DISCRETE SYMMETRIES OF CONFORMAL THEORIES

J.-B. ZUBER

Service de Physique Théorique, CEN Saclay, F-91191 Gif-sur-Yvette Cedex, France

Received 28 May 1986

The discrete symmetries of conformal theories are related to the various twisted boundary conditions that may be imposed in a toroidal geometry. The corresponding partition functions are only invariant under a specific subgroup of the modular group. This is illustrated on the Ising, Potts and RSOS models. Conversely, some of the discrete symmetries should be recovered from the various submodular invariants that may be constructed as bilinears in the Virasoro characters.

Identification of each conformal invariant 2D theory [1] with some underlying critical statistical model has so far been mainly empirical, and relies on the coincidence of critical exponents. Clearly some information about the possible symmetries would be very helpful in this identification. It has been recently shown [2,3] that the constraint of modular invariance of the partition function of a conformal unitary [4] theory constructed on a torus restricts the operator content of the theory a great deal and may lead to a classification of the consistent conformal theories. In this note, I want to show that similar considerations also tell us something about symmetries. It is known [4] that unitary $c < 1$ conformal theories cannot admit any continuous symmetry, as testified by the absence of $h = 1$ (corresponding to a conserved current) in the table of conformal weights. Hence, we look for *discrete* symmetries. These symmetries must manifest themselves by the various twisted boundary conditions (or frustration lines) that may be imposed on the system: this was recently exemplified on the case of the three-state Potts model [5].

Consider a statistical model on a finite $M \times N$ rectangular lattice of hamiltonian

$$\mathscr{H} = \sum_{i=1}^{M} \sum_{j=1}^{N} h(\sigma_{i,j}\sigma_{i+1,j}) + h(\sigma_{i,j}\sigma_{i,j+1}).$$

The spin–spin interaction h is invariant under a discrete symmetry group G: $H({}^{g}\sigma, {}^{g}\sigma') = h(\sigma, \sigma')$. Under these circumstances, any twisted boundary condition in G may be imposed:

$$\sigma_{M+1,j} = {}^{g_1}\sigma_{1,j}, \quad \sigma_{i,N+1} = {}^{g_2}\sigma_{i,1}, \quad g_1, g_2 \in \mathrm{G}, \tag{1}$$

and is consistent with the translation symmetry of the torus: the lines along which the identification (1) are performed may be shifted on the torus, using the G-invariance of $\mathscr{H}$.

Let us assume that in the continuous critical theory, the conformal fields φ may be assigned similar twisted boundary conditions. Restricting ourselves for simplicity to the case of $\mathrm{G} = \mathbf{Z}_N$ cyclic group:

$$\varphi(z + n_1\omega_1 + n_2\omega_2) = \exp[(2\mathrm{i}\pi/N)(n_1 p_1 + n_2 p_2)]\,\varphi(z), \tag{2}$$

where ω_1 and ω_2 are two (complex) periods of the torus and p_1, p_2 are mod N integers. By the mapping $w = \exp(2\pi\mathrm{i}z/\omega_1)$ translations along ω_1 are mapped on the 2π rotations of the plane, under which the field φ of conformal weights h, $\bar{h}$ behaves as

$$\varphi(z + n_1\omega_1) = \exp[2\mathrm{i}\pi(h - \bar{h})n_1]\,\varphi(z) = \exp(2\mathrm{i}\pi p_1/n)\,\varphi(z).$$

In the unitary model of central charge $c_m = 1 - 6/m(m+1)$, $h - \bar{h}$ is rational, of denominator $4m(m+1)$. Taking p_1 and N coprimes, we conclude that N *must divide* $4m(m+1)$. This is a first necessary condition on the possible values of N to

0370-2693/86/$03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

have a $\mathbf{Z}_N$ symmetry in the model of central charge c_m.

The partition function $Z_{p_1,p_2}(\tau)$ in the presence of the boundary conditions (2) is not invariant under the full modular group Γ [2,3] but only under the subgroup that leaves (2) invariant (see ref. [6] for similar ideas in the context of strings). In particular the partition function $Z_{p_1,0}$ must be invariant under the subgroup $\hat{\Gamma}^0(N)$ of unimodular transformations $\tau \to (a\tau+b)/(c\tau+d)$ such that $a=d=\pm 1 \bmod N$, $b=0 \bmod N$. (Here I assume that $Z_{p_1,0}$ is real: $Z_{p_1,0}=Z_{-p_1,0}$, and p_1, N coprimes) $Z_{p_1,0}$ can also be expanded on characters of the Virasoro algebra as

$$Z_{p_1,0}=(\mathrm{q}\bar{\mathrm{q}})^{-c/24}\sum_{h,\bar{h}} N_{h,\bar{h}}\chi_h(\mathrm{q})\chi_{\bar{h}}(\bar{\mathrm{q}}), \tag{3}$$

where the coefficients $N_{h,\bar{h}}$ are non-negative integers because $Z_{p_1,0}$ may be regarded as the critical limit of the trace of the transfer matrix of a frustrated system, to which the usual arguments [2,3] apply. Therefore, the existence of positive bilinear combinations (3) invariant under $\hat{\Gamma}^0(N)$ is a second necessary condition for a $\mathbf{Z}_N$ symmetry.

Conversely, if $Z(\tau)$ is invariant under $\hat{\Gamma}^0(N)$, the action of the full group Γ on Z defines new functions ${}^{[g]}Z(\tau)\equiv Z({}^g\tau)$, invariant under the conjugate group $g^{-1}\hat{\Gamma}^0 g$; there are as many such functions as there are equivalence classes $[g]$ in the coset $\Gamma/\hat{\Gamma}^0(N)$, i.e. their number is the index of $\hat{\Gamma}^0$ in Γ [7]

$$I_N=\frac{N^2}{2}\prod_{p/N}(1-1/p^2) \quad \text{if } N\geqslant 3,$$

$$p \quad \text{prime divisors of } N, \tag{4}$$

$$I_2=3.$$

All these functions are invariant under $\Gamma(N)$, the largest subgroup common to all the conjugate groups $g^{-1}\hat{\Gamma}^0 g$: $\Gamma(N)$ is the principal congruence subgroup of level N: $a=d=\pm 1 \bmod N$, $b=c=0 \bmod N$, and it is normal in Γ [7]. Therefore $\Gamma/\Gamma(N)=\mathrm{PSL}(2,\mathbf{Z}_N)$ is a finite group, which permutes the various types of boundary conditions and the functions ${}^{[g]}Z$. The I_N functions ${}^{[g]}Z$ represent the I_N real partition functions $Z_{p_1,p_2}=Z_{-p_1,-p_2}$ with a non-trivial twist (p_1 or $p_2\neq 0$). Each individual function ${}^{[g]}Z=Z_{p_1,p_2}$ is actually left invariant by a group $G^{-1}\hat{\Gamma}^0 g/\Gamma(N)$ isomorphic to $\mathbf{Z}_N$: in each sector, the $\mathbf{Z}_N$ symmetry is still present. The functions Z_{p_1,p_2} all have the general form $(\mathrm{q}\bar{\mathrm{q}})^{-c/24}\sum N_{h,\bar{h}}\chi_h(\mathrm{q})\chi_{\bar{h}}(\bar{\mathrm{q}})$, with coefficients which are not necessarily positive integers if $p_1\neq 0$.

This general discussion will be illustrated on the case of the $m=3$ (Ising), $m=5$ or 6 (Potts) and generic m (RSOS model [8]), with $N=2, 3, 2$ respectively. For $N\leqslant 4$, the group $\hat{\Gamma}^0(N)$ is generated by the two transformations $\tau\to\tau+N$ and $\tau\to\tau/(\tau+1)$. For the Ising model ($m=3$), the most general invariant under $\hat{\Gamma}^0(2)$ reads

$$\begin{aligned} Z_{1,0}=(\mathrm{q}\bar{\mathrm{q}})^{-1/48} & \\ \times\big[& (N_1-N_2)\big(|\chi_0|^2+|\chi_{1/2}|^2+|\chi_{1/16}|^2\big) \\ & +N_2\big(|\chi_0+\chi_{1/2}|^2+2|\chi_{1/16}|^2\big)\big], \end{aligned} \tag{5}$$

where the first term is the Z_{00} function and the second one contains the spin $\frac{1}{2}$ contribution of the Ising fermion. By the action, of $\tau\to -1/\tau$, this gives

$$\begin{aligned} Z_{0,1}=(\mathrm{q}\bar{\mathrm{q}})^{-1/48}\big[& (N_1+N_2)\big(|\chi_0|^2+|\chi_{1/2}|^2\big) \\ & +(N_1-N_2)|\chi_{1/16}|^2\big]. \end{aligned} \tag{6}$$

The latter is the critical limit of $\mathrm{tr}(\mathscr{C}^{T}\mathscr{F})$ with $\mathscr{C}$ the transfer matrix of the Ising model, and $\mathscr{F}$ the operator that reverses spins. In the large imaginary $\tau=\mathrm{i}T$ limit, the effect of $\mathscr{F}$ is negligible, and we expect $Z_{01}\to Z_{00}$. This fixes $N_1+N_2=1$ with $N_2\neq 0$, hence $N_1=0$, $N_2=1$. Consider the ratio $Z_{10}/Z_{00}=\exp(-Tf)$, with f the interface self-energy in a finite geometry. In a selfdual theory where duality exchanges the spin–spin correlation function with the interface energy f, one expects $f\sim\pi\eta/L$ for a system of finite width $L=|\omega_1|$ [9]. In the present case, this gives $\eta=\frac{1}{4}$.

A similar procedure is applied to the three-state critical and tricritical Potts model, $m=5$ (respectively $m=6$). Modular invariants are constructed in terms of characters χ_{rs} of odd s (respectively odd r) only [2,3].

For $m=5$:

$$Z_{1,0}=Z_{2,0}=(q\bar{q})^{-c/24}\times\sum_{r=1}^{2}\left[(\chi_{r1}+\chi_{r5})\chi^*_{r3}+\text{c.c.}+|\chi_{r3}|^2\right]$$

$$=(q\bar{q})^{-1/30}\left[(\chi_0+\chi_3)\chi^*_{2/3}+(\chi_{2/5}+\chi_{7/5})\chi^*_{1/15}+\text{c.c.}+|\chi_{2/3}|^2+|\chi_{1/15}|^2\right], \tag{7}$$

and for $m=6$:

$$Z_{10}=(q\bar{q})^{-1/28}\left[(\chi_0+\chi_5)\chi^*_{4/3}+(\chi_{1/7}+\chi_{22/7})\chi^*_{10/21}+(\chi_{5/7}+\chi_{12/7})\chi^*_{1/21}+\text{c.c.}+|\chi_{4/3}|^2+|\chi_{10/21}|^2+|\chi_{1/21}|^2\right]. \tag{8}$$

Both are invariant under $\hat{\Gamma}^0(3)$, in agreement with the Z_3 symmetry and exhibit the spin $\frac{1}{3}$ contribution of the Potts pseudofermion. The operator content of Z_{10} ($m=5$) agrees with the results of ref. [5]. Dividing Z_{10} by Z_{00}, one extracts $\eta=\frac{4}{15}$ for $m=5$ and $\eta=\frac{4}{21}$ for $m=6$.

Finally for the generic model of the main sequence [3], interpreted as a multicritical model [8]:

$$Z_{1,0}=\tfrac{1}{2}(q\bar{q})^{-c/24}\sum_{r=1}^{m-1}\sum_{s=1}^{m}\chi_{rs}\cdot\chi^*_{r\,m+1-s} \tag{9}$$

is invariant under the group $\hat{\Gamma}^0(2)$, in agreement with the $\mathbf{Z}_2$ symmetry of the RSOS model [8]. This case encompasses the critical and tricritical Ising models ($m=3$ and 4). The lowest dimension operator in (9) gives the behavior of the interface energy:

$$f=2\pi\min(h_{rs}+h_{r,\bar{s}})$$

$$=2\pi\frac{(m-1)(m+3)}{8m(m+1)}\quad m\text{ odd},$$

$$=2\pi\frac{(m-2)(m+2)}{8m(m+1)}\quad m\text{ even}. \tag{10}$$

This is consistent with $\eta=\frac{3}{20}$ for the tricritical Ising model.

This procedure possibly extended to arbitrary subgroups of the modular group should now be applied to other models: those of the "complementary sequence" [3] that generalize the Potts model, or others newly discovered [10]. The generic model of the complementary series for example admit $[N/2]$ invariants under $\tau\to\tau/(1+\tau)$ and $\tau\to\tau+N$ where $N=[\frac{1}{2}(m+1)]$: their implication on symmetries will be discussed elsewhere.

Note added. As this letter was being completed, I received a paper by Cardy [11] which overlaps strongly with the present work. Cardy's work is more systematic in his study of the various kinds of boundary conditions for the Ising and Potts models, and of their implications on the operator content. My approach, which emphasizes the role of the invariance under subgroups of the modular group, seems more general in view of the discussion of other cases.

It is a pleasure to acknowledge interesting discussions with A. Cappelli and H. Saleur. I want to thank particularly Claude Itzykson, for several illuminating and crucial suggestions. Finally, I thank G. von Gehlen and V. Rittenberg for sending me before publication their paper, which played a seminal role in this work.

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov Nucl. Phys. B241 (1984) 333.
[2] J. Cardy, Santa Barbara preprint UCSB-TH-75-1985.
[3] C. Itzykson and J.-B. Zuber, Saclay preprint 86/019, Nucl. Phys., to be published.
[4] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; in: Vertex operators in mathematics and physics, eds. J. Lepowsky, S. Mandelstam and I. Singer (Springer, Berlin, 1985).
[5] G. von Gehlen and V. Rittenberg, Bonn preprint HE-86-03.
[6] W. Nahm, Nucl. Phys. B114 (1976) 174;
N. Seiberg and E. Witten, Princeton preprint (February 1986);
C. Imbimbo and A. Schwimmer, to be published.
[7] M.I. Knopp, Modular functions in analytic number theory (Markham Publishing Co., Chicago, 1970);
R.C. Gunnings, Lectures on modular forms (Princeton U.P., Princeton, NJ, 1962).
[8] D.A. Huse, Phys. Rev. B30 (1984) 3908.
[9] J.-M. Luck, J. Phys. A15 (1982) L169;
J.L. Cardy, J. Phys. A17 (1984) L385, L961.
[10] A. Cappelli, C. Itzykson and J.-B. Zuber, to be published.
[11] J. Cardy, Effect of boundary conditions on the operator content of two-dimensional conformally invariant theories, Santa Barbara preprint 78-1986.