# The planar approximation. II

C. Itzykson and J.-B. Zuber
*Division de la Physique, CEN-Saclay, Boite Postale No. 2, 91190 Gif-Sur-Yvette, France*

(Received 18 July 1979; accepted for publication 11 October 1979)

The planar approximation is reconsidered. It is shown that a saddle point method is ineffective, due to the large number of degrees of freedom. The problem of eliminating angular variables is illustrated on a simple model coupling two $N \times N$ matrices.

## 1. INTRODUCTION

The idea that a large-$N$ expansion in the theory of SU($N$) gauge fields is a means to generate an approximation to the true system remains an attractive one. Since the work of 't Hooft[1] on two-dimensional QCD, there have been various attempts at developing a systematic treatment. Recent claims have been made that this limit enables one to understand the connection with the string formulation of the dual model[2] and that it provides a semiquantitative understanding of various selection rules in the framework of quark interactions.[3]

The zero-dimensional counting problem in the same approximation is related to the theory of random matrices and might find applications in models of disordered media.[4] The question has been considered by mathematicians[5] and physicists[6] using combinatorial methods, or analytical ones.[7-9] In Sec. 2 we shall present a short review of this subject. In [A] the method was applied to quantum mechanics, where it was shown to give accurate approximations.

The existence of a large parameter $N$, namely the order of the invariance group, suggests at first that some form of the saddle point method might apply to the path-integral representing the transition amplitude. This seems further confirmed by the observation that expectation values of products of invariant operators $A,B,\cdots$ factorize in the large $N$ limit:

$$\langle AB\cdots\rangle \rightarrow \langle A\rangle\langle B\rangle\cdots. \tag{1.1}$$

Could there exist a classical fluctuationless configuration to describe the situation?

Unfortunately, this turns out to be rather illusory, as will be illustrated in the following. To be specific, we shall study a simple model involving finitely many degrees of freedom, each one represented by a Hermitian $N \times N$ matrix $M$. These finitely many degrees of freedom might be thought of as a finite lattice approximation to a genuine $d$-dimensional continuum. The integrals to be considered have the form:

$$Z = \int \prod_1^p dM_i \exp\Big[-\sum_{i=1}^{p} V(M_i) + \sum_{i,j=1}^{p} \beta_{ij}\,\mathrm{tr} M_i M_j\Big], \tag{1.2}$$

with $V(M)$ a potential term, typically

$$V(M) = \tfrac{1}{2}\,\mathrm{tr}M^2 + \frac{g}{N}\,\mathrm{tr}M^4, \tag{1.3}$$

inducing a quartic anharmonic term with strength $g/N$.

Here $\beta_{ij}$ is a short-range "kinetic" coupling among sites, for instance, $\beta_{ij} = \beta$ if $i$ and $j$ are nearest neighbors and zero otherwise. Finally $dM$ is a U($N$)-invariant volume element. By allowing a finite number of space–time points and letting $N \rightarrow \infty$, we interchange, of course, the infinite volume (thermodynamic) limit and the restriction to planar diagrams (large-$N$ limit). However, as long as this procedure is thought of as a means of generating Feynman diagrams in a series expansion in $g$ corresponding to processes without infrared divergences, it seems without danger.

Rescaling $M$ into $N^{1/2}M$, we can look upon (1.2) as an integral involving an action multiplied by the large number $N$, which calls for the saddle-point method of evaluation. This is obviously too naive since it omits two aspects of crucial importance. The first is the contribution of the measure itself and the second the large degeneracy due precisely to the invariance group, here U($N$) with $N^2$ parameters. The search for a saddle point can only be undertaken once these degenerate degrees of freedom have been eliminated. When this is done, one deals with a basis of group invariants. A sharp distinction appears here between the above planar problem, and the one encountered in a seemingly analogous situation involving vector instead of matrix variables, such as the classical Heisenberg O($N$) ferromagnet. In this case, the variables attached to each of the $p$ points of the lattice are $N$-dimensional vectors $S_i^a$, $a = 1,\ldots,N$, $i = 1,\ldots,p$. A basis of invariants under the real orthogonal group O($N$) is given in terms of the $p(p+1)/2$ scalar products $\mathbf{S}_i\cdot\mathbf{S}_j$, $i \leqslant j$. (Since $p \ll N$ no quantity involving a determinant does occur.) In the measure, the O(N) degrees of freedom can be factored out, leaving as a result

$$\prod_{i\leqslant j} d(\mathbf{S}_i\cdot\mathbf{S}_j)\,|\det(\mathbf{S}_i\cdot\mathbf{S}_j)|^{(N-p-1)/2}.$$

As a consequence,

$$\begin{aligned} Z_{\text{vector}} &= \int \prod_{i=1}^{p} d^N S_i \exp\Big[-\sum_i V(\mathbf{S}_i^2) + \sum_{i,j}\beta_{ij}\mathbf{S}_i\cdot\mathbf{S}_j\Big] \\ &= Z_0 \int \prod_{i\leqslant j} d(\mathbf{S}_i\cdot\mathbf{S}_j)\,|\det(\mathbf{S}_i\cdot\mathbf{S}_j)|^{(N-p-1)/2} \\ &\quad\times \exp\Big[-\sum_i V(\mathbf{S}_i^2) + \sum_{i,j}\beta_{ij}\mathbf{S}_i\cdot\mathbf{S}_j\Big], \end{aligned} \tag{1.4}$$

with $Z_0$ a normalization constant independent of $V$ and the last integral extending over the positivity domain of the matrix $(\mathbf{S}_i\cdot\mathbf{S}_j)$. This expression is suited to the application of the saddle-point method, which will lead in this case to the usual $1/N$ expansion of the classical ferromagnet. This success

 0022-2488/80/030411-11$1.00  

may be attributed to the fact that we have found a choice of $p(p+1)/2$ invariants much smaller in number than the original $Np$ variables, more specifically much smaller than the large parameter $N$. As in thermodynamics, each degree of freedom will contribute to a connected quantity a finite amount. If the total number of degrees of freedom is vanishingly small as compared to the large parameter, the saddle-point method is useful and will be the starting point of a systematic expansion.

The situation is not as good in the matrix case. Except in the case of a single matrix where the space of invariants is $N$-dimensional and hence much smaller than the original space (of dimension $N^2$), it is sufficient to look at the set of invariants for two matrices to see that its size is comparable to the size of the original space. Consequently, fluctuations to all orders will be essential in the evaluation of the integral and no simple saddle-point method will work. Does this really mean that the planar problem is totally untractable in general? We have no answer to this question, but the successful applications of the planar approximation to quantum mechanics (see [A]) leaves some hope that an appropriate trick works for each specific case.

It is therefore of interest to confront the type of difficulty discussed above on the first nontrivial instance, namely when the integral $Z$ involves two matrices only. This is the main part of the present investigation, to which we devote the last two sections. The result of the integration over angular variables corresponding to the unitary group transformations is presented in Sec. 3, while in Sec. 4 we discuss two expansions of the planar limit.

The outcome of this investigation seems a little disappointing. We feel, nevertheless, that it is worth being reported since it illustrates the nontrivial character of the planar approximation. Moreover, some of the expressions derived below might turn out to be useful in another context. Finally, our incomplete solution might raise other people's interest in finding a more complete answer.

## 2. THE COUNTING PROBLEM REVISITED

We recall the results obtained elsewhere[7-9] on the counting of diagrams with a definite topology. We shall generalize the theory to include an arbitrary polynomial interaction $V(M)$, which we assume even for simplicity:

$$V(M)=\tfrac{1}{2}\,\mathrm{tr}M^2+\sum_{p\geqslant 2}\frac{g_p}{N^{p-1}}\,\mathrm{tr}M^{2p}. \tag{2.1}$$

Let $dM$ be the unitary invariant measure on Hermitian $N\times N$ matrices

$$dM=\prod_{i=1}^{N}dM_{ii}\prod_{i<j}2d\,\mathrm{Re}M_{ij}\,d\,\mathrm{Im}M_{ij}. \tag{2.2}$$

We define

$$Z(g)=\int dM\,e^{-V(M)}, \tag{2.3}$$

which makes sense in some appropriate (complex) domain for the coupling constants. In Eq. (2.1), the coefficients of higher-order terms are weighted with inverse powers of $N$ in such a way that the perturbative expansion of

$$E\left(g,\frac{1}{N}\right)=-\frac{1}{N^2}\ln\left(\frac{Z(g)}{Z(0)}\right), \tag{2.4}$$

will produce contributions of the form

$$E\left(g,\frac{1}{N}\right)=\sum_{H\geqslant 0}\frac{1}{N^{2H}}E_{(H)}(g), \tag{2.5}$$

with $E_{(H)}$ defined in terms of diagrams drawn on a surface with $H$ handles ($H$ = genus of the surface). $E_{(0)}$ corresponds to the planar (or spherical) topology, $E_{(1)}$ to the torus, and so on. In the sequel, we shall concentrate on the vacuum "energy" $E$ and leave aside questions dealing with Green's functions. This generalization can be done along the lines of [A]. In the measure $dM$, the angular factors corresponding to the unitary transformation $U$ to a diagonal form:

$$M=U\Lambda U^{\dagger},\quad \Lambda\equiv\begin{pmatrix}\lambda_1 & & & 0\\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_n\end{pmatrix}, \tag{2.6}$$

can be factored out. When integrating over an invariant function $f(M)$, i.e., such that $f(M)\equiv f(UMU^{\dagger})$, we have

$$\int dM f(M)=\frac{(2\pi)^{N(N-1)/2}}{\Pi_1^N p!}\int\prod_1^N d\lambda_i\,\Delta(\lambda)^2 f(\Lambda), \tag{2.7}$$

with $\Delta(\lambda)$ the Vandermonde determinant

$$\Delta(\lambda)=\prod_{i>j}(\lambda_i-\lambda_j)=\det(\lambda_i^{\,j-1}). \tag{2.8}$$

The numerical factor in (2.7) will follow from our subsequent arguments. The structure of this relation suggests a connection between the calculation of $Z$ and the theory of orthogonal polynomials, as discussed by Bessis[8] and Parisi.[9] This goes as follows. First define $\bar g_p=g_p/N^{p-1}$ and, for the time being, let $\bar g_p$ be considered as fixed, real, and such that the measure

$$d\mu(\lambda)=d\lambda\,e^{-V(\lambda)}, \tag{2.9}$$

is integrable. Here $V(\lambda)$ stands for

$$V(\lambda)=\frac{\lambda^2}{2}+\sum_{p\geqslant 2}\bar g_p\lambda^{2p}. \tag{2.10}$$

We call $Z_n(g)$ what was previously called $Z(g)$ in the case $n=N$, i.e.,

$$Z_n(g)=\int d\mu(\lambda_1)\cdots d\mu(\lambda_n)[\Delta(\lambda_1,\ldots,\lambda_n)]^2, \tag{2.11}$$

and define the polynomial of degree $n$

$$\begin{aligned}P_n(\lambda)&=(-1)^n P_n(-\lambda)\\&=Z_n^{-1}(g)\int d\mu(\lambda_1)\cdots d\mu(\lambda_n)[\Delta(\lambda_1,\ldots,\lambda_n)]^2\\&\quad\times\prod_{s=1}^{n}(\lambda-\lambda_s).\end{aligned} \tag{2.12}$$

The term of highest degree has a coefficient equal to 1. The polynomials $P_n(\lambda)$ are orthogonal with respect to the measure $d\mu(\lambda)$. Indeed,

$$Z_n\int d\mu(\lambda_{n+1})P_n(\lambda_{n+1})\lambda_{n+1}^{\,s}$$

$$= \int \prod_1^{n+1} d\mu(\lambda_k) \Delta(\lambda_1,...,\lambda_{n+1}) \Delta(\lambda_1,...,\lambda_n) \lambda_{n+1}^s$$

$$= \frac{1}{n+1} \int \prod_1^{n+1} d\mu(\lambda_k) \Delta(\lambda_1,...,\lambda_{n+1})$$

$$\times \sum_{k=1}^{n+1} (-1)^{n+1-k} \lambda_k^s \Delta(\lambda_1,...,\hat{\lambda}_k,...,\lambda_{n+1}).$$

The sum inside the integrand is the expansion of the determinant

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} & \lambda_1^s \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} & \lambda_2^s \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_{n+1} & \cdots & \lambda_{n+1}^{n-1} & \lambda_{n+2}^s \end{vmatrix},$$

with respect to its last column. It vanishes for $s = 0,1,...,n-1$, which proves the assertion. For $s = n$, we find

$$Z_n \int d\mu(\lambda) P_n(\lambda) \lambda^n = Z_n \int d\mu(\lambda) P_n^2(\lambda)$$

$$= Z_{n+1}/(n+1).$$

Hence

$$h_n = \int d\mu(\lambda) P_n^2(\lambda) = Z_{n+1}/(n+1) Z_n. \quad (2.13)$$

This relation shows that the knowledge of the orthogonal polynomials yields a handle on $Z$. A statistical interpretation can be given to $Z$ as a partition of a one-dimensional repulsive Coulomb gas of particles interacting with a potential $V$.

The polynomials $P_n$ satisfy a three-term recursion relation

$$\lambda P_n = P_{n+1} + R_n P_{n-1}. \quad (2.14)$$

Since

$$h_{n+1} = \int d\mu(\lambda) P_{n+1} \lambda P_n$$

$$= \int d\mu(\lambda)(P_{n+2} + R_{n+1} P_n) P_n = R_{n+1} h_n,$$

we have

$$R_n = \frac{h_n}{h_{n-1}} = \frac{n}{n+1} \frac{Z_{n+1} Z_{n-1}}{Z_n^2}. \quad (2.15)$$

Consequently,

$$Z_n = n!\, h_{n-1} h_{n-2} \cdots h_1 h_0 = n!\, R_{n-1} R_{n-2}^2 \cdots R_1^{n-1} h_0^n, \quad (2.16)$$

with

$$Z_1 = h_0 = \int d\mu(\lambda). \quad (2.17)$$

Incidentally, this provides a justification for the factor occurring in Eq. (2.7). For choose there $f(M) = \exp(-\frac{1}{2}\operatorname{tr} M^2)$. The left-hand side is equal to $(2\pi)^{N^2/2}$. On the right-hand side, the integral is $Z_N(0)$, corresponding to Hermite polynomials with the measure $d\mu(\lambda) = e^{-\lambda^2/2} d\lambda$. In this case, $h_N = (2\pi)^{1/2} N!$ and hence $Z_N = N!\, h_{N-1} h_{N-2} \cdots h_0 = \Pi_1^N p!(2\pi)^{N/2}$. The factor in the right-hand side of (2.7) is just fitted to match these two results.

The preceding development follows from the standard textbook treatment. We now use an argument due to Bessis and Parisi to obtain a recursion formula on $R_n$. From Eqs. (2.13) and (2.14), it follows that

$$n\, h_n = \int d\lambda\, e^{-V} \lambda P_n' P_n$$

$$= \int d\lambda\, e^{-V} P_n'(P_{n+1} + R_n P_{n-1})$$

$$= R_n \int d\lambda\, e^{-V} P_n' P_{n-1}$$

$$= R_n \int d\lambda\, e^{-V} V' P_n P_{n-1},$$

where an integration by parts has been used to obtain the last equality. Now

$$\int d\lambda\, e^{-V} P_n (V' P_{n-1})$$

$$= \int d\lambda\, e^{-V} P_n \Big[\lambda + \sum_{p \geqslant 1} 2(p+1) \bar{g}_{p+1} \lambda^{2p+1}\Big] P_{n-1}$$

$$= h_n \Big[1 + \sum_{p \geqslant 1} 2(p+1) \bar{g}_{p+1} \sum_{\text{paths}} R_{\alpha_1} \cdots R_{\alpha_p}\Big]. \quad (2.19)$$

In this expression the coefficient of $2(p+1)\bar{g}_{p+1}$ is a sum over the $(2p+1)!/p!(p+1)!$ paths along a "staircase" from the stair at height $n-1$ to the one at height $n$, in $2p+1$ steps of one unit, $p+1$ up, $p$ down. A factor $R_\alpha$ occurs when descending from the stair $\alpha$ down to stair $\alpha - 1$. For instance, we have

$$p = 1 \quad \sum_{\text{paths}} = R_{n-1} + R_n + R_{n+1},$$

$$p = 2 \quad \sum_{\text{paths}} = R_{n-2} R_{n-1} + R_{n-1}^2 + 2R_{n-1} R_n$$

$$+ R_{n-1} R_{n+1} + R_n^2 + 2 R_n R_{n+1}$$

$$+ R_{n+1}^2 + R_{n+1} R_{n+2}, \quad (2.20)$$

and so on. We can, of course, express this result in terms of the $(n,n-1)$ matrix element of the Jacobi matrix $\lambda$ in Eq. (2.14) raised to the power $(2p+1)$. Inserting this expression into (2.19) yields:

$$n = R_n \Big(1 + \sum_{p \geqslant 1} 2(p+1)\, \bar{g}_{p+1} \sum_{\text{paths}} R_{\alpha_1} \cdots R_{\alpha_p}\Big). \quad (2.21)$$

Since we are only interested here in the leading term of $Z(g)$, we shall only use the dominant estimate of $R_n$ for $n$ of order $N$. From Eq. (2.21), $R_n$ is of order $N$, and we set

$$x = n/N, \quad R_n = N a^2(x). \quad (2.22)$$

This entails for $\bar{g}_{p+1} = g_{p+1}/N^p$,

$$x = a^2(x) + \sum_{p \geqslant 2} g_p \frac{(2p)!}{p!(p-1)!} a^{2p}(x). \quad (2.23)$$

The quantity of interest, namely the generating function for the number of planar diagrams, is $E_{(0)}(g)$ given by Eqs. (2.4), (2.5). Dropping the index (0), we find

$$E(g) = \lim_{N \to \infty} -\frac{1}{N^2} \Big[\sum_1^{N-1} (N-n) \ln\Big(\frac{R_n(g)}{R_n(0)}\Big)$$

$$+N\ln\left(\frac{h_0(g)}{h_0(0)}\right)\Big]$$
$$= -\int_0^1 dx\,(1-x)\ln\left(\frac{a^2(x)}{x}\right). \tag{2.24}$$

To analyze these relations, we define the functions

$$v(\lambda) = \frac{\lambda^2}{2} + \sum_{p\geqslant 2} g_p \lambda^{2p},$$
$$w(\lambda) = \lambda^2 + \sum_{p\geqslant 2} \frac{(2p)!}{p!(p-1)!} \lambda^{2p}. \tag{2.25}$$

They are related through

$$w(\lambda) = \frac{1}{2\pi}\int_{-2\lambda}^{2\lambda} d\xi\,(4\lambda^2-\xi^2)^{1/2} v''(\xi),$$
$$\frac{v(\lambda)}{\lambda} = \int_0^{\lambda/2} \frac{d\xi}{(\lambda^2/4-\xi^2)^{1/2}}\frac{w(\xi)}{\xi}. \tag{2.26}$$

$E(g)$ can be written as

$$E(g) = \int_0^a d\lambda w'(\lambda)[1-w(\lambda)]\ln\left(\frac{w(\lambda)}{\lambda^2}\right)$$
$$= -\tfrac{1}{2}\ln a^2 + \int_0^a \frac{d\lambda}{\lambda}(2-w)w - (g=0), \tag{2.27}$$

with $a\equiv a(1)$ defined through

$$1 = w(a) = \int_{-2a}^{2a} d\xi \frac{(4a^2-\xi^2)^{1/2}}{2\pi} v''(\xi). \tag{2.28}$$

These expressions coincide, of course, with those given in [A] for the quartic potential. One can note that the condition $w(a)=1$ follows from a variational principle. If $a$ is left arbitrary in (2.27) without the subtraction term, then the relation between $a$ and $g$ expresses the stationnarity of $E$ with respect to $a$.

The preceding development avoids completely the use of the saddle-point method as presented in [A]. Nevertheless, it contains implicitly the asymptotic distribution of eigenvalues of the matrix $M$. We recall that the original eigenvalues have been scaled down by a factor $\sqrt{N}$ to obtain the reduced variables. The density of eigenvalues, i.e., the distribution of roots of the polynomial $P_n(\lambda)$, is readily related to the Jacobi matrix for $\lambda$ in a basis of orthonormal states

$$\mathscr{P}_n(\lambda) = \frac{1}{\sqrt{h_n}} P_n(\lambda). \tag{2.29}$$

From (2.14) and (2.15)

$$\lambda\mathscr{P}_n(\lambda) = \left(\frac{h_{n+1}}{h_n}\right)^{1/2}\mathscr{P}_{n+1} + R_n\left(\frac{h_{n-1}}{h_n}\right)^{1/2}\mathscr{P}_{n-1}$$
$$= \sqrt{R_{n+1}}\,\mathscr{P}_{n+1} + \sqrt{R_n}\,\mathscr{P}_{n-1}. \tag{2.30}$$

Consider now the quantity

$$\lim_{N\to\infty}\frac{1}{N^{p+1}}\int d\mu(\lambda)\sum_0^{N-1}\mathscr{P}_n(\lambda)\lambda^{2p}\mathscr{P}_n(\lambda) = \langle\lambda^{2p}\rangle. \tag{2.31}$$

In the reduced variables ($\lambda\to\lambda/\sqrt{N}$), we look for a positive density $u(\lambda)$ such that

$$\langle\lambda^{2p}\rangle = \int d\lambda\,u(\lambda)\lambda^{2p} = \int_0^1 dx\,a^{2p}(x)\sum_{q=0}^{p}\left[\binom{p}{q}\right]^2, \tag{2.32}$$

where the last expression results in this limit $N\to\infty$ from Eq. (2.30), using notations introduced in (2.22). This relation takes the form

$$\int d\lambda\,u(\lambda)\lambda^{2p}$$
$$= \int_0^1 dx\,a^{2p}(x)\int_{-1}^1 \frac{dy}{\pi}\frac{(2y)^{2p}}{(1-y^2)^{1/2}}$$
$$= \int_{-2a(1)}^{2a(1)} d\lambda\,\lambda^{2p}\int_{|\lambda|/2}^{a(1)}\frac{d\mu}{\pi}\frac{w'(\mu)}{(4\mu^2-\lambda^2)^{1/2}},$$

and yields an even measure concentrated on the interval $(-2a,2a)$, where $a\equiv a(1)$, equal to

$$u(\lambda) = \frac{1}{\pi}\int_{|\lambda|/2}^{a} d\xi\frac{w'(\xi)}{(4\xi^2-\lambda^2)^{1/2}}$$
$$= \frac{(4a^2-\lambda^2)^{1/2}}{\pi}\frac{1}{2\pi}\fint_{-2a}^{2a}\frac{d\eta}{(4a^2-\eta^2)^{1/2}}\frac{v'(\eta)}{\eta-\lambda}. \tag{2.33}$$

This gives a distribution of the form $(1/\pi)(4a^2-\lambda^2)^{1/2}$ times a polynomial in $\lambda$, equal for $\lambda^2<4a^2$ to the real part of an even analytic function

$$\frac{1}{2\pi}\int_{-2a}^{2a}\frac{d\eta}{(4a^2-\eta^2)^{1/2}}\frac{v'(n)}{\eta-\lambda},$$

vanishing faster than $1/\lambda$ as $|\lambda| \to \infty$, and with a discontinuity on the finite interval $\lambda^2 > 4a^2$ equal to $iv'(\lambda)(4a^2 - \lambda^2)^{-1/2}$. The condition $w(a) = 1$, is equivalent to the statement $\int_{-2a}^{2a} u(\lambda)d\lambda = 1$. This reproduces, of course, the result for the quartic interactions given in [A]. The extension of the previous analysis to include functions $v(\lambda)$ not necessarily even is, of course, possible. One can also proceed[8] to the systematic study of the corrections in powers of $1/N$, starting from the exact expression (2.21).

## 3. INTEGRATION OVER THE UNITARY GROUP

We return to the investigation of integrals of the type (1.2) over several $N \times N$ Hermitian matrices, in fact, to the simplest one involving two matrices

$$Z = \int dM_1\, dM_2 \exp\{ -[V(M_1) + V(M_2) - \beta \operatorname{tr}(M_1 M_2)]\}. \tag{3.1}$$

As explained in the Introduction it is important to integrate first over the angular variables. We are therefore led to study the expression

$$I(M_1,M_2;\beta) = \int dU \exp[\beta \operatorname{tr}(M_1 U M_2 U^\dagger)], \tag{3.2}$$

where $dU$ is the normalized Haar measure on the unitary group U($N$). We can, in fact, restrict the integration to SU($N$) since this is the only part which acts effectively on the Hermitian matrices $M$ in the adjoint representation. If $\Lambda_1$ and $\Lambda_2$ stand for the diagonal matrices of eigenvalues of $M_1$ and $M_2$ as in (2.6), $I$ depends only on $\Lambda_1$ and $\Lambda_2$ and is, in fact, a symmetric function of each set. Then $Z$ reduces to the form

$$Z = \frac{(2\pi)^{N(N-1)}}{(\Pi_1^N p!)^2} \int \prod^N d\lambda_{1,i} d\lambda_{2,i} \Delta^2(\Lambda_1)\Delta^2(\Lambda_2) \times \exp\{-[V(\Lambda_1) + V(\Lambda_2)]\} I(\Lambda_1,\Lambda_2), \tag{3.3}$$

due to the invariance of the measure $dM\, e^{-V(M)}$ under unitary transformations.

We will now show that

$$I(\Lambda_1,\Lambda_2;\beta) = \beta^{-N(N-1)/2} \prod_1^{N-1} p!\, \frac{\det(e^{\beta\lambda_{1,i}\lambda_{2,j}})}{\Delta(\Lambda_1)\Delta(\Lambda_2)}. \tag{3.4}$$

Let $D$ be the unitary invariant Laplacian operator on Hermitian matrices

$$D \equiv \sum_i \frac{\partial^2}{\partial M_{ii}^2} + \frac{1}{2}\sum_{i<j}\left[\frac{\partial^2}{(\partial \operatorname{Re} M_{ij})^2} + \frac{\partial^2}{(\partial \operatorname{Im} M_{ij})^2}\right]. \tag{3.5}$$

Consider the propagator

$$f(t;M_1,M_2) = \langle M_1 | e^{-tD/2} | M_2\rangle = \frac{1}{(2\pi t)^{N^2/2}} \exp\left[-\frac{1}{2t}\operatorname{tr}(M_1 - M_2)^2\right], \tag{3.6}$$

a solution for $t$ positive of the heat equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}D_1\right) f(t;M_1,M_2) = 0, \tag{3.7}$$

which reduces when $t \to 0$ to a $\delta$ function with respect to the measure introduced above. If $g(t,M)$ is a solution of the above equation for $t > 0$, which coincides for $t = 0$ with a given function $g(M)$ invariant under unitary transformations, i.e., a symmetric function of the eigenvalues of $M$, then

$$g(t,\Lambda_1) = C\int dU \int d\Lambda_2\, \Delta^2(\Lambda_2) f(t;\Lambda_1, U\Lambda_2 U^\dagger) g(\Lambda_2). \tag{3.8}$$

The constant $C$ corresponds to the value appearing on the right-hand side of Eq. (2.7)

$$C = (2\pi)^{N(N-1)/2} \prod_1^N p!. \tag{3.9}$$

Consequently,

$$\Delta(\Lambda_1) g(t,\Lambda_1) = \int d\Lambda_2\, K(t;\Lambda_1,\Lambda_2)[\Delta(\Lambda_2) g(\Lambda_2)], \tag{3.10}$$

$$K(t;\Lambda_1,\Lambda_2) = C\Delta(\Lambda_1)\Delta(\Lambda_2)\int dU f(t;\Lambda_1, U\Lambda_2 U^\dagger),$$

which means that $K$ is the evolution kernel for antisymmetric functions of the form

$$\xi(\Lambda) = \Delta(\Lambda) g(\Lambda). \tag{3.11}$$

The function $\xi$ satisfies the equation

$$\begin{aligned}\frac{\partial \xi}{\partial t} &= \frac{1}{2}\left(\frac{1}{\Delta(\Lambda)}\sum_{i=1}^N \frac{\partial}{\partial\lambda_i}\Delta^2(\Lambda)\frac{\partial}{\partial\lambda_i}\frac{\xi}{\Delta(\Lambda)}\right)\\ &= \frac{1}{2}\sum_i\left(\frac{\partial}{\partial\lambda_i} + \sum_{k\neq i}\frac{1}{\lambda_i - \lambda_k}\right)\left(\frac{\partial}{\partial\lambda_i} - \sum_{k\neq i}\frac{1}{\lambda_i - \lambda_k}\right)\xi\\ &= \frac{1}{2}\sum_i \frac{\partial^2\xi}{\partial\lambda_i^2} - \sum_{k\neq l\neq i}\frac{1}{\lambda_i - \lambda_k}\frac{1}{\lambda_i - \lambda_l}\xi.\end{aligned} \tag{3.12}$$

The last sum vanishes owing to the identity

$$\frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{1}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} + \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} = 0.$$

Therefore, $\xi$ fulfills

$$\frac{\partial\xi}{\partial t} = \frac{1}{2}\sum_{i=1}^N \frac{\partial^2}{\partial\lambda_i^2}\xi,$$

and is required to be antisymmetric. The kernel $K$ of the corresponding evolution is then

$$\begin{aligned}K(t;\Lambda_1,\Lambda_2) &= \frac{1}{(2\pi t)^{N/2}}\frac{1}{N!}\sum_{\mathscr{P}}(-1)^{\mathscr{P}} \times \exp\left[-\frac{1}{2t}\sum_i(\lambda_{1,i} - \lambda_{2,\mathscr{P}_i})\right]\\ &= \frac{1}{(2\pi t)^{N/2}}\frac{1}{N!}\det\left\{\exp\left[-\frac{1}{2t}(\lambda_{1,i} - \lambda_{2,j})^2\right]\right\}.\end{aligned} \tag{3.13}$$

If we compare this with (3.10) and (3.6), we find

$$\int dU \exp\left[-\frac{1}{2t}\operatorname{tr}(\Lambda_1 - U\Lambda_2 U^\dagger)^2\right] = t^{N(N-1)/2}\prod_1^N p!\,\frac{\det[\exp - (1/2t)(\lambda_{1,i} - \lambda_{2,j})^2]}{\Delta(\Lambda_1)\Delta(\Lambda_2)}, \tag{3.14}$$

a formula equivalent to (3.4). The reader will recognize in this derivation the features that made the planar approximation to quantum mechanics very simple in terms of fermionic wavefunctions (see [A]).

Let us now derive a series expansion for $I(M_1,M_2;\beta)$ in terms of the characters of the linear (or unitary) group, using a device due to Weyl. We recall that the irreducible representations of the group $U(N)$ are characterized by a sequence of nondecreasing integers $n_0 \leqslant n_1 \leqslant \cdots n_{N-1}$, which for $n_0 \geqslant 0$ can be attached to a Young tableau.[10] We will consider here only polynomial representations, i.e., those such that the group factor $U(N)/SU(N)$ is represented by $(\det U)^{n_0}$, $n_0 \geqslant 0$. The complete set of representations is obtained by relaxing the positivity condition on $n_0$. Let $U \to \mathscr{D}^{\{n\}}_{aa'}(U)$ be the corresponding representation with character $\chi_{\{n\}}$

$$\chi_{\{n\}}(U) = \sum_a \mathscr{D}^{\{n\}}_{aa}(U) = \frac{\det(\delta_i^{\,n_j + j})}{\det(\delta_i^{\,j})}, \tag{3.15}$$

where $\delta_i$ are the eigenvalues of $U$. Denote by $d_{\{n\}}$ the dimension of this representation:

$$d_{\{n\}} = \chi_{\{n\}}(I). \tag{3.16}$$

Let first restrict our attention to the $SU(N)$ group. This is, anyhow, the only part that enters the integral (3.2). In this case $n_0 = 0$. We have the orthogonality and completeness relations

$$\int dU\, \mathscr{D}^{\{n\}}_{a_1 a_1'}(U)\mathscr{D}^{\{n'\}*}_{a_2 a_2'}(U) = \frac{1}{d_{\{n\}}}\delta^{\{n\},\{n'\}}\delta_{a_1 a_2}\delta_{a_1' a_2'},$$
$$\sum_{\{n\},a,a'} d_{\{n\}}\mathscr{D}^{\{n\}}_{aa'}(U)\mathscr{D}^{\{n\}*}_{aa'}(U') = \delta(U,U'). \tag{3.17}$$

Let $U_1$ and $U_2$ stand for two arbitrary elements in $SU(N)$. We have

$$\int dU \exp(\beta \operatorname{tr} U_1 U U_2 U^\dagger)$$
$$= \int dV e^{\beta \operatorname{tr} V} \int dU \delta(V, U_1 U U_2 U^\dagger)$$
$$= \int dV e^{\beta \operatorname{tr} V} \sum_{\{n\},a,a'} \mathscr{D}^{\{n\}}_{aa'}(U_1)\chi_{\{n\}}(U_2)\mathscr{D}^{\{n\}*}_{aa'}(V).$$

The integral over $V$ being invariant under the adjoint action $V \to UVU^\dagger$, we can replace $\mathscr{D}^{\{n\}*}_{aa'}(V)$ by $(\delta_{aa'}/d_{\{n\}})\chi^*_{\{n\}}(V)$. Now

$$\int dV \operatorname{tr} V)^p \chi^*_{\{n\}}(V) = \delta_{p,|n|}\sigma_{\{n\}}, \tag{3.18}$$

where $|n| = \Sigma_0^{N-1} n_i$ and $\sigma_{\{n\}}$ is the number of times the representation $\mathscr{D}^{\{n\}}(U)$ occurs in the tensor product $\overset{|n|}{\otimes} U$. This can also be interpreted as the number of distinct ways of constructing piece by piece the Young tableau for the representation $\mathscr{D}^{\{n\}}$ while respecting the rules for such tableaux. (Therefore $\sigma_{\{n\}}$ is nothing but the dimension of the representation of the permutation group on $|n|$ objects pertaining to the same tableau. For a proof see Ref. 10.) It follows that

$$= \sum_{\{n\}} \frac{\beta^{|n|}}{|n|!}\frac{\sigma_{\{n\}}}{d_{\{n\}}}\chi_{\{n\}}(U_1)\chi_{\{n\}}(U_2). \tag{3.19}$$

This result has been derived for $U_1$, $U_2 \in SU(N)$, and the sum on the right-hand side runs only over representations with $n_0 = 0$. It can readily be extended to $U_1$, $U_2 \in U(N)$ provided we reintroduce all representations with $n_0 \geqslant 0$. $\chi_{\{n\}}(U)$ is a polynomial in the matrix elements of $U$ and therefore can be continued as a function of an arbitrary $N \times N$ matrix. By analytic continuation, we therefore reach the conclusion that

$$I(M_1,M_2;\beta) = \sum_{\{n\}} \frac{\beta^{|n|}}{|n|!}\frac{\sigma_{\{n\}}}{d_{\{n\}}}\chi_{\{n\}}(M_1)\chi_{\{n\}}(M_2). \tag{3.20}$$

A similar formula could in fact be directly obtained by expanding the numerator of the right-hand side of Eq. (3.4) in powers of the eigenvalues. Comparison with (3.20) yields

$$\sigma_{\{n\}} = |n|!\, d_{\{n\}} \prod_0^{N-1} p! \Big/ \prod_0^{N-1} (n_p + p)!, \tag{3.21}$$

and we recall the Weyl formula

$$d_{\{n\}} = \Delta(n_{N-1} + N - 1, n_{N-2} + N - 2, \ldots, n_0) \Big/ \prod_0^{N-1} p!,$$

where $\Delta$ is the discriminant used througout our previous discussion. Table I gives explicit formulas for the characters up to $|n| = 4$ in terms of traces of powers of the matrix $M$. We check, of course, that

$$d_{\{n\}} = \chi_{\{n\}}(I) \quad \text{and} \quad (\operatorname{tr} M)^p = \sum_{\substack{\{n\} \\ |n| = p}} \sigma_{\{n\}}\chi_{\{n\}}(M). \tag{3.22}$$

TABLE I. Characters of the linear group up to $|n| = 4$.

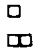
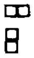
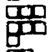
| Young tableau | $\chi_{\{n\}}(A)$ | $d_{\{n\}}$ | $\sigma_{\{n\}}$ |
|---|---|---|---|
|  | $\operatorname{tr}A$ | $N$ | 1 |
|  | $\frac{1}{2}[(\operatorname{tr}A)^2 + \operatorname{tr}A^2]$ | $\frac{1}{2}N(N+1)$ | 1 |
|  | $\frac{1}{2}[(\operatorname{tr}A)^2 - \operatorname{tr}A^2]$ | $\frac{1}{2}N(N-1)$ | 1 |
|  | $\frac{1}{6}[(\operatorname{tr}A)^3 + 2\operatorname{tr}A^3 + 3\operatorname{tr}A\operatorname{tr}A^2]$ | $\frac{1}{6}N(N+1)(N+2)$ | 1 |
|  | $\frac{1}{3}[(\operatorname{tr}A)^3 - \operatorname{tr}A^3]$ | $\frac{1}{3}N(N+1)(N-1)$ | 2 |
|  | $\frac{1}{6}[(\operatorname{tr}A)^3 + 2\operatorname{tr}A^3 - 3\operatorname{tr}A\operatorname{tr}A^2]$ | $\frac{1}{6}N(N-1)(N-2)$ | 1 |
|  | $\frac{1}{24}[(\operatorname{tr}A)^4 + 6\operatorname{tr}A^4 + 3(\operatorname{tr}A^2)^2 + 6\operatorname{tr}A^2(\operatorname{tr}A)^2 + 8\operatorname{tr}A^3\operatorname{tr}A]$ | $\frac{1}{24}N(N+1)(N+2)(N+3)$ | 1 |
|  | $\frac{1}{8}[(\operatorname{tr}A)^4 - 2\operatorname{tr}A^4 - (\operatorname{tr}A^2)^2 + 2\operatorname{tr}A^2(\operatorname{tr}A)^2]$ | $\frac{1}{8}N(N+1)(N+2)(N-1)$ | 3 |
|  | $\frac{1}{12}[(\operatorname{tr}A)^4 - 4\operatorname{tr}A^3\operatorname{tr}A + 3(\operatorname{tr}A^2)^2]$ | $\frac{1}{12}N^2(N+1)(N-1)$ | 2 |
|  | $\frac{1}{8}[(\operatorname{tr}A)^4 + 2\operatorname{tr}A^4 - (\operatorname{tr}A^2)^2 - 2\operatorname{tr}A^2(\operatorname{tr}A)^2]$ | $\frac{1}{8}N(N+1)(N-1)(N-2)$ | 3 |
|  | $\frac{1}{24}[(\operatorname{tr}A)^4 - 6\operatorname{tr}A^4 + 3(\operatorname{tr}A^2)^2 - 6\operatorname{tr}A^2(\operatorname{tr}A)^2 + 8\operatorname{tr}A^3\operatorname{tr}A]$ | $\frac{1}{24}N(N-1)(N-2)(N-3)$ | 1 |

TABLE II. The coefficients $X_k(A,B)$.

| $k$ | $X_k(A,B)$ |
|---|---|
| 1 | $\langle A\rangle\langle B\rangle$ |
| 2 | $f_2(A)f_2(B)$ |
| 3 | $f_3(A)f_3(B)$ |
| 4 | $f_4(A)f_4(B)-4\frac{f_2^2(A)}{2}\frac{f_2^2(B)}{2}$ |
| 5 | $f_5(A)f_5(B)-5f_2(A)f_3(A)f_2(B)f_3(B)$ |
| 6 | $f_6(A)f_6(B)-6\Big\{f_4(A)f_2(A)\Big[f_4(B)f_2(B)+\frac{f_3^2(B)}{2}\Big]$ |
| | $+\frac{f_3^2(A)}{3!}\Big[f_4(B)f_2(B)+2\frac{f_3^2(B)}{2!}+2\frac{f_2^3(B)}{3!}\Big]$ |
| | $+\frac{f_3^2(A)}{3!}\Big[2\frac{f_2^3(B)}{2!}-12\frac{f_2^3(B)}{3!}\Big]\Big\}$ |
| 7 | $f_7(A)f_7(B)-7\Big\{f_5(A)f_2(A)[f_5(B)f_2(B)+f_4(B)f_3(B)]$ |
| | $+f_4(A)f_3(A)\Big[f_5(B)f_2(B)+2f_4(B)f_3(B)+2f_3(B)\frac{f_2^2(B)}{2!}\Big]$ |
| | $+f_3(A)\frac{f_2^2(A)}{2!}\Big[2f_4(B)f_3(B)-12f_3(B)\frac{f_2^2(B)}{2!}\Big]\Big\}$ |
| 8 | $f_8(A)f_8(B)$ |
| | $-8\Big\{f_6(A)f_2(A)\Big[f_6(B)f_2(B)+f_5(B)f_3(B)+\frac{f_4^2(B)}{2!}\Big]$ |
| | $+f_5(A)f_3(A)\Big[f_6(B)f_2(B)+2f_5(B)f_3(B)+2\frac{f_4^2(B)}{2!}+2\frac{f_3^2(B)}{2!}f_2(B)+2f_4(B)\frac{f_2^2(B)}{2}\Big]$ |
| | $+\frac{f_4^2(A)}{2!}\Big[f_6(B)f_2(B)+2f_5(B)f_3(B)+3\frac{f_4^2(B)}{2!}+4\frac{f_3^2(B)}{2!}f_2(B)+2f_4(B)\frac{f_2^2(B)}{2!}+6\frac{f_2^4(B)}{4!}\Big]$ |
| | $+\frac{f_3^2(A)}{2!}f_2(A)\Big[2f_5(B)f_3(B)+4\frac{f_4^2(B)}{2!}-20\frac{f_3^2(B)}{2!}f_2(B)-12f_4(B)\frac{f_2^2(B)}{2!}-48\frac{f_2^4(B)}{4!}\Big]$ |
| | $+f_4(A)\frac{f_2^2(A)}{2!}\Big[2f_5(B)f_3(B)+2\frac{f_4^2(B)}{2!}-12\frac{f_3^2(B)}{2!}f_2(B)-14f_4(B)\frac{f_2^2(B)}{2!}\Big]$ |
| | $+\frac{f_4^2(A)}{4!}\Big[6\frac{f_4^2(B)}{2!}-48\frac{f_3^2(B)}{2!}f_2(B)+360\frac{f_2^4(B)}{4!}\Big]\Big\}$ |

We have now two exact expressions for the kernel $I(M_1,M_2;\beta)$ given in Eqs. (3.4) and (3.20). For our purpose, we are also interested in an expansion of $\ln[I(M_1,M_2;\beta)]$ for large $N$ assuming the eigenvalues of $M_1$ and $M_2$ to be of order $N^{1/2}$. Without loss of generality we take $M_1$ and $M_2$ diagonal and rescale them as $M_1=\sqrt{N}A$ and $M_2=\sqrt{N}B$ with $A$ and $B$ of order unity. We look for the dominant term

$$X(A,B;\beta)=\lim_{N\to\infty}\frac{1}{N^2}\ln\big[I(\sqrt{N}\,A,\sqrt{N}\,B,\beta)\big]$$
$$=\lim_{N\to\infty}\frac{1}{N^2}\ln\Big[\int dU\,e^{N\beta\,\mathrm{tr}(AUBU^\dagger)}\Big], \tag{3.23}$$

with

$$A=\begin{pmatrix}a_1 & & 0\\ & \ddots & \\ 0 & & a_n\end{pmatrix},\quad B=\begin{pmatrix}b_1 & & 0\\ & \ddots & \\ 0 & & b_n\end{pmatrix}.$$

The quantity $X$ admits a series expansion in powers of $\beta$

$$X(A,B;\beta)=\sum_1^\infty\frac{\beta^k}{k}X_k(A,B), \tag{3.24}$$

where $X_k(A,B)=X_k(B,A)$ is a symmetric function of the $a_i$ and $b_i$, homogeneous of degree $k$. It is given in terms of the quantities $\langle A^p\rangle\equiv(1/N)\,\mathrm{tr}A^p$, $\langle B^p\rangle$. By singling out $\langle A\rangle$ which can readily be factored in $I$, it will be useful to use rather the mean values $e_{(n)}(A)$, $e_n(B)$:

$$e_0(A)=1,\quad e_1\equiv 0,$$
$$e_p(A)=\langle(A-\langle A\rangle)^p\rangle$$
$$=\frac{1}{N}\mathrm{tr}\Big(A-\frac{1}{N}\mathrm{tr}A\Big)^p,\quad p\geqslant 2, \tag{3.25}$$

or even better the "connected" ones $f_n(A)$, $f_n(B)$. The relation between these two basis, already discussed in [A] for Green's functions is most easily expressed through the generating functions

$$\phi(j;A)=1+\sum_2^\infty j^k e_k(A),$$
$$\psi(z;A)=1+\sum_2^\infty z^k f_k(A),$$
$$\phi(j;A)=\psi(z[j;A];A),$$
$$z[j;A]=j\phi(j;A), \tag{3.26}$$

or, more explicitly,

$$e_k(A)=\sum_{\substack{\{r_q\geqslant 0\}\\ \Sigma qr_q=k}}\frac{k!}{(k+1-\Sigma r_q)!}\frac{[f_2(A)]^{r_2}}{r_2!}\frac{[f_3(A)]^{r_3}}{r_3!}\cdots,$$
$$f_k(A)=-\sum_{\substack{\{r_q\geqslant 0\}\\ \Sigma qr_q=k}}\frac{(k+\Sigma r_q-2)!}{(k-1)!}\frac{[-e_2(A)]^{r_2}}{r_2!}\times\frac{[-e_3(A)]^{r_3}}{r_3!}\cdots. \tag{3.27}$$

We can start grinding the coefficients $X_k(A,B)$ using the expansion given in (3.20). The results up to order 8 are displayed in Table II. To expose some properties of this expansion, we shall write for $X$ differential equations similar to those discussed at the beginning of this section. Let $X^{(N)}$ be equal to $(1/N^2)\ln[I(\sqrt{N}A,\sqrt{N}B;\beta)]$, i.e., to the same quantity as $X$ before going to the limit $N\to\infty$. We have

$$e^{N^2X^{(N)}}=\frac{\Pi_1^{N-1}p!}{(\beta N)^{N(N-1)/2}}\frac{\det(e^{N\beta a_ib_j})}{\Delta(A)\Delta(B)}. \qquad (3.28)$$

The quantity

$$[\Delta(A)e^{N^2X^{(N)}}]^{-1}\sum_{i=1}^N\left(\frac{\partial}{\partial a_i}\right)^p\Delta(A)e^{N^2X^{(N)}},$$

is obviously equal to $N^p\beta^p\Sigma b_i^p$ and we therefore derive the identity

$$\beta^p\langle B^p\rangle=\frac{1}{\Delta(A)}\frac{1}{N}\sum_i\left(\frac{1}{N}\frac{\partial}{\partial a_i}+N\frac{\partial X^{(N)}}{\partial a_i}\right)^p\Delta(A).$$

Since $N\partial X^{(N)}/\partial a_i$ is of order unity, we may omit in the large-$N$ limit the action of derivatives on it, when expanding this $p$th power. To leading order,

$$\beta^p\langle B^p\rangle=\sum_{s=1}^p\frac{p!}{s!(p-s)!}\frac{1}{N^{p-s+1}}\times\sum_{i\neq j_1\neq\cdots\neq j_{p-s}}\frac{(N\partial X/\partial a_i)^s}{(a_i-a_{j_1})\cdots(a_i-a_{j_{p-s}})}, \qquad (3.29)$$

where the term $s=0$ is absent since $\Delta^{-1}\Sigma_i\partial_i^p\Delta\equiv0$ for $p<N$. For $p=1$ this yields

$$\beta\langle B\rangle=\frac{1}{N}\sum_i N\frac{\partial X}{\partial a_i},$$

which means

$$X_1=\beta\langle A\rangle\langle B\rangle,$$
$$\frac{1}{N}\sum_i N\frac{\partial X_k}{\partial a_i}=0,\quad k>1. \qquad (3.30)$$

Thus, for $k>1$, $X_k$ which is a homogeneous function of $A$ of degree $k$ may be written in terms of the $e_s(A)$, $s\leqslant k$, which all satisfy

$$\frac{1}{N}\sum_i N\frac{\partial e_s(A)}{\partial a_i}=0.$$

Defining $\tilde X$ through

$$X=\tilde X+\beta\langle A\rangle\langle B\rangle, \qquad (3.31)$$

we find

$$\beta^pe_p(B)=\sum_{s=1}^p\frac{p!}{s!(p-s)!}\frac{1}{N^{p-s+1}}\times\sum_{i\neq j_1\neq\cdots\neq j_{p-s}}\frac{(N\partial\tilde X/\partial a_i)^s}{(a_i-a_{j_1})\cdots(a_i-a_{j_{p-s}})}. \qquad (3.32)$$

This infinite set of equations determines the functions $X_k$ recursively. The algebra becomes rapidly quite cumbersome, and we did not succeed in finding a simple algorithm. We shall, however, indicate some simple features. Let us first focus on the first terms ($s=1$) of the right-hand side of Eq. (3.32). It reads

$$\Delta^{(p)}\tilde X\equiv\frac{p}{N^p}\sum_{i\neq j_1\neq\cdots\neq j_{p-1}}\frac{N\partial\tilde X/\partial a_i}{(a_i-a_{j_1})\cdots(a_i-a_{j_{p-1}})}. \qquad (3.33)$$

One may show that

$$\Delta^{(p)}e_k(A)=k\sum_{r_1+r_2+\cdots+r_p=k}e_{r_1}\cdots e_{r_p},\quad k=2,3,\cdots, \qquad (3.34)$$

where the right-hand side is zero for $k<p$. On the generating function $\phi(j;A)$ of Eq. (3.26)

$$\Delta^{(p)}\phi(j;A)=j\frac{\partial}{\partial j}(j^p\phi^p). \qquad (3.35)$$

Let us now show that $\Delta^{(p)}$ has a very simple action on the connected $f_k(A)$. Since it is a derivative, we have

$$\Delta^{(p)}\phi(j;A)=\Delta^{(p)}\psi(z[j;A];A)=\frac{\partial\psi}{\partial z}\Delta^{(p)}z[j;A]+\Delta^{(p)}\psi(z;A)\Big|_{z=z[j;A]},$$

but from (3.26)

$$\Delta^{(p)}z[j;A]=j\Delta^{(p)}\phi(j;A)$$

and

$$j\frac{\partial z}{\partial j}\left(1-j\frac{\partial\psi}{\partial z}\right)=j\frac{\partial z}{\partial j}\frac{\partial}{\partial z}(z-j\psi)+j\psi=z.$$

Hence

$$\Delta^{(p)}\psi(z;A)=\left(1-j\frac{\partial\psi}{\partial z}\right)\Delta^{(p)}\phi(j;A)=\left(1-j\frac{\partial\psi}{\partial z}\right)j\frac{\partial}{\partial j}(j\phi)^p=z\frac{\partial}{\partial z}z^p=pz^p, \qquad (3.36)$$

or equivalently

$$\Delta^{(p)}f_k(A)=p\delta_{kp}. \qquad (3.37)$$

This now suggests to rewrite Eq. (3.32) as

$$\beta^p\sum_{\substack{\{r_q\}\\ \Sigma qr_q=p}}\frac{p!}{(p+1-\Sigma r_q)!}\prod_{q\geqslant2}\frac{(f_q(B))^{r_q}}{r_q!}=\Delta^{(p)}\tilde X+\sum_{s=2}^p\binom{p}{s}\frac{1}{N^{p-s+1}}\times\sum_{i\neq j_1\neq\cdots\neq j_{p-s}}\frac{(N\partial\tilde X/\partial a_i)^s}{(a_i-a_{j_1})\cdots(a_i-a_{j_{p-s}})},$$

and so solve for $\tilde X$ according to its increasing degree in the $f_k(B)$. To lowest order (linear terms) one has

$$\Delta^{(p)}\tilde X^{(1)}=\beta^pf_p(B), \qquad (3.38)$$

and hence

$$\tilde X^{(1)}=\frac{\beta^p}{p}f_p(A)f_p(B)+\cdots,\quad p=2,3,\cdots,$$

where the triple dots stand for terms independent of $f_p(A)$. $\tilde X^{(1)}$ is necessarily of the form

$$\tilde X^{(1)}=\sum_{p=2}\frac{\beta^p}{p}f_p(A)f_p(B). \qquad (3.39)$$

$\beta^2$ can be extracted self-consistently from the above equations.

An alternative method of evaluation reveals the connection of our problem with matrix elements of the free evolution operator between Slater determinants, i.e., wavefunctions for large Fermi systems. To this end, we make the following change of variables:

$$\beta = \frac{1}{1+t/2}, \quad g_p = \frac{g'_p}{2(\frac{1}{2}+1/t)^p}, \qquad M_{1,2} = (\tfrac{1}{2}+1/t)^{1/2} M'_{1,2}. \tag{4.9}$$

In this way $Z(g,\beta)$ takes the form

$$Z(g,\beta) = (\tfrac{1}{2}+1/t)^{N^2}(2\pi t)^{N^2/2}\int dM\, e^{-V(M)}W(g',t), \tag{4.10}$$

$$W(g',t) = (2\pi t)^{-N^2/2}\left(\int dM\, e^{-V(M)}\right)^{-1}\int dM_1\, dM_2 \times\exp\{-\tfrac{1}{2}[V(M_1,g')+V(M_2,g')] - (1/2t)\,\mathrm{tr}(M_1-M_2)^2\}. \tag{4.11}$$

Henceforth we drop the prime on the coupling constants. After integration over the unitary group we have

$$\begin{aligned} W(g,t) &= K\frac{1}{(2\pi t)^{N/2}}\int d\Lambda_1\, d\Lambda_2\, \Delta(\Lambda_1)\Delta(\Lambda_2) \\ &\quad\times\exp[-\tfrac{1}{2}V(\Lambda_1)-\tfrac{1}{2}V(\Lambda_2)] \\ &\quad\times\det\left\{\exp\left[-\frac{1}{2t}(\lambda_{1,i}-\lambda_{2,j})^2\right]\right\} \\ &= K'\frac{1}{(2\pi t)^{N/2}}\int d\Lambda_1\, d\Lambda_2\, \det[\psi_k(\lambda_{1,l})] \\ &\quad\times\det\left\{\exp\left[-\frac{1}{2t}(\lambda_{1,i}-\lambda_{2,j})^2\right]\right\} \\ &\quad\times\det[\psi_m(\lambda_{2,n})], \end{aligned} \tag{4.12}$$

with constants $K$ and $K'$ independent of $t$ adjusted to insure that $W(g,0)=1$. We have introduced the orthonormal functions of Sec. 2

$$\psi_k(\lambda) = \mathscr{P}_k(\lambda)e^{-1/2V(\lambda)}, \tag{4.13}$$

with $V$ as in (2.10), and the normalized polynomials $\mathscr{P}_k$ defined in (2.29). The determinants are $N\times N$ with the index of the orthogonal functions running from 0 to $N-1$. Finally we find

$$\begin{aligned} W(g,t) &= \det\int d\lambda_1\, d\lambda_2\, \psi_k(\lambda_1) \\ &\quad\times\frac{\exp[-(\lambda_1-\lambda_2)^2/2t]}{(2\pi t)^{1/2}}\psi_l(\lambda_2) \\ &= \det(\langle k|e^{-ht}|l\rangle). \end{aligned} \tag{4.14}$$

Here $h$ is the free Hamiltonian

$$h = -\frac{1}{2}\frac{d^2}{d\lambda^2}, \tag{4.15}$$

$W(g,t)$ has been written as the matrix element of the free evolution operator in the ground state of $N$ "fermions" occupying the levels $\psi_0,\psi_1,\ldots,\psi_{N-1}$. In the large-$N$ limit, we define

$$\varphi(g,t) = -\lim_{N\to\infty}\frac{1}{N^2}\ln[W(g,t)] \equiv \sum_1^\infty\frac{(-1)^{n-1}}{n!}\varphi_n(g)t^n. \tag{4.16}$$

For $t\to\infty$, $\varphi$ behaves as $\frac{1}{2}\ln t$, while for $g=0$

$$\varphi(0,t) = \tfrac{1}{2}\ln(1+t/4). \tag{4.17}$$

Again, we are unable to obtain $\varphi(g,t)$ except as a power series in $t$. To see this, we introduce the projectors

$$P = \sum_0^{N-1}\psi_k\otimes\psi_k, \quad Q = I-P, \tag{4.18}$$

in the Hilbert space $\mathscr{L}^2(R)$, which enable us to express $W$ as an infinite determinant

$$W = \det(Q+Pe^{-ht}P). \tag{4.19}$$

We then decompose $h$ in a block form adapted to the $\psi_k$ basis:

$$h = PAP + PBQ + QB^\dagger P + QCQ. \tag{4.20}$$

Thus[11]

$$\begin{aligned} -\ln W &= -\mathrm{tr}\{\ln[I-P(I-e^{-ht})P]\} \\ &= t\,\mathrm{tr}A - \frac{t^2}{2!}\mathrm{tr}BB^\dagger + \frac{t^3}{3!}\mathrm{tr}B(CB^\dagger - B^\dagger A) \\ &\quad - \frac{t^4}{4!}\mathrm{tr}\{B[C(CB^\dagger - B^\dagger A) \\ &\quad - (CB^\dagger - B^\dagger A)A] - 2BB^\dagger BB^\dagger\} + \cdots. \end{aligned} \tag{4.21}$$

Except for the first, all terms in this series involve for large $N$ the matrix elements of $h$ close to the "Fermi level" $N$. To obtain $\varphi(g,t)$, we divide the above expression by $N^2$ and look for the limiting behavior. With an implicit limit sign, the first term reads

$$\begin{aligned} \varphi_1(g) &= \frac{1}{N^2}\mathrm{tr}A = \frac{1}{N^2}\sum_{k=0}^{N-1}\langle k|h|k\rangle \\ &= \frac{1}{N^2}\sum_{k=0}^{N-1}\frac{1}{2}\int d\lambda\left[\frac{d}{d\lambda}(e^{-1/2V(\lambda)}\mathscr{P}_k(\lambda))\right]^2 \\ &= \frac{1}{8N^2}\sum_{k=0}^{N-1}\int d\lambda\,(V'\psi_k)^2. \end{aligned} \tag{4.22}$$

With the notations of (2.33), it follows that

$$\varphi_1(g) = \tfrac{1}{8}\int_{-2a}^{2a} d\lambda\, u(\lambda)[v'(\lambda)]^2, \tag{4.23}$$

where $Nv(\lambda) = V(N^{1/2}\lambda)$. Explicitly for the quartic interaction

$$\begin{aligned} \varphi_1(g) &= \frac{1}{8}\int_{-2a}^{2a}\frac{d\lambda}{2\pi}(4a^2-\lambda^2)^{1/2} \\ &\quad\times(1+8ga^2+4g\lambda^2)(\lambda+4g\lambda^3)^2 \\ &= \frac{1}{8}\frac{(1-a^2)(4-a^2)}{36a^2}. \end{aligned} \tag{4.24}$$

The computation can be carried further. For instance, for the same interaction to second order in $t$, we find after tedious calculations

$$\varphi_2(g) = \frac{1}{32}\left\{1+\frac{1-a^2}{a^4}\left[1-\frac{1}{3}a^2+\left(\frac{1-a^2}{3}\right)^3\right]\right\}. \tag{4.25}$$

This program may be pursued order by order: The term $\tilde{X}^{(n)}$ of degree $n$ in $f(B)$ satisfies a set of equations

$$\Delta^{(p)}\tilde{X}^{(n)} = \mathscr{F}^{(p,n)}(\tilde{X}^{(1)},...\tilde{X}^{(n-1)},f(A),f(B)),$$

which may be integrated owing to (3.37). In particular a compact expression may be given to the terms quadratic in *both* $f(A)$ and $f(B)$. To summarize

$$X(A,B;B)$$

$$= \beta\langle A\rangle\langle B\rangle + \sum_{n=2}^{\infty}\beta^n\Big\{\frac{1}{n}f_n(A)f_n(B) - \sum_{\substack{\{r_p\geq 0\}\\ \Sigma pr_p=n\\ \Sigma r_p=2}}\sum_{\substack{\{s_q\geq 0\}\\ \Sigma qs_q=n\\ \Sigma s_q=2}}\Big[\prod_{p\geq 2}\frac{f_p^{r_p}(A)}{r_p!}\times\prod_{q\geq 2}\frac{f_q^{s_q}(B)}{s_q!}\min(p-1,q-1)\Big]+\cdots\Big\}, \qquad (3.40)$$

where $\min(p-1,q-1)$ runs over the indices $p$ or $q$ appearing in the term at hand, and the three dots stand for terms at least cubic in $f(A)$ or $f(B)$. Of course, this general expression coincides with the first few terms listed in Table II.

As a last remark, we observe that the Cauchy determinant

$$\det\Big(\frac{1}{1-x_i y_j}\Big) = \frac{\Delta(x)\Delta(y)}{\Pi_{i,j}(1-x_i y_j)}, \qquad (3.41)$$

can be used to obtain a reproducing kernel for $I(A,B;\beta)\sim\exp[N^2X(A,B;\beta)]$ in the form

$$I(A,B;\beta) = \frac{1}{N!}\oint\prod_1^N\Big(\frac{dz_k}{2\pi i z_k}\Big)\Delta(Z)\Delta(Z^{-1})\times\exp\Big\{N^2\sum_1^\infty\frac{1}{s}\langle Z^{-s}\rangle\langle A^s\rangle\Big\}I(Z,B;\beta),$$

where $Z$ is a diagonal matrix. The reader will recognize that the integral runs over the equivalence classes of the unitary group U($N$).

## 4. THE TWO-MATRIX PROBLEM

We now focus our attention on the quantity $Z$ of Eqs. (3.1) or (3.3) using the closed form obtained in (3.4) for the integral over the unitary group

$$Z = \int dM_1\,dM_2\exp[-V(M_1)-V(M_2)+\beta\,\mathrm{tr}M_1M_2]$$
$$= \frac{(2\pi)^{N(N-1)}}{N\Pi_1^N p!}\beta^{-N(N-1)/2}\int d\Lambda_1\,d\Lambda_2\,\Delta(\Lambda_1)\Delta(\Lambda_2)\times\exp[-V(\Lambda_1)-V(\Lambda_2)]\det[\exp(\beta\lambda_{1,i}\lambda_{2,j})], \quad (4.1)$$

with

$$V(\Lambda) = \frac{1}{2}\sum_i\lambda_i^2 + \sum_{p\geq 2}\frac{g_p}{N^{p-1}}\sum_i\lambda_i^{2p}. \qquad (4.2)$$

We can deal with this expression in two ways. The first one is a small $\beta$ expansion where we substitute in the exponent the series in $\beta$ discussed at the end of the previous section. The alternative strong coupling expansion will be presented afterwards.

Thus we write

$$Z = \frac{(2\pi)^{N(N-1)}}{(\Pi_i^N p!)^2}\int d\Lambda_1\,d\Lambda_2\,\Delta^2(\Lambda_1)\Delta^2(\Lambda_2)\times\exp\Big[-V(\Lambda_1)-V(\Lambda_2)+N^2X\Big(\frac{\Lambda_1}{N^{1/2}},\frac{\Lambda_2}{N^{1/2}};\beta\Big)\Big]. \qquad (4.3)$$

For fixed $\Lambda_2$, this is an integral over $\Lambda_1$ with an "effective potential" of a generalized type involving only symmetric functions. We can therefore use the techniques of Sec. 2, which are equivalent to the saddle-point method of [A]. Symmetry under the interchange $\Lambda_1\longleftrightarrow\Lambda_2$ implies that the coefficients of this generalized potential which depends only on $\Lambda_2$ can be determined self-consistently by requiring that the symmetric functions of both matrices be equal at the saddle point. We use the work "generalized potential" since it contains arbitrary powers of the symmetric functions, in contrast with the original one (4.2). We may speak in that case of "nonlocality" in the index of eigenvalues. To illustrate this point, we shall compute

$$\mathscr{E}(g,\beta) = -(1/N^2)\ln[Z(g,\beta)/Z(g,0)], \qquad (4.4)$$

to fourth order in $\beta$. Rescaling $\Lambda$ into $\Lambda\sqrt{N}$, we find that $\mathscr{E}(\beta)$ is the saddle-point value of the functional:

$$\mathscr{E} = \Big\{\int_0^1 dx\,v[\lambda(x)] - \int_0^1\int_0^1 dx\,dy\ln|\lambda(x)-\lambda(y)|\Big\} + [\lambda(x)\longleftrightarrow\mu(x)] - \beta\langle\lambda\rangle\langle\mu\rangle - \frac{\beta^2}{3}f_3(\lambda)f_3(\mu) - \frac{\beta^3}{3}f_3(\lambda)f_3(\mu) - \frac{\beta^4}{4}\times[f_4(\lambda)f(\mu) - f_2^2(\lambda)f_2^2(\mu)] - \cdots. \qquad (4.5)$$

The rescaled eigenvaues have been rearranged as increasing functions of the reduced index $x = i/N$. A continuous limit as $N\to\infty$ is understood.

The notations of Sec. 3 have been generalized to mean $\langle\lambda^s\rangle = \int_0^1 dx\lambda^s(x)$, $e_s(\lambda) = \langle(\lambda-\langle\lambda\rangle)^s\rangle$, and $f_s(\lambda)$ is related to $e_s(\lambda)$ as in (3.26)–(3.27). If $u(\lambda)$ denotes the density of eigenvalues, we obtain the saddle-point equation

$$0 = -v'(\lambda) + 2\,⨍\frac{d\lambda' u(\lambda')}{\lambda-\lambda'} + \beta\langle\mu\rangle + \beta^2 f_2(\mu)(\lambda-\langle\lambda\rangle) + \beta^3 f_3(\mu)[(\lambda-\langle\lambda\rangle)^2 - \langle(\lambda-\langle\lambda\rangle)^2\rangle] + \beta^4 f_4(\mu)[(\lambda-\langle\lambda\rangle)^3 - \langle(\lambda-\langle\lambda\rangle)^3\rangle] - \beta^4[2f_4(\mu)+f_2^2(\mu)]f_2(\lambda)[\lambda-\langle\lambda\rangle] - \cdots.$$

For definiteness, let us consider the $\varphi^4$ theory with $v(\lambda) = \lambda^2/2 + g\lambda^4$. A consistent Ansatz assumes the odd mean values to vanish, viz., $\langle\lambda^{2s+1}\rangle = \langle\mu^{2s+1}\rangle = 0$. The lowest order in $\beta$ for $\mathscr{E}(g,\beta)$ is $\beta^2$, and we readily find

$$\mathscr{E}(g,\beta) = -\frac{\beta^2}{2}\Big[1-4g\frac{dE(g)}{dg}\Big]^2 + O(\beta^4) = -\frac{\beta^2}{8}[a^2(4-a^2)]^2+\cdots, \qquad (4.7)$$

with $E(g)$ given by (2.27)

$$E(g) = \tfrac{1}{24}(a^2-1)(9-a^2) - \tfrac{1}{2}\ln a^2,$$
$$12ga^4 + a^2 - 1 = 0. \qquad (4.8)$$

This can be checked diagrammatically to the first few orders in $g$. With more algebra, the coefficients of higher powers in

Even though this direct method lacks some elegance, it is, however, very effective.[12]

The conclusions to be drawn from this large amount of algebra were already presented in the Introduction. The planar approximation seems a very nontrivial one, and even in the simplest case discussed in this paper, no simple algorithm was found. But it could well be that, for deeper geometric reasons, the same approximation is more tractable in the case of gauge fields.


## ACKNOWLEDGMENTS

We gratefully acknowledge numerous discussions with G. Parisi and E. Brézin, who informed us of similar considerations by E. Witten.



[1] G.'t Hooft, Nucl. Phys. B **72**, 461 (1974); B **75**, 461 (1974).
[2] G. Veneziano in Proc. 19th Int. Conf. High Energy Physics, Tokyo 1978, A. Migdal, unpublished.
[3] E. Witten, Harvard Preprint HUTP 79/A007.
[4] H. Wegner, to be published.
[5] W.T. Tutte, Can. J. Math. **14**, 21 (1962).
[6] J. Koplik, A. Neveu, and S. Nussinov, Nucl. Phys. B **123**, 109 (1977).
[7] E. Brézin, I. Itzykson, G. Parisi, and J.-B. Zuber, Commun. Math. Phys. **59**, 35 (1978), hereafter referred to as [A].
[8] D. Bessis, Saclay Preprint 79, 23, to be published in Comm. Math. Phys.
[9] G. Parisi, Private communication.
[10] H. Weyl, *The Classical Groups* (Princeton Univ. Press, Princeton, N.J., 1946).
[11] A complete expression may be found in R. Balian and E. Brézin, Nuov. Cimento B **64**, 37 (1969).
[12] Note added in proof: The two matrix problem has finally been solved by M.L. Mehta, Saclay preprint **79**, 124.