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Asymptotic estimates in quantum electrodynamics

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We discuss various aspects of the estimate of large orders of perturbation theory in quantum electrodynamics. Gauge-invariant subclasses of diagrams corresponding to a fixed number of fermion loops have a behavior dominated by solutions to the coupled Maxwell and Dirac equations for complex values of the charge. We present numerical evidence for the existence of such solutions. The complete theory involving an arbitrary number of fermion loops is expected to exhibit strong cancellations. We show the relation of this problem to the Thomas-Fermi approximation and raise some related mathematical questions.

I. INTRODUCTION

The nature of the perturbation series in quantum electrodynamics (QED) is a long-standing problem. Illuminating comments were made by Dyson in a paper in 1952.[1] We discuss here some methods to estimate the contributions of a very high order (denoted by k) to a given Green's function. We do not claim to have settled all the questions arising in this investigation and will limit ourselves to a first survey. We hope to clarify some of the issues in a future publication.

In a series of works initiated by Langer[2] in statistical mechanics, by Bender and Wu[3] in quantum mechanics, and by Lipatov[4] for scalar quantum field theory, methods have been developed and tested to obtain such estimates. Brézin, Le Guillou, and Zinn-Justin[5] and Parisi[6] have reviewed and extended these techniques. In particular, they have shown numerical applications of this new information to the computation of critical exponents in statistical mechanics.

In a previous letter we have extended these considerations to scalar electrodynamics[7] and one of us has, furthermore, studied the introduction of Fermi fields.[8] On the other hand, a number of rigorous results have been derived on the Borel summability of perturbation series in quantum mechanics,[9] in field theories involving Bose fields,[10] and, more recently, for a two-dimensional Yukawa model,[11] this being certainly a very incomplete list.

We shall assume the reader familiar with the techniques presented in some of the above references, especially Refs. 4–7.

Investigations in QED have also been carried from a slightly different point of view in a series of papers by Adler.[12]

The dominant contribution to the main estimates is obtained by studying solutions to the corresponding classical equations from which a WKB vacuum decay amplitude is obtained in a regime of complex coupling. Renormalizability only enters at this stage to dictate the solubility of these equations in terms of well-behaved fields when massive parameters can be neglected. At a later stage of course, when one computes higher terms of these estimates (starting with the multiplicative, k-independent, and constant terms) subtractions have to be taken into account.

In simple cases such as four-dimensional ϕ^4 theory it was shown in Refs. 6 and 7 that the main term of these estimates is connected with an inequality in nonlinear analysis due to Sobolev. The same type of inequalities were applied by Glaser, Grosse, Martin, and Thirring[13] to study the spectrum of Schrödinger's equation. As we shall show, their methods are also useful for our purpose in the presence of Fermi fields.

We present the general formalism using Euclidean path integrals in Sec. II. Integration over fermionic degrees of freedom introduces a Fredholm determinant discussed in this context by Matthews and Salam[14] and Schwinger.[15] For well-behaved potentials it is an entire function of the coupling constant, and it is elementary to prove for nonvansihing fermion mass that it has no zeros for a real coupling constant. We expand this determinant according to the number of fermion loops and consider the corresponding contributions to Green's functions. For large orders these are dominated by nearby zeros of the determinant leading to a classical extremal problem. It is physically clear that the limitation to a fixed number of charged particle loops eliminates the restrictions implied by Fermi statistics. It is therefore to be expected that apart from technical details the same type of estimates should apply to these subsets of diagrams as in the Bose case.

In Sec. III we study Dirac's equation in an external field. On a simple example we show explicitly that it admits regular solutions for purely imaginary values of the coupling constant.

Section IV extends these considerations to the "coupled problem" which arises for the asymptotic evaluation of Green's functions with a fixed number of fermionic loops. With a particular ansatz for the angular dependence of the vector potential A_μ and the corresponding spinor field, we obtain well-behaved solutions (i.e., corresponding to a finite action) corresponding to a purely imaginary value of the coupling constant. This part is akin to the treatment of scalar electrodynamics showing great similarities in the numerical behavior of solutions. We apply these results to some existing data. We point out that the method is not limited to the study of photon amplitudes, but applies as well to functions involving external fermion lines. Our numerical applications are very preliminary. In order to trust the results quantitatively, a careful investigation of fluctuations around the classical solution is necessary. This should also shed light on whether we have picked the correct extremum of the functional integral.

In the following section (V) we discuss the implication of taking the exclusion principle fully into account. We present the problem first in the simpler framework of a Yukawa theory. The physical idea is again that for complex values of e the vacuum becomes unstable. The technique of computation of the decay amplitude requires a generalization of the WKB method to fermionic integrals. This is achieved by using a Thomas-Fermi approximation borrowed from atomic physics. The latter is reviewed in Sec. VI, where it is shown to reproduce the asymptotic distribution of levels for a Schrödinger-type problem and seems therefore suited to study the behavior of the entire functions arising as Fredholm determinants.

In the final section (VII) we attempt to use this insight for Yukawa interactions as well as QED. In the first case, we find estimates which are in perfect agreement with all expectations. Except perhaps for subtle effects due to renormalization they imply a much softer divergence of perturbation theory than simple diagram counting would naively seem to indicate. The QED case is of a more difficult nature due to the effects of gauge invariance. Simple-minded arguments would imply that the Fredholm determinant is an entire function of order two while Adler claims to have found cases where it is of order four, as one would find neglecting the effects of current conservation. We were not yet able to settle this question and discuss the alternative. This has, of course, a bearing on the growth of the integrand in the functional integral and in turn affects the nature of the perturbation series. We point out that the application of the Thomas-Fermi approximation would favor order two, which would imply a very slow increase of higher orders as $(\ln k)^k$. In both cases, however, we find a reduction with respect to simple combinatorial counting.

II. EUCLIDEAN FORMALISM

We study the dynamical system of a charged Fermi field coupled minimally to an electromagnetic potential. A Wick rotation to Euclidean space is performed so that for real Euclidean momenta Green's functions are smooth analytic functions, at least perturbatively (for a discussion, see Ref. 12), except perhaps at zero momentum in a massless theory. These Green's functions will be defined by the functional integrals

$$G_{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n;y_1,\ldots,y_p;z_1,\ldots,z_p) = \int \mathfrak{D}(A)\mathfrak{D}(\psi)\mathfrak{D}(\bar\psi)A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\psi(y_1)\cdots\psi(y_p)\bar\psi(z_1)\cdots\bar\psi(z_p)\exp[-(S_A+S_F)]. \quad (2.1)$$

The fermionic part of the integral uses anticommuting ψ and $\bar\psi$ variables. The photon and fermion contributions to the action read

$$S_A = \int d^4x\left[\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \tfrac{1}{2}\lambda(\partial\cdot A)^2\right],$$
$$S_F = \int d^4x\,\bar\psi(i\gamma\cdot\partial - e\gamma\cdot A - M)\psi. \quad (2.2)$$

We choose anti-Hermitian γ matrices fulfilling

$$\{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu} \quad (2.3)$$

A convenient representation is given in terms of Pauli matrices as

$$\gamma^0 = -i\begin{pmatrix}0 & I\\ I & 0\end{pmatrix} = -i\tau_1\otimes I,$$
$$\vec\gamma = -i\begin{pmatrix}0 & -i\vec\sigma\\ i\vec\sigma & 0\end{pmatrix} = -i\tau_2\otimes\vec\sigma. \quad (2.4)$$

For real eA a natural choice of $\bar\psi$ leading to a real S_F is $\bar\psi = \psi^\dagger\gamma_5$, with an Hermitian γ^5,

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} = \tau_3\otimes I, \quad (2.5)$$

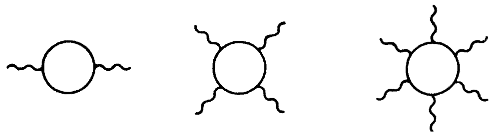

FIG. 1. One-charged-particle loop diagrams corresponding to the perturbative expansion of the Fredholm determinant.

although in the functional integral ψ and $\bar\psi$ are to be considered as independent variables.

The set of matrices I, $i\vec\sigma$ which generate a quaternion algebra will play a special role in the sequel. We use the notations (this "slash" notation for quaternions and their conjugate was adopted for typographical reasons; it should not be confused with the one generally used in the context of γ matrices which appears in the text as $\gamma\cdot n$ for $\gamma_\mu n_\mu$)

$$\backslash\!\!\!\sigma_\mu=(I,i\vec\sigma),\quad \not\sigma_\mu=(I,-i\vec\sigma)=\backslash\!\!\!\sigma_\mu^\dagger, \tag{2.6}$$

and for any four-vector

$$\backslash\!\!\!n=n^\mu\backslash\!\!\!\sigma_\mu=n^0+i\vec n\cdot\vec\sigma,\quad \not n=n^\mu\not\sigma_\mu=n^0-i\vec n\cdot\vec\sigma. \tag{2.7}$$

Relations between these matrices may be written in terms of symbols introduced by 't Hooft[16]

$$\backslash\!\!\!\sigma_\mu\not\sigma_\nu=\delta_{\mu\nu}+i\eta_{\mu\nu a}\sigma^a,\quad \not\sigma_\mu\backslash\!\!\!\sigma_\nu=\delta_{\mu\nu}+i\bar\eta_{\mu\nu a}\sigma^a, \tag{2.8}$$

with $\eta_{\mu\nu a}$ and $\bar\eta_{\mu\nu a}$ antisymmetric in (μ,ν) and a running from 1 to 3. Explicitly

$$\begin{aligned}&\eta_{\mu\nu a}=\epsilon_{\mu\nu a},\quad \mu,\nu=1,2,3\\&\eta_{\mu 0 a}=\delta_{\mu a},\\&\bar\eta_{\mu\nu a}=(-1)^{\delta_{\mu 0}+\delta_{\nu 0}}\eta_{\mu\nu a}.\end{aligned} \tag{2.9}$$

If we first concentrate on the Green's functions without external fermion lines, the Fermi fields may be integrated over and we obtain

$$G_{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n)=\int\mathfrak{D}(A)A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\times\frac{\operatorname{Det}(i\gamma\cdot\partial-e\gamma\cdot A-M)}{\operatorname{Det}(i\gamma\cdot\partial-M)}e^{(-S_A)}. \tag{2.10}$$

A simple generalization of this formula enables one to express the Green's functions with external fermion lines as an integral over fields $A_\mu(x)$ (see Sec. IV). The (infinite) normalization factor $\operatorname{Det}(i\gamma\cdot\partial-M)$ subtracts the vacuum loops. Using the previous notation, the Fredholm determinant $\operatorname{Det}(i\gamma\cdot\partial-e\gamma\cdot A-M)$ can be written formally (i.e., before any subtractions)

$$\begin{aligned}\Delta(eA)&=\operatorname{Det}\begin{pmatrix}-M & -i(i\not\partial-e\not A)\\ -i(i\backslash\!\!\!\partial-e\backslash\!\!\!\!A) & -M\end{pmatrix}\\&=\operatorname{Det}[(i\not\partial-e\not A)(i\backslash\!\!\!\partial-e\backslash\!\!\!\!A)+M^2]\\&=\operatorname{Det}[(i\backslash\!\!\!\partial-e\backslash\!\!\!\!A)(i\not\partial-e\not A)+M^2].\end{aligned} \tag{2.11}$$

Since

$$(i\backslash\!\!\!\partial-e\backslash\!\!\!\!A)^\dagger=(i\not\partial-e^*\not A^*), \tag{2.12}$$

these expressions exhibit the positivity of $\Delta(eA)$ for eA real together with the reality property

$$\Delta(eA)^*=\Delta(e^*A^*). \tag{2.13}$$

The existence of a matrix $C\,(=\sigma_2)$ such that

$$C\sigma^TC^{-1}=-\sigma,\quad C\not A C^{-1}=\backslash\!\!\!\!A^T \tag{2.14}$$

implies that $\Delta(eA)$ is an even function of e (Furry's theorem), if one notices that the derivative operator is odd under transposition in the configuration-space variables.

For most of the following we shall neglect the fermion mass M and thus study Euclidean massless QED. It is expected that the dominant contributions to high orders are mass independent.

The determinant $\Delta(eA)$ requires subtractions for a proper definition. These include not only the vacuum loops explicited in (2.10) but also a constant in the vacuum polarization term, i.e., the term quadratic in eA in $\ln\Delta(eA)$. Let $\Delta_2(eA)$ be the quantity thus defined. Its logarithm has a perturbation expansion (depicted in Fig. 1) as

$$-\ln\Delta_2(eA)=e^2\bar\omega(A)+\sum_{n=2}^{\infty}\frac{e^{2n}}{2n}\int dx_1\cdots dx_{2n}A_{\mu_1}(x_1)\cdots A_{\mu_{2n}}(x_{2n})\operatorname{tr}\left(\gamma^{\mu_1}\left\langle x_1\left|\frac{1}{i\gamma\cdot\partial}\right|x_2\right\rangle\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}\left\langle x_{2n}\left|\frac{1}{i\gamma\cdot\partial}\right|x_1\right\rangle\right), \tag{2.15}$$

where

$$e^2\bar\omega(A)=\frac{e^2}{2}\int dx_1dx_2A_{\mu_1}(x_1)A_{\mu_2}(x_2)\left[\operatorname{tr}\left(\gamma^{\mu_1}\left\langle x_1\left|\frac{1}{i\gamma\cdot\partial}\right|x_2\right\rangle\gamma^{\mu_2}\left\langle x_2\left|\frac{1}{i\gamma\cdot\partial}\right|x_1\right\rangle\right)-\sigma(\partial^{\mu_1}\partial^{\mu_2}-\delta^{\mu_1\mu_2}\Box)\delta(x_1-x_2)\right]$$

and the constant σ is adjusted to compensate for the logarithmic infinity in the two-photon amplitude. We shall show below that for a well-behaved vector potential the series in (2.15) should converge in a circle of finite radius in the complex e plane.

An alternative computation of $\Delta(eA)$ is based on the study of the eigenmodes of the Dirac equation for fixed real A_μ. We shall assume that the potential is a smooth enough function. Clearly this is not the case for most of the integration domains in functional space. But this hypothesis is on the same footing as the expression for the action (2.2) when one defines the functional integral in the first place.

For vanishing M, chirality is a good quantum number so that the Dirac equations

$$(i\gamma\cdot\partial - e\gamma\cdot A)\psi = 0\,, \quad \bar{\psi}(-i\gamma\cdot\overleftarrow{\partial} - e\gamma\cdot A) = 0 \tag{2.16}$$

decouple in the representation (2.4) for ψ and $\bar{\psi}$ written

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \bar{\psi} = (\bar{u}, \bar{v})\,. \tag{2.17}$$

We obtain the four equations

$$(i\partial\!\!\!/ - eA\!\!\!/)u = 0\,, \quad \bar{u}(-i\overleftarrow{\partial\!\!\!/} - eA\!\!\!/) = 0\,, \tag{2.18a}$$

$$(i\partial\!\!\!/ - eA\!\!\!/)v = 0\,, \quad \bar{v}(-i\overleftarrow{\partial\!\!\!/} - eA\!\!\!/) = 0\,. \tag{2.18b}$$

In these equations the constant e is no more to be interpreted as the perturbative parameter but rather runs over the discrete set of values $e_n(A)$ for which suitably regular solutions exist. While in general the two equations of each set (2.18a) and (2.18b) are independent, a transformation enables one to obtain the solutions of one of the set in terms of those of the other in terms of the same $e_n(A)$. Taking into account that eigenvalues generally occur in quartets, $e_n, -e_n, +e_n^*, -e_n^*$, we can write for $\Delta_2(eA)$ a representation

$$\Delta_2(eA) = \exp[-e^2\bar{\omega}(A)]\prod_n\left[\left(1-\frac{e^2}{e_n^2(A)}\right)\exp\left(\frac{e^2}{e_n^2(A)}\right)\right], \tag{2.19}$$

where $e_n(A)$ runs over the eigenvalues with positive imaginary parts, say. Since we have subtracted in the infinite product the full second-order term of its logarithm we have to reinstate the correctly subtracted contribution $\exp[-e^2\bar{\omega}(A)]$ defined in (2.15).

This formula is suggestive of Hadamard's canonical form of entire functions provided that we could prove that $\sum_n[1/|e_n(A)|^{2+\epsilon}]$ converges for any positive ϵ. This would make the infinite product well defined and would produce an entire function of order 2.[17] As we shall see in Sec. VII this is indeed the central problem when one studies the global properties of the theory.

If we restrict ourselves to one-fermion loop diagrams, i.e., if we replace in (2.10) $\Delta_2(eA)$ by its logarithm, we expect its nearest singularities in the coupling e, for fixed S_A, to play a dominant role. These singularities are the nearest eigenvalues. The same holds for contributions corresponding to a fixed number of fermion loops and not merely one loop. In both cases the location of the nearest $e_n(A)$ is of importance.

This set $\{e_n(A)\}$ is of course gauge independent. Let us mention two general results on these eigenvalues:

(i) As already noticed above in the massive case, $e_n(A)$ cannot be real. The same seems to hold when $M=0$, as already pointed by Adler.[12] We shall verify this property below using a specific structure of A_μ.

(ii) For A real, given the value of $\int dx\,\frac{1}{4}F^2$ there exists a zero-free neighborhood of the origin in the complex e plane. This is seen using Sobolev's inequality which states that for any real, smooth function ϕ, such that the integrals exist, one has in four-dimensional space

$$\int d^4x\,\phi^4 \le \frac{3}{32\pi^2}\left(\int d^4x(\partial\phi)^2\right)^2. \tag{2.20}$$

From either Eq. (2.18a) or (2.18b) we deduce

$$\int d^4x\,|\partial u|^2 = |e(A)|^2\int d^4x\,|A|^2|u|^2\,. \tag{2.21}$$

Combining with (2.20) we obtain

$$|e(A)|^2\int d^4x\sum_{\mu,\nu}(\partial_\mu A_\nu)^2 \ge \frac{32\pi^2}{3}\,. \tag{2.22}$$

Since

$$\int d^4x\sum_{\mu,\nu}(\partial_\mu A_\nu)^2 = \int d^4x[\tfrac{1}{2}F^2 + (\partial\cdot A)^2]$$

we can minimize the left-hand side of (2.22) with respect to the choice of gauge and obtain

$$|e_n(A)|^2\int d^4x\,\tfrac{1}{4}F^2 \ge \frac{16\pi^2}{3}\,. \tag{2.23}$$

Consequently, if the infinite product in (2.19) converges with finitely many subtractions [that is to say, if $\sum_n|e_n(A)|^{-p}$ converges for p large enough] the series for $\ln\Delta_2(eA)$ given by (2.15) will have a finite radius of convergence for A real, as already stated.

Our first goal is therefore to study the Dirac equation in a given external field.

III. EUCLIDEAN DIRAC EQUATION IN AN EXTERNAL FIELD. EXPLICIT SOLUTIONS

Let us investigate the zero modes of the Dirac equation

$$(i\partial\!\!\!/ - eA\!\!\!/)u = 0\,. \tag{3.1}$$

We want to exhibit explicit solutions for a particular external field $A_\mu(x)$, which we take to be of the form

$$A_\mu(x) = M_{\mu\nu}x_\nu a(x^2)\,, \tag{3.2}$$

with M a 4×4 antisymmetric constant matrix with square equal to -1:

$$M = -M^T = -M^{-1}\,. \tag{3.3}$$

The function $a(x)^2$ is supposed to be a regular well-behaved function such that S_A is finite.

Such configurations were already used in our previous study of scalar electrodynamics. It was shown that they lead to a simple closed algebra, with A_μ fulfilling the Lorentz condition $\partial \cdot A = 0$, together with a transversity property $x \cdot A = 0$. The action S_A is therefore independent of the gauge parameter λ and reduces to

$$S_A = \int d^4x \, \tfrac{1}{4} F^2 = 2\pi^2 \int_0^\infty dx^2 x^2 \left[a^2 + \left(\frac{d}{dx^2} x^2 a(x^2) \right)^2 \right]. \quad (3.4)$$

A matrix M satisfying (3.3) is either self-dual or anti-self-dual. With the convention that the Levi-Civita symbol is such that $\epsilon_{0123} = 1$, we choose M anti-self-dual, which means that $M_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M_{\rho\sigma}$ so that it can be parametrized as

$$M_{\mu\nu} = \eta_{\mu\nu a} n_a , \quad (3.5)$$

where $\vec{n}$ is a unit three-dimensional vector [$\vec{n}$ plays the same role as the isospin Pauli matrices $\vec{\tau}$ in the similar parametrization of the non-Abelian SU(2) pseudoparticles[16]].

There is a corresponding ansatz for the spinor u involving two scalar functions of x^2 as

$$u = [f(x^2) - ig(x^2) \not{x} \not{M} x] U , \quad (3.6)$$

with $(Mx)_\mu \equiv M_{\mu\nu} x_\nu$ orthogonal to x and U a constant spinor. Note that $-i \not{x} \not{M} x = x_\mu (Mx)_\nu \bar{\eta}_{\mu\nu a} \sigma^a$ is an Hermitian traceless matrix. Other possible choices for u such that $[f(x^2) + \vec{\sigma} \cdot \vec{n} g(x^2)] U$ lead to singular solutions. Given $a(x^2)$ there ought, of course, to exist higher-angular-momenta spinor solutions than those described by (3.6). Inserting these structures into (3.1) we are led to

$$\{ \not{x} [2f' - e(A) a x^2 g] - i \not{M} x [2x^2 g' + 6g - e(A) a f] \} U = 0 . \quad (3.7)$$

Here the prime denotes differentiation with respect to x^2 and both terms in square brackets are functions of this variable only. We shall find a solution irrespective of U if we fulfill the two conditions

$$\begin{aligned} &2f' - e(A) x^2 a g = 0 , \\ &2x^2 g' + 6g - e(A) a f = 0 . \end{aligned} \quad (3.8)$$

We are looking for solutions regular at the origin and vanishing fast enough at infinity (typically we want such integrals as $\int d^4x \, |\partial u|^2$, $\int d^4x \, A^2 |u|^2$ to exist).

After a change of variable

$$t = \ln x^2 \quad (3.9)$$

and of functions

$$\begin{aligned} f(x^2) &= e^{-t} \phi(t) , \\ g(x^2) &= e^{-2t} \gamma(t) , \\ a(x^2) &= e^{-t} \alpha(t) , \end{aligned} \quad (3.10)$$

the system (3.8) reads

$$\begin{aligned} \dot{\phi} - \phi &= \tfrac{1}{2} e(A) \alpha \gamma , \\ \dot{\gamma} + \gamma &= \tfrac{1}{2} e(A) \alpha \phi . \end{aligned} \quad (3.11)$$

The overdot represents the derivative with respect to t. It is easy to see that these equations have no regular solution for $e(A)\alpha$ real. Indeed, from (3.11) we learn that

$$\frac{d}{dt} (\phi^2 - \gamma^2) = \phi^2 + \gamma^2 . \quad (3.12)$$

Clearly, because of the linearity of the system, (ϕ, γ) can be chosen real if $e(A)\alpha$ is real, and (3.12) would then imply that $\int_{-\infty}^{+\infty} dt (\phi^2 + \gamma^2) = 0$. This is, of course, in agreement with the general statement of Sec. II. On the other hand, these equations admit solutions for $e(A)$ purely imaginary and real. Choosing, for instance, ϕ real and γ imaginary we set

$$\begin{aligned} e(A) &= -2i\lambda , \\ \eta_1 &= \phi + i\gamma , \\ \eta_2 &= \phi - i\gamma . \end{aligned} \quad (3.13)$$

The real functions η_1, η_2 must verify

$$\begin{aligned} \dot{\eta}_1 &= (1 + \lambda\alpha) \eta_2 , \\ \dot{\eta}_2 &= (1 - \lambda\alpha) \eta_1 . \end{aligned} \quad (3.13')$$

If $\alpha(t)$ is an even function of t (which means that the field $A_\mu(x)$ is invariant under inversion) η_1 may be chosen even and η_2 may be chosen odd or vice versa. The set of equations (3.13) is an Hamiltonian system, η_1 and η_2 playing the role of conjugate variables.

A numerical investigation shows that for smooth $\alpha(t)$ there do exist solutions vanishing at $|t| \to \pm\infty$ for discrete values of λ.

With the particular choice

$$\alpha(t) = \frac{1}{\cosh 2t} \quad (3.14)$$

we find explicitly in terms of Legendre polynomials

$$\begin{aligned} &\lambda_n = 2n , \quad n = \pm 1, \pm 2, \ldots \\ &\eta_1 = \cosh t \, P_{n-1}(\tanh 2t) - \sinh t \, P_n(\tanh 2t) , \\ &\eta_2 = \sinh t \, P_{n-1}(\tanh 2t) - \cosh t \, P_n(\tanh 2t) . \end{aligned} \quad (3.15)$$

The corresponding trajectories in the (η_1, η_2) plane are algebraic curves some of which are represented on Fig. 2. For $n = 1$, for instance, one obtains a branch of Bernouilli's lemniscate.

Although they are not going to play a crucial role in the following, we find it gratifying and instructive to obtain these explicit solutions. They may serve as an example of the distribution of zeros of

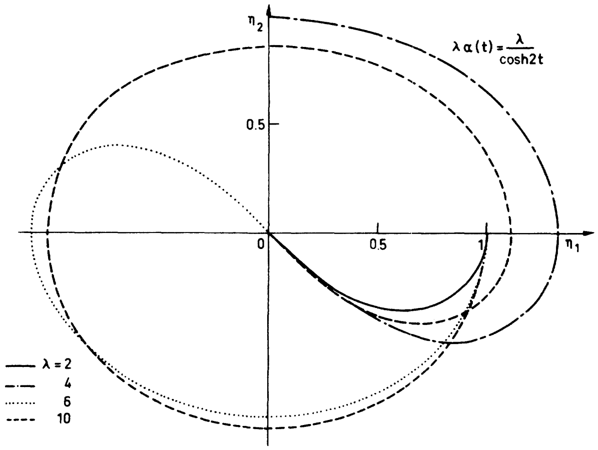

FIG. 2. The (η_1, η_2) curves corresponding to the explicit solutions of the Euclidean Dirac equation described in Sec. III.

fermionic determinants. Also they confirm a conjecture made by Adler that there might be no solution of the Dirac equation for purely imaginary eA (Adler's conjecture was verified for a slightly different configuration of the external field). One should clearly try to obtain the remaining modes corresponding to (3.14).

IV. ONE-FERMION-LOOP DIAGRAMS

Our purpose in this section is to study the set of one-fermion-loop diagrams for a given process at large orders. The method can be extended to any fixed number of such loops. The finite number of subtractions needed to define $\Delta(eA)$ does not play any role since we now integrate $\ln\Delta(eA)$. A typical problem will be for instance to investigate the asymptotic behavior of the vacuum polarization, the kth order of which may be written as

$$\langle A_\mu(x)A_\nu(y)\rangle^{(k)}_{1\text{loop}} = -\frac{(e^2)^k}{k}\int \mathfrak{D}(A)A_\mu(x)A_\nu(y) \times \sum_n \frac{1}{e_n(A)^{2k}}e^{-S_A}. \quad (4.1)$$

Within the original real integration domain we expect the nearest zeros (the smallest $|e_n(A)|^2$) to dominate for large k, with exponentially small corrections arising from the successive terms in the sum over n. This argument does not take into account renormalization effects. However, in the case at hand we know[18] that the set of diagrams of order k has only a single logarithmic divergence, which therefore will only occur in the integral over dilatations at the final stage of the evaluation of (4.1). Hence we feel safe on that side. For large k we therefore estimate (4.1) as

$$\langle A_\mu(x)A_\nu(y)\rangle^{(k)}_{1\text{loop}} \simeq -\frac{(-e^2)^k}{k}\int \mathfrak{D}(A)A_\mu(x)A_\nu(y) \times \exp(-\{S_A + k\ln[-e^2(A)]\}), \quad (4.2)$$

anticipating the fact that the effective $e^2(A)$ will be negative and for fixed A, $e^2(A)$ stands for the smallest eigenvalue.

Since k is large we use the steepest-descent method by looking for stationary points in the A functional integral. These are given as solutions of

$$\frac{\delta}{\delta A_\mu(x)}\{S_A + k\ln[-e^2(A)]\} = 0$$

or equivalently

$$[-\Box\delta_{\mu\nu} + (1-\lambda)\partial_\mu\partial_\nu]A_\nu(x) + \frac{k}{e^2(A)}\frac{\delta e^2(A)}{\delta A_\mu(x)} = 0. \quad (4.3)$$

This is Maxwell's equation with an effective current given in terms of $e^2(A)$ as $[k/e^2(A)]\delta e^2(A)/\delta A_\mu(x)$. In turn, $e(A)$ is given by a solution to the first or second set of Dirac equations (2.18). This is what we call the "coupled problem."

Even though we have no proof that we have found the best possible answer we are able at least to exhibit a numerical solution. It has the following features.

(i) The potential A_μ is real with a structure given by (3.2) and (3.5).

(ii) The eigenvalue $e(A)$ is purely imaginary.

(iii) We can describe the solutions of (2.18) in terms of a two-spinor u satisfying (3.1) with a structure given by (3.6). The amplitudes f and g are relatively imaginary as in Sec. III and depend only on x^2. This behavior of A_μ and u is the most "symmetric" one. It is, of course, clear that A and u can be conformally transformed to generate equivalent solutions corresponding to the same $e(A)$.

Within these choices we can carry the same analysis leading to Eqs. (3.13). Assuming we have a solution to the latter, we construct the spinors u, $\bar{u}$, v, $\bar{v}$, taking into account

$$e^*(A) = -e(A), \quad fg^* + f^*g = 0. \quad (4.4)$$

The following expressions hold with $U, V, \bar{U}, \bar{V}$ arbitrary fixed spinors and $f, g, e(A)$ given by (3.10), (3.12), and (3.13):

$$u = (f - ig\not{x}\not{M}x)U, \quad \bar{u} = \bar{U}(f^* - ig^*\not{M}x\not{x}),$$
$$v = \not{M}x(f + ig\not{x}\not{M}x)V, \quad \bar{v} = \bar{V}(f^* + ig^*\not{M}x\not{x})\not{M}x. \quad (4.5)$$

We have now to compute the current, i.e., the quantity $[1/e^2(A)]\delta e^2(A)/\delta A_\mu(x)$. This is achieved by noting that the equations fulfilled by the spinors follow from an extremal principle applied to the action S_F(2.2) which reads

$$S_F = i\int d^4x[\bar{v}(i\overleftarrow{\partial\!\!\!/} - eA\!\!\!/)u - \bar{u}(i\partial\!\!\!/ - eA\!\!\!/)v], \qquad (4.6)$$

interpreted now as a c number. This action vanishes together with its variation at fixed eA if we insert the solutions to Dirac equations. After summation over all degenerate solutions of (4.6), i.e., over the spinors $U, V, \bar{U}, \bar{V}$, we therefore find that

$$\frac{1}{e^2(A)}\frac{\delta e^2(A)}{\delta A_\mu(x)} = -2\,\frac{\sum i(\bar{v}\gamma_\mu u - \bar{u}\gamma_\mu v)}{d^4x\sum i(\bar{v}A\!\!\!/ u - \bar{u}A\!\!\!/ v)}$$
$$= -2\,\frac{N_\mu}{D}. \qquad (4.7)$$

In order for our ansatz to be a coherent one we have to find that the current has the same structure as A, i.e., is of the form $M_{\mu\nu}x_\nu$ times a function of x^2. Inserting (4.5) we find

$$N_\mu = \sum i(\bar{v}\gamma_\mu - \bar{u}\gamma_\mu v)$$
$$= \sum[i\bar{V}(f^* + ig^* M\!\!\!/ x x\!\!\!/)M x\gamma_\mu(f - ig x\!\!\!/ M\!\!\!/ x)U$$
$$- i\bar{U}(f^* - ig^* M\!\!\!/ x x\!\!\!/)\gamma_\mu M\!\!\!/ x(f + ig x\!\!\!/ M\!\!\!/ x)V].$$

Since for real e from (4.6) the equations for u and $\bar{v}$ are adjoint equations, we can take $i\bar{V} = U^\dagger$ not to overcount the degrees of freedom. Similarly, $-i\bar{U} = V^\dagger$, so that the above expression is in fact a trace. Any extra factor disappears in the ratio (4.7). Hence using (4.4)

$$N_\mu = [ff^* + (x^2)^2 g^* g]\mathrm{tr}(M\!\!\!/ x\gamma_\mu + \gamma_\mu M\!\!\!/ x)$$
$$- if^* g\,\mathrm{tr}(M\!\!\!/ x\gamma_\mu x\!\!\!/ M\!\!\!/ x - \gamma_\mu M\!\!\!/ x x\!\!\!/ M\!\!\!/ x$$
$$+ M\!\!\!/ x x\!\!\!/ M\!\!\!/ x\gamma_\mu - M\!\!\!/ x x\!\!\!/\gamma_\mu M\!\!\!/ x).$$

Since $M\!\!\!/ x M\!\!\!/ x = x^2$ and, from the orthogonality of x and Mx

$$M\!\!\!/ x\, x\!\!\!/ M\!\!\!/ x = -x^2 x\!\!\!/, \quad M\!\!\!/ x\, x\!\!\!/ M\!\!\!/ x = -x^2 x\!\!\!/,$$

the coefficient of $-if^*g$ is seen to vanish. Finally, from (2.8)

$$N_\mu = 4[ff^* + (x^2)^2 gg^*]M_{\mu\nu}x_\nu.$$

Similarly,

$$D = \int d^4x \sum i(\bar{v}A\!\!\!/ u - \bar{u}A\!\!\!/ v)$$
$$= \int d^4x\, A_\mu N_\mu$$
$$= \int d^4x\, a(x^2)x^2[ff^* + (x^2)^2 gg^*].$$

Finally,

$$\frac{1}{e^2(A)}\frac{\delta e^2(A)}{\delta A_\mu(x)}$$
$$= -2\,\frac{[ff^* + (x^2)^2 gg^*]M_{\mu\nu}x_\nu}{\int d^4x\, a(x^2)x^2[ff^* + (x^2)^2 gg^*]}. \qquad (4.8)$$

This structure is indeed of the type expected. We insert this expression together with (3.2) into Maxwell's equation, after observing that

$$[-\Box\delta_{\mu\nu} + (1-\lambda)\partial_\mu\partial_\nu]A_\nu(x) = -M_{\mu\nu}x_\nu(4x^2a'' + 12a'),$$

so that we obtain

$$4x^2a'' + 12' + \frac{2k}{\int d^4x\, a(x^2)x^2[ff^* + (x^2)^2 gg^*]}ff^*$$
$$+ (x^2)^2 gg^* = 0. \qquad (4.9)$$

We turn to the variable t as in (3.9) and the associated functions α, η_1, η_2 all assumed to be real. The last equation reads

$$\ddot{\alpha} - \alpha + \frac{k}{2\pi^2\int_{-\infty}^{+\infty}dt\;\alpha(\eta_1^2 + \eta_2^2)}(\eta_1^2 + \eta_2^2) = 0. \qquad (4.10)$$

A final rescaling will produce fields of order unity if we set

$$A(t) = \lambda\alpha(t) = \frac{ie(A)}{2}\alpha(t), \qquad (4.11)$$

with $e(A)$ given by

$$-e^2(A) = \frac{8\pi^2}{k}\int_{-\infty}^{+\infty}dt A(t)[\eta_1^2(t) + \eta_2^2(t)] \qquad (4.12)$$

in such a way that the coupled system takes its final form

$$\ddot{A} = A - (\eta_1^2 + \eta_2^2),$$
$$\dot{\eta}_1 = (1 + A)\eta_2, \qquad (4.13)$$
$$\dot{\eta}_2 = (1 - A)\eta_1.$$

These reduced equations also follow from an extremal principle applied to an effective action

$$S_{\text{eff}} = \int_{-\infty}^{+\infty}dt\left(\frac{A^2 + \dot{A}^2}{2} + \dot{\eta}_1\eta_2 - \dot{\eta}_2\eta_1 + \eta_1^2 - \eta_2^2\right.$$
$$\left. - A(\eta_1^2 + \eta_2^2)\right). \qquad (4.14)$$

The η-dependent part could have already been written as we were studying Dirac's equation with our choice for A_μ while the term independent of η is the Maxwell action evaluated for this A_μ up to rescalings implied by (4.11) and (4.12). One could have therefore derived the coupled system right away had it not been necessary to check the consistency of the ansatz in vector form.

We set I equal to the integral appearing in (4.12) in such a way that

$$-e^2(A)=\frac{8\pi^2}{k}I\ . \tag{4.15}$$

We look for solutions of (4.13) such that $I<\infty$ and our conventions are positive. Indeed, from (4.14) we derive an "energy"-conservation theorem:

$$\frac{A^2-\dot{A}^2}{2}+\eta_1{}^2-\eta_2{}^2-A(\eta_1{}^2+\eta_2{}^2)=\text{const}=0\,, \tag{4.16}$$

where the value zero follows from the boundary conditions at large $|t|$. Therefore one has the following virial equalities:

$$\begin{aligned}I&=\int_{-\infty}^{+\infty}dtA(\eta_1{}^2+\eta_2{}^2)\\&=\int_{-\infty}^{+\infty}dt(A^2+\dot{A}^2)\\&=\tfrac{2}{3}\int_{-\infty}^{+\infty}dt(A^2+\eta_1{}^2-\eta_2{}^2)\\&=\int_{-\infty}^{+\infty}dt(\dot{\eta}_1\eta_2-\dot{\eta}_2\eta_1+\eta_1{}^2-\eta_2{}^2)\\&=2\int_{-\infty}^{+\infty}dt(\dot{A}^2+\dot{\eta}_1\eta_2-\dot{\eta}_2\eta_1)\,.\end{aligned} \tag{4.17}$$

We have studied numerically the system (4.13) under the assumption that $t=0$ was the reflection point, i.e., $A(t)$ was an even function of t, and have found a solution (corresponding to the smallest possible value of I) depicted on Fig. 3. On this figure we have only represented A and η_1 but of course η_2 has also a regular behavior implying that it vanishes at $|t|\to\infty$. The curves were obtained by varying $A(0)$ with $\dot{A}(0)=\eta_2(0)=0$ and $\eta_1(0)$ given by (4.16). The equalities (4.17) were used to check the accuracy of our solution. The value obtained for I is

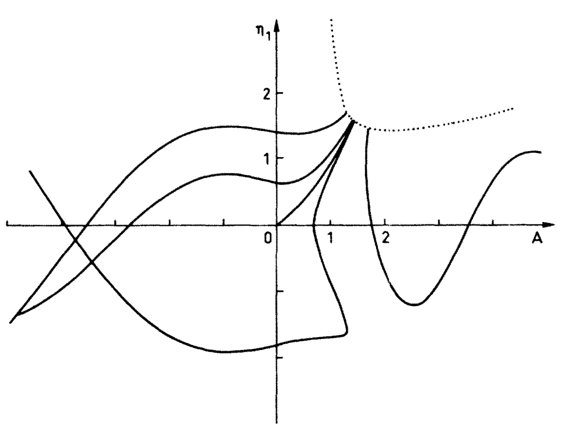

FIG. 3. Plot in the (η_1,A) plane of various trajectories corresponding to the coupled problem of Sec. IV. The desired one reaches the origin.

$$I=5.9201 \tag{4.18}$$

corresponding to $A(0)=1.4214$.

One notes the similarity of our results to the corresponding ones obtained in the case of scalar electrodynamics.

The degeneracies of these solutions contribute only to the k-independent overall constants of the asymptotic estimates. Let us now attempt to use the above results for these estimates. Of course, we should study the effects of fluctuations around the saddle point to obtain the multiplicative constants and make sure that at least locally we have guessed the correct answer. Let us assume optimistically that this is the case and proceed without further apology. In any case we expect our results to be at least qualitatively correct.

Because of the scaling properties (4.11) and (4.15) each field $A_\mu(x)$ corresponding to an external photon line will contribute a factor $k^{1/2}$. The same contribution will occur for each one-parameter set of continuous transformations leaving $e^2(A)$ invariant which will be used to separate a collective coordinate before integrating over fluctuations. These transformations are generated by ordinary translations, dilatations (corresponding to shifts in the origin of the variable t) to which one should add rotations of the unit vector n occurring in the expression of $M_{\mu\nu}$ used to construct A_μ. The total factor corresponding to these effects will therefore be

$$k^{(n_\gamma+7)/2}\,, \tag{4.19}$$

with n_γ standing for the number of external photon lines.

Parenthetically we remark that when we perform the integration over dilatations we should recover the dominant logarithm of the cutoff, where dominant means the one with the largest coefficient in k.

Let us turn now to the vacuum polarization to be specific and extract the coefficient F of its unique logarithmic divergence. We use our solution to compute the integral at the saddle point, multiply by the coefficient (4.19) for $n_\gamma=2$, and obtain, with α standing now for the fine-structure constant $e^2/4\pi$,

$$\begin{aligned}F(\alpha)&=\sum_{k=1}^{\infty}F_k(\alpha)\,,\\F_k(\alpha)&\simeq\left(-\frac{\alpha}{2\pi I}\right)^k k!k^3f\,.\end{aligned} \tag{4.20}$$

Of course the choice $k!k^3$ is arbitrary and could be replaced by any value $\Gamma(k+a)k^b$ provided $a+b=4$, without affecting the main estimate.

This is to be compared with the first three known terms of this series[19]

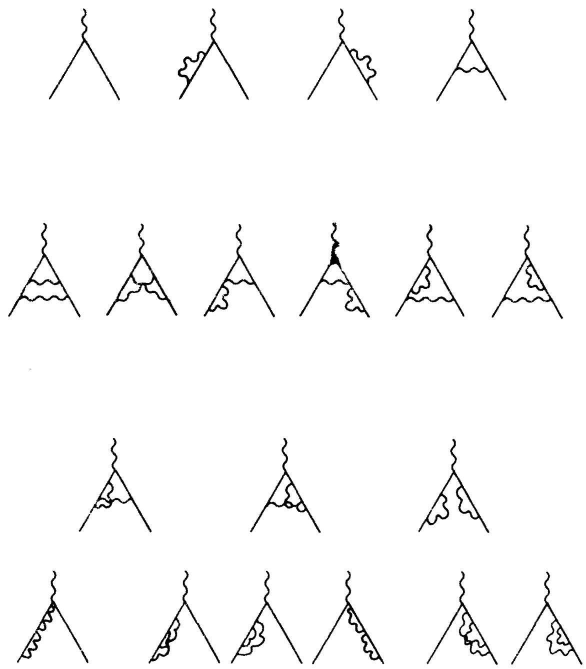

FIG. 4. The set of zero-electron-loop diagrams corresponding to the vertex function in QED.

$$F(\alpha)=\frac{2}{3}\left(\frac{\alpha}{2\pi}\right)+\left(\frac{\alpha}{2\pi}\right)^2-\frac{1}{4}\left(\frac{\alpha}{2\pi}\right)^3+\cdots. \tag{4.21}$$

We observe that the first two terms do not alternate in sign but of course this alternance of sign is only predicted as an asymptotic effect. If we take (4.20) at face value without attempting to improve it and apply it blindly we would predict for the size of F_4

$$F_4(\alpha)\sim+0.4\left(\frac{\alpha}{2\pi}\right)^4, \tag{4.22}$$

with an undetermined theoretical uncertainty.

As pointed out repeatedly, the above method can be extended to Green's functions involving external fermion lines. To illustrate this, consider for instance the electron-photon vertex in the approximation where one keeps only diagrams without internal electron loops (Fig. 4).

The truncated Green's function will be given by

$$\Gamma(x_1, x_2; y)|_{0\,\text{loop}} = \int \mathfrak{D}(A)\left(x_1\left|(i\gamma\cdot\partial-M)\frac{1}{i\gamma\cdot\partial-e\gamma\cdot A-M}\right|y\right)\gamma_\mu \times\left(y\left|\frac{1}{i\gamma\cdot\partial-e\gamma\cdot A-M}(i\gamma\cdot\partial-M)\right|x_2\right)e^{-S(A)}. \tag{4.23}$$

The behavior of the coefficients of its expansion in powers of e will again be dominated by the nearest singularities of the propagators. If we assume again that apart from providing a scale factor the fermion mass is irrelevant in the large-k limit we find similar estimates as before. The prefactor (4.19) is now replaced by $k^{7/2}$ with two extra powers of k, with one coming from the absence of the logarithm and the second from the double denominator in (4.23). We may attempt to apply this, for instance, to the electron anomalous moment with

$$\begin{aligned} a_e^{(0)} &= \tfrac{1}{2}(g_e-2)^{(0)} \\ &=\sum_1^\infty a_k, \\ a_k &\simeq \left(-\frac{\alpha}{2\pi I}\right)^k k!k^4 a, \end{aligned} \tag{4.24}$$

with the same I as above.

The known results for the zero-electron-loop contributions can be extracted from the review[20]

$$\begin{aligned} a_e^{(0)} &=\frac{\alpha}{2\pi}-\left(\frac{\alpha}{2\pi}\right)^2\left[\frac{31}{4}-\frac{5\pi^2}{3}-3\zeta(3)+2\pi^2\ln 2\right] \\ &\quad+\left(\frac{\alpha}{2\pi}\right)^3 8\times(0.915\pm0.015)+\cdots \\ &=\frac{\alpha}{2\pi}-1.377\left(\frac{\alpha}{2\pi}\right)^2+(7.32\pm0.12)\left(\frac{\alpha}{2\pi}\right)^3+\cdots. \end{aligned} \tag{4.25}$$

This shows an alternance in sign already. Using (4.24) one would predict the size of a_4 to be

$$a_4\simeq -15.6\left(\frac{\alpha}{2\pi}\right)^4. \tag{4.26}$$

The numbers quoted in (4.22) and (4.26) are only indicative. More work has to be done along these lines to give serious credit to these estimates. We have not yet had to cope seriously with ultraviolet nor infrared divergences. The latter should certainly show up to enhance the size of subclasses of diagrams in specific kinematical regions.

V. ROLE OF FERMI STATISTICS

As we have seen in the previous sections, Fermi statistics has no serious effect on the large-order behavior of the perturbative expansion, if we restrict ourselves to the study of diagrams involving only a finite number of fermionic loops. The situation changes drastically if we consider the complete theory. Diagrams with different number of fermionic loops oscillate in sign [there is a factor $(-1)^{N_f}$, N_f being the number of these loops]. They interfere destructively and the asymptotic behavior of the sum is different from that of each particular term.

To understand what is going on, let us first consider a Yukawa theory in (Euclidean) space-time dimension $D<4$, and for simplicity let us restrict our attention to the sum of all vacuum-to-vacuum diagrams:

$$Z(g)=\int \mathfrak{D}(\sigma)\mathfrak{D}(\psi)\mathfrak{D}(\bar\psi)\times\exp\left\{-\int d^Dx\left[\bar\psi i\gamma\cdot\partial\left(1-\frac{\Box}{\Lambda_f^2}\right)\psi+M\bar\psi\psi+\tfrac{1}{2}(\partial\sigma)^2+\tfrac{1}{2}m^2\sigma^2+\frac{1}{\Lambda_b^2}\sigma\Box^2\sigma+\lambda\bar\psi\psi\sigma\right]\right\},$$
$$g=\lambda^2,\quad Z(g)=\sum_k Z_k g^k. \tag{5.1}$$

Λ_f and Λ_b are ultraviolet cutoffs which will be removed later on. The coupling constant g is proportional to $\hbar$. If we were to neglect Fermi statistics we would find

$$Z_k\sim k!R^{-k}k^aA. \tag{5.2}$$

In the bosonic case this type of result is obtained for instance by considering a particular direction σ_0 in functional space, performing an exact integration in this direction and retaining only quadratic terms for the integration of the Lagrangian $\mathfrak{L}$ in the transverse directions:

$$\sigma(x)=t\sigma_0(x)+\delta\sigma(x)$$
$$\mathfrak{L}(\sigma)\simeq\mathfrak{L}(t\sigma_0)+\frac{\partial\mathfrak{L}}{\partial\sigma}(t\sigma_0)\delta\sigma+\frac{\partial^2\mathfrak{L}}{\partial\sigma^2}(t\sigma_0)\delta\sigma^2, \tag{5.3}$$

Independently of the direction σ_0, one always finds for Z_k the asymptotic behavior (5.2) and the best estimate is obtained by maximizing Z_k. It can easily be seen that σ_0 becomes k independent in the large-k limit and in this way one can recover all the results of Refs. 4 and 5. (It happens, in fact, that the explicit integration is done not in a single direction, but on a finite-dimensional manifold.) The integration over transverse directions ($\delta\sigma$) is essential to compute the constants a and A. If, however, we are interested only in finding the factorial increase and the radius R, the integration over small fluctuations may be neglected and one has only to evaluate a one-dimensional integral.

If we try to apply the same method in the case of the integral (5.1) we find a deceiving result. If we approximate the functional integral over the fermions by a finite N-dimensional integral, $\psi=\sum_{q=1}^N\alpha_q\psi_q$, we find $Z_k=0$ for $2k>N$. The situation is even worse. It has been proved by Caianiello[21] that one has then

$$Z_k<\Lambda_f^{\,k}. \tag{5.4}$$

The limits $k\to\infty$ and $\Lambda_f\to\infty$ do not seem to commute (as they do in the bosonic case) and we must first eliminate the fermion cutoff from the Lagrangian if we want to obtain a nontrivial result.

If we try blindly to extend the same techniques used in the bosonic case to the fermionic one, we are inevitably led to consider at once the integration over an infinite number of fermionic modes, which does not seem to be a great simplification as compared to the original problem. It seems that we have started in the wrong direction. We must go back to the origins of the asymptotic estimates, understand the implications of Fermi statistics, and find the best formalism to implement them.

The estimates of large orders of the perturbation expansion can be deduced from the analytic properties of the Green's functions for complex values of the coupling constant. We accept the standard hypothesis of analyticity around the origin except for a cut along the negative real axis. The nonconvergence of the expansion implies that the cut reaches the origin and the problem can be transmuted in an asymptotic estimate for the discontinuity $A(g)$ on the cut in the limit $g\to0$,

$$Z_k\sim\Gamma(\alpha k),\quad A(g)=\mathrm{Im}\,Z(g)\sim\exp\left(-\frac{1}{|g|^{1/\alpha}}\right). \tag{5.5}$$

Physically the existence of an imaginary part of the Green's functions for negative values of the coupling constant stems from the fact that the Hamiltonian is unbounded from below under these circumstances. Indeed, for negative g the forces among the particles, in a Minkowskian picture, become strongly attractive. If we put N particles in a box with a size comparable to the range of the forces, the potential energy will be proportional to gN^2 while the kinetic energy will be proportional to N for bosons and to $N^{1+1/d}$ for fermions, if $d=D-1$ is the number of spatial dimensions. The difference among bosons and fermions is a combined effect of the Pauli exclusion principle and of the Heisenberg's uncertainty relations. In both cases the leading term is the potential one for large N when $d>1$ ($N>1/|g|$ for bosons, $N>1/|g|^{d/d-1}$ for fermions) so that we can construct collapsed collective states of arbitrary negative energy. The particle density of these states is very large when $|g|\to0$; it is proportional to a negative power of $|g|$. In this situation the vacuum is unstable against decay into these objects. It becomes a metastable state with an exponentially large mean life time and consequently its energy acquires an exponentially small imaginary part.

It is remarkable to note that the techniques used to estimate the large-order behavior of perturbation theory are the same used by Langer[2] in his study of metastability. At this stage the crude argument presented above shows a clear difference among bosons and fermions. In the first case the collapse is always possible, while in the second it is forbidden if $d<1$. The difference is even more striking as we attempt to study the collapsed state. In both instances we deal with a high-density system. In the first one, a Bose-Einstein condensation is likely to occur, leading to a coherent state which is well described by its wave function, i.e., a classical field. It seems, therefore, natural to expect that the effects of these collapsed states can be taken into account by considering the contribution of a classical field configuration to the functional integral. This is unfortunately not true for fermions which do not undergo such a condensation and cannot be described in the high-density limit by a coherent state and the associated classical field. The correct description of such a system is the one given by the Thomas-Fermi approximation.

This analysis suggests that in order to obtain the correct estimates for large perturbative orders of the complete fermionic theory, we must extend the Thomas-Fermi approximation to Euclidean quantum field theory. Before tackling this problem we shall first review this approximation and the related WKB methods by casting them in such a formalism that the extension to our case will be rather immediate.

A final remark is in order concerning QED. In this case the long-range character of the Coulomb forces together with the Fermi statistics could have some bearing on the lower dimensionality for which perturbation theory becomes marginally divergent. While $1+1$ is the relevant dimension of space-time of massive Yukawa couplings, as we saw above, a similar argument yields $1+3$ as the corresponding dimension for QED. Indeed, the potential term would be reduced from gN^2 (g is proportional to e^2) to $gN^{2(1-1/d)}$. Equating the exponents of the kinetic and potential parts of the energy gives $d=D-1=3$ as the lowest divergent dimension.

VI. THOMAS-FERMI APPROXIMATION

The nonrelativistic Thomas-Fermi approximation is used to study the behavior of a Fermi system at high density in the presence of strong external fields (e.g., the electrons around a high-Z nucleus, or highly compressed nuclear matter in a neutron star).

A typical problem which is solved by this approximation is the following: Let us, for simplicity, consider a system of spinless fermions interacting with an external potential $U(x)$ ($U \leqslant 0$). Assume all negative-energy levels are filled and we want to compute the total energy of the system and the fermion density. The total energy $E(U)$, i.e., the sum of the binding energies can be written as

$$E(U)=\int dx\, \epsilon(x|U)\,, \tag{6.1}$$

where the energy density $\epsilon(x|U)$ is a nonlocal, nonlinear functional of the external potential U. Similarly, the fermion density $\rho(x|U)$ will be a complicated functional of U. We want to evaluate these two functionals when the external field is very strong, i.e., when the number of bound states is very large. In this situation most of the bound states have large quantum numbers and we can use for that part of the spectrum the Bohr-Sommerfeld quantization rules. If the problem is three-dimensional, we can associate to each quantum state a region in six-dimensional phase space of size $(2\pi\hbar)^3$. The number N of bound states is the volume in units of $(2\pi\hbar)^3$ of the region of phase space corresponding to negative one-particle energy. All quantities with a classical limit may be obtained as integrals over this allowed classical region. This holds, for instance, for the energy or particle density.

In the case of a simple scalar potential with the one-particle energy given by $(1/2m)p^2+U(x)$, one finds

$$\begin{aligned} N &\sim \int \frac{d^3x\, d^3p}{(2\pi\hbar)^3}\, \theta\left(-\frac{p^2}{2m}-U(x)\right) \\ &= \frac{1}{6\pi^2\hbar^3}\int d^3x[-2mU(x)]^{3/2}\,, \\ E &\sim \int \frac{d^3x\, d^3p}{(2\pi\hbar)^3}\left(\frac{p^2}{2m}-U(x)\right)\theta\left(\frac{p^2}{2m}-U(x)\right) \\ &= -\frac{1}{m30\pi^2\hbar^3}\int d^3x[-2mU(x)]^{5/2}\,. \end{aligned} \tag{6.2}$$

We can use a slightly different language by saying that around each point x we fill the Fermi sphere up to a momentum

$$p_F(x)=[-2m\,U(x)]^{1/2} \tag{6.3}$$

and from (6.3) we recover the approximations (6.2).

Equations (6.2) can be directly derived by observing that the functionals $\rho(x|U)$ and $\epsilon(x|U)$ are nonlocal (by which we mean that they are not really expressed as functions of U and a finite number of its derivatives at x) because the true quantum states are not localized in space. When we use semiclassical methods we do in fact approximate the states by localized ones, and ρ and ϵ become local functions of U. Therefore they can be computed by making a comparison with a situation with (locally) constant U. We use the notation $U(x)=u$ in the vicinity of x and set

$$\rho(x|U)=\rho(u)\,, \quad \epsilon(x|U)=\epsilon(u) \tag{6.4}$$

and we find that the Thomas-Fermi (TF) approximation amounts to writing

$$N_{\rm TF}[U] = \int d^d x\, \rho(U(x)), \qquad E_{\rm TF}[U] = \int d^d x\, \epsilon(U(x)). \tag{6.5}$$

We can therefore regard the Thomas-Fermi approximation as a quasiconstant external field approximation. It has the remarkable property of becoming exact in the limit of very strong potentials. This, as we have seen, follows from the fact that states with large quantum numbers become dominant in such a way that WKB approximation can be used to estimate the individual levels.

In other words, the WKB approximation, the Thomas-Fermi approximation, and the quasiconstant field are strongly related as they are based on the same physical picture. We shall now present a unified framework where all these ideas apply.

Our goal will first be to study a d-dimensional Schrödinger operator with a positive confining potential $V(x)$, i.e., $V(x)$ is unbounded at infinity. We would like to give a proper definition to the Fredholm determinant,

$$\frac{\mathrm{Det}[-\Delta + V(x) - E]}{\mathrm{Det}[-\Delta + V(x)]} = \mathrm{Det}\left(1 - E\frac{1}{(-\Delta + V)}\right) = \prod_i \left(1 - \frac{E}{E_i}\right) \equiv D(E). \tag{6.6}$$

The infinite product in (6.6) runs over all (positive) eigenvalues of the Schrödinger operator. Let us assume for the moment that this infinite product converges (we shall see later what to do if this condition is not satisfied). In this case the function $D(E)$ is well defined. It is an entire function of E, i.e., analytic in the whole complex E plane. Only $\mathrm{Det}[1 - E(-\Delta + V)^{-1}]$ is well defined, the first two determinants are both infinite. Let us define the quantities

$$\nu(E) = \sum_i \delta(E - E_i), \quad N(E) = \int_0^E dE'\, \nu(E'). \tag{6.7}$$

$\nu(E)$ is the density of levels counted with their multiplicity, while $N(E)$ is the number of levels with energy less than E [for the purists $N(E)$ is given by a Stieltjes integral and $\nu(E)$ is its derivative].

An elementary computation shows that in the complex-E-plane cut along the spectrum

$$L(E) = \ln D(E) = E\int_0^\infty dE'\, \frac{N(E')}{E'(E - E')}, \qquad N(E) = -\frac{1}{\pi}\mathrm{Im}L(E). \tag{6.8}$$

The discontinuity equation is, in fact, independent of the Cauchy representation as it can be directly derived by computing the phase of (6.6). Our aim is to obtain information on the asymptotic behavior of $N(E)$ for positive E from the asymptotic behavior of $L(E)$ for negative E. Applying the identities

$$\ln \mathrm{Det}A = \mathrm{Tr}\ln A, \qquad \mathrm{Tr}\ln\frac{A}{B} = \mathrm{Tr}\int_0^\infty \frac{dt}{t}(e^{-tB} - e^{-tA}) \tag{6.9}$$

to the case

$$A = -\Delta + V(x) - E,$$

$$B = -\Delta + V(x)$$

we find

$$L(E) = -\int_0^\infty \frac{dt}{t}\mathrm{Tr}\{\exp[-t(-\Delta + V(x) - E)] - \exp[-t(-\Delta + V(x))]\}, \tag{6.10}$$

where it is understood that the t integration is to be done at the end after explicit subtractions at $t=0$. In the (WKB) limit $\hbar \to 0$, p and x commute so that the trace in (6.10) can be readily evaluated in dimension D as

$$L(E) \sim \frac{-\Gamma(-D/2)}{(4\pi\hbar^2)^{D/2}} \times \int d^D x\{[V(x) - E)^{D/2} - V(x)^{D/2}\}[1 + O(\hbar^2)],$$

$$N(E) \sim \frac{1}{(4\pi\hbar^2)^{D/2}\Gamma(1 + D/2)} \times \int d^D x\, \theta(E - V(x))[E - V(x)]^{D/2}[1 + O(\hbar^2)]. \tag{6.11}$$

In dimension one for convex potentials there are no oscillations in the density of eigenvalues and the WKB quantization condition reads

$$N(E_n) = n + \tfrac{1}{2}. \tag{6.12}$$

The added $\frac{1}{2}$ guarantees that the function $N(E)$ computed from the levels given by (6.12) approximates in the mean the function $N(E)$ given by (6.11) [the true expression $N(E)$ is of course integer-valued while our approximation defines a continuous function].

Corrections to the WKB quantization condition arise when we recall that p and x have a commutator of order $\hbar$ and use the Baker-Hausdorff formula for the exponential of noncommuting operators. An expansion in powers of $\hbar^2$ for $L(E)$ follows:

$$L(E)=\frac{-\Gamma(-D/2)}{(4\pi\hbar^2)^{D/2}}\int d^Dx\left\{[(V(x)-E)^{D/2}-V(x)^{D/2}]-\frac{\hbar^2}{12}\frac{\Gamma(3-D/2)}{\Gamma(-D/2)}(\vec\partial V(x))^2[(V(x)-E)^{(D/2)-3}-V(x)^{(D/2)-3}]+O(\hbar^4)\right\}. \quad (6.13)$$

If the infinite product (6.5), and consequently the dispersive integral (6.8), is not convergent, the determinant of the operator is no longer defined. However, we can still use the same arguments if we generalize the classical Fredholm theory, introducing the concept, familiar in quantum field theory, of renormalized determinant of an operator.

Given an operator A with only a discrete spectrum and eigenvalues of finite multiplicity, such that their only accumulation point is at infinity, a renormalized determinant $D(z)=\mathrm{Det}_R(z-A)$ is an entire function of z which has zeros at the location of the eigenvalues of order equal to the multiplicity of the corresponding eigenvalue. We further require the minimal growth of $|D(z)|$ at infinity consistent with the actual distribution of zeros.

Let us briefly recall some classical results of the theory of entire functions.[17] An entire function $E(z)$ is of order ρ if

$$\overline{\lim}\frac{\ln|\ln|E(z)||}{\ln|z|}=\rho, \quad (6.14)$$

which means that for any positive ϵ $|\ln E(z)|$ is bounded by $(|z|^{\rho+\epsilon}+\text{const})$. Given a point set $\{\lambda_i\}$ with only one accumulation point at infinity, let p be the smallest value such that

$$\sum|\lambda_i|^{-p}<\infty. \quad (6.15)$$

One can construct all the entire functions of order $\rho=p$ which have simple zeros at the points $z=\lambda_i$ (the extension to multiple zeros is straightforward). They are given by

$$E(z)=\exp[P_{\rho_0}(z)]\times\prod_i\left\{\left(1-\frac{z}{\lambda_i}\right)\exp\left[\sum_1^{\rho_0}\frac{(-1)^{s+1}}{s}\left(\frac{z}{\lambda_i}\right)^s\right]\right\}, \quad (6.16)$$

where ρ_0 is the integer part of ρ and $P_{\rho_0}(z)$ is an arbitrary polynomial of degree ρ_0 representing the remaining normalization arbitrariness.

It is evident that in the Fredholm case the order of the entire function is less than 1, so that it is essentially unique and the renormalized determinant is proportional to the standard Fredholm determinant. The renormalized determinant is the natural generalization of the Fredholm theory. The Hadamard prescription for the infinite product just corresponds to the minimal subtractions at $E=0$ in the dispersive integral (6.8) to insure convergence. The origin of these extra subtractions will be clearer in the next example where we use the same line of arguments to recover the results of the Thomas-Fermi approximation.

Let us consider the following determinant:

$$\mathfrak{D}(g)=\frac{\mathrm{Det}(-\Delta+m^2+gV)}{\mathrm{Det}(-\Delta+m^2)}=\mathrm{Det}[1+g(-\Delta+m^2)^{-1/2}V(-\Delta+m^2)^{-1/2}], \quad (6.17)$$

where V is a smooth positive potential with fast decrease at infinity. The self-adjointness of the operator $(-\Delta+m^2)^{-1/2}V(-\Delta+m^2)^{-1/2}$ implies that the spectrum of $-\Delta+gV$ is concentrated on the negative g real axis. The eigenvalues of finite multiplicity correspond to those values $1/\lambda_i<0$ for which a bound state of energy $-m^2$ appears in the potential $\lambda_i V$.

If we call $\mathfrak{N}(g)$ the number of bound states in the potential gV having a binding energy larger than m^2, we have

$$\ln\mathfrak{D}(g)\equiv\mathfrak{L}(g)=g\int_{-\infty}^0\frac{dg'}{g'}\frac{\mathfrak{N}(g')}{(g-g')}=-\int_0^\infty\frac{dt}{t}\mathrm{Tr}(e^{-t(-\Delta+gV+m^2)}-e^{-t(-\Delta+m^2)})=\sum_i\ln\left(1-\frac{g}{\lambda_i}\right). \quad (6.18)$$

If we neglect terms of order $\hbar$ we find

$$\mathfrak{L}(g)=-\frac{1}{(4\pi)^{D/2}}\int_0^\infty\frac{dt}{t^{1+D/2}}\mathrm{Tr}e^{-t(m^2+gV)} \quad (6.19)$$

and when D, the space dimension, is less than two, the t integral requires only one subtraction corresponding to the normalization of the determinant for zero coupling ($g=0$). Irrespective of this subtraction the large-g behavior of (6.19) is

$$\mathfrak{L}(g)\to-\frac{\Gamma(-D/2)}{(4\pi)^{D/2}}\int d^Dx(gV)^{D/2}, \quad (6.20)$$

which implies the Thomas-Fermi result

$$\mathfrak{N}(g)\to|g|^{D/2}\frac{V_D}{(2\pi)^D}\int d^Dx\,V(x)^{D/2}, \quad (6.21)$$

where $V_D=2\pi^{D/2}/D\Gamma(D/2)$ is the volume of the unit sphere in D-dimensional space.

In quantum field theory the same determinant stands for the sum of all vacuum diagrams in the external potential $gV(x)$, $\mathcal{L}(g)$ is the sum of all connected vacuum diagrams. From the representation (6.18) we have the sum rules

$$\sum \frac{1}{(\lambda_i)^s} = (-1)^s \int d^D x_1 \cdots d^D x_S \Delta(x_1 - x_2) \cdots \times \Delta(x_{S-1} - x_S)\Delta(x_S - x_1) \times V(x_1) \cdots V(x_S), \tag{6.22}$$

where the right-hand side is the expression for the s-order Feynman diagram in the external potential V (Δ is the free propagator). The diagrammatic representation of Fig. 1 is valid here also. The fact that diagrams of order $S < \frac{1}{2}D$ are ultraviolet divergent is deeply connected with the asymptotic properties of the number of bound states for large coupling in a smooth potential. Moreover, the subtraction procedure of quantum field theory, which consists in subtracting the divergent part of a diagram in order to define a renormalized finite vacuum to vacuum amplitude, has a counterpart the introduction of subtractions in the Hadamard infinite product representation, which are needed to define a renormalized determinant.

This correspondence between field-theoretical subtractions and representations of entire functions is well known in constructive theory; see for instance the Yukawa case, Ref. 11.

A slightly different approach leading to the same results uses the identity

$$\frac{d\mathcal{L}(g)}{dg} = \int d^D x\, G(x, x|gV)V(x), \tag{6.23}$$

where $G(x, y|gV)$ is the Green's function of the operator $-\Delta + m^2 + gV$. In order to compute $G(x, x|gV)$ we expand V around its value at the point x,

$$V(y) = V(x) + (y-x)_\mu \partial_\mu V(x) + \tfrac{1}{2}(y-x)_\mu (y-x)_\nu \frac{\partial^2 V(x)}{\partial x_\mu \partial x_\nu} + \cdots, \tag{6.24}$$

and proceed following our preceding remarks by taking a leading quasiconstant V approximation, i.e., we keep $V = V(x)$ as the zero-order approximation treating the remaining terms as perturbations. This is equivalent to the reasoning leading to Eqs. (6.4) and (6.5). We can control the consistency of this procedure. The higher-order corrections will be negligible only if the propagator $G(x, y|gV)$ is essentially concentrated in a sphere $|x - y|^2 \lesssim O((gV)^{-1})$. When this condition is satisfied, the zeroth-order propagator will be equal to the free one with an effective mass $m^2 + g^2V^2$ and it will be exponentially damped for $|x-y|^2|gV(x)| \gg 1$.

The reader who has gone through this unfamiliar review of semiclassical methods is ready for tackling the large-order behavior of the perturbative expansion for fermionic interactions.

VII. LARGE ORDERS OF FERMIONIC INTERACTIONS

Let us return to our original problem, the evaluation of the functional integral (5.1) or equivalently (2.1) in the QED case. The integration over the anticommuting Fermi fields can be exactly done as in Sec. II (the integral is Gaussian), yielding a renormalized determinant:

$$\begin{aligned} Z(g) &= \int \mathfrak{D}(\sigma)\, \mathrm{Det}_R(i\gamma\cdot\partial + M + g) \times \exp\left(-\int d^D x \tfrac{1}{2}[(\partial\sigma)^2 + m^2\sigma^2]\right) \\ &= \int \mathfrak{D}(\sigma) \exp\left(-\int d^D x \tfrac{1}{2}[(\partial\sigma)^2 + m^2\sigma^2] - \mathcal{L}_E(g\sigma)\right), \end{aligned} \tag{7.1}$$

$$\mathcal{L}_E(g\sigma) = -\ln \mathrm{Det}_R(i\gamma\cdot\partial + M + g\sigma).$$

If $2 < D < 4$ only one subtraction is needed, corresponding to the first diagram of Fig. 1, which gives a divergent contribution to the boson self-energy. Both cutoffs have been removed. The Fermi fields have been eliminated and we can now apply the standard techniques of Refs. 4–6 to the functional integral (7.1).

As discussed in Sec. V we can reduce the functional integral to a one-dimensional one along the direction σ_0, and only at the end will we take care of the integration in the transverse direction. We therefore find

$$\begin{aligned} Z(g) &\simeq \int_{-\infty}^{+\infty} da\, E(ag)e^{-a^2}, \\ E(z) &= \mathrm{Det}_R(i\gamma\cdot\partial + M + z\sigma_0), \end{aligned} \tag{7.2}$$

where the function σ_0 has been normalized in such a way that its contribution to the free action is unity:

$$S(\sigma_0) \equiv \int d^D x \tfrac{1}{2}[(\partial\sigma_0)^2 + m^2\sigma_0^2] = 1. \tag{7.3}$$

The function E looks similar to the Borel transform of $Z(g)$. Indeed, the following relations hold:

$$\begin{aligned} Z(g) &= \sum_k Z_k g^k, \quad E(z) = \sum_k E_k z^k \\ Z_k &\simeq \tfrac{1}{2}\Gamma(k + \tfrac{1}{2})E_{2k}. \end{aligned} \tag{7.4}$$

Assume for a moment that the spin-$\frac{1}{2}$ particles were bosons. Then instead of the determinant we would have found its inverse in Eq. (7.1) and the function $E^{-1}(z)$ in (7.2) and (7.4). Since the determinant is an entire function with infinitely many zeros its inverse is a meromorphic function with poles. Therefore one would have obtained

$$E_{2k}^{-1} \sim (-r)^{-2k}, \quad Z_k \sim k!(-r)^{-2k}, \tag{7.5}$$

where r is the distance to the nearest zero of $E(z)$ (we suppose for simplicity that the zeros occur by complex-conjugate pairs on the imaginary axis). The correct estimates would have been again obtained by minimizing r, i.e., finding the regular field σ_0 such that $(i\gamma\cdot\partial + M + z\sigma_0)\psi = 0$ yields a minimal $|z|$ for $S(\sigma_0)=1$. The same approach could have been used in the standard ϕ^4 theory, at the price of introducing a field σ conjugate to ϕ^2, and would of course have produced the known results. We have seen that it was essentially the strategy which produces the one-loop estimates. It is, by now, clear why the techniques used in the study of asymptotic estimates in perturbation theory have so much in common with those devised in the search of optimal conditions for the existence of bound states in potential theory. In the case of fermions, i.e., if the entire function $E(z)$ is used in (7.2) and (7.4), this mechanism for producing a factorial increase fades away. Indeed, if the integral over a in (7.2) was on a bounded domain, $Z(g)$ would be analytic in g for small g. The only possible origin of singularity at $g=0$ comes therefore from possible divergences of the integral at infinity.

When we compute the kth order of the perturbation expansion, the functional integral (7.1) will be dominated by the integration region where $|g\sigma| \gg 1$. In this region the behavior of the determinant is controlled by the asymptotic distribution of its zeros [while in the bosonic case the corresponding region is $|g\sigma| \sim O(1)$ and the location of only one zero is crucial].

In this striking difference between fermions and bosons, the reader will recognize the counterpart in Euclidean quantum field theory of the phenomena discussed in Sec. V. In the case of boson collapse only one state is relevant and has macroscopic $[O(1/g)]$ occupation number (Bose-Einstein condensation), while in the collapse of a fermionic system a very large number of different levels become important.

It is crucial to note that the behavior of the entire function $E(z)$ is connected to its order ρ. Roughly

$$E_k \sim \frac{1}{\Gamma(k/\rho)}\,. \tag{7.6}$$

More precise estimates follow from the representation

$$E_k = \frac{1}{2\pi i}\oint \frac{dz}{z^{k+1}}\,E(z)$$

and looking for a saddle point for large k using the bounds on $E(z)$ for $|z|\to\infty$. In this way one finds

$$\begin{aligned} E_k &\sim \frac{1}{\Gamma(k/\rho)}\,\mathrm{Re}\left(\frac{1}{R(\sigma_0)}\right)^k, \\ Z_k &\sim \Gamma\left(k\left(1-\frac{2}{\rho}\right)\right)\mathrm{Re}\left(\frac{1}{R(\sigma_0)}\right)^{2k}, \end{aligned} \tag{7.7}$$

where the functional $R(\sigma_0)$ can be obtained from the asymptotic behavior of $E(z)$. The same result may be directly inferred by writing

$$Z_k = \frac{1}{2\pi i}\oint \frac{dz}{z^{k+1}}\int_{-\infty}^{+\infty} da\,E(za)e^{-a^2}$$

and applying the steepest-descent method to the double (z,a) integral.

Up to now we have not used the particular structure of the integral (7.1). The discussion applies to an arbitrary trilinear interaction among bosons and fermions independently of the detailed form of the Langrangian. We can summarize the steps as follows:

(i) Integrate over the fermions.

(ii) Obtain a relation as (7.2) where $E(z)$ is the fermionic determinant.

(iii) Find the order ρ of the entire function $E(z)$.

(iv) Compute $R[\sigma]$ and find the field $\bar\sigma$ which minimizes it (this leads to the stationarity condition $\delta R/\delta\bar\sigma = 0$).

(v) Consider the effects of quadratic transverse fluctuations.

The dominant asymptotic behavior will then be given by

$$Z_k \sim \Gamma\left(k\left(1-\frac{2}{\rho}\right)\right)\mathrm{Re}\left(\frac{1}{R(\bar\sigma)}\right)^{2k}. \tag{7.8}$$

For the Yukawa interaction, we find that the function $E^2(z)$ is given in terms of a determinant $\mathfrak{D}(z)$:

$$E^2(z) = \mathfrak{D}(z) = \det{}_R[-\Delta + (M+z\sigma)^2 - iz\gamma\cdot\partial\sigma]\,. \tag{7.9}$$

Using the method described in the previous section we discover that the spin term (in $\gamma\cdot\partial\sigma$) is irrelevant for large z and that

$$E(z) \sim \exp\left(\frac{-\Gamma(-D/2)}{2(4\pi)^{D/2}}\int d^Dx\{[M+z\sigma(x)]^2\}^{D/2}\right), \quad z \text{ real}. \tag{7.10}$$

For large values of the field σ, and consequently for large orders of the perturbation expansion the effective interaction among bosons becomes local and if we write it as in (7.1), it takes the form

$$\mathfrak{L}_E \sim \int d^Dx\,|g\sigma(x)|^D \tag{7.11}$$

up to a coefficient. Similarly, $R^{-1}(\sigma_0)$ is easily computed and is proportional to the integral

$$\int d^D x |\sigma_0(x)|^D . \tag{7.12}$$

We remember that we must minimize $R[\sigma_0]$ at fixed free action $S(\sigma_0)$. Therefore using a standard normalization, $\bar{\sigma}$ satisfies the differential equation

$$(-\Delta + m^2)\bar{\sigma} = \bar{\sigma}^{D-1} . \tag{7.13}$$

The evaluation of $R(\bar{\sigma})$ is now easy and the final results read

$$\begin{aligned} &Z_k \sim C^k , \quad D<2 \\ &Z_k \sim \cos\left(\frac{2\pi k}{D}\right)\Gamma\left(\frac{D-2}{D}\,k\right)R(D)^{-k} , \quad 2<D<4 . \end{aligned} \tag{7.14}$$

In less than two dimensions the perturbative expansion is convergent. In even dimensions additional complications are present. Some diagrams are logarithmically divergent and additional logarithms appear in the exponent of (7.10). In dimensions 2 and 4 we find respectively

$$\begin{aligned} &Z_k \sim (\ln k)^k \left(-\frac{1}{R(2)}\right)^k , \quad D=2 \\ &Z_k \sim \Gamma\left(\frac{k}{2}\right)(\ln k)^k \left(-\frac{1}{R(4)}\right)^k , \quad D=4 . \end{aligned} \tag{7.15}$$

These estimates depend crucially on the ultraviolet behavior of the theory, which in the renormalizable case may differ from that of the free theory. The estimates (7.15) should therefore be considered valid in four dimensions provided the cutoff Λ_b pertaining to the boson system is kept finite. We are not going to elaborate here any further on the subtle effects possibly induced by renormalization on the asymptotic estimates (7.15). This important point should clearly be further carefully studied.

It may be interesting to observe what happens when a cutoff is introduced in the kinetic energy of the fermions. The order of the fermionic determinant decreases from $D/2$ to $D/4$ and the theory remains convergent up to $D=4$:

$$\begin{aligned} &Z_k \sim C^k , \quad D<4 \\ &Z_k \sim \Gamma\left(\frac{D-4}{2}\,k\right) , \quad D>4 . \end{aligned} \tag{7.16}$$

The above results are in perfect agreement with Caianiello's theorem. When the Fermi propagator is bounded in x space [$D<1$ without cutoff, $D<3$ with the cutoff given by (7.1)] the perturbation series is convergent. A necessary and sufficient condition for a convergent expansion in the Yukawa case is presumably

$$\lim_{x\to y} \operatorname{tr} G_F(x-y) < \infty , \tag{7.17}$$

with the trace taken over spin and internal degrees of freedom, a condition which generalizes the one used by Caianiello $[\lim_{x\to y} G_F(x-y)<\infty]$.

The extension of the method to QED should appear without difficulty. This is, unfortunately, not the case due to new difficulties related to gauge invariance and current conservation. We shall close this section by presenting these problems. For simplicity, let us restrict our attention to the vacuum-to-vacuum amplitude $Z(e)$. Instead of (2.10) we therefore consider

$$Z(e) = \int \mathfrak{D}(A)\,\mathrm{Det}(i\gamma\cdot\partial - e\gamma\cdot A| - M)e^{-S_A} \tag{7.18}$$

and for the sake of the argument we drop the gauge dependence parameter λ so that $S_A = \int dx\,\frac{1}{4}F^2$.

In the one-mode approximation $A_\mu = aA_\mu^{(0)}$ we find

$$Z(e) = \int_{-\infty}^{+\infty} da\, C_{(2)}(eA)e^{-a^2} , \tag{7.19}$$

where

$$\begin{aligned} &C_{(2)}{}^2(z) = H_{(2)}(z) , \\ &H_{(g)}(z) = \mathrm{Det}\left[+(\partial + izA^{(0)})^2 - M^2 + \frac{zg}{2}\sigma_{\mu\nu}F_{\mu\nu}\right] , \\ &\int dx\,\tfrac{1}{4}F^{(0)2} = 1 . \end{aligned} \tag{7.20}$$

We have explicitly generalized the determinant $H(z)$ to an arbitrary magnetic coupling g for reasons which will appear clear below. According to the previous arguments Eq. (7.8) will hold if ρ is the order of $H_{(2)}(z)$. We are of course considering such fields $A^{(0)}$ decreasing fast enough at infinity to make the determinant in (7.18) well defined. Relaxing this hypothesis might enable one to understand the origin of infrared problems.

If the zeros λ_i of $H_{(2)}(z)$ are concentrated near the imaginary axis (as it happens in the Yukawa case), there is no difference between a condition $\sum_{i=0}^{\infty}\lambda_i{}^{-k}<\infty$, which implies the finiteness of the kth perturbative order of the determinant and the condition $\sum_{i=0}^{\infty}|\lambda_i|^{-k}<\infty$ which implies that the order ρ is smaller than or equal to k. From gauge invariance the photon self-energy is logarithmically and not quadratically divergent and the scattering of light by light convergent. If we were to stick to the hypothesis on the location of the λ_i's close to the imaginary axis, we would find that the order of the entire function $H_{(2)}(z)$ is $\rho=2$ (its type would be infinite, meaning that $\sum_i |\lambda_i|^{-2-\epsilon}$ would converge for any positive ϵ but not for $\epsilon=0$) and we would conclude that

$$Z_k \sim (\ln k)^k . \tag{7.21}$$

In fact, we are able to derive this result without any hypothesis on the distribution of the zeros only when $g<2$, while some doubt exists on its validity when $g=2$. Let us briefly sketch the argument.

If we apply the semiclassical methods of Sec. VI to the function $H_g(z)$, we find at first that the spin term looks negligible in the limit $h\to 0$. Furthermore, it is certainly insufficient to consider a constant A_μ field which could be gauged away. One must rather take $F_{\mu\nu}$ as locally constant. The main term of the effective interaction

$$\int dx\, \mathcal{L}_{(g)}(zF) = -\tfrac{1}{2}\ln H_{(g)}(z) \tag{7.22}$$

is therefore given by the Euler-Heisenberg (EH) Lagrangian

$$\mathcal{L}^{\mathrm{EH}}_{(g)}(F) = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-M^2 s} \times\left[\lambda_1\lambda_2\,\frac{\cosh\frac{1}{2}gs\lambda_1}{\sinh s\lambda_1}\,\frac{\cosh\frac{1}{2}gs\lambda_2}{\sinh s\lambda_2}\right], \tag{7.23}$$

where two subtractions in the Laurent expansion in s of the integrand are understood and $\lambda_{1,2}$ are built out of the two invariants $F^2\equiv F_{\mu\nu}F_{\mu\nu}$ and $F\tilde{F}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$:

$$\lambda_{1,2}{}^2 = \tfrac{1}{4}\{F^2 \pm [(F^2)^2-(F\tilde{F})^2]^{1/2}\}\,. \tag{7.24}$$

In the quasiconstant field method of Sec. VI the expression (7.23) though computed for constant F is now used in the action (7.22) for the effective Lagrangian

$$\int dx\, \mathcal{L}_{(g)}(zF) \sim \int dx\, \mathcal{L}_{(g)}[zF(x)]\,. \tag{7.25}$$

This computation seems to confirm that $H_{(g)}(z)$ is of order 2 for $g<2$ by analyzing the large-field behavior of (7.23). However, for $g>2$ the Euler-Heisenberg Lagrangian diverges for strong enough fields and the physical value $g=2$ is marginal for a consistent application of the quasiconstant field method. In a sense it is gratifying to see that spin and not only Fermi statistics comes into play through magnetic effects, even though it makes life a little harder.

As was explained in the previous section, the validity of the method requires that the propagator in the presence of strong external fields is damped outside a sphere with vanishing radius as the field tends to infinity. A simple computation shows that this is the case for all $g<2$ but fails for $g=2$. This appears clearly on (7.23) where the region $s\neq 0$ is exponentially suppressed when $g<2$ but not when $g=2$. This fact can be traced to a well-known phenomenon: In the presence of a constant magnetic field the energy of a spinning particle in its ground state behaves as

$$E_H \propto -|H|(g-2)\,; \tag{7.26}$$

it goes to infinity with H as $g<2$ but is independent of the strength of the field if $g=2$.

To conclude, if we take correctly into account Fermi statistics but if we neglect the effects of spin the asymptotic estimate (7.21) holds (we assume a cutoff photon propagator to avoid the effects of renormalization). In order to decide whether (7.21) is still valid in the actual theory ($g=2$) it is necessary to proceed to a more careful, but mathematically well-defined analysis.

In the last article quoted in Ref. 12 Adler has investigated the set of eigenvalues pertaining to a specific $A_\mu(x)$. Using symmetry arguments he was able to reduce Dirac's equation to a simple radial equation and to obtain WKB estimates for the large eigenvalues. His result would support an $H_{(2)}(z)$ of order 4. If this were the general case the asymptotic estimates (7.21) would be replaced by

$$Z_k \sim \Gamma\left(\frac{k}{2}\right). \tag{7.27}$$

The zeros of $H_{(2)}(z)$ would then spread in the complex plane, and the convergence of $\sum_i \lambda_i{}^{-4}$ would give no information on the series $\sum_i |\lambda_i|^{-4}$.

The problem of estimating the asymptotic behavior of the perturbation series in QED in terms of the bare coupling constant is therefore left open. However, its solution has been reduced to a clearly stated mathematical problem: to find the order of an entire function or, even more specifically, to study the distribution of bound states in a strong magnetic field.

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