A NOTE ON U(N) INTEGRALS IN THE LARGE N LIMIT

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The U(N) integral $\int \mathcal{D}U \exp[N\,\mathrm{tr}(UJ + U^\dagger J^\dagger)] = \exp(N^2 W)$ is reconsidered in the large N limit and the coefficients of the expansion of W in the moments of the eigenvalues of $(JJ^\dagger)$ explicitly computed.

It is possible to interpret the strong coupling expansion of large-N lattice QCD in terms of planar surfaces [1,2]. A central role in this process is played by the one-link integral:

$$\exp[N^2 W(JJ^\dagger)] = \int \mathcal{D}U \exp[\beta N \mathrm{tr}(UJ + U^\dagger J^\dagger)]. \tag{1}$$

Indeed the weights attached to configurations where several plaquettes cross along the same link are given by the expansion of W in powers of the moments,

$$\rho_n = N^{-1}\mathrm{tr}(JJ^\dagger)^n. \tag{2}$$

The integral (1) has been the object of much work, especially in the large-N limit [3]. A few years ago, Brézin and Gross were able to compute exactly W in the large-N limit as the solution of the coupled equations

$$W = \frac{2}{N}\sum_a (\beta^2\lambda_a + c)^{1/2} - \frac{1}{2N^2}\sum_{a,b} \ln[(\beta^2\lambda_a + c)^{1/2} + (\beta^2\lambda_b + c)^{1/2}] - c - \tfrac{3}{4}, \tag{3}$$

with

$$c = \frac{1}{N}\sum_a (\beta^2\lambda_a + c)^{-1/2} \quad \text{for } \frac{1}{N}\sum(\beta^2\lambda_a)^{-1/2} \geqslant 2$$

(strong coupling regime), (4a)

or

$$c = 0 \quad \text{for } \frac{1}{N}\sum(\beta^2\lambda_a)^{-1/2} \leqslant 2$$

(weak coupling regime). (4b)

Here, the λ denote the real eigenvalues of the hermitean matrix $JJ^\dagger$; of course β^2 might be absorbed into λ, but is kept for later convenience.

In the strong coupling regime, the coefficients of the expansion of W in powers of ρ were not explicitly available so far. We write:

$$W = \sum_{n=1} \beta^{2n} \sum_{\substack{\alpha_k \geqslant 0 \\ \Sigma k\alpha_k = n}} W_{[\alpha]}\, \rho_1^{\alpha_1} \rho_2^{\alpha_2} \ldots \rho_n^{\alpha_n}, \tag{5}$$

where the sum runs over the partitions $[\alpha]$ of the integer n:

$$n = (\alpha_1)\cdot 1 + (\alpha_2)\cdot 2 + (\alpha_3)\cdot 3 + \ldots (\alpha_n)\cdot n.$$

The first W_n have been given by Samuel [3] (in his notations $C^c_{[\alpha]} N^{2n-2} = n!\, W_{[\alpha]} + O(1/N^2)$).

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Our new result is that in general

$$W_{[\alpha]} = (-1)^n \frac{(2n+\Sigma\alpha_q-3)!}{(2n)!} \prod_q \left(-\frac{(2q)!}{(q!)^2}\right)^{\alpha_q} \frac{1}{\alpha_q!} . \tag{6}$$

We induced this formula from a study of recursive relations between the $W_{[\alpha]}$, introduced recently by Kazakov [2]. For example, for the trivial partition $\alpha_n = 1$, if we set $f_{[n]} = nW_{[n]}$ Kazakov has shown that

$$f_{[n]} = -\sum_{k=1}^{n-1} f_{[k]} f_{[n-k]} , \tag{7}$$

which leads to $f_{[n]} = (-1)^{n+1}(2n-2)!/n!(n-1)!$ in agreement with (6). By the same method, one may show that for a general partition $[\alpha]$ of n: $n = k_1 + k_2 + ... + k_m$)

$$f_{[k_1,k_2,...,k_m]} \equiv \prod_q (q^{\alpha_q}\alpha_q!) W_{[\alpha]} \tag{8}$$

satisfies

$$f_{[k_1,...,k_m]} = -\sum_{j=2}^{m} k_j f_{[k_1+k_j,k_2,...,\hat{k}_j,...,k_m]} - \sum_{l=1}^{k_1-1} \sum_{\alpha\in P(L)} f_{[k_1-l,L\backslash\alpha]} f_{[l,\alpha]} , \tag{9}$$

where $L = \{k_2, ..., k_m\}$ and $P(L)$ is the set of all subsets of L, including the empty set. These identities result from the use of the equation of motion in (1), which was also the starting point of Brézin and Gross.

Solving these recursion relations should lead to the general expression (6). However to establish (6), it seems more convenient to insert it into (5), and to identify the sum with the expression (3), (4). Let $R(z)$ stand for:

$$R(z) = \sum_{k=1} (-1)^k z^{2k} \rho_k \frac{(2k)!}{(k!)^2} = N^{-1} \mathrm{tr}(1 + 4z^2 JJ^\dagger)^{-1/2} - 1, \tag{10}$$

and $z(\beta)$ be the solution of

$$z(1 + R(z)) = \beta. \tag{11}$$

Using Lagrange's formula, eqs. (5), (6) may be resummed to:

$$W = \frac{1}{2} \int_\beta^{z(\beta)} \frac{(z-\beta)^2}{z^3} \mathrm{d}z + \frac{1}{2} \int_0^{z(\beta)} \mathrm{d}z \, [R^2(z)/z + 2(z-\beta)R(z)/z^2] . \tag{12}$$

The quantity c in eq. (4a) is related to $z(\beta)$ by:

$$2c^{1/2} = \beta/z(\beta). \tag{13}$$

This makes the identification of eqs. (3) and (12) easy, the less trivial step being the proof that:

$$-\frac{1}{2N^2} \sum_{a,b} \ln[\tfrac{1}{2}(1 + 4z^2\lambda_a)^{1/2} + (1 + 4z^2\lambda_b)^{1/2}] = \frac{1}{2} \int_0^z \mathrm{d}z' \frac{R^2(z') + 2R(z')}{z'} , \tag{14}$$

for any z, which may be established by differentiating the two sides of the equation with respect to z.

Finally, it may be worth noticing that $W''(\beta)$ has a simple expression:

$$W''(\beta) = \tfrac{1}{2}[1/\beta^2 - 1/z^2(\beta)] . \tag{15}$$

This may of course be verified from eq. (3), but it would be interesting to derive it directly from the definition (1) of W.

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