over the down spinor component is only a customary notation, and both components are anticommuting. The gamma matrices may be taken to be the Pauli matrices

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \gamma^1\gamma^2 = i\gamma^3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \qquad (18)$$

(Note that γ^3 is diagonal, so that up- and down-components of (17) describe opposite chiralities, and that returning to real space-time, a Wick rotation would make γ^2 real: this allows to choose solutions of the Dirac equation (see below) with reality properties and justifies calling the two components of (17) Majorana-Weyl fermions.)

We can write the Dirac Lagrangian explicitely,

$$\begin{aligned} \frac{1}{2}\bar{\Psi}\gamma^\mu\partial_\mu\Psi &= \frac{1}{2}\Psi^t\gamma^1\gamma^\mu\partial_\mu\Psi = \\ &= \Psi^t \begin{pmatrix} \frac{1}{2}(\partial_1 + i\partial_2) & 0 \\ 0 & \frac{1}{2}(\partial_1 - i\partial_2) \end{pmatrix} \Psi = \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} \ . \end{aligned} \tag{19}$$

Therefore, the action for massless two-dimensional fermions is

$$S = \frac{1}{2\pi}\int d^2z \ (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) \ . \tag{20}$$

The factor 2π is introduced for later convenience. Dirac equations of motion are $\bar{\partial}\psi = \partial\bar{\psi} = 0$, their solutions show that the spinor components are holomorphic and antiholomorphic functions respectively, namely

$$\psi = \psi(z) \ , \quad \bar{\psi} = \bar{\psi}(\bar{z}) \ . \tag{21}$$

The fermionic system decomposes into a holomorphic (analytic) part and an antiholomorphic (antianalytic) part.

The kinetic Lagrangian term (20) can be inverted to derive propagators,

$$\bar{\partial} < \psi(z)\psi(z') >= \partial < \bar{\psi}(\bar{z})\bar{\psi}(\bar{z'}) >= \pi\delta^{(2)}(\vec{r} - \vec{r'}) \ . \tag{22}$$

Due to the normalization chosen in (20) we obtain the following simple results

$$< \psi(z)\psi(w) >= \frac{1}{z-w} \ , \tag{23}$$

$$< \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) >= \frac{1}{\bar{z}-\bar{w}} \ , \tag{24}$$

$$< \psi(z)\bar{\psi}(\bar{w}) >= 0 \ . \tag{25}$$

The above model can be generalized to incorporate the internal symmetry group $O(N)$. We consider N Majorana-Weyl fermions with the following action

$$S = \frac{1}{2\pi}\sum_{\alpha=1}^{N}\int d^2z \ (\psi_\alpha\bar{\partial}\psi_\alpha + \bar{\psi}_\alpha\partial\bar{\psi}_\alpha) \ . \tag{26}$$

The action is invariant under the $O(N)$ global tranformations, with the set of conserved currents $J^a_\mu = \frac{1}{2}\Psi^t\gamma^1\gamma_\mu T^a\Psi$. We will consider their complex components,

$$J^a(z) \equiv J^a_z = \frac{1}{2}(J^a_1 - iJ^a_2) = \frac{1}{2}\psi_\alpha(z)T^a_{\alpha\beta}\psi_\beta(z) \ , \qquad (27)$$

$$\bar{J}^a(\bar{z}) \equiv J^a_{\bar{z}} = \frac{1}{2}(J^a_1 + iJ^a_2) = \frac{1}{2}\bar{\psi}_\alpha(\bar{z})T^a_{\alpha\beta}\bar{\psi}_\beta(\bar{z}) \ . \qquad (28)$$

The holomorphicity (resp. anti-holomorphicity) of the currents J (resp. $\bar{J}$) that follow from the equation of motion imply the conservation law $\partial^\mu J_\mu = 2(\partial\bar{J} + \bar{\partial}J) = 0$. In fact this holomorphicity of J and antiholomorphicity of $\bar{J}$ are equivalent to the conservation of both the vector currents J_μ *and* the axial currents $J^a_{\text{Axial}\,\mu} = \frac{1}{2}\Psi^t\gamma^1\gamma_\mu\gamma^3T^a\Psi$ whose $z, \bar{z}$ components are $(J, -\bar{J})$.

The only change to the above formulae due the field quantization is the normal ordering of field operators, $J^a = \frac{1}{2} : \psi_\alpha T^a_{\alpha\beta}\psi_\beta :$ etc. Now, let us calculate the operator product $J^a(z)J^b(w)$ in the limit where z approaches w. Using the Wick theorem and (23-25) we calculate

$$\begin{aligned} J^a(z)J^b(w) &= \frac{1}{4} : \psi(z)T^a\psi(z) :: \psi(w)T^b\psi(w) : \\ &= \frac{1}{2(z-w)^2}\delta^{ab} + if^{abc}\frac{J^c(w)}{z-w} + \text{reg} \ . \end{aligned} \qquad (29)$$

The last ('reg') term is finite in the limit $z \to w$. In the same way, we obtain

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) = \frac{1}{2(\bar{z}-\bar{w})^2}\delta^{ab} + if^{abc}\frac{\bar{J}^c(\bar{w})}{\bar{z}-\bar{w}} + \text{reg} \ . \qquad (30)$$

$$J^a(z)\bar{J}^b(\bar{w}) = \text{reg} \ . \qquad (31)$$

These 'short distance expansions' (29-31) have to be understood in the sense of insertions in correlation functions: in the presence of other fields located at points different from z and w, one may write

$$\langle J^a(z)J^b(w)\cdots\rangle = \frac{1}{2(z-w)^2}\delta^{ab}\langle\cdots\rangle + if^{abc}\frac{1}{z-w}\langle J^c(w)\cdots\rangle + \text{reg} \ . \quad (32)$$

Finally, we can compare the above relations with the generic formula (9). The second term on the right hand side of (29) can be recognized as the

Cauchy kernel, so that it matches the first term in (9). We have determined also the Schwinger term, of the form $\delta^{ab}\delta'(\vec{x}-\vec{y})$. This will be exposed more clearly in the next lecture.

One lesson to be remembered from this first lecture is the importance of complex coordinates when dealing with massless fields in two dimensions: the holomorphic (z) and antiholomorphic ($\bar{z}$) dependences have decoupled.

2nd Lecture

Radial ordering

As is well known, there are two main quantization procedures in field theory. One appeals to functional integration, where the basic observables, the correlation functions of fields, result from the integration with a certain measure $\int \mathcal{D}\,\phi e^S$ of the field functionals. For example the two-point function of the current that we have been considering reads

$$< J^a(z)J^b(w)\ldots >= \left(\int \mathcal{D}\phi\, e^S\right)^{-1} \int \mathcal{D}\phi\, e^S J^a(z)J^b(w)\ldots \tag{33}$$

The second procedure emphasizes the role of observables as operators acting in the Hilbert space of the theory. The non commutation of the field operators and their ordering in the correlation functions is an important feature of that quantization procedure. Thus the correlation functions are to be computed as the vacuum expectation values of suitably ordered products of field operators. Usually, the physical observables are expressed in terms of correlation functions made of time ordered products of fields. In conformal field theory, it is more convenient to order the fields radially outward from the origin. The radially ordered product of two operators is defined as

$$\mathcal{R}\hat{X}(z,\bar{z})\hat{Y}(w,\bar{w}) = \begin{cases} \hat{X}(z,\bar{z})\hat{Y}(w,\bar{w})\ , & |z| > |w| \\ \pm\hat{Y}(w,\bar{w})\hat{X}(z,\bar{z})\ , & |z| < |w| \end{cases} \tag{34}$$

where the plus (minus) sign is for bosonic (fermionic) operators. The procedure for calculating radially ordered correlation functions, 'the radial quantization scheme', is very powerful because it facilitates the use of complex analysis and contour integrals.

In fact the radial ordering appears in a natural way in a conformally invariant two–dimensional field theory. Suppose the space direction periodic, i.e. let it be a circle of a given length L. Euclidean space–time is thus a cylinder, a situation familiar in the context of string theory when one looks at time evolution of *closed* strings, or of statistical mechanics when one works with a finite strip with periodic boundary conditions. We denote the complex coordinates of that cylinder by $\zeta, \bar{\zeta}$ (the real part of ζ is the space coordinate).

As we shall see soon, a conformal field theory has a certain covariance under conformal changes of coordinates. In particular, we can consider the following mapping,

$$z = e^{2i\pi\frac{\zeta}{L}} \ , \quad \bar{z} = e^{-2i\pi\frac{\bar{\zeta}}{L}} \ , \tag{35}$$

that maps the cylinder onto the plane (punctured, i.e. with the origin removed). Equal time lines on the cylinder correspond to constant radius circles on the plane. Our radial ordering on the plane thus corresponds to the usual time ordering on the cylinder.

Let us now rephrase the results that we have obtained on the short distance product of two currents in the operator language. To distinguish the two approaches, we shall put again a hat on fields to stress their operator interpretation. Thus (29) reads

$$\mathcal{R}\left(\hat{J}^a(z)\hat{J}^b(w)\right) = \frac{1}{2(z-w)^2}\delta^{ab} + f^{abc}\frac{\hat{J}^c(w)}{z-w} + reg \ . \tag{36}$$

Affine current algebra

As it has been already mentioned, the conservation laws reexpressed in complex coordinates lead to the (anti)holomorphic dependence of the current components (see (27,28)). Holomorphic fields can be expanded in Laurent series,

$$J^a(z) = \sum_{n\in Z} J^a_n z^{-n-1} \ , \quad \bar{J}^a(\bar{z}) = \sum_{n\in Z} \bar{J}^a_n \bar{z}^{-n-1} \ , \tag{37}$$

$$J^a_n = \oint_O \frac{dz}{2i\pi} J^a(z) z^n \ , \quad \bar{J}^a_n = \oint_O \frac{d\bar{z}}{2i\pi} \bar{J}^a(\bar{z})\bar{z}^n \ , \tag{38}$$

where the integrals are along contours encircling the origin.

Let us now derive the commutator between the Laurent modes,

$$\begin{aligned} [\hat{J}^a_n, \hat{J}^b_m] &= \oint_O \frac{dz}{2i\pi} z^n \oint_O \frac{dz}{2i\pi} w^m \hat{J}^a(z)\hat{J}^b(w) - \oint_O \frac{dz}{2i\pi} z^n \oint_O \frac{dz}{2i\pi} w^m \hat{J}^b(w)\hat{J}^a(z) \\ &= \oint_O \frac{dw}{2i\pi} w^m \left[\oint_{|z|>|w|} \frac{dz}{2i\pi} z^n - \oint_{|z|<|w|} \frac{dz}{2i\pi} z^n\right] \mathcal{R}\left(\hat{J}^a(z)\hat{J}^b(w)\right) \end{aligned} \tag{39}$$

The difference between the two z-contour integrals, one inwards, one outwards with respect to the w-contour, combines into a single integration along a contour around the point w (see Fig. 1).

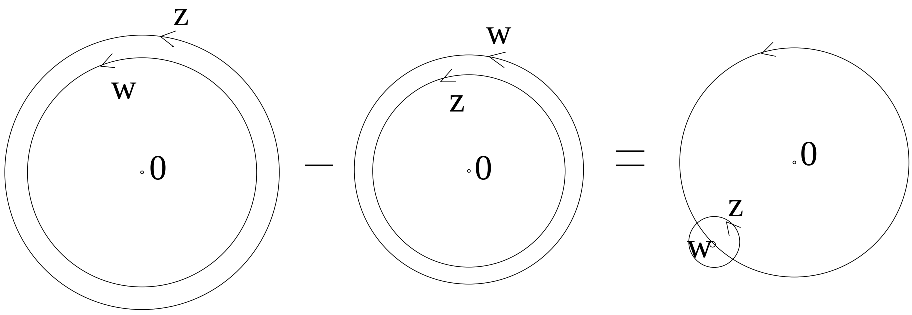

Fig 1 : The difference between two z-contour integrals may be reexpressed as a contour integral around w

Then, if we insert the short distance product (36), only singular terms contribute to the final result.

$$
\begin{aligned}
[\hat{J}^a_n, \hat{J}^b_m] &= \oint_O \frac{dw}{2i\pi} w^m \oint_w \frac{dz}{2i\pi} z^n \mathcal{R}\left(\hat{J}^a(z)\hat{J}^b(w)\right) \\
&= \oint_O \frac{dw}{2i\pi} w^m \oint_w \frac{dz}{2i\pi} z^n \left[\frac{1}{2(z-w)^2}\delta^{ab} + if^{abc}\frac{T^c(w)}{z-w} + reg\right] \\
&= \frac{n}{2}\delta^{ab}\delta_{n+m,0} + if^{abc}J^c_{n+m} \ . \qquad (40)
\end{aligned}
$$

The current algebra of the modes $\hat{J}^a_n$ is called an affine Lie algebra:

$$
[\hat{J}^a_n, \hat{J}^b_m] = if^{abc}\hat{J}^c_{n+m} + \frac{n}{2}\hat{k}\delta^{ab}\delta_{n+m,0} \ . \qquad (41)
$$

It is infinite dimensional: there is an infinite number of generators, $\hat{J}^a_n$ and $\hat{k}$. The finitely many modes $\hat{J}^a_0$ form the ordinary Lie algebra with structure constants f^{abc}. The extra term commutes with all generators, $[\hat{k}, \hat{J}^a_n] = 0$, whence the name 'central term'. This ensures that the Jacobi identity is satisfied. For irreducible representations, Schur's lemma implies that the $\hat{k}$-operator must be proportional to the identity, $\hat{k} = k\hat{I}$. The constant k thus depends on the specific representation of the affine algebra. We have found that for N free Majorana fermions $k = 1$: it is a 'level $k = 1$' representation of the affine $SO(N)$ algebra. Later, we will see that for all 'good' representations of current algebras, k is integer, with appropriate normalizations of the generators.

In the following we shall drop the hat above operators.

Conformal (Virasoro) algebra

Another important infinite dimensional algebra appears if we consider the local changes of coordinates, $x^\mu \to x^\mu + \varepsilon^\mu(x)$. The infinitesimal change of the action defines the energy-momentum tensor $T_{\mu\nu}$

$$\delta S = \frac{1}{2\pi}\int d^2x\ T_{\mu\nu}\partial^\mu\varepsilon^\nu \tag{42}$$

(the choice of normalization with $\frac{1}{2\pi}$ will be convenient in the following). Let us concentrate again on the example of the free massless Majorana fermion. The complex components of the energy–momentum tensor read

$$\begin{aligned} T(z) \equiv T_{zz} &= -\frac{1}{2} : \psi\partial\psi : , \\ \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}} &= -\frac{1}{2} : \bar{\psi}\bar{\partial}\bar{\psi} : , \\ T_{z\bar{z}} &= T_{\bar{z}z} = 0 \ . \end{aligned} \tag{43}$$

If we return to Cartesian tensor components, the vanishing of off-diagonal complex components means that the energy–momentum tensor is symmetric and traceless, while the holomorphicity of the diagonal components amounts to the conservation law $\partial^\mu T_{\mu\nu} = 0$. As in the previous section, we can evaluate the short distance product expansions,

$$\begin{aligned} T(z)T(w) &= \frac{1}{4(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg} , \\ \bar{T}(\bar{z})\bar{T}(\bar{w}) &= \frac{1}{4(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \text{reg} , \\ T(z)\bar{T}(\bar{w}) &= \text{reg} \ . \end{aligned} \tag{44}$$

The Laurent modes are defined by:

$$T(z) = \sum_{n\in Z} L_n z^{-n-2} \ , \quad \bar{T}(\bar{z}) = \sum_{n\in Z} \bar{L}_n \bar{z}^{-n-2} \ , \tag{45}$$

$$L_n = \oint_O \frac{dz}{2i\pi} T(z) z^{n+1} \ , \quad \bar{L}_n = \oint_O \frac{d\bar{z}}{2i\pi} \bar{T}(\bar{z}) \bar{z}^{n+1} \ . \tag{46}$$

Following the same procedure as above for the J's, it is now straightforward to derive the following algebra,

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{m+n} + \frac{1}{24}n(n^2-1)\delta_{n+m,0} \quad , \\
\left[\bar{L}_n, \bar{L}_m\right] &= (n-m)\bar{L}_{m+n} + \frac{1}{24}n(n^2-1)\delta_{n+m,0} \quad , \\
\left[L_n, \bar{L}_m\right] &= 0 \quad .
\end{aligned} \tag{47}
$$

In general the Virasoro algebra is defined as

$$
[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1) \tag{48}
$$

and c is the central charge. We thus see that the operators L_m and $\bar{L}_m$ of (47) form two commuting Virasoro algebras of central charge $c = \frac{1}{2}$. Equation (46) shows that L_n, resp. $\bar{L}_n$, is the generator of the change $\delta z = z^{n+1}$ (resp. $\delta \bar{z} = \bar{z}^{n+1}$) in the quantum field theory. It is interesting to confront these operators with their classical counterparts, namely

$$
\mathcal{L}_n = -z^{n+1}\frac{\partial}{\partial z} \ , \ \ \bar{\mathcal{L}}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}} \ , \tag{49}
$$

which satisfy the following classical algebra

$$
[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}, \tag{50}
$$

together with similar relations for the antiholomorphic sector. We see now that the 'central term' in (47) is due to quantum effects.

Note also that L_0, $\bar{L}_0$ are the rotation/dilatation generators, whereas L_{-1}, $\bar{L}_{-1}$ are those of translations.

3d Lecture

Conformal invariance

Let us first discuss briefly the general features of conformally invariant field theories, in a generic space–time dimension D. A conformal transformation is defined as an angle-preserving local change of coordinates.

If $g_{\mu\nu}$ is the metric tensor ($ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$), a transformation that leaves the metric invariant up to a local scale change,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = (1+\alpha(x))\, g_{\mu\nu}(x) \tag{51}$$

is conformal. For an infinitesimal coordinate transformation $x^\mu \to x^\mu + \varepsilon^\mu(x)$, the condition reads

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \varepsilon^\rho \partial_\rho g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu \varepsilon^\rho + g_{\nu\rho}(x)\partial_\mu \varepsilon^\rho = (1+\alpha(x))g_{\mu\nu}(x) \ . \tag{52}$$

Thus in Euclidean space the transformation is conformal if and only if the following equations are satisfied,

$$g_{\mu\rho}\partial_\nu \varepsilon^\rho + g_{\nu\rho}\partial_\mu \varepsilon^\rho = \alpha(x) g_{\mu\nu} \ , \tag{53}$$

Contracting with $g^{\mu\nu}(x)$, one identifies $\alpha = 2\partial_\rho \varepsilon^\rho$.

In a classical local field theory, the infinitesimal change of the action under a local change of coordinates is defined by the energy-momentum tensor $T_{\mu\nu}$, see (42). Equation (42) implies the invariance of the action under constant translations $\varepsilon(x) = a$. If we assume morover that the energy-momentum tensor is both symmetric and traceless, then the action is also invariant under infinitesimal rotations $\varepsilon^\mu = \omega^{\mu\nu} x_\nu$, (with $\omega^{\mu\nu}$ antisymmetric), and dilatations $\varepsilon^\mu = \lambda x^\mu$. (Conversely with adequate assumptions, invariance under rotations and dilatations implies the symmetry and tracelessness of $T_{\mu\nu}$.)

If we combine the fact that $T_{\mu\nu}$ is symmetric and traceless together with equation (53),

$$T_{\mu\nu}\partial^\mu \varepsilon^\nu = T_{\mu\nu}\frac{1}{2}\left(\partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu\right) = \frac{1}{2}\alpha(x) T_{\mu\nu} g^{\mu\nu} = 0 \ , \tag{54}$$

then we draw the striking conclusion that the action S is left invariant under arbitrary conformal transformations! (Polyakov, 1970).

In the quantized conformally invariant field theory, equ. (42) should be understood as inserted in the functional integral and implies Ward identities for correlation functions. Consider some correlation function,

$$< \phi_1 \dots \phi_N >= \frac{1}{Z} \int \mathcal{D}\phi \; e^{S[\phi]} \phi_1 \dots \phi_N \;, \tag{55}$$

where $Z = \int \mathcal{D}\phi \; e^{S[\phi]}$. Denote by $\delta\phi$ the change of the field ϕ under the conformal transformation $x \to x + \varepsilon$. Writing that the functional integral in the numerator is invariant under that change, we get

$$\sum_{i=1}^{N} < \phi_1 \dots \delta\phi_i \dots \phi_N > + \frac{1}{2\pi} \int d^D x \; \partial^\mu \varepsilon^\nu < T_{\mu\nu}(x) \phi_1 \dots \phi_N >= 0 \;. \tag{56}$$

In particular, if the $\delta\phi(x)$ are local expressions depending only on $\phi(x)$, $\varepsilon(x)$ and a finite number of their derivatives,

$$\delta\phi_i(x) = P_i(\partial, \varepsilon)\phi_i(x) \tag{57}$$

we find after functional differentiation with respect to $\epsilon_\nu(x)$

$$\partial_x^\mu \langle T_{\mu\nu}(x)\phi_1(x_1) \cdots \phi_N(x_N)\rangle = \sum_{i=1}^{N} \widetilde{P}_{\nu\, i}(\partial_i) \delta^{(D)}(x - x_i) \langle \phi_1 \cdots \phi_N \rangle \;. \tag{58}$$

In particular the conservation law $\partial^\mu T_{\mu\nu} = 0$ holds everywhere except at coinciding points $x = x_i$.

Conformal invariance in two dimensions

From now on, we shall restrict ourselves to two–dimensional theories. In complex coordinates, equation (53) reads

$$\partial_{\bar{z}} \varepsilon^z = \partial_z \varepsilon^{\bar{z}} = 0 \;. \tag{59}$$

Thus conformal transformations correspond to holomorphic changes of the complex coordinates,

$$z \to z + \varepsilon(z) \;, \quad \bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z}) \;. \tag{60}$$

There exists a subset of conformal transformations that form a group,

$$z \to \frac{az+b}{cz+d} \ . \tag{61}$$

Those are the only one-to-one applications of the complex plane with a point at infinity (or Riemann sphere) onto itself. In general we may only demand analyticity of ϵ in a bounded region.

Assume that T is traceless and symmetric (hence $T_{z\bar{z}} = T_{\bar{z}z} = 0$) and rewrite the Ward identities (56) in complex coordinates

$$\begin{aligned} &\delta < \phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N) >= \\ &-\int \frac{d\bar{z}\wedge dz}{2i\pi}\bar{\partial}\varepsilon(z,\bar{z}) < T_{zz}(z,\bar{z}) \quad \phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N) > +\text{c.c.} \ . \end{aligned} \tag{62}$$

Assume moreover that ε vanishes fast enough at large distances from the origin to allow integration by parts, say outside a domain $\mathcal{D}'$ and is analytic in a domain $\mathcal{D} \subset \mathcal{D}'$ containing the points $z_1,\ldots,z_N$. Moreover, as we have just seen in the previous subsection, $T_{\mu\nu}$ is conserved, i.e., in $z,\bar{z}$ components, $T_{zz} \equiv T(z)$ is a holomorphic function of z (and *mutatis mutandis* for $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$). More precisely, the correlation function

$$< T(z)\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N) > \tag{63}$$

is analytic everywhere except at the positions of inserted fields. Similarly,

$$< \bar{T}(\bar{z})\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N) > \tag{64}$$

is antianalytic except at $z = z_1, ..., z_N$.

Using this analyticity and Stokes theorem, we can transform the right hand side of (62), originally an integral over the domain $\mathcal{D}'$ where ϵ is non vanishing (see Fig. 2)

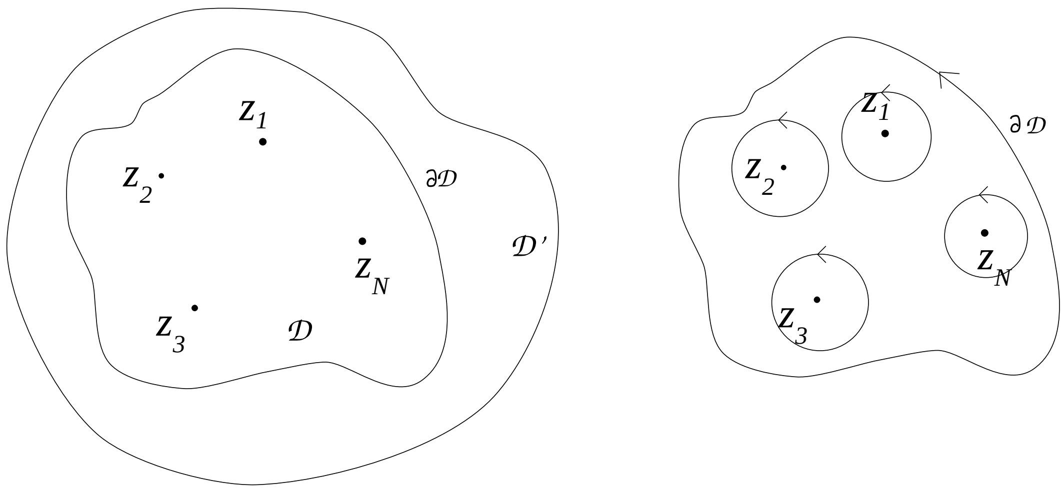

Fig 2 : Transforming the integral in (62) into
a z contour integral around $z_1, \cdots z_N$.

$$
\begin{aligned}
\text{r.h.s.} &= \int_{\mathcal{D}'} \frac{d\bar{z} \wedge dz}{2i\pi} \epsilon(z,\bar{z}) \bar{\partial} < T_{zz} \phi_1 \cdots \phi_N > + \text{c.c.} && (65)\\
&= \int_{\mathcal{D}} \frac{d\bar{z} \wedge dz}{2i\pi} \epsilon(z) \bar{\partial} < T_{zz} \phi_1 \cdots \phi_N > + \text{c.c.} && (66)\\
&= \int_{\mathcal{D}} d\Big(\frac{dz}{2i\pi} \epsilon(z) < T_{zz} \phi_1 \cdots \phi_N > \Big) + \text{c.c.} && (67)\\
&= \oint_{\partial\mathcal{D}} \frac{dz}{2i\pi} \epsilon(z) < T(z) \phi_1 \cdots \phi_N > + \text{c.c.} && (68)\\
&= \sum_{i=1}^{N} \oint_{z_i} \frac{dz}{2i\pi} \epsilon(z) < T(z) \phi_1 \cdots \phi_N > + \text{c.c.} && (69)
\end{aligned}
$$

that is, into a sum over small contours encirling each of the points z_i. The left hand side of (62) is also a sum of local contributions of each $\delta\phi_i$, thus we may identify each with the corresponding contour integral

$$
\delta\phi(z_1, \bar{z}_1) = \oint_{z_1} \frac{dz}{2i\pi} \varepsilon(z) T(z) \phi(z_1, \bar{z}_1) + c.c. \; . \tag{70}
$$

This shows that analytical properties of the product $T\phi$ encode the variation of the field.

Primary fields

When we describe a system which possesses some symmetry, it is generally appropriate to pick objects that obey 'tensorial' transformation laws. In the case of conformal field theory, this role is played by 'primary fields'. Under an arbitrary conformal change of complex coordinates $z \to z'(z)$, $\bar{z} \to \bar{z}'(\bar{z})$ a primary field operator transforms by definition according to

$$\phi(z,\bar{z}) = \left(\frac{dz'}{dz}\right)^h \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\bar{h}} \phi'(z',\bar{z}') \ . \tag{71}$$

The real numbers h and $\bar{h}$ are called conformal dimensions (or conformal weights). Note that the form $\phi(z,\bar{z})(dz)^h(d\bar{z})^{\bar{h}}$ is invariant. For an infinitesimal transformation $z \to z + \varepsilon(z)$, $\bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})$ this reduces to

$$\delta\phi(z,\bar{z}) = \left[\varepsilon(z)\partial + h\varepsilon'(z) + \bar{\varepsilon}(\bar{z})\bar{\partial} + \bar{h}\bar{\varepsilon}'(\bar{z})\right]\phi(z,\bar{z}) \ . \tag{72}$$

Formulae (70) and (72) for $\delta\phi$ are consistent if we have the following short distance expansion,

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{z-w} + \text{reg} \ , \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &= \frac{\bar{h}\phi(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\phi(w,\bar{w})}{\bar{z}-\bar{w}} + \text{reg} \ . \end{aligned} \tag{73}$$

The operator product expansions (73) can be used as an alternative definition of the primary fields.

As a short exercise, let us return to our example of free massless Majorana fermions. Taking (43), we can calculate the singular parts of the operator products,

$$\begin{aligned} T(z)\psi(w) &= \frac{\psi(w)}{2(z-w)^2} + \frac{\partial\psi(w)}{z-w} + \text{reg} \ , \\ \bar{T}(\bar{z})\psi(w) &= \text{reg} \ . \end{aligned} \tag{74}$$

It means that the fermionic field $\psi(z)$ is a primary field of conformal weights $(h,\bar{h}) = (\frac{1}{2}, 0)$. In the same way, one can show that $\bar{\psi}(\bar{z})$ is a primary field of weights $(0, \frac{1}{2})$.

As another example, the reader may treat the case of a free massless boson field $\phi(z)$ for which the two-point function is $< \phi(z)\phi(0) >= -\ln z$ and the energy-momentum tensor $T(z) = -\frac{1}{2}(\partial\phi)^2$. Using Wick theorem, she (or he) will verify that $\exp i\alpha\phi(z)$ is a primary field of conformal weight $h = \alpha^2/2$. Those 'vertex operators' play a prominent role in Shankar's lectures.

Of course, not all fields satisfy the simple transformation law (71) under conformal changes of coordinates. For example, we see from (73) that derivatives of primary fields have more complicated transformation properties. Let us also check the properties of the energy–momentum tensor under conformal transformations. For massless fermions, we see from (44) that the order of singularities is higher than what is allowed by the definition formulae (73). One can prove that the most general form of short distance products between components of the energy–momentum tensor is

$$\begin{aligned} T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg} , \\ \bar{T}(\bar{z})\bar{T}(\bar{w}) &= \frac{c}{2(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \text{reg} , \\ T(z)\bar{T}(\bar{w}) &= \text{reg} . \end{aligned} \tag{75}$$

It leads to the following infinitesimal transformation law,

$$\delta T(z) = [\varepsilon(z) + 2\varepsilon'(z)]\, T(z) + \frac{c}{12}\varepsilon''(z), \tag{76}$$

which can be integrated to yield the law for finite conformal transformations,

$$T(z) = T'(z') \left(\frac{dz'}{dz}\right)^2 + \frac{c}{12}\{z', z\} , \tag{77}$$

which involves the Schwartzian derivative,

$$\{z', z\} \equiv \frac{\frac{d^3z'}{dz^3}}{\frac{dz'}{dz}} - \frac{3}{2}\left(\frac{\frac{d^2z'}{dz^2}}{\frac{dz'}{dz}}\right)^2 . \tag{78}$$

Finally, if we refer to Laurent modes defined by (46) the following pair of Virasoro algebras arises

$$[L_n, L_m] \;=\; (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad ,$$

$$
\begin{aligned}
\left[\bar{L}_n, \bar{L}_m\right] &= (n-m)\bar{L}_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad , \\
\left[L_n, \bar{L}_m\right] &= 0 \quad .
\end{aligned} \tag{79}
$$

The real central charge is a very important characteristics of the conformal field theory. As we have seen, the model of free massless fermions has $c = \frac{1}{2}$ (see (44)). Can the reader compute the value of c for the boson field just mentionned?

4th Lecture

Physical interpretation of the conformal weights

To expose the meaning of conformal weights of primary fields (71), let us consider the 2-point correlation function,

$$< \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2) > \;=\; \frac{1}{(z_1 - z_2)^{2h}} \frac{1}{(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} < \phi'(1)\phi'(0) > \;, \quad (80)$$

where we have made use of a change of variable $z \to z' = \frac{z - z_2}{z_1 - z_2}$. Furthermore, we choose the normalization $< \phi'(1)\phi'(0) >= 1$ and denote $z_1 - z_2 = r_{12}\, e^{i\mathrm{Arg}(z_1 - z_2)}$.

$$< \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2) >= \frac{1}{r_{12}^{2h+2\bar{h}}} e^{-2i(h-\bar{h})\mathrm{Arg}(z_1 - z_2)} \;. \quad (81)$$

The number $h + \bar{h}$ is the scaling dimension of the field ϕ, while the number $h - \bar{h}$ is the spin of the field ϕ

$$\langle \phi(z e^{i2\pi})\phi(0)\rangle = e^{-4i\pi(h-\bar{h})}\langle \phi(z)\phi(0)\rangle \;. \quad (82)$$

A short tour through the representation theory

We shall make now a brief survey of the representation theory of the Virasoro and affine algebras. The only representations that will concern us are the so-called 'highest weight' representations. The simplest example of a highest weight representation is provided by the familiar example of the $SU(2)$ algebra,

$$[J_+, J_-] = 2J_z \;, \quad [J_z, J_\pm] = \pm J_\pm \;. \quad (83)$$

A highest weight is a state $|j, j>$ satisfying the conditions

$$J_+|j, j >= 0 \;, \quad J_z|j, j >= j|j, j > \;. \quad (84)$$

The descendant states are produced by acting with the operator J_-,

$$|j, j - p >= J_-^p|j, j > \;, \quad J_z|j, j - p >= (j - p)|j, j - p > \;. \quad (85)$$

Linear combinations of the states $\{|j,j>, |j,j-1>, |j,j-2>, \ldots\}$ form the space of the representation of spin j. If the representation is finite dimensional, $2j$ has to be an integer and

$$J_-^{2j+1}|j,j>=0 \ . \tag{86}$$

The same construction can be applied to the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \ . \tag{87}$$

We follow the same procedure as for $SU(2)$ with the following correspondances $L_0 \to J_z$, $L_{n>0} \to J_+$, $L_{n<0} \to J_-$. A highest weight (h.w.) state is thus defined by the following conditions,

$$L_0|h>=h|h> \ , \quad L_{n>0}|h>=0 \ , \tag{88}$$

and the representation space $\mathcal{M}_{h,c}$ is generated by 'descendant' states of the form

$$L_{-1}^{\alpha_1} L_{-2}^{\alpha_2} \cdots L_{-p}^{\alpha_p}|h> \ . \tag{89}$$

However, there exists a big difference in comparison with the $SU(2)$ case: the representations of the Virasoro algebra are always infinite dimensional, being generated by an infinite number of independent states. The representations of the Virasoro algebra are 'graded', i.e. within a representation, the eigenvalues of L_0 are integrally spaced

$$L_0 \left(L_{-1}^{\alpha_1} L_{-2}^{\alpha_2} \ldots L_{-p}^{\alpha_p}|h> \right) = \left(h + \sum_{j=1}^{p} j\alpha_j \right) L_{-1}^{\alpha_1} L_{-2}^{\alpha_2} \ldots L_{-p}^{\alpha_p}|h> \ . \tag{90}$$

The h.w. state has the lowest eigenvalue h, and its descendants form the 'conformal tower', see Fig. 3. The integer $\sum_{j=1}^{p} j\alpha_j$ is called the level of the state $L_{-1}^{\alpha_1} L_{-2}^{\alpha_2} \ldots L_{-p}^{\alpha_p}|h>$ in the tower.

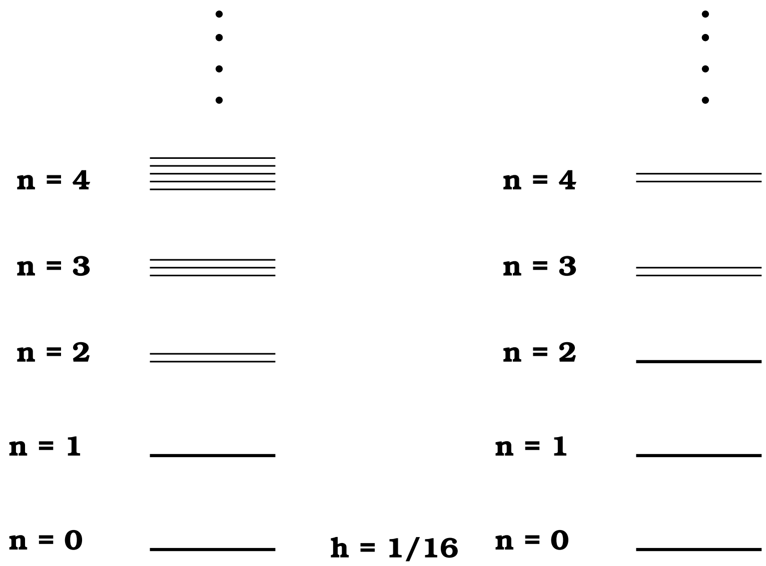

Fig 3 : The 'conformal tower' of descendants above the h.w. state $h = \frac{1}{16}$
in the representation $\mathcal{M}_{c=\frac{1}{2},h=\frac{1}{16}}$ (left) and
in the irreducible representation $\mathcal{V}_{c=\frac{1}{2},h=\frac{1}{16}}$ (see below).
The multiplicity is depicted for each level $n = 0, 1, \ldots, 4$.

In conformal field theory, we may have to deal with either a finite or an infinite number of representations of the Virasoro algebra. Because there are two copies of the Virasoro algebra (one for the holomorphic part L_n and one for the antiholomorphic part $\bar{L}_n$), a physical h.w. state is characterized by two weights $|h, \bar{h}>$. Among these representations, the one built from the vacuum state is singled out. The vacuum state is defined as the h.w. state possessing vanishing conformal weights,

$$|0> \equiv |h = 0, \bar{h} = 0> \quad . \tag{91}$$

The vacuum state has the following properties,

$$L_{n\geq -1}|0> = \bar{L}_{n\geq -1}|0> = 0 \ , \tag{92}$$

and is thus invariant under translations. For a 'unitary' representation, the central charge c is a positive real number, the same for the left and right copies of the Virasoro algebra. (The meaning of 'unitary' is that the space of states is a Hilbert space, i.e. has a positive norm, and the Virasoro algebra is consistent with this norm in the sense that $L_n^\dagger = L_{-n}$. This property is not satisfied by all representations of the Virasoro algebra.)

Let us now turn to a short discussion of the representation theory of affine (current) algebras,

$$[J_n^a, J_m^b] = if^{abc}J_{n+m}^c + \frac{1}{2}kn\delta^{ab}\delta_{n+m,0} \ , \tag{93}$$

where f^{abc} are the structure constants associated to a simple Lie algebra, k is some coefficient. The zero modes J_0^a forms an ordinary Lie algebra ('horizontal Lie subalgebra'), while the full set of J_n^a constitutes an 'affinization' of the horizontal subalgebra.

Consider the affinization of $SU(2)$ algebra, denoted by $\widehat{SU}(2)$, spanned by the generators J_n^+, J_n^-, J_n^z $(n \in Z)$. A h.w. state is defined as follows,

$$J_{n>0}^+|j>= J_{n>0}^-|j>= J_{n>0}^z|j>= 0 \ , \ J_0^z|j>= j|j> \ , \ J_0^+|j>= 0 \ . \tag{94}$$

A tower of states is created by acting on the h.w. state with J_0^- or any of the $J_{n<0}$.

For 'good' (i.e. unitary) representations k and $2j$ must be integers and satisfy the following relation,

$$0 \leq 2j \leq k \ . \tag{95}$$

Originally introduced in elementary particle physics, current algebras are now regarded as relevant in many contexts, including condensed matter physics, as illustrated by Ian Affleck in his lectures.

There is in fact an interesting connection between current algebras and the Virasoro algebra.

Sugawara construction

Let us start from a representation of a current algebra $\hat{g}$ by currents J_n^a and let us form the following combination

$$T(z) = \frac{1}{\kappa}\sum_{a=1}^{n} : J^a(z)J^a(z) : \ , \tag{96}$$

The claim is that, for a proper choice of the constant κ, $T(z)$ qualifies as an energy momentum tensor, or equivalently, that its Laurent moments satisfy the Virasoro algebra,

$$L_n = \frac{1}{\kappa}\sum_{a=1}^{n}\sum_{m=-\infty}^{m=+\infty} : J^a_{n-m} J^a_m : \ . \tag{97}$$

The normal ordering is defined as the requirement that the operators $J^a_{n>0}$ stand at the right. (Note that thanks to eq. (93) two currents J^a_m and J^a_n with m, n of the same sign and with *the same* a do commute). Thus

$$\kappa L_n = \sum_{m<n} J^a_m J^a_{n-m} + \sum_{m\geq n} J^a_{n-m} J^a_m \ . \tag{98}$$

To fix the constant κ, we require that the fields $J^a(z)$ transform as primary fields of conformal weights (1,0),

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} + \text{reg} \ . \tag{99}$$

The above operator product leads to the following relations,

$$[L_n, J^a_m] = -mJ^a_{n+m} \ . \tag{100}$$

It is now easy to show that κ should be adjusted in such a way that

$$\kappa = k + g, \tag{101}$$

where g is the quadratic Casimir of the adjoint representation of the Lie algebra,

$$\sum_{a,b} f^{abc} f^{abd} = g\delta^{cd} \ . \tag{102}$$

Recall that for $SU(N)$, $g = N$.

Now, it is straightforward (though tedious!) to check the Virasoro algebra. By explicit calculations, we conclude that the L_n defined in (97) satisfy (87) with the following value of the central charge

$$c = \frac{k \dim g}{k+g} \ , \tag{103}$$

where $\dim g$ is the dimension of the Lie algebra (recall $\dim SU(N) = (N^2 - 1)$).

The above construction is known as the Sugawara construction. If we start from the $\widehat{SU(2)}$ current algebra, then we find a Virasoro algebra with the central charge $c = \frac{3k}{k+2}$. The highest weight state $|j\rangle$ transforming as the spin-j representation of the horizontal $SU(2)$ is also a highest weight of Virasoro with

$$L_0|j\rangle = \frac{j(j+1)}{k+2}|j\rangle \ . \tag{104}$$

5th Lecture

Finite size effects

The transformation laws developed for conformal field theory may be also applied to conformal changes corresponding to true changes in the geometry, not only to changes of the system of coordinates. Below, we give an example of how we can use the conformal theory in the plane to solve it on a cylinder.

Correlation function on a cylinder

Let us consider a cft on a cylinder of perimeter L. As was already mentionned, the conformal transformation $w \to z = e^{\frac{2i\pi w}{L}}$ maps the cylinder on a (punctured) plane. A primary field operator ϕ transforms from the plane to the cylinder according to (71),

$$\phi_{plane}(z, \bar{z}) = \left(\frac{dw}{dz}\right)^{h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}} \phi_{cyl}(w, \bar{w}) \quad . \tag{105}$$

Taking the result for the 2-point correlation function on the plane (81), we determine its counterpart on the cylinder,

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle_{cyl} = \left(\frac{2i\pi}{L}\right)^{2h} \left(\frac{-2i\pi}{L}\right)^{2\bar{h}} \frac{(z_1 z_2)^{h} (\bar{z}_1 \bar{z}_2)^{\bar{h}}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \ . \tag{106}$$

Let us restrict ourselves to 'spinless' fields, i.e. $h = \bar{h}$,

$$< \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2) >_{cyl} = \frac{1}{\left(\frac{L}{\pi} \sin \frac{\pi(w_1 - w_2)}{L}\right)^{4h}} \ . \tag{107}$$

Now, it is interesting to look at two extreme opposite regimes. First, assume that the distance between the points of field insertions is much smaller than the size of the system, i.e. $|w_1 - w_2| << L$. In this case, the correlation function (107) can be approximated by $\frac{1}{|w_1 - w_2|^{4h}}$, i.e. one recovers the result of the plane (81). In other words, in this limit finite-size effects on correlation functions can be ignored and (81) describes a *universal* behavior. The opposite limit, $l \equiv \mathrm{Im}(w_1 - w_2) >> L$, probes the correlation function for large 'time' separations. This is useful for applications to statistical systems at criticality. Then, the correlation function (107) behaves like $\exp\left(-\frac{l}{\xi_L}\right)$, where the correlation length is defined by $\xi_L = \frac{L}{4h\pi}$. Using CFT, we have (or rather Cardy has !) thus justified a finite size scaling law that had been observed empirically [Cardy 1984 and further references therein].

Partition function on a cylinder

Let us now compute the partition function on a cylinder, from a Hamiltonian point of view. The time direction corresponds as above to the imaginary part of the complex variable $w = w_1 + iw_2$. Then the partition function is given by the following formula,

$$Z = \mathrm{tr}\zeta^T \quad , \quad \zeta = e^{-H} \quad , \tag{108}$$

where the Hamiltonian is defined by,

$$H = \int dw_1 T_{22}(w) = i(\partial_w - \partial_{\bar{w}}) = i(L_{-1}^{(w)} - \bar{L}_{-1}^{(\bar{w})}) \ . \tag{109}$$

Using the transformation law (77) we obtain the relation,

$$T_{cyl}(w) = -\left(\frac{2\pi}{L}\right)^2 \left(z^2 T_{plane}(z) - \frac{c}{24}\right) \ . \tag{110}$$

It enables us to rewrite the Hamiltonian using operators defined on the plane,

$$H = i(L_{-1}^{(w)} - \bar{L}_{-1}^{(w)}) = \frac{2\pi}{L}\left(L_0 - \frac{c}{24} + \bar{L}_0 - \frac{c}{24}\right) \ . \tag{111}$$

The partition function can be now expressed as the trace in the Hilbert space $\mathcal{H}$ that describes the cft in the plane,

$$Z = \mathrm{tr}_{\mathcal{H}} e^{-HT} = \mathrm{tr}_{\mathcal{H}} e^{-\frac{2\pi T}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)} \ . \tag{112}$$

The Hilbert space $\mathcal{H}$ decomposes into a sum of representations of the product of the two (left and right) Virasoro algebras. Let us denote by $d_n^{(h)}$ (the degeneracy) the number of independent states at the level n of the conformal tower of highest weight h. The numbers $d_{\bar{n}}^{(\bar{h})}$ are defined analogously. Using this notation,

$$Z = \sum_{(h,\bar{h})} \sum_{n,\bar{n}} d_n^{(h)} d_{\bar{n}}^{(\bar{h})} \exp\left[-2\pi \frac{T}{L}\left(h + \bar{h} + n + \bar{n} - \frac{c}{12}\right)\right] \ . \tag{113}$$

Assume now that $\frac{T}{L} >> 1$. Let λ_0 denotes the eigenvalue of largest modulus of the operator ζ. In the limit under study, the partition function can be approximated by $Z = \lambda_0^T$.

Usually, the largest eigenvalue is provided by the vacuum state of conformal weights $h = \bar{h} = 0$ (this is true for the 'unitary' physical models), so that we can set $\lambda_0 = e^{\frac{2\pi}{L}\frac{c}{12}}$. It gives the following value of the free energy per unit 'time' length,

$$F = \frac{1}{T} \ln Z = \frac{\pi c}{6L} \ . \tag{114}$$

The above result can be interpreted as a finite-size correction to the free energy, i.e. a 'Casimir effect' (Note that Z has been normalized in such a way that the 'bulk' free energy $\lim_{L,T\to\infty}\frac{1}{TL}\ln Z$ vanishes at the critical point where we are standing). It is remarkable that in the cft this Casimir effect depends only on the geometry of the system and the value of the central charge [Affleck; Blöte , Cardy, Nightingale].

More on the representations of the Virasoro algebra

The decomposition of the Hilbert space $\mathcal{H}$ and the resulting calculation of the degeneracies $d_n^{(h)}$ and $d_n^{(\bar{h})}$ have to be carried out in *irreducible* representations of the Virasoro algebra. It thus important to know when a highest weight representation of the Virasoro algebra is irreducible. Let us parametrize the central charge of the Virasoro algebra using a (real or complex) parameter x

$$c = 1 - \frac{6}{x(x+1)} \ . \tag{115}$$

It can be proved (Kac; Feigin, Fuchs) (and it is highly non trivial!) that the representation $\mathcal{M}_{h,c}$ is *reducible* if and only if the highest weight can be written as

$$h = h_{rs} = \frac{(r(x+1) - sx)^2 - 1}{4x(x+1)} \ , \tag{116}$$

where r and s are positive integers. Moreover the discussion by Feigin and Fuchs tells us how to construct an irreducible representation $\mathcal{V}_{h,c}$ when $\mathcal{M}_{h,c}$ is not irreducible.

Suppose furthermore that x is a positive fractional number,

$$x = \frac{p'}{p - p'} \ , \tag{117}$$

where p, p' are coprime integers (i.e. without common divisor). It is then consistent (in a sense to be explained soon) to restrict to h_{rs} such that:

$$1 \le r \le p' - 1 \qquad 1 \le s \le p - 1 \ . \tag{118}$$

Thus, under these circumstances, for a given value of $c = 1 - \frac{6(p-p')^2}{pp'}$ there exists a *finite* number of possible h_{rs} and all these weights are fractional numbers. We shall refer to these representations as the *minimal* ones.

Further strong restrictions emerge if we require the unitarity of the representation. It was proved (Friedan, Qiu, Shenker, one more highly non trivial result !) that the necessary and sufficient conditions for highest weight representations of the Virasoro algebra to be unitary are either

$$c \ge 1 \ \ , h \ge 0, \tag{119}$$

or

$$c = 1 - \frac{6}{m(m+1)} \ , \ h \in \left\{ h_{rs} = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)} \right\} \ , \tag{120}$$

where m, r, s are integers, $m \ge 3$, $1 \le r \le m - 1$ and $1 \le s \le m$.

Examples

Critical Ising and Potts models

Let us show how well known models of statistical mechanics fit in this scheme. I guess everybody knows the Ising model. The Potts model is a simple generalization of the Ising model in which (in two dimensions) 'spins' σ are assigned to the sites of a square lattice and may take Q distinct values, denoted by $\sigma = 1, \cdots Q$. The interaction energy of a configuration depends on whether at the ends of each edge, the two spins are or are not in the same state. Thus this energy reads

$$H = J \sum_{\text{edges } ij} \delta_{\sigma_i \sigma_j} \tag{121}$$

Clearly, if $Q = 2$, we recover the Ising model (up to the addition of a constant term in H). In two dimensions, the Potts model is known to undergo a second order phase transition (thus has a critical conformal point) if $Q \leq 4$. This means that there is a low temperature phase in which the symmetry between all the possible groundstates is spontaneously broken, and as $T \to T_c$, the 'magnetization' $\langle\sigma\rangle$ vanishes as a certain power $\beta(Q)$ of $(T_c - T)$. Right at T_c, the correlation function $\langle\sigma(r)\sigma(0)\rangle$ has a power law decay $\approx \frac{1}{r^\eta}$. Beside the $Q = 2$ (Ising) case, a case of interest is $Q = 3$. As a matter of a fact, they are described at criticality by cft's with central charges obeying the formula (120) with respectively $m = 3$ and $m = 5$, hence $c = \frac{1}{2}$ resp $\frac{4}{5}$. That the central charge of the Ising model is 1/2, i.e. the same as that we found above for free fermions is by no means an accident. We all know since the work of Onsager that free fermions are hidden in the Ising model; these free fermions are massless at $T = T_c$, and they build the relevant cft. Now in the $m = 3$ minimal cft, the conformal weights may only take three values: $h = 0, \frac{1}{2}$ and $\frac{1}{16}$. With them we may make various fields of integer or half integer spin

$h = \bar{h} = 0$, the identity field I

$h = \frac{1}{2}\,,\ \bar{h} = 0$, the Majorana fermion ψ

$h = 0\,,\ \bar{h} = \frac{1}{2}$, the fermion $\bar{\psi}$

$h = \bar{h} = \frac{1}{2}$, the composite $\bar{\psi}\psi$, i.e. a mass term for the fermion: this is indeed the 'relevant' operator that drives the system out of its conformal point at $T = T_c$;

$h = \bar{h} = \frac{1}{16}$: this is another relevant term, nothing else than the spin operator: it describes the response of the system to an external magnetic field. From that value of $h = \bar{h}$ for the spin, we get for the 2-spin function the critical behavior $< \sigma(0)\sigma(r) > \approx 1/r^{\frac{1}{4}}$, i.e. the well-known value of the Ising exponent $\eta = \frac{1}{4}$.

For the 3-state Potts model, similar considerations apply. The little subtlety is that only a subset of the allowed conformal weights (eq (120)) are used in the description of the model under normal circumstances. For example, the weight $h_{33} = \frac{1}{15}$ yields the conformal dimension of the 'spin', from which the exponent η above follows as 4/15.

6th Lecture

The partition function on the torus

We shall now see that all the information about the operator content of the conformal field theory is contained in the partition function of the cft on a torus, and that the latter is subject to strong constraints.

A torus can be regarded as a parallelogram whose opposite edges have been identified. Let us adopt the convention that the parallelogram vertices lie at the following points on the complex plane:

$$0 \ , \quad 2\pi \ , \quad 2\pi\tau \ , \quad 2\pi(1+\tau) \ , \tag{122}$$

where τ is some complex number called the modular (or aspect) ratio of the torus, and chosen to satisfy $\mathrm{Im}\,\tau > 0$.

We have computed above in (112) the partition function on a cylinder, but in fact by taking a trace in $\mathcal{H}$ we have implicitly identified the two ends of the cylinder and made a torus of modular ratio $\tau = i\frac{T}{L}$. Thus by a slight modification of the above discussion, we find that for arbitrary τ

$$Z = \mathrm{tr}_{\mathcal{H}} e^{2i\pi\tau\left(L_0 - \frac{c}{24}\right) - 2i\pi\bar{\tau}\left(\bar{L}_0 - \frac{c}{24}\right)} \ . \tag{123}$$

Let us introduce the following parameters,

$$q = e^{2i\pi\tau} \quad , \quad \bar{q} = e^{-2i\pi\bar{\tau}} \quad , \tag{124}$$

and define the character of the irreducible representation $\mathcal{V}_{h,c}$ of highest weight h of the Virasoro algebra,

$$\chi_{h,c}(q) = \mathrm{tr}_{\mathcal{V}_{h,c}} \left(q^{L_0 - \frac{c}{24}} \right) = q^{h - \frac{c}{24}} \sum_{n=0}^{\infty} d_n^{(h)} q^n \ . \tag{125}$$

From the detailed analysis of the irreducible representations follows the knowledge of these characters as explicit functions of q.

We conclude that the partition function on a torus can be decomposed into a bilinear form of characters [Cardy 1986],

$$Z = \sum_{(h,\bar{h})} N_{h\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}) \ , \tag{126}$$

where the integer $N_{h\bar{h}}$ (a multiplicity) tells us how many times the representation $(h, \bar{h})$ enters. As the identity operator must be present and non degenerate in any sensible theory, we have also the constraint that $N_{00} = 1$.

Modular invariance on the torus

Now it is important to note that the shape of the torus does not uniquely determine τ, namely we can perform arbitrary 'modular' transformations,

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \ , \quad ad - bc = 1 \ , \tag{127}$$

where a, b, c, d are integers, and τ' describes the same torus. Modular transformations form a group. Any modular transformation can be obtained as a composition of two basic transformations,

$$T : \tau \to \tau + 1 \quad , \qquad S : \tau \to -\frac{1}{\tau} \quad . \tag{128}$$

The crucial point is that we want the partition function Z to be intrinsically attached to the torus, i.e. to be invariant under modular transformations. This modular invariance of the partition function, together with the form (126), turns out to yield very strong constraints on the operator content.

T-transformation

We have here $q \to e^{2i\pi} q$, and consequently

$$\chi_h(q) \to e^{2i\pi\left(h - \frac{c}{24}\right)} \chi_h(q) \ , \quad \chi_{\bar{h}}(q) \to e^{-2i\pi\left(\bar{h} - \frac{c}{24}\right)} \chi_{\bar{h}}(q) \ , \tag{129}$$

$$\chi_h(q)\chi_{\bar{h}}(q) \to e^{2i\pi(h - \bar{h})} \chi_h(q)\chi_{\bar{h}}(q) \ . \tag{130}$$

Thus, the invariance under the T-transformation requires spins $h - \bar{h}$ of field operators to be integers.

S-transformation

It is more difficult to find the S-transformation, because the characters transform among themselves under it.

Let us concentrate on the family of representations of the Virasaro algebra (116). Denote for short $\chi_{rs} = \chi_{h_{rs}}$. It can be shown that

$$\chi_{rs}\left(-\frac{1}{\tau}\right) = \sum_{r',s'} S_{rs,r's'} \chi_{r's'}(\tau) \ , \tag{131}$$

with r', s' running over the same range as in (118). The matrix S is symmetric and unitary

$$S_{rs,r's'} = (-1)^{rs'+r's+1} \sqrt{\frac{2}{pp'}} \sin \frac{\pi r r' p}{p'} \sin \frac{\pi s s' p'}{p} \ . \tag{132}$$

Likewise for the representations of the $\widehat{SU}(2)$ current algebra of level k, labeled by a spin j, with $0 \leq 2j \leq k$, formula (97) gives representations of the Virasoro algebra and the corresponding characters transform under the S transformation according to the unitary matrix

$$S_{jj'} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(2j+1)(2j'+1)}{k+2} \ . \tag{133}$$

Now we have all the ingredients to discuss the following problem: find all modular invariant partition functions Z of the form (126) with N's non negative integers, $N_{00} = 1$. Solving this problem for a given class of representations amounts to classifying conformal field theories of that class. I won't dwell on that any longer. Suffice it to say that this programme has been carried out for the 'minimal' representations of Virasoro and for the (related) cft's with a $SU(2)$ current algebra. The solution exhibits a beautiful structure that had not been anticipated: we refer the reader to the literature [Cappelli et al.]. This classification programme has been pursued lately for theories with a higher rank current algebra ($SU(3)$ in particular: see the recent work of T. Gannon).

The Operator Product Expansion

This short guided tour would be very incomplete without some discussion of another fundamental feature of conformal field theories, namely their consistency under operator product expansions. In any quantum field theory, we have learnt from the work of Wilson the importance of the short distance expansion of a product of two fields. In a conformal field theory, we **postulate** that the set of fields that we have discussed so far, the primaries and their descendants, form a closed set under the Operator Product Expansion (OPE). Thus for two fields Φ_I, Φ_J, assumed to be primary fields of weights

$(h_I, \bar{h}_I)$, $(h_J, \bar{h}_J)$, we write

$$\Phi_I(z_1, \bar{z}_1)\Phi_J(z_2, \bar{z}_2) = \sum_K C_{IJK}(z_1 - z_2)^{h_K - h_I - h_J}(\bar{z}_1 - \bar{z}_2)^{\bar{h}_K - \bar{h}_I - \bar{h}_J}$$
$$\sum_{n,\bar{n}} \beta_{IJK}^{(n,\bar{n})}(z_1 - z_2)^{|n|}(\bar{z}_1 - \bar{z}_2)^{|\bar{n}|}\Phi_K^{(n,\bar{n})}(z_2, \bar{z}_2)\ . \tag{134}$$

This means simply that the product of Φ_I and Φ_J may be expanded on all other primaries Φ_K and their descendants denoted here $\Phi_K^{(n,\bar{n})}$ with coefficients $C_{IJK}\beta_{IJK}^{(n,\bar{n})}$. The notation $|n|$ denotes the level of the descendant and it is understood that $\Phi_K^{(0,0)} \equiv \Phi_K$ and $\beta_{IJK}^{(0,\bar{0})} \equiv 1$. The relative coefficients $\beta_{IJK}^{(n,\bar{n})}$ are easy to find using the Ward identities of the Virasoro algebra. In constrast, the *structure constants* C_{IJK} of the OPE are important and non trivial data of the cft. They give for example the three-point function of the three primaries Φ_I, Φ_J, Φ_K

$$< \Phi_I(z_1, \bar{z}_1)\Phi_J(z_2, \bar{z}_2)\Phi_K(z_3, \bar{z}_3) > \tag{135}$$
$$= \frac{C_{IJK}}{(z_1 - z_2)^{(h_I + h_J - h_K)}(\bar{z}_1 - \bar{z}_2)^{(\bar{h}_I + \bar{h}_J - \bar{h}_K)} \times \text{cyclic perm.}}\ . \tag{136}$$

These structure constants may be extracted from a separate discussion of the consistency of the OPE. We refer to the original paper by Belavin, Polyakov and Zamolodchikov for that matter.

The fusion algebra

Rather than computing all these C_{IJK}, we may content ourselves in a first step with finding the selection rules that apply to them. In other words, what are the fields Φ_K that couple to a given pair Φ_I and Φ_J? This important question can be given the appropriate precise meaning. It is more suited to look at how the holomorphic (or antiholomorphic) components of fields combine, i.e. to discuss the *fusion* of representations (of Virasoro, current,… algebras). For this fusion operation, the 'minimal' representations that we have introduced above, or those of $\widehat{SU}(2)$, form a closed algebra. This is the 'consistency' alluded to in lecture 5. We write

$$\mathcal{V}_{h_I,c}.\mathcal{V}_{h_J,c} = \oplus_{h_K} N_{IJ}^K \mathcal{V}_{h_K,c} \tag{137}$$

with coefficients N_{IJ}^K that are multiplicities, hence integers. An amazing discovery of E. Verlinde is that the determination of these multiplicities follows from the knowledge of the modular matrix S

$$N_{IJ}^{K} = \sum \frac{S_{IL}S_{JL}S_{KL}}{S_{0L}} \tag{138}$$

with 0 referring to the identity field (or vacuum representation). The mere fact that with the S matrices of (132) and (133) these numbers are non negative integers is not trivial and the general validity of formula (138) reflects the beautiful consistency of Conformal Field Theory.

References

A.M. Polyakov *J.E.T.P. Lett.* **12** (1970) 381.
A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333.
J. Cardy, *J. Phys.* **A17** (1984) L385.
H.W. Blöte, J.L. Cardy and M.P. Nightingale, *Phys. Rev. Lett.* **56** (1986) 742.
I. Affleck, *Phys. Rev. Lett.* **56** (1986) 746.
J. Cardy, *Nucl. Phys.* **B270** (1986) 186.
A. Cappelli, C. Itzykson and J.-B. Zuber, *Nucl. Phys.* **B280** [FS18] (1987) 445; *Comm. Math. Phys.* **113** (1987) 1.
T. Gannon, *Comm. Math. Phys.* **161** (1995) 233.
E. Verlinde, *Nucl. Phys.* **300** (1988) [FS22] 360.

Collected reprints:
C. Itzykson, H. Saleur and J.-B. Zuber, *Conformal Invariance and Applications to Statistical Mechanics*, World Scientific 1988.

Lectures of J. Cardy and P. Ginsparg at the 1988 Les Houches Summer School, *Fields, strings and critical phenomena*, eds E. Brézin and J. Zinn-Justin, North Holland 1990.

Textbook :
J.-M. Drouffe and C. Itzykson, *Statistical Field Theory*, Cambridge Univ. Press 1988.

Monographs
P. Goddard and D. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, World Scientific 1988.
S. Ketov, *Conformal Field Theory*, World Scientific 1995.
P. Christe and M. Henkel, *Introduction to Conformal Invariance and Its Applications to Critical Phenomena*, Lecture Notes in Physics, Springer Verlag 1993.
P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory* , Springer Verlag, to appear 1996.