Renormalization group and the infrared behavior of gauge theories

Enrico C. Poggio*†
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138 and Department of Physics, Brandeis University, Waltham, Massachusetts 02154

Helen R. Quinn*‡§ and Jean Bernard Zuber*‖
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 1 November 1976)

This paper discusses the application of renormalization-group techniques to the study of the near-mass-shell behavior in gauge theories with fermions. We critically review past work for the Abelian theory of quantum electrodynamics (QED), and also summarize what is known from perturbative analysis. We reformulate the renormalization-group equation in a way which is acceptable to the infrared region and discuss the subsidiary requirements necessary to obtain useful results from this equation. Our analysis is cast in a framework which avoids any dependence on the simple group structure of QED and is thus suitable for generalization to the non-Abelian case. A full discussion of that case is reserved for a later paper, but we present here some discussion of the differences which arise and the modifications which they necessitate in our analysis.

I. INTRODUCTION

This paper is a study of the application of renormalization-group techniques to the understanding of the near-mass-shell behavior of the fermion propagator and the fermion-fermion-vector vertex in theories with massless vector mesons. We begin by critically reviewing what has been done in the case of QED. There, powerful perturbative methods can exactly sum all infrared-singular contributions to obtain a factorizable exponential term.[1-3] Previous renormalization-group treatments[3-6] achieve the result only by making (implicit) strong assumptions which can only be justified by the perturbative treatment.

Here we present a reformulation of the renormalization-group equation applicable to the infrared region and discuss the supplementary information necessary to obtain useful results from this equation. There is a clear parallel with the application of renormalization-group techniques to massive-particle theories in the ultraviolet region. There, one had to argue that, for external momenta $p_i^2 \gtrsim \mu^2 \gg m_i^2$, where μ^2 is an arbitrary renormalization point and m_i^2 are the particle masses, one can neglect the dependence of Green's functions on the variables m_i^2/μ^2 and m_i^2/p_i^2 [that is, the errors made in so doing are corrections of $O(m^2/\mu^2, m^2/p^2)$]. In the infrared case, we argue that, at least for QED, if we renormalize at $p_i^2 - m_i^2 = -\mu^2$, then for

$$p_i^2 - m^2 \lesssim \mu^2 \ll m_i^2$$

one can neglect the dependence on the variable μ^2/m^2. This is a nontrivial statement. Rather than use methods which are very much linked to the simple group structure and form of the Ward identity of QED, we instead prove this property by more general arguments. We do not obtain any result for QED that is not already well known from the exact perturbative analysis. Our intention in this work is rather to present a method which could be extended to the more complicated non-Abelian theories, and in so doing to analyze as far as possible their infrared structure. We will see clearly how differences arise and how these more general methods could be extended to the non-Abelian case. As extensive analysis of this case will be reported in a future publication.

The remainder of this paper is organized as follows. In Sec. II, we review briefly the perturbative method of Grammer and Yennie[1] and its application to the fermion-fermion-photon proper vertex function[2] and the fermion propagator. We summarize the relevant results obtained by this method, and remark on the features that make it peculiar to the QED (Abelian) case.

Section III A presents our renormalization-group equation. The form of the equation is, of course, a consequence of our particular choices of renormalization and differentiation prescription. We argue that our choice is a convenient one for the discussion of the on-shell infrared behavior, since it allows us to isolate a single relevant dimensionless ratio, $(p^2 - m^2)/\mu^2$. We discuss the necessary characteristic behavior of the vertex function Γ_μ and the renormalization-group functions β and γ that allow this property to hold. We then solve the equation and find that the method provides a proof of exponentiation and factorization of infrared logarithms, but does not provide as complete a result as was obtained from the detailed perturbative analysis. The discussion of previous work,[4,5] which claims to obtain stronger results from the renormalization-group method alone, is given in Appendix A, where we show that the results arise, in fact, from assumptions which are not justified, except by the full perturbative method. In Sec. III B, we outline the arguments for the required properties of Γ_μ, β, and γ. A detailed proof, which makes extensive use of recent techniques developed by Cvitanovic and Kinoshita,[7] is presented in Appendix B. From the analysis of Sec. III B and Appendix B one can establish a criterion to determine the power of the infrared-logarithmic-divergent contribution of any graph of Γ_μ in the near-on-shell region. The details of this criterion are given in Appendix C. Section III C contains a brief discussion of massless QED, where the elaborate arguments of Sec. III B (and, of course, Sec. II) are unnecessary as there is, *a priori*, only a single mass scale. In Sec. IV we consider the application of the method to non-Abelian gauge theories. We find that the successful procedure used for QED should be modified if one hopes to reach well-defined conclusions about the non-Abelian case by this method. Section V is a brief summary statement of what we have done. Appendix D contains various calculational examples that illustrate the arguments of Secs. III, IV, and Appendix B.

II. REVIEW OF THE PERTURBATIVE ANALYSIS

Grammer and Yennie[1] have presented a method of isolating infrared-divergent terms in perturbation theory for QED. Each photon propagator which connects a fermion line which leaves a diagram with momentum p' to a fermion line which leaves it with momentum p is divided into two parts, according to the prescription

$$D_{\mu\nu}(k)=\left(g_{\mu\nu}+\lambda\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}=\frac{G_{\mu\nu}(k)+K_{\mu\nu}(k)}{k^2}, \tag{2.1}$$

where

$$K_{\mu\nu}(k)=k_\mu k_\nu b(k,p,p'). \tag{2.2}$$

By a clever choice of the function b, it can be arranged that only diagrams involving K-type photons are infrared-singular. This is possible since photons connecting to fermions in internal loops can never give infrared-singular contributions, in particular, vacuum polarization contributions are infrared-finite by one power of k^2.[6] The summation over all possible insertions of a single K photon into any vertex diagram containing any number of K- and G-type photons can readily be seen to change the amplitude only by a simple overall factor $B(p,p')$,

$$B(p,p')=\int d^4k\, b(k,p,p')\sim\int d^4k\frac{1}{k^2[(k+p)^2-m^2][(k+p')^2-m^2]}. \tag{2.3}$$

The exponentiation of the infrared-singular pieces is obtained from a simple combinatorics exercise.

This method has been applied to the case of the proper vertex function by Korthals-Altes and de Rafael. They obtain, when $p^2-m^2\simeq p'^2-m^2\simeq q^2$, for the vertex function

$$\Gamma_\mu(p,p')=\gamma_\mu\exp\left[\frac{\alpha}{2\pi}(2-\lambda)\ln\frac{p^2-m^2}{-m^2}\right]f(p^2/m^2)+O(q_\mu), \tag{2.4}$$

where λ is the usual covariant gauge parameter defined in Eq. (2.1) and $f(1)$ is finite. As far as we know, theirs is the first reliable derivation of (2.4) that does not rely on the eikonal approximation[8] and therefore includes, exactly, all nonleading as well as leading infrared logarithms. For later convenience, we restate this result using an off-mass-shell subtraction: $p^2-m^2=p'^2-m^2=Q^2=-\mu^2$. The derivation is exactly parallel to that given by Korthals-Altes and de Rafael.[2] The result is with $\Gamma^R_\mu=\gamma_\mu\Gamma^R+O(q_\mu)$,

$$\Gamma^R\left(\frac{p^2-m^2}{\mu^2};\ \frac{p'^2-m}{\mu^2};\ \frac{q^2}{\mu^2};\ g_\mu\right)=\exp\left[\frac{\alpha}{2\pi}(2-\lambda)(B(p^2,p'^2)-B(m^2-\mu^2,m^2-\mu^2))\right]f(p^2/m^2;p'^2/m^2;q^2/m^2;g_\mu), \tag{2.5}$$

with $f(1;1;1;g_\mu)=1$. B is a given in (2.3) with

$$b(p,p',k)=\frac{m^2}{k^2[(k+p)^2-m^2][(k+p')^2-m^2]}. \tag{2.6}$$

The function f is infrared-finite for all p^2, p'^2, and q^2. Furthermore, as we will show in the next section, as $\mu^2/m^2\to 0$,

$$g_\mu{}^2=4\pi\alpha+O(\mu^2/m^2). \tag{2.7}$$

We should stress that there are several features which are peculiar to QED and are essential for the application of this method:

(1) As we have already mentioned, the contributions from photons with subtracted self-energy insertions is infrared-finite [Eq. (2.7) is a con-

sequence of this fact]. This follows from the fact that all such insertions involve an overall massive fermion loop, where the fermion mass scale is larger than any momenta scale in the infrared regime.[6]

(2) The simplicity of the sum over all possible insertions of a K photon in a given diagram and of the counting of diagrams with different numbers of G and K photons is peculiar to the Abelian gauge group; an immediate manifestation of this is the simple Ward identity. In the non-Abelian theory, changing the position of a given gluon line may, and usually does, change the group-theoretic factor with which the diagram enters. The cancellation between diagrams which is crucial to the simplicity of the Grammer-Yennie method is lost.

III. INFRARED STRUCTURE OF THE NEAR-MASS-SHELL REGION

A. The renormalization-group equations and the infrared asymptotic behavior

Near the mass shell, we can define the renormalized fermion-fermion photon proper vertex function

$$\Gamma_\mu(p,p')=\gamma_\mu\Gamma(p^2/\mu^2;p'^2/\mu^2;m^2/\mu^2;q^2/\mu^2)+\cdots . \tag{3.1}$$

By the dots we include the magnetic moment terms proportional to $\sigma_{\mu\nu}q^\nu$, which in QED are infrared-finite, and terms of the order of (p^2-m^2). We will choose the subtraction momentum such that

$$p^2-m^2=p'^2-m^2=q^2=-\mu^2, \tag{3.2}$$

where m is the physical mass of the fermion. Notice that the simple Ward-Takahashi identity connects (3.1) with the fermion propagator $S(p)$:

$$\Gamma_\mu(p,p)=\frac{\partial}{\partial p^\mu}S^{-1}(p). \tag{3.3}$$

We will be interested in the region $\mu^2\ll m^2$; the limit $\mu^2\to 0$ is clearly on-mass-shell renormalization and is subject to the usual infrared-singularity problems. For any finite μ^2 the subtraction is well defined, without having to introduce a photon mass, and the renormalized Γ satisfies the equations

$$\Gamma\left(\frac{p^2-m^2}{\mu^2};\frac{p'^2-m^2}{\mu^2};\frac{q^2}{\mu^2};\frac{m^2}{\mu^2};\lambda_\mu;g_\mu\right)=Z\left(\frac{\bar\mu^2}{\mu^2};\frac{m^2}{\bar\mu^2};g_{\bar\mu}\right)\Gamma\left(\frac{p^2-m^2}{\bar\mu^2};\frac{p'^2-m^2}{\bar\mu^2};\frac{q^2}{\bar\mu^2};\frac{m^2}{\bar\mu^2};\lambda_{\bar\mu};g_{\bar\mu}\right) \tag{3.4}$$

and

$$\Gamma\left(1;1;1;\frac{m^2}{\mu^2};\lambda_\mu;g_\mu\right)=1. \tag{3.5}$$

$1/(1+\lambda)$ is the usual gauge-fixing parameter. Equations (3.4) and (3.5) are simply the statement that Γ is multiplicatively renormalizable. The coupling constant g_μ is here defined in the usual way by subtracting the photon self-energy $\Pi_{\mu\nu}(q^2)=(g_{\mu\nu}q^2-q_\mu q_\nu)d(q^2)$ at $q^2=\mu^2$,[9] so that

$$g_\mu{}^2d\left(\frac{q^2}{\mu^2};\frac{m^2}{\mu^2};\lambda_\mu;g_\mu\right)=g_\mu{}^2d\left(\frac{m^2}{\bar\mu^2};\lambda_{\bar\mu};g_{\bar\mu}\right) \tag{3.6}$$

and

$$d\left(1;\frac{m^2}{\mu^2};\lambda_\mu;g_\mu\right)=1. \tag{3.7}$$

Equation (3.4) can now be converted into a differential renormalization-group equation. Differentiating with respect to μ^2 and setting $\bar\mu^2=\mu^2$, we obtain

$$\left[\mu^2\frac{\partial}{\partial_\mu{}^2}+\frac{1}{2}\beta\left(g_\mu;\frac{m^2}{\mu^2}\right)\frac{\partial}{\partial g_\mu{}^2}+\delta\left(g_\mu;\frac{m^2}{\mu^2};\lambda_\mu\right)\frac{\partial}{\partial\lambda_\mu{}^{-1}}-\gamma\left(g_\mu;\frac{m^2}{\mu^2};\lambda_\mu\right)\right]\Gamma=0, \tag{3.8}$$

where

$$\Gamma\equiv\Gamma\left(\frac{p^2-m^2}{\mu^2};\frac{p'^2-m^2}{\mu^2};\frac{q^2}{\mu^2};\frac{m^2}{\mu^2};\lambda_\mu;g_\mu\right), \tag{3.9}$$

and, as usual,

$$\beta(g_\mu;m^2/\mu^2)=\mu^2\frac{\partial}{\partial\mu^2}\left[g_{\bar\mu}{}^2d\left(\frac{\mu^2}{\bar\mu^2};\frac{m^2}{\mu^2};g_{\bar\mu}\right)\right]_{\bar\mu=\mu}, \tag{3.10a}$$

$$\delta(g_\mu;m^2/\mu^2;\lambda_\mu)=-\frac{(1+\lambda_\mu)^{-1}}{g_\mu{}^2}\beta\left(\frac{m^2}{\mu^2};g_\mu\right), \tag{3.10b}$$

$$\gamma(g_\mu;m^2/\mu^2;\lambda_\mu)=\mu^2\frac{\partial}{\partial\mu^2}\left[\ln Z\left(\frac{\mu^2}{\bar\mu^2};\frac{m^2}{\bar\mu^2};g_{\bar\mu};\lambda_{\bar\mu}\right)\right]_{\bar\mu=\mu}. \tag{3.10c}$$

We emphasize that (3.8) is merely a reformulation of (3.4). As such, it contains no further information than the statement that Γ is multiplicatively renormalizable. Furthermore, it is not a unique differential form of that statement. Its form is determined by our renormalization and differentiation prescriptions, both of which are not unique. We choose these prescriptions because, as we shall show, they allow us to eliminate the dependence on the variable μ^2/m^2 in the region $\mu^2 \ll m^2$, leaving only the interesting variables $(p^2-m^2)/(-\mu^2)$ and $(p'^2-m^2)/(-\mu^2)$ and thus allowing Eq. (3.8) to give information about the behavior of the theory near the mass shell. Previous attempts[4,5] to study this region via the renormalization-group approach have used different prescriptions, which do not explicitly have this feature. The results which they claim to obtain are consequences of certain strong assumptions which are made without explicit justification. We discuss this further in Appendix A.

The need to eliminate one class of variables is not peculiar to the study of the near-mass-shell region. Let us remind the reader of the more familiar case of the application of the renormalization group to massive-particle Green's functions in the deep Euclidean region. There, all external momenta, p_i^2, satisfy

$$p_i^2 \gtrsim \mu^2 \gg m_i^2,$$

where m_i^2 are the physical masses of the theory. The renormalization-group equations yield new and useful results only after one has shown that, for $m^2 \ll \mu^2$,

$$\Gamma(p_i^2/\mu; m_i^2/\mu^2; q^2/m^2; g_\mu; \lambda_\mu) = \Gamma(p_i^2/\mu^2; 0; q^2/m^2; g_\mu; \lambda_\mu) + O(m_i^2/\mu^2; m_i^2/p_i^2), \tag{3.11}$$

and, in particular, that for $m^2 \ll \mu^2$

$$\begin{aligned}\delta(m_i^2/\mu^2; g_\mu; \lambda_\mu) &= \delta_0(g_\mu; \lambda_\mu) + O(m_i^2/\mu^2),\\ \beta(m_i^2/\mu^2; g_\mu) &= \beta_0(g_\mu) + O(m_i^2/\mu^2),\\ \gamma(m_i^2/\mu^2; \lambda_\mu; g_\mu) &= \gamma_0(g_\mu; \lambda_\mu) + O(m_i^2/\mu^2),\end{aligned} \tag{3.12}$$

where β_0 and γ_0 are the corresponding functions for the fully massless theory. The proof of (3.11) and (3.12) is a *nontrivial* matter and rests, basically, on a power-counting analysis of perturbation-theory diagrams.[10,11] Furthermore, our *only* knowledge of β_0, γ_0 is that obtained from *perturbative* calculations.

For QED, in the near-mass-shell region, with our subtraction convention, we shall show that the property analogous to (3.11), for the infrared (IR) region

$$p_i^2 - m_i^2 \lesssim \mu^2 \ll m_i^2,$$

$$q^2 \lesssim \mu^2 \ll m_i^2,$$

is

$$\Gamma\left(\frac{p_i^2-m_i^2}{\mu^2}; \frac{m_i^2}{\mu^2}; \frac{q^2}{m_i^2}; g_\mu; \lambda_\mu\right) = \Gamma_{\rm IR}\left(\frac{p_i^2-m_i^2}{\mu^2}; g_\mu; \lambda_\mu\right) + O\left(\frac{\mu^2}{m_i^2}; \frac{p_i^2-m_i^2}{m_i^2}; \frac{q^2}{m_i^2}\right), \tag{3.13}$$

where, clearly,

$$\Gamma_{\rm IR}\left(\frac{p_i-m_i}{\mu^2}; g_\mu; \lambda_\mu\right) = \lim_{\mu^2/m_i^2\to 0} \Gamma\left(\frac{p_i^2-m_i^2}{\mu^2}; \frac{m_i^2}{\mu^2}; g_\mu; \lambda_\mu\right). \tag{3.14}$$

Similarly, in analogy with (3.12), we have, for $\mu^2 \ll m^2$,

$$\beta\left(\frac{m^2}{\mu^2}; g_\mu\right) = \frac{\mu^2}{m^2}\tilde\beta_{\rm IR}\left(\frac{\mu^2}{m^2}; g_\mu\right), \tag{3.15}$$

$$\delta\left(\frac{m^2}{\mu^2}; g_\mu; \lambda_\mu\right) = -\frac{(1+\lambda_\mu)^{-1}}{g_\mu^2}\beta\left(\frac{m^2}{\mu^2}; g_\mu\right) = O\left(\frac{\mu^2}{m^2}\right), \tag{3.16}$$

and

$$\gamma\left(\frac{m^2}{\mu^2}; g_\mu; \lambda_\mu\right) = \gamma_{\rm IR}(g_\mu; \lambda_\mu) + O(\mu^2/m^2), \tag{3.17}$$

where $\tilde\beta_{\rm IR}$ is, order by order in perturbation theory, at worst, logarithmically behaved as $\mu^2/m^2 \to 0$. The proof of (3.13), (3.15), (3.16), and (3.17), which will be discussed in Sec. III B and Appendix B, is the new contribution of our paper. Let us first investigate their consequences. Substituting (3.13) through (3.17) in (3.8), one finds that $\Gamma_{\rm IR}$ satisfies the equation

$$\left[\mu^2\frac{\partial}{\partial\mu^2} - \gamma_{\rm IR}(g_\mu; \lambda_\mu)\right]\Gamma_{\rm IR}\left(\frac{p^2-m^2}{\mu^2}; \frac{p'^2-m^2}{\mu^2}; \frac{q^2}{\mu^2}; g_\mu; \lambda_\mu\right) = O\left(\frac{\mu^2}{m^2}; \frac{p_i^2-m_i^2}{m^2}\right). \tag{3.18}$$

Furthermore, as $q^2 \to 0$, where q is the photon momentum, it is well known that there are no q^2 singularities so that

$$\Gamma_{\rm IR}\left(\frac{p^2-m^2}{\mu^2};\frac{p'^2-m^2}{\mu^2};\frac{q^2}{\mu^2}\right)=\Gamma_{\rm IR}\left(\frac{p^2-m^2}{\mu^2};\frac{p'^2-m^2}{\mu^2};0\right)+O\left(\frac{q^2}{m^2}\right). \tag{3.19}$$

This last result is a consequence of the fact that the photon does not couple directly to any massless particle. Using the normalization condition (3.8), Eq. (3.18) can be solved to give, with $p=p'$,

$$\Gamma_{\rm IR}\left(\frac{p^2-m^2}{\mu^2};g_\mu;\lambda_\mu\right)=\left(\frac{\mu^2}{p^2-m^2}\right)^{\gamma_{\rm IR}(g_\mu;\lambda_\mu)}\times\left[1+O\left(\frac{\mu^2}{m^2};\frac{p^2-m^2}{m^2}\right)\right] \tag{3.20}$$

and, using (3.3),

$$S_{\rm IR}(p)=\frac{1}{\gamma\cdot p-m}\left\{\left(\frac{p^2-m^2}{\mu^2}\right)^{\gamma_{\rm IR}}\left[1+O\left(\frac{\mu^2}{m^2};\frac{p^2-m^2}{m^2}\right)\right]\right\}. \tag{3.21}$$

This is as much as can be determined from using the renormalization group alone. The value of the coefficient $\gamma_{\rm IR}$ must be obtained by some other method; the only one we know is perturbation theory. A direct calculation yields[12]

$$\gamma_{\rm IR}(g_\mu;\lambda_\mu)=-\frac{{g_\mu}^2}{8\pi^2}(2-\lambda_\mu)+O({g_\mu}^6). \tag{3.22}$$

In fact, comparing (3.20) with the exact perturbative result (2.5), we see that

$$\gamma_{\rm IR}=-{g_\mu}^2/8\pi^2(2-\lambda_\mu) \tag{3.23}$$

is correct to all orders in perturbation theory. Again, we stress that this result depends strongly on the detailed perturbative analysis reviewed in Sec. II, and that we are not aware of any valid derivation based on renormalization-group arguments alone.

The justification for comparing result (2.5) in terms of α, the physical coupling constant, to that of (3.20), in terms of g_μ, depends on a property particular to massive QED, namely, the behavior of the invariant charge in the infrared region. We would like to comment briefly on this point here. When $q^2 \ll m^2$, the renormalization-group equation regulating the behavior of the invariant charge $g(q^2)$, (3.10a), becomes

$$q^2\frac{\partial}{\partial q^2}g^2(q^2)=\frac{q^2}{m^2}\tilde{\beta}\left(\frac{q^2}{m^2};g(q^2)\right), \tag{3.24}$$

where we have used (3.15). In lowest order

$$\tilde{\beta}=-\frac{g^4(q^2)}{60\pi^2}. \tag{3.25}$$

Equation (3.24) is solved to give

$$g^2(q^2)=\frac{g^2(\mu^2)}{1-[g^2(\mu^2)/60\pi^2](q^2/m^2-\mu^2/m^2)}. \tag{3.26}$$

The physical coupling constant is usually defined to be

$$\alpha\equiv\frac{g^2(0)}{4\pi}\cong\tfrac{1}{137}.$$

Therefore

$$g^2(\mu^2)\equiv 4\pi\alpha(\mu^2)=4\pi\alpha+O(\mu^2/m^2). \tag{3.27}$$

$g(0)$ is effectively a fixed point.

B. Outline of the proof of infrared properties (3.13), (3.15), and (3.17)

The properties claimed in Eqs. (3.13)–(3.17) can be summaried in the statement that *there is no singular dependence on the variable* μ^2/m^2, $\mu^2 \ll m^2$ when the theory is subtracted according to our prescription. Equation (3.15), which describes the behavior of the β function, is a consequence of the Appelquist-Carazzone[6] argument that the contribution of massive-particle loops in a diagram involving massless external lines is suppressed by at least one power of k^2/m^2, for $k^2 \ll m^2$, where k^2 is a typical external momentum. Since all vacuum polarization diagrams in QED involve at least one overall fermion loop, the theorem applies. For the functions $\Gamma_{\rm IR}$ and $\gamma_{\rm IR}$ the argument is somewhat more involved. One method of proof would be to appeal to the full perturbative results discussed in Sec. II, which clearly have the required properties. However, as we have previously stressed, these methods are very much tied to QED and we wish to frame our arguments in such a way that we can investigate, at least to some extent, how the situation differs in non-Abelian theories. Hence, we present here an argument based on the examination of the infrared properties of Feynman diagrams.

Let us first define some useful terms. We analyze the photon-fermion-fermion vertex function. We refer to the zero-loop vertex, γ_μ, as the *bare vertex*, and to a two-fermion-irreducible fermion-fermion scattering amplitude as the *kernel* K. We denote a kernel with no internal vertex corrections or propagator corrections by K_B, *the bare kernel*.

Our argument follows from two statements, which we shall prove in detail in Appendix B.

(1) Consider the unsubtracted vertex contribution from any graph of the type shown in Fig. 1, consisting of a bare vertex and a bare kernel. We show that this contribution (which we shall call bare vertex graph) can be divided into two terms. One term is ultraviolet-singular and infrared-finite. The other is ultraviolet-finite but diverges in the infrared region as both fermion momenta go on the mass shell. Furthermore, *the bare vertex graph diverges as a single power of a logarithm*. Specifically, we can parametrize the contribution of this term, before subtractions, as[13]

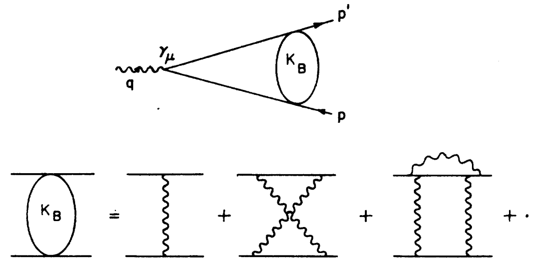

FIG. 1. Bare vertex graphs. K_B is the bare two-fermion irreducible kernel.

$$\Gamma_0^{\rm uns}(p^2/m^2;p'^2/m^2)=\ln\left(\max\left\{\frac{|m^2-p^2|}{m^2};\frac{|m^2-p'^2|}{m^2}\right\}\right)F\left(\frac{p^2}{m^2};\frac{p'^2}{m^2}\right)+\text{IR-finite terms} \qquad (3.28)$$

for $p^2\approx m^2\approx p'^2;q^2\lesssim m^2$. $F(1,1;0)\equiv F_0$ is a well-defined constant. The logarithmically divergent term arises from that part of the integration where all internal photon momenta vanish simultaneously. The single overall subtraction required to make the diagrams ultraviolet-finite at $p^2-m^2=p'^2-m^2=q^2=-\mu^2\ll m^2$ converts (3.28) into

$$\Gamma_0^{\rm sub}(p^2/m^2;p'^2/m^2)=F_0\ln\max\left\{\frac{|m^2-p^2|}{\mu^2};\frac{|m^2-p'^2|}{\mu^2}\right\}+O(\mu^2/m^2;q^2/m^2) \qquad (3.29)$$

in the near-shell IR region

$$q^2\simeq p^2-m^2\simeq p'^2-m^2\lesssim\mu^2\ll m^2\,. \qquad (3.30)$$

(2) Consider the effect of including, in all possible ways, properly subtracted vertex and propagator insertions in diagrams in the class of Fig. 1. We shall call these diagrams dressed-vertex graphs. Since the insertions do not modify the overall divergence structure of the "bare" integrals except by logarithms, we need only concern ourselves with the region where these integrals give the infrared-singular contributions, namely where *all internal photon momenta are vanishingly small*. The insertions are, as we shall argue, of the form

$$\prod_i \ln^{n_i}\left(\frac{p_i^{\,2}-m^2}{\mu^2}\right),$$

where p_i is some internal fermion momentum. We then show that insertions of this form, after the required single overall subtraction, always result in an amplitude, in the region (3.30), which can be written as[13]

$$\Gamma=\sum_n F_n\left(\ln\max\left\{\frac{|m^2-p^2|}{\mu^2};\frac{|m^2-p'^2|}{\mu^2}\right\}\right)^n+O(\mu^2/m^2;q^2/m^2)\,, \qquad (3.31)$$

where the coefficients F_n are finite numbers and the maximum value that n can take can be determined for every graph contributing to Γ, as will be discussed in Appendix C. The proof of (3.31) is given in Appendix B.

Our argument for the behavior of the full vertex function has then the following iterative-inductive form:

(a) We assume that the renormalized $\Gamma(p,p')$ and $\Sigma(p)$, the irreducible fermion self-energy, have the required form [i.e., only logarithms of $(p^2-m^2)/\mu^2$, $(p'^2-m^2)/\mu^2$ but not of (μ^2/m^2)], up to L-loop order of perturbation theory (a vertex graph with L loops gives a g^{2L+1} contribution). From statement (1) above, or by direct computation, this clearly holds for the one-loop contribution to the vertex and by (3.3) for $\Sigma(p)$ in this order.

(b) Then it is also true for $L+1$ loops. The argument is as follows:

From the Schwinger-Dyson equation, Fig. 2, any vertex contribution can be either written as a diagram of the class discussed in (1) modified by the insertion of properly subtracted vertex and propagator corrections, or it is just one of the bare vertex graphs shown in Fig. 1. Because of

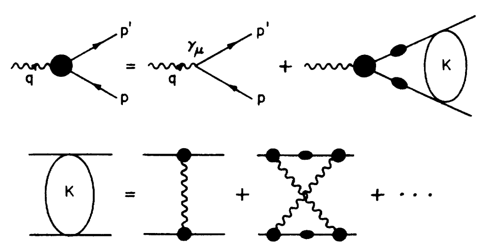

FIG. 2. Schwinger-Dyson equation for the dressed-vertex diagrams. The blobs represent fermion self-energy insertions.

property (1) these latter diagrams contributing at the $(L+1)$-loop level satisfy (3.31) with $L=1$. For an $(L+1)$-loop diagram, any insertion cannot be necessarily of more than L loopes. Since the most sensitive infrared region is when all internal photon momenta are small, the L-loop insertions are in the near-shell IR region (3.30) and therefore of the form (3.31), by statement (a). Because of property (2), the $(L+1)$-loop diagrams must also be of the form (3.31).

To obtain (3.17), the infrared behavior of γ, recall that from (3.10c)

$$\gamma\left(\frac{\mu^2}{m^2}; g_\mu; \lambda_\mu{}^{-1}\right) = \mu^2 \frac{\partial}{\partial_\mu{}^2}\left[\Gamma\left(\frac{p^2-m^2}{\mu^2}; \frac{p'^2-m^2}{\mu^2}; \frac{q^2}{\mu^2}; \frac{\mu^2}{m^2}; g_\mu; \lambda_\mu{}^{-1}\right)\right]_{p^2=m^2+\mu^2,\, p'^2=m^2+\mu^2,\, q^2=\mu^2}, \tag{3.32}$$

and thus as $\mu^2/m^2 \to 0$, $\gamma = \gamma_{\rm IR}(g_\mu) + O(\mu^2/m^2)$.

C. Massless-fermion QED

We remark that the arguments of Sec. III B are all unnecessary in the case of massless-fermion QED.[14] The main problem in applying scaling argument (i.e., the renormalization group) to the massive case was due to the existence of two distinct mass scales, p^2-m^2 and m^2. In the massless case, there is only one scale, that set by the external momenta. There is, in fact, no *a priori* distinction between the infrared and the ultraviolet regions in this case. Analyzing the region $k^2 \ll \mu^2$, where k and μ are the external and renormalization momentum, respectively, one finds, as with all massless Abelian theories, that the origin $g=0$ is an infrared-stable fixed point, and the invariant charge behaves as

$$g^2(k^2) = \frac{g^2(\mu^2)}{1+F_0 g^2(\mu^2)\ln(\mu^2/k^2)} \underset{k^2/\mu^2\to 0}{\sim} \frac{1}{F_0 \ln(\mu^2/k^2)}. \tag{3.33}$$

The solution of (3.33) for the vertex and fermion propagator is straightforward:

$$\Gamma \sim [\ln(\mu^2/k^2)]^{\frac{3}{16}(1+\lambda)}.$$

IV. NON-ABELIAN THEORIES

We turn now to the more complicated non-Abelian theories. If we try to analyze them using our subtraction procedure we find two clear differences. The first has to do with the infrared behavior of $\beta(m^2/\mu^2; g_\mu)$. Because of gluon self-interaction contributions, Eq. (3.15) does not hold anymore. In fact, in perturbation theory,

$$\lim_{\mu^2/m^2\to 0} \beta_{\rm NA}(g_\mu; m^2/\mu^2) = \beta_{\rm YM}(g_\mu) + O(\mu^2/m^2), \tag{4.1}$$

where $\beta_{\rm YM}(g_\mu)$ is the β function corresponding to a pure Yang-Mills theory.[6] Unlike massless QED, the origin in coupling-constant space for a pure Yang-Mills theory is not infrared-free, and thus perturbative calculations do not provide a reliable estimate of $\beta(g_\mu)$ as $\mu \to 0$, since g_μ may become quite large in this limit.

The second difference has to do with the dependence of the vertex function on the variable μ^2/m^2. We recall that in QED, Eqs. (3.13) and (3.17), there is no significant dependence on this variable. This allows rescaling techniques to give useful results. In the non-Abelian case, there are new graphs containing three-gluon and four-gluon interactions, which could conceivably have a different IR behavior under our rescalings. In fact, as demonstrated in Appendix D, the analysis of graphs with gluon self-energy insertions demonstrates the presence of arbitrary powers of $\ln(\mu^2/m^2)$, when use of our subtraction convention is made. Specifically, the analysis suggests that if we subtract the gluon propagator at $q^2=-M^2$, large logarithms that only depend on the ratio μ^4/m^2M^2 are present together with logarithms of $(p^2-m^2)/\mu^2$. Thus choosing a renormalization scheme with $M^2=\mu^4/m^2$ would allow the vertex functions to depend only on the single ratio $(p^2-m^2)/\mu^2$, as in QED, and consequently there would exist a corresponding $\gamma_{\rm IR}^{\rm NA}(g_\mu)$. We should remark that for QED we have arbitrarily chosen $M^2=\mu^2$ as the photon subtraction momentum. In fact, the results stated in Sec. III will hold as long as we choose any M^2 such that $M^2 \ll m^2$, because of Eq. (3.26), which reflects the simple behavior of the QED IR-invariant charge. The fact that there exists a particular choice of IR renormalization scheme for the non-Abelian case is again a reflection of the particular behavior of the long-distance invariant charge of these theories.[15]

Finally, the IR solution of Eq. (3.8) will now depend crucially on the behavior of $\gamma_{\rm IR}^{\rm NA}(g_\mu)$ and $\beta_{\rm YM}(g_\mu)$, in the region $\mu^2/m^2 \to 0$. A full discussion of this and of the renormalization scheme which allows us to determine the existence of $\gamma_{\rm IR}^{\rm NA}(g_\mu)$ will be given in a subsequent paper.[16]

V. SUMMARY

The principal purpose of this paper was to provide a correct application of the renormalization-group method to study the infrared behavior of gauge theories. For QED we present what we believe to be the first correct formulation of the problem. Even though this method does not teach us anything new about QED, where the same problem can be analyzed more explicitly by powerful perturbative methods, it can also be applied to the investigation of the more complex non-Abelian theories.

ACKNOWLEDGMENTS

We would like to thank B. W. Lee and the theory group at Fermilab, where portions of this work were carried out, for their hospitality and interest.

One of us (E.C.P) would also like to thank the theory group at the University of Rome and the Aspen Center for Physics for their kind hospitality. He is also grateful to K. Symanzik for a stimulating conversation that sparked his interest in this problem. Another of us (H.R.Q.) would like to thank the Institute for Theoretical Physics at Stanford University for its hospitality.

APPENDIX A: CRITICAL ANALYSIS OF PREVIOUS WORK

Previous attempts[3-6] to discuss the near-shell region using the renormalization group have been based on the analysis of Bogoliubov and Shirkov which we repeat in the following. Starting from (3.4), differentiating with respect to p^2, and setting $\overline{\mu^2}=p^2$, one obtains

$$p^2\frac{d}{dp^2}\ln\Gamma\left(\frac{p^2}{\mu^2};\frac{m^2}{\mu^2};g_\mu\right)=\xi\frac{\partial}{\partial\xi}\ln\Gamma\left(\xi;\frac{m^2}{p^2};g_p\right)\bigg|_{\xi=1}. \tag{A1}$$

Furthermore, renormalizing at $\mu^2=m^2$ in order to eliminate possible rapid variations of g_{p^2} as compared to g_{μ^2}, when p^2 is near the mass shell, one obtains the exact equation

$$p^2\frac{d}{dp^2}\ln\Gamma\left(\frac{p^2}{m^2};1;g_m\right)=\xi\frac{\partial}{\partial\xi}\ln\Gamma\left(\xi;\frac{m^2}{p^2};g_p\right)\bigg|_{\xi=1}. \tag{A2}$$

Equation (A2), like Eq. (3.8), contains *no* further information than Eq. (3.4). As $p^2\to m^2$, (A2) is approximated by

$$x\frac{\partial}{\partial x}\ln\Gamma(x;1;g_m)=\sigma(x,g_m)\,, \tag{A3}$$

where

$$\sigma(x,g_m)\equiv\xi\frac{\partial}{\partial\xi}\ln\Gamma\left(\xi;\frac{1}{x};g_m\right) \tag{A4}$$

and $x=p^2/m^2$, which approaches unity. In order for (A3) to furnish some useful results about the behavior of $\Gamma(x,1,g_m)$ as $x\to1$, we must have some additional information about the behavior of $\sigma(x,g_m)$, $x\to1$. Bogoliubov and Shirkov assume, without giving any detailed justification, that

$$\lim_{x\to1}\sigma(x,g_m)\sim\frac{A(g_m)}{1-x}+\text{finite terms}\,. \tag{A5}$$

The form of Eq. (A5) excludes terms of the form $[\ln^p(1-x)]/(1-x)$. This assumption corresponds to what is calculated in lowest-order perturbation theory, where of course no $\ln(1-x)$ terms can appear. However, it must be emphasized that, *a priori, nothing prevents the appearance of* $\ln^p(1-x)$ *in the right-hand side of (A3)*. Indeed, such logarithms do appear term by term in the two-loop calculation, though they cancel in the full result.[12] It is this cancellation which must be proved to *all* orders in order to justify (A5). This can be done by the perturbative analyses of Sec. II, where $A(g_m)$ is explictly determined, or by Sec. III, where $A(g_m)$ is undetermined. In other words, the result of Refs. 3–6 can only be obtained by making an assumption which is tantamount to assuming the desired result. There is no information in (A3) about the limit $x\to1$ without some such further assumption. Finally, it should be remarked that Refs. 4 and 5 do not explicitly state any limitations on their arguments which would preclude their application to non-Abelian theories. As discussed in Sec. IV, the form of (A5) can be shown to be incorrect by an explicit two-loop calculation in such theories.

APPENDIX B: PROOF OF THE REQUIRED INFRARED PROPERTIES

1. The bare-vertex graphs

Consider any diagram of the class of Fig. 1. We will first show that it is divergent in the near-shell IR region by at most one power of $\ln(p^2-m^2)$ using a heuristic argument based on naive power counting in momentum space. An L-loop contribution can be written as a Feynman integral of the form

$$I_B=\int\prod_{i=1}^{L}\frac{d^4k_i}{k_i^2}\prod_{j=1}^{M}S_j(p,\{k_m\})\prod_{j=m+1}^{2L}S_j(p',\{k_m\})N(p,p',\{k_m\})\,, \tag{B1}$$

where

$$S_j^{-1}(p,\{k_m\}) \equiv \left(\sum_{i=1}^{L} \eta_i^j k_i + p\right)^2 - m^2\,, \tag{B2}$$

and the $\eta_i^j(=0,\pm 1)$ reflect the flow of photon momenta through the fermion lines. The indices $j=1,2,\ldots,M$ and $j=M+1,\ldots,L$ label the p and p' fermion lines, respectively. Because the bare graph has only one two-fermion cut,

$$\sum_{i=1}^{L} \eta_i^j k_i = \sum_{i=1}^{L} \eta_i^{j'} k_i\,, \quad j=\{1,\ldots,M\},\quad j'=\{M+1,\ldots,L\}$$

only for the values of j and j' corresponding to the p and p' fermion lines attached to the incoming photon. All other sums must be distinct. By naive power counting, only the terms containing no photon momenta in the numerator are infrared-divergent, and these are ultraviolet-finite. The relevant infrared-singular integral is proportional to $N(p,p',0)$, the numerator evaluated at vanishing internal momenta. For small $\{k_i\}$ and $p^2=p'^2=m^2$, it can be written

$$I_B^{\mathrm{IR}} = \text{constant} \times \int \prod_i \frac{d^4k_i}{k_i^2} \prod_{j=1}^{M} \frac{1}{\sum_1 (\eta_i^j k_i)\cdot p} \prod_{j=M+1}^{2L} \frac{1}{\sum_1 (\eta_i^j k_i)\cdot p'}\,. \tag{B3}$$

This integral is overall logarithmically divergent. We want to show that there is no divergence in any subintegration. Let us then consider any subset of the integration variables, say $k_1, k_2, \ldots, k_n$, for $n<L$. We consider the region where all other $(L=n)$ k_i's are hard compared to this subset and construct a reduced diagram by contracting to a point any line containing a "hard" photon momentum. This of course includes any fermion line for which $\eta_i^j \neq 0$ for any $i>n$. The superficial degree of IR divergence for the reduced diagram is

$$D_n = 2n - f\,, \tag{B4}$$

where f is the number of fermion lines in the reduced diagram. However, for bare vertex diagrams $2n>f$ for all possible n's smaller than L. In fact, D_n can be zero, $2n=f$, only if the reduced diagram is in itself a vertex diagram. But this could only happen if the hard-photon lines were all vertex and/or self-energy corrections, which, by definition, are not included in bare diagrams. Thus the infrared singularity comes only from the final integration $(n=L)$, giving a single logarithm of $(p^2-m^2)/m^2 \simeq (p'^2-m^2)/m^2$.

This result[17] can be made rigorous in the context of a parametric representation of (B1) using, say, the techniques and arguments developed by Cvitanovic and Kinoshita (CK).[18,7] We summarize these here since they will be of use in proving the iterative argument for the dressed-vertex diagrams. For convenience, let us consider the case with $q=0$. As we have already said, the infrared behavior of the vertex is insensitive to the small-q regime. Assign parameters y_i, $i=1,\ldots,L$, for the photon lines and parameters x_i, $i=1,\ldots,2L$, for the fermion lines. Then an integral of the type (B3) can be written as

$$I_F = \int \prod_{i=1}^{2L} dx_i \prod_{i=1}^{L} dy_i\, \delta\Big(1-\sum_i x_i - \sum_i y_i\Big) \frac{1}{U^2 V^L}\,. \tag{B5}$$

U is a polynomial in the x_i's and y_i's, while

$$V = \sum_{i=1}^{2L} x_i(m^2-p^2) + V(x,y)p^2 \tag{B6}$$

and

$$V(x,y) = \frac{1}{u}\sum_{ij} x_i x_j B_{ij}(x,y)\,, \tag{B7}$$

where the B_{ij} are constructed according to the formula $B_{ij}=\sum_C U_c$, where the sum runs over all possible loops containing both the lines i and j, and U_c is the corresponding U for a diagram obtained by contracting one such loop to a point. As $p^2 \to m^2$, the region of maximal IR divergence, as we said, corresponds to all the k_i's $\to 0$. This is equivalent, in parametric space, to a region where all x_i's approach zero and all y_i's are of order unity. From the analysis of Cvitanovic and Kinoshita (which our heuristic argument in momentum space closely parallels) one finds that, in this region, for graphs of the class of Fig. 1,

(i) $U(x,y) \to U(y) = \text{constant}$,

(ii) none of the U_c vanishes,

(iii) I_F has only one overall log divergence in p^2-m^2, and

(iv) $V(x,y) \to f(x)$.

By doing a standard scaling transformation

$$z = \sum_{i=1}^{2L} x_i\,, \qquad x_i \to z x_i\,, \tag{B8}$$

this divergence can be isolated by the z integration. Specifically integral (B5) becomes

$$I_F \sim \int_0^1 dz \frac{z^{2L-1}}{[z(m^2-p^2)+z^2 f(x_i)p^2]^L} \sim \ln\frac{p^2-m^2}{m^2}. \tag{B9}$$

2. Dressed-vertex graphs

We consider now the contributions obtained by inserting in all possible ways *renormalized* vertex and self-energy corrections into the bare vertex diagrams. We remind the reader that the unsubtracted insertions contain two pieces, one is UV-divergent but IR-finite and the other is UV-finite but IR-divergent. Clearly, by the nature of our subtraction procedure, *in the near-shell region*, the IR-finite piece is of $O(\mu^2/m^2)$. The effects of these terms, when inserted, will then always be of, at least, $O(\mu^2/m^2)$. In the region where all photon momenta are small and of the same order, by our induction assumption, the behavior of each subtracted insertion is given by a polynomial in $\ln[S_j^{-1}(p;\{k_i\})/\mu^2]$, where S_j^{-1} has been defined in (B2). Then a typical (unsubtracted) integral contributing to dressed-vertex diagrams is of the form (for $q=0$)

$$I(b_1, b_2, \ldots, b_{2L}) = \int \prod_{i=1}^{L} \frac{d^4k_i}{k_i^2} \prod_{j=1}^{2L} S_j(p;k_i) \times \ln^{b_j}\left[\frac{S_j^{-1}(p;k_i)}{\mu^2}\right]. \tag{B10}$$

Define the generating function of the integral in (B2) as

$$I(\alpha_1, \alpha_{2L}) = \sum_{\{b_i\}} I(b_1, \ldots, b_{2L}) \prod_{i=1}^{2L} \frac{\alpha_i^{b_i}}{b_i!}, \tag{B11}$$

which, in parametric form, becomes

$$I(\alpha_1, \ldots, \alpha_{2L}) = F(\alpha_i) \int \prod_{i=1}^{2L} dx_i \prod_{i=1}^{L} dy_i\, \delta\Big(1-\sum_i x_i - \sum y_i\Big) \prod_{i=1}^{2L} \left(\frac{V}{\mu^2 x_i}\right)^{\alpha_i} \frac{1}{U^2 V^L}, \tag{B12}$$

where x_i, y_i and U, V have been defined previously. By the CK arguments, only the region where all $x_i \sim 0$ contribute, and using the scaling (B8), Eq. (B12) becomes, in the near-mass-shell region,

$$I(\alpha_1, \ldots, \alpha_{2L}) \sim \int \pi dx_i \pi dy_i \prod_{i=1}^{2L} \left(\frac{1}{x_i}\right)^{\alpha_i} \int_0^1 dz\, J_L(z;x_i;y_i) = F\left(\alpha_1, \ldots, \alpha_{2L}; \frac{p^2-m^2}{-\mu^2}\right) - F\left(\alpha_1, \ldots, \alpha_{2L}; \frac{m^2}{-\mu^2}\right) \tag{B13}$$

where

$$J_L(z;x_i;y_i) \equiv \frac{z^{L-1}(\mu^2)^{-\alpha}}{[(m^2-p^2)+zf(x_i,y_i)p^2]^{L-\alpha}} \tag{B14}$$

and $\alpha = \sum_{i=1}^{2L} \alpha_i$. Equation (B13) reflects the fact that as $p^2 \to m^2$, the IR singularities only come from doing the z integration. By inverting (B13), $I(b_1, \ldots, b_{2L})$ is clearly also of the form

$$I\left(\frac{p^2-m^2}{\mu^2}\right) - I\left(\frac{m^2}{\mu^2}\right),$$

where $I(x)$ is a polynomial in $\ln x$. The subtracted contribution then has the desired form.

APPENDIX C: COUNTING LOGARITHMIC INFRARED DIVERGENCES

The results of Sec. III B, as derived in Appendix B, allow us to determine the maximum power of logarithmic divergence that any given graph can have in the near-mass-shell region. The crucial result is the fact, expressed by property (1) in Sec. III B, that bare vertex graphs, those of Fig. 1, are at most divergent as *one* power of $\ln(p^2-m^2)$. Therefore any graph formed by iterating any bare kernel contribution N times can have no more than a $\ln^N(p^2-m^2)$ behavior. By the Schwinger-Dyson equation, any vertex graph is built by iterating the bare kernel and dressing all fermion self-energy and internal photon-fermion vertices in all possible ways. With this in mind, and noticing that graphs with N kernel iterations also have N two-fermion cuts, the following procedure can be used to determine the maximum degree of log divergence of any graph:

(1) Count the *total* number of fermion self-energy loops, S, and shrink to a point all the self-energy insertions.

Then for the resulting reduced graph

(2) count the number of two-fermion cuts, F,

(3) isolate all vertex insertions arising from internal photon lines.

(4) For each vertex correction repeat (2) and (3) until all two-fermion cuts from all possible subinsertions, $\sum_{\text{sub in}} Fi$, are counted. Then the maximum number of $\ln(p^2-m^2)$ of the graph is

given by

$$N_{\max}=S+F+\sum_{\text{subinsertions}} Fi\,.$$

For example, the graph in Fig. 3 which contributes at the 17-loop level has at most nine powers of $\ln(p^2-m^2)$. Clearly, for graphs at the n-loop level, N can vary from n, for graphs made up exclusively of self-energy insertions and single-photon ladder exchange vertex insertions, to 1 for the bare vertex contributions.

APPENDIX D: CALCULATION OF THE INFRARED BEHAVIOR OF VERTEX GRAPHS WITH ONE INSERTION

In this appendix we will present an explicit calculation of the infrared behavior of the vertex function with vertex or self-energy insertions. This will illustrate the arguments of Secs. III, IV, and Appendix B. For simplicity we will consider the vertex at $q=0$, and graphs with a one-photon-exchange bare kernel. The behavior of any of the diagrams of Fig. 4 is determined by the sum of integrals of the form

$$I_n=\int\frac{d^4k}{(2\pi)^4}m^2\ln^n\left[\frac{(p+k)^2-m^2}{\mu^2}\right]\times\frac{1}{\mu^2[(p+k)^2-m^2]}\,. \tag{D1}$$

Consider the generating function

$$I(\alpha)\equiv\sum_{n=0}^{\infty}\frac{\alpha^n I_n}{n!}=\frac{m^2}{(\mu^2)^\alpha}\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(p+k)^2-m^2]^{2-\alpha}k^2}\,. \tag{D2}$$

Equation (D2) is an example of the functions introduced in (B11). By standard methods

$$I(\alpha)\sim\frac{(1-\alpha)}{(2-\alpha)}\frac{m^2}{(\mu^2)^\alpha}\int_0^1 dx\frac{x^{1-\alpha}}{[xm^2-p^2x(1-x)]^{1-\alpha}} \tag{D3}$$

$$=\frac{1}{\alpha(1-\alpha)}\frac{m^2}{p^2}\left\{\exp\left(\alpha\ln\frac{m^2}{\mu^2}\right)-\exp\left[\alpha\ln\left(\frac{m^2-p^2}{\mu^2}\right)\right]\right\} \tag{D4}$$

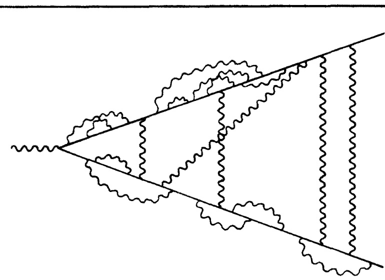

FIG. 3. A 17-loop vertex graph which diverges as $\ln^9(p^2-m^2)$ in the near-mass-shell region.

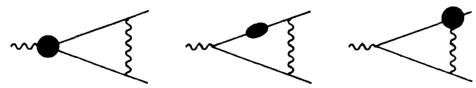

FIG. 4. Typical dressed diagrams computed in Appendix D.

and therefore

$$I_n\sim\frac{m^2}{p^2}\sum_{S=1}^{n+1}\frac{n!}{S!}\left[\ln^S\frac{m^2}{\mu^2}-\ln^S\frac{(m^2-p^2)}{\mu^2}\right]. \tag{D5}$$

Subtracting at $p^2-m^2=\mu$,

$$I_n\sim\sum_{S=1}^{n+1}\frac{n!}{S!}\ln^S\left(\frac{m^2-p^2}{\mu^2}\right), \tag{D6}$$

in agreement with (3.29).

A typical non-Abelian diagram is given in Fig. 5. Its infrared behavior is determined by sums of integrals of the form

$$J_n(\mu^2)=\int\frac{d^4k}{(2\pi)^4}m^2\frac{\ln^n(k^2/M^2)}{k^2[(k+p)^2-m^2]^2}, \tag{D7}$$

where M^2 is the subtraction momenta of the photon propagator. Using the generating function (D2) for J_n

$$J(\alpha)\equiv\sum_{n=0}^{\infty}\frac{\alpha^n J_n}{n!}=\frac{m^2}{(M^2)^\alpha}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^{1-\alpha}[(k+p)^2-m^2]^2}, \tag{D8}$$

which, by Feynman parametrization, gives

$$J(\alpha)=m^2\int_0^1 dx\frac{1}{m^2-p^2(1-x)}\times\left[\frac{m^2x-p^2x(1-x)}{(1-x)M^2}\right]^\alpha \tag{D9}$$

and therefore

$$J_n=\int_0^1 dx\frac{m^2}{m^2-p^2(1-x)}\times\ln^n\left[\frac{m^2-p^2(1-x)}{M^2}\left(\frac{x}{1-x}\right)\right], \tag{D10}$$

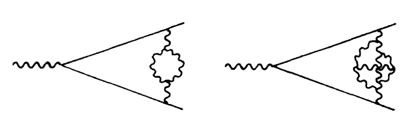

FIG. 5. Typical non-Abelian diagrams with gluon self-energy corrections that can be computed as in Appendix D.

which gives

$$J_n = \frac{m^2}{p^2}\int_{(m^2-p^2)/M^2}^{m^2/M^2} \frac{dy}{y}\ln^n\left[\frac{y(y-(m^2-p^2)/\mu^2)}{(m^2/\mu^2-y)}\right] \tag{D11}$$

as $p^2 \to m^2$, the subtracted integral gives

$$J_n^S \equiv \int_{(m^2-p^2)/M^2}^{m^2/M^2} \frac{dy}{y}\ln^n\left[\frac{y(y-(m^2-p^2)/M^2)}{(m^2/M^2-y)}\right] - \int_{\mu^2/M^2}^{m^2/M^2}\frac{dy}{y}\ln^n\left[\frac{y(y-\mu^2/M^2)}{(m^2/M^2-y)}\right] \tag{D12}$$

by changing variables $z=(M^2/\mu^2)y$, we obtain

$$J_n^S = \int_{(m^2-p^2)/\mu^2}^{m^2/\mu^2}\frac{dz}{z}\ln^n\left[\left(\frac{\mu^2}{M^2}\right)\frac{z(z-(m^2-p^2)/\mu^2)}{(m^2/\mu^2-z)}\right] - \int_1^{m^2/\mu^2}\frac{dz}{z}\ln^n\left[\left(\frac{\mu^2}{m^2}\right)\frac{z(z-1)}{(m^2/\mu^2-z)}\right] \tag{D13}$$

as $\mu^2/m^2 \to 0$,

$$J_n^S = \int_{(m^2-p^2)/\mu^2}^{m^2/\mu^2}\frac{dz}{z}\ln^n\left[\frac{\mu^4}{m^2M^2}z\left(z-\frac{m^2-p^2}{\mu^2}\right)\right] - \int_1^{m^2/\mu^2}\frac{dz}{z}\ln^n\left[\frac{\mu^2}{m^2M^2}z(z-1)\right] + \text{IR-finite pieces}. \tag{D14}$$

Clearly, for $M^2=\mu^2$, J_n^S will contain terms of the form $\ln^{n_1}(m^2/\mu^2)\ln^{n_2}[(p^2-m^2)/\mu^2]$, $n_1+n_2=n+1$. Furthermore, if $\mu^4/m^2M^2=1$, $J_n^S=F_n((m^2-p^2)/\mu^2)$ only.

---

*Research supported in part by the National Science Foundation under Grant No. MPS75-20427.

†Research supported in part by ERDA under Contract No. E(11-1)3230.

‡Work supported in part by the Alfred Sloan Foundation.

§Presently on leave at: Dept. of Physics, Stanford University, Stanford, California.

||Present (permanent) address: C.E.N. de Saclay, Service de Physique Théoretique, Gif-sur-Yvette, France. Work supported in part by NATO.

[1]G. Grammer and D. R. Yennie, Phys. Rev. D 8, 4332 (1973) and references of previous work cited therein.

[2]C. P. Korthals-Altes and E. de Rafael, Nucl. Phys. B106, 237 (1976).

[3]For a recent review of infrared problems in QED, see N. Papanicolaou, Phys. Rep. 24C, 229 (1976).

[4]N. N. Bogoliubov and D. V. Shirkov, in *Introduction to the Theory of Quantum Fields* (Interscience, New York, 1959).

[5]C. Nash and R. L. Stuller, Imperial College Report No. KTP174/17, 1975 (unpublished).

[6]T. W. Appelquist and J. Carazzone, Phys. Rev. D 11, 2865 (1975).

[7]P. Cvitanovic and T. Kinoshita, Phys. Rev. D 10, 3978 (1974); D 10, 3991 (1974) and references to previous works cited therein.

[8]L. D. Landau, A. Abrikosov, and I. Khalatnikov, Nuovo Cimento Suppl. 3, 80 (1956).

[9]The subtraction of the photon self-energy at $q^2=\mu^2$ is not compulsory, but it has been done for convenience. See Sec. IV and Appendix D for further discussions about this choice.

[10]See, for example, M. Baker and K. Johnson, Phys. Rev. D 3, 2516 (1971) for QED, and E. C. Poggio, *ibid.* 8, 2431 (1973) for the $\lambda\phi^4$ theory. Both proofs are, essentially, corrolaries of Weinberg's power-counting theorem: S. Weinberg, Phys. Rev. 118, 838 (1960).

[11]This is, of course, equivalent to showing the vanishing of the right-hand side of the Callan-Symanzik equation.

[12]L. D. Soloviev, Dokl. Akad. Nauk SSSR 110, 203 (1956) [Sov. Phys.—Dokl. 1, 536 (1956)].

[13]This is a convenient parametrization of typical vertex integrals. We have dropped the dependence on q^2, since as we have mentioned the $q=0$ behavior of QED vertices is well defined, giving rise to IR-finite terms. For the one-loop case, we have, for example

$$\Gamma_0^{(1)} \sim \int_0^1 dx\frac{1}{p^2x+p'^2(1-x)} \times \ln\frac{(p^2-m^2)x+(p'^2-m^2)(1-x)}{m^2},$$

which, in the near-shell region defined by (3.29), behaves as

$$\frac{p^2-m^2}{p^2-p'^2}\ln\frac{p^2-m^2}{m^2}+\frac{p'^2-m^2}{p'^2-p^2}\ln\frac{p'^2-m^2}{m^2}.$$

This expression is singular only if both p^2 and p'^2 are equal to m^2.

[14]As far as we know, the IR behavior of massless QED was discussed first by K. Symanzik, in *Proceedings of the Colloquim on the Renormalization of Yang-Mills Fields and Applications to Particle Physics, Marseille, France*, 1972, edited by C. P. Korthals-Altes (C.N.R.S., Marseille, France, 1972).

[15]E. C. Poggio, Phys. Rev. Lett. 36, 1511 (1976).

[16]E. C. Poggio and J. B. Zuber, Brandeis-Saclay report, 1977 (unpublished).

[17]This result can also be deduced from a different heuristic argument given by J. T. Cornwall and G. Tiktopoulos, Phys. Rev. D 13, 3370 (1976).

[18]Even though not explicitly stated, the result is apparent in both Refs. 1 and 7.