Nuclear Physics B220[FS8] (1983) 415–433
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DISTRIBUTION OF ZEROS IN ISING AND GAUGE MODELS

C. ITZYKSON, R.B. PEARSON* and J.B. ZUBER

Service de Physique Théorique, CEN SACLAY, 91191 GIF-sur-YVETTE Cedex, France

Received 5 April 1983

We discuss some features of Ising and gauge systems in the complex temperature plane. The distribution of zeros of the partition function enables one to study critical properties in a way complementary to the methods using real values. Data on small lattices confirm this picture. Nearby complex singularities seem to exhibit a universal behavior which might have some relation with a model of random surfaces.

1. Introduction

The partition function of a simple statistical system on a finite lattice can often be expressed as a polynomial in a finite number of variables. One way to visualize the properties of this polynomial is to study the location of its zeros. This is a subject with a long history, a landmark of which is the Lee-Yang [1] circle theorem in the activity plane (e^{-2h}, h magnetic field) for an Ising ferromagnetic system, but it has also been extensively studied in other variables like the temperature by Fisher and others [2].

With the advent of large computing facilities a number of difficult analytical problems can now be tackled by numerical methods [3]. We suggest that the location of these complex zeros and their behavior in the thermodynamic limit is an interesting subject, enabling one to confirm the scaling picture, to give independent measurements of critical indices and universal ratios of amplitudes and possibly to discover new phenomena.

In sect. 2 we present such an analysis of the scaling behavior of zeros and define two critical angles φ and ψ. The former describes the slope of the line of zeros with the real axis in the temperature complex plane in the vicinity of the critical point in the absence of magnetic field. The second is the angle at which these zeros depart in a real magnetic field. If α is the specific heat critical exponent, A_-/A_+ the ratio of

* Institute of Theoretical Physics, Santa-Barbara, Ca. 93106, USA.

specific heat amplitudes below and above the critical temperature, then we find that

$$\tan[(2-\alpha)\varphi] = \frac{\cos(\pi\alpha) - A_{-}/A_{+}}{\sin \pi\alpha}, \tag{1}$$

with φ predicted to be $\frac{1}{4}\pi$ in a mean field approximation. Similarly the angle ψ is related to the critical exponents β (of the spontaneous magnetization) and δ of the relation between field and magnetization at critical temperature ($h \sim m^{\delta}$) as

$$\psi = \frac{\pi}{2\beta\delta}. \tag{2}$$

Of course these angles are conformal invariant and therefore independent of a regular parametrization of the vicinity of the critical point. These considerations are complementary to those pertaining to the divergence of the moduli of the various quantities in the scaling region.

The occurrence of complex zeros that eventually stabilize along lines (or regions in more complicated cases which are not studied in this paper) does not in general indicate singularities in the thermodynamic quantities. They are rather to be interpreted as Stokes lines which separate different asymptotic behaviors of the partition function in the thermodynamic limit. In general along these lines the real part of the analytic continuation of the free energy will be continuous while the discontinuity of the imaginary part is proportional to the density of zeros. The singularities of the free energy will in fact be located at the ends of these lines of zeros. We do not present mathematical proofs of the existence of these analytic continuations but we show numerical evidence from small lattices supporting these claims.

The same data suggest a global structure of these lines for the three-dimensional Ising model taken as a typical example which may be partly analysed in terms of simple approximations such as the Bethe approximation [4] which we study in sect. 3. One interpretation among others of this approach is to consider that it approximates the geometry, i.e. the lattice, rather than the physics. Its precise form is lattice dependent which is also the case of the global distribution of zeros. The Bethe approximation enables one therefore to understand some gross features of the latter and in particular the occurrence of nearby complex singularities which may control the low-temperature series. Although the pattern of these complex singularities is geometry dependent, they exhibit some critical behavior which seems to have a certain degree of universality as already noticed in the late sixties [5]. We have analysed the longest available low-temperature series on various lattices. Although not very precise, the values for the exponents of these singularities are incompatible in three dimensions with their mean field value. In particular the free energy is predicted by mean field to have a $(u-u_c)^{3/2}$ singularity while the corresponding measured critical exponent ranges around 0.9 instead of 1.5. A possible relation between these singularities and the Lee-Yang edge singularity remains problematic. We argue that similar complex singularities are also expected for lattice gauge theories, irrespective of the gauge group, in their high-temperature expansions. This follows from duality in three dimensions for an abelian symmetry group and extends in any dimension for an arbitrary group in some appropriate temperature like parameter. The analog of the Bethe approximation can be identified for infinite dimension with the one studied by Drouffe, Parisi and Sourlas [6]. Such approximations lead to similar complex singularities. Analysis of the much shorter Wilson series available in four dimensions shows that the type of singularity is much closer to the mean field prediction.

These facts suggest a connection with a model of random surfaces, but we are unable to substantiate this claim quantitatively.

Our investigations are only preliminary. Further numerical and analytical work would clearly be necessary to clarify some of the points raised in this paper.

2. Analytic structure near the critical point

In this section we concentrate on the properties of the Ising model near to the critical point while in sect. 3 we consider the global structure. In order to correctly reproduce the non-trivial scaling behaviour near the critical point we must incorporate the renormalization group properties of the model into our description.

To set the notation define the partition function by

$$Z_L = \sum_{(\sigma)=\pm 1} \exp\left\{ \beta \sum_{\langle ij \rangle} (\sigma_i\sigma_j - 1) + h \sum_i (\sigma_i - 1) \right\}, \tag{2.1}$$

where we take for simplicity interactions between nearest-neighbor spins on a regular lattice of linear dimension L which may be taken infinite in the end. The set of all configurations can be divided into classes with a fixed total number of spins with $\sigma_i = -1$ and a fixed total number of pairs of spins with $\sigma_i\sigma_j = -1$. If the number of configurations in such a class is denoted by $C_{m,n}$ then

$$Z_L = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{m,n} y^m t^n, \tag{2.2}$$

where

$$y = e^{-2h}, \tag{2.3}$$

$$t = e^{-2\beta}, \qquad (= \tanh(\beta^*)), \tag{2.4}$$

and M and N are the total numbers of sites and bonds respectively in the lattice. The important point is that Z_L is a polynomial in two variables, and so we may parametrize its analytic structure entirely in terms of its zeros. There are two complementary viewpoints which are most natural. If we fix β, or equivalently t, then Z is a polynomial of degree M in y. The zeros in y are often called Lee-Yang zeros because of the analysis of critical behaviour in terms of them given by these authors and for the remarkable theorem stating that for real positive β, or for t between 0 and 1 on the real axis, all of the zeros of Z as a function of y occur on the unit circle [1]. On the other hand if we fix h, or y, then Z is a polynomial of degree N in t. In two dimensions all of these zeros occur on a pair of circles when there is no magnetic field, but in the general case there is no simple result for the locus of zeros. Fisher emphasized the study of these zeros [2] quite some time ago, (so we might call them Fisher zeros).

In spite of the absence of a simple expression for the locus of zeros in the general case, there are empirically some observed regularities. In particular the zeros seem to fall naturally on smooth arcs (see fig. 1) which we are tempted to identify with cuts in the thermodynamic limit. This sort of behaviour can be easily understood. Because of the very large number of terms in the polynomials which make up the partition function the behaviour in different regions of the complex plane tends to become dominated by some set of the coefficients and is essentially independent of others. This becomes exact in the thermodynamic limit. Thus we have different analytic functions (or types of qualitative behaviour) in different regions of the complex plane. These functions have oscillating phases, but smoothly varying amplitudes. In the general case one type of behaviour will dominate, but there will be boundary regions where two types of behaviour have comparable magnitude. In these regions there can be cancellations and thus zeros. These boundary regions are called Stokes boundaries [7] and in the thermodynamic limit they become cuts. It is clear from this discussion that the natural boundary condition across these cuts is that the real part of the free energy is continuous.

If L is reasonably large, but not infinite, then in the space of parameters we may identify three regions. (i) There are points well away from the critical point where the finite lattice is indistinguishable from the infinite-volume system. That is to say $L \gg \xi \simeq 1$ where ξ is the correlation length and the lattice spacing is taken as a unit. (ii) There are points in the critical or scaling region for which $L \gg \xi \gg 1$ so that the finite lattice exhibits the same scaling behaviour as the infinite system. And (iii) there are points for which $L \simeq \xi \gg 1$ for which one observes significant finite lattice or rounding effects. The renormalization group may be used to characterize the behaviour of both regions (ii) and (iii) which are properly called the scaling region. The result of this full analysis is called finite size scaling following Barber and Fisher [8]. Since for L finite Z is described in terms of its zeros there must be a scaling theory for their location.

If we assume that we know the identity of the scaling fields and operators we can express the RG transformation properties of the hamiltonian under a change of

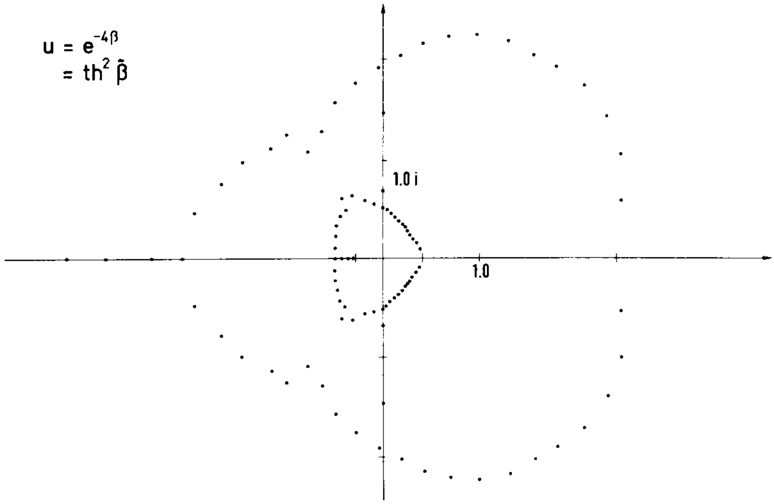


Fig. 1. The zeros of the partition function in the complex $u = t^2$ plane for a $4 \times 4 \times 4$ cubic lattice (see ref. [3]).

length scale b as:

$$H_L = H_L^* + K_1 O_1^L + K_2 O_2^L + \cdots$$
$$\to H_{L/b}^* + K_1 b^{y_1} O_1^{L/b} + K_2 b^{y_2} O_2^{L/b} + \cdots + \varepsilon_0(K_1, K_2 \ldots), \tag{2.5}$$

where O_i^L is a local operator summed over a lattice of size L and ε_0 is a new constant term which is a regular smooth function of $K_1, K_2 \ldots$. Expressed in terms of the total free energy this gives the familiar scaling equation

$$F_L(K_1, \ldots) = F_{L/b}(K_1 b^{y_1}, \ldots) + \varepsilon_0(K_1, \ldots). \tag{2.6}$$

For the partition function we have simply

$$Z_L(K_1, \ldots) = Z_{L/b}(K_1 b^{y_1}, \ldots) e^{\varepsilon_0}. \tag{2.7}$$

If we choose to scale by a fixed fraction, say λ, of L and if we notice that since ε_0 is bounded, e^{ε_0} cannot vanish, then the zeros of Z_L are the same as the zeros of

$$Z_{1/\lambda}(K_1 L^{y_1} \lambda^{y_1}, \ldots) = Q(x_1, x_2, \ldots), \tag{2.8}$$

where we have introduced scaling variables

$$x_i = K_i L^{y_i}, \tag{2.9}$$

and the scaling function Q is an analytic function of its arguments when they are small. The first two operators are the leading thermal and magnetic operators with

$$y_t = 1/\nu, \tag{2.10}$$

$$y_h = \beta\delta/\nu = (\beta + \gamma)/\nu. \tag{2.11}$$

For large enough L it is reasonable to neglect the effect of other operators which will give rise to corrections to scaling. In this approximation we have a very simple scaling result for the zeros of the partition function

$$Q(x_t, x_h) = Q(KL^{1/\nu}, hL^{\beta\delta/\nu}) = 0, \tag{2.12}$$

where

$$K = (u - u_c), \qquad u = t^2. \tag{2.13}$$

Because the partition function is even in h, Q can only be a function of h^2 which is smooth by our hypothesis, so we may formally solve the equation (2.12) to give

$$h^2 L^{2\beta\delta/\nu} = f_i(KL^{1/\nu}), \tag{2.14}$$

where i labels the ith root. By hypothesis $f_i(x)$ is an analytic function in x and we can invert this relation to give

$$KL^{1/\nu} = f_i^{-1}(h^2 L^{2\beta\delta/\nu}). \tag{2.15}$$

If we take $K = 0$ or $u = u_c$ and L large we have the closest Lee-Yang zeros behaving as

$$h_i^2 = L^{-2\beta\delta/\nu} f_i(0) < 0, \tag{2.16}$$

where the sign of $f_i(0)$ is determined by the Lee-Yang theorem. If $h = 0$ then

$$K_i = L^{-1/\nu} f_i^{-1}(0), \tag{2.17}$$

where $f_i^{-1}(0)$ is a complex number in general. If we hold K fixed and take L to infinity then, for any finite i, h tends to the position of the Lee-Yang singularity which is L independent so that

$$f_i(x_t) \underset{x_t \to \infty}{\to} -Cx_t^{2\beta\delta}, \tag{2.18}$$

or

$$h_i^2 \to -CK^{2\beta\delta}, \tag{2.19}$$

which is the well known scaling result for the gap in the Lee-Yang singularity [9].

The inverse of this relationship implies, (for a non-zero field and large L or fixed L and for a strong field, while still in the scaling region) that

$$K_i \to e^{i\pi/2\beta\delta}\left(h/\sqrt{C}\right)^{1/\beta\delta}. \tag{2.20}$$

This predicts that the trajectories of the zeros close to the critical point in a magnetic field will have an angle

$$\psi \equiv \pi/2\beta\delta \simeq 58°, \tag{2.21}$$

and this is observed, see fig. 2. For very strong fields one expects a breakdown of scaling and a crossover to a trivial limiting behaviour. The large-h limit is described by mean field exponents and gives an angle of exactly 60° which is unfortunately quite close to the scaling prediction.

The Lee-Yang theorem implies the fairly strong result that $f_i(x_t)$ is real and negative for arbitrary real $x_t = KL^{1/\nu}$. This state of affairs might appear as in fig. 3a.

The behaviour for large positive x_t has already been argued (eq. (2.18)). For large negative x_t, which corresponds to the low-temperature limit, there is no longer any

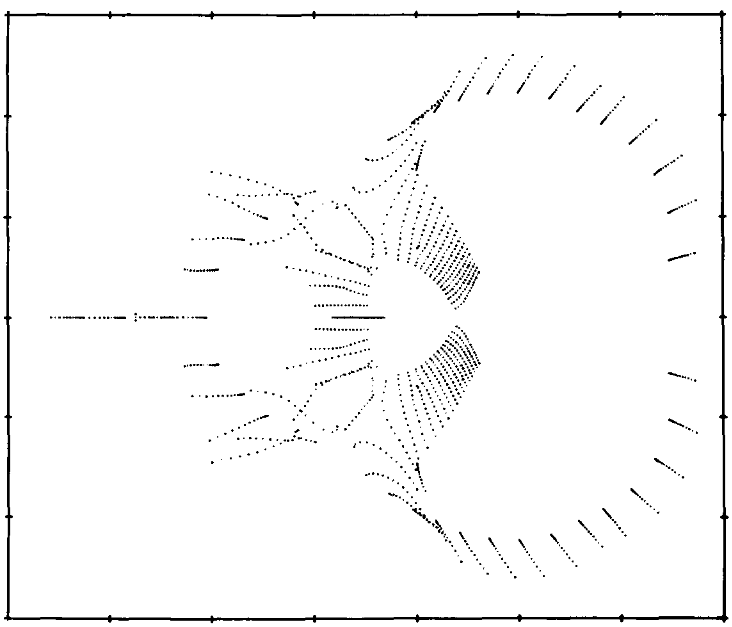


Fig. 2. Motion of the zeros in a magnetic field: h varies by 0.1 from 0 to 1.5.

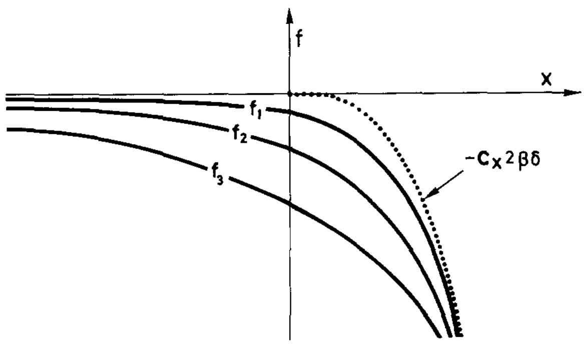
gap and scaling breaks down to a situation where there is a uniform distribution of zeros in h. This implies that $f_i(x_t) \to 0$ for $x_t \to -\infty$ for all i. On fig. 3b we display the actual trajectory of the first nine zeros on the 4^3 lattice, for real β $0.16 \leqslant \beta \leqslant 0.29$ ($\beta_c = 0.2217$). This agrees with the qualitative behaviour of fig. 3a in its central region.

At $x_t = 0$ the distribution must reproduce the behaviour of the magnetization

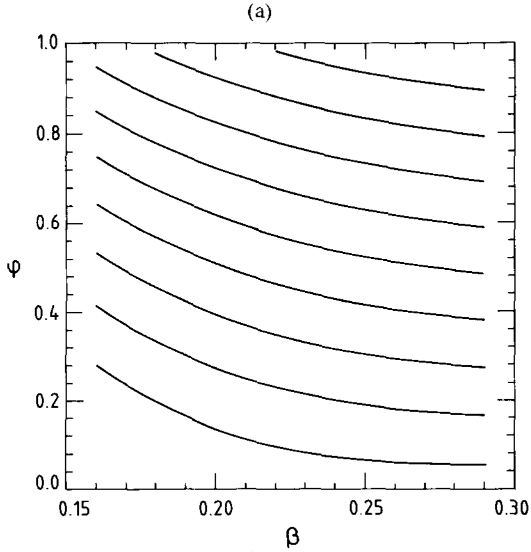
$$M \sim h^{1/\delta}, \tag{2.22}$$

for large L. The density of zeros should therefore be proportional to $x_h^{1/\delta}$ which implies

$$f_i(0) \sim -Ci^{2\delta/\delta+1}. \tag{2.23}$$

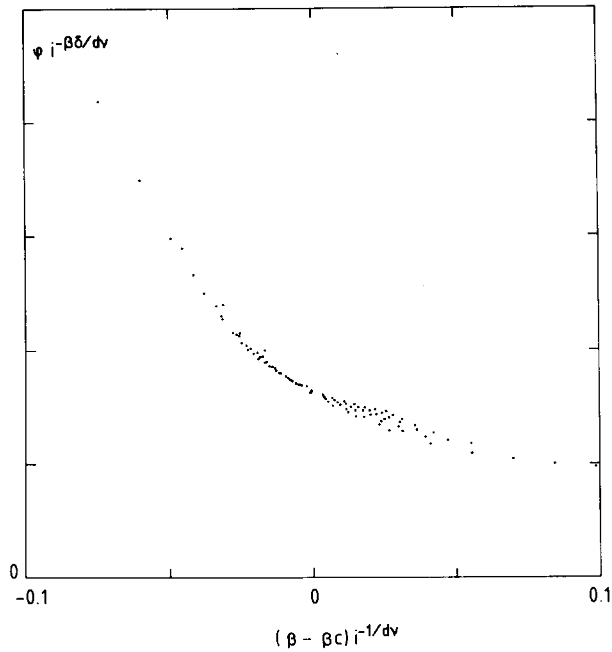


(a)



(b)

Fig. 3. (a) Expected shapes of the functions $f_i(x)$. (b) The phases φ of the first nine Lee-Yang zeros $y = e^{-2h} = e^{i\varphi}$ for the 4^3 lattice of ref. [3] at real β ($\beta_c = 0.222$). (c) Scaling behaviour of these zeros for β varying from 0.16 to 0.29 by 0.01.



(c)

Fig. 3 (continued).

This behaviour is observed on fig. 3b where the measured value of $\delta/\delta+1$ is 0.9 to be compared with 0.83 from series expansions. There could be a further scaling in the number of the root i such that for large i

$$h_i^2 L^{2\beta\delta/\nu} = f_i(x_t) = i^{2\delta/(\delta+1)} F\left(x_t i^{-1/\beta(\delta+1)}\right). \tag{2.24}$$

This compactly describes the pattern of Lee-Yang zeros for real x_t. When we set $h=0$ we obtain a stronger version of eq. (2.17)

$$K_i = L^{-1/\nu} i^{1/d\nu} F^{-1}(0). \tag{2.25}$$

This introduces a new scaling variable depending on the fraction of the total number of zeros

$$\lambda = i/L^d. \tag{2.26}$$

The assumed scaling relation (2.24) can then be written as

$$h_\lambda^2 = \lambda^{2\beta\delta/d\nu} F\left(K\lambda^{-1/d\nu}\right). \tag{2.27}$$

In fig. 3c we show this universal behaviour which is indeed already surprisingly observed on data from a rather small lattice. The Lee-Yang edge singularity is to be related to the first correction to the leading behaviour of F for large argument.

Indeed if

$$F(w) \underset{w\to\infty}{\to} -Cw^{2\beta\delta}[1+Bw^{-\varepsilon}+\cdots]. \tag{2.28}$$

Then the edge singularity in the magnetization written as

$$M_{\text{sing}} = \text{cst}\left(h^2-h_c^2\right)^{\sigma}, \tag{2.29}$$

is related to ε by

$$\frac{1}{1+\sigma}=\frac{\varepsilon}{d\nu}. \tag{2.30}$$

Note that ε is close to 2

$d=2$	$\nu=1$	$\sigma=-0.155$	$\varepsilon=2.37$
$d=3$	$\nu=0.63$	$\sigma=0.086$	$\varepsilon=1.74$
$d=6$	$\nu=0.5$	$\sigma=0.5$	$\varepsilon=2.$

The data for σ are from ref. [9]*.

The expression (2.27) also gives correctly the density of zeros needed to produce the specific heat singularity, i.e. $2-\alpha=d\nu$, and predicts that all of the zeros in the complex u plane near u_c should be lying on a single straight line with the phase of $F^{-1}(0)$. This agrees with the observed pattern of zeros in fig. 1.

This phase is in fact universal. From the representation of the specific heat in terms of a Cauchy integral derived from the density of zeros of the partition function, one easily finds a relationship between the specific heat index α, the amplitude ratio for the specific heat A_+/A_- and the phase of $F^{-1}(0)\equiv\pi-\varphi$. This is

$$\tan[(2-\alpha)\varphi]=\frac{\cos(\pi\alpha)-A_-/A_+}{\sin(\pi\alpha)}. \tag{2.31}$$

The known ratio $A_+/A_-\simeq 0.48-0.51$ [10] predicts that $\varphi\simeq 57°$ in excellent agreement with the observed pattern of zeros. Mean field predicts in this case $\alpha=0$, $A_+/A_-=0$ and therefore an angle $\varphi=45°$.

* For a renormalisation group calculation of σ, see also ref. [17].

3. Complex singularities

3.1. BETHE APPROXIMATION

Fig. 1 exhibits an accumulation of zeros along the real negative axis for a s.c. lattice suggesting the presence of another singular point. This fits nicely with the evidence from low-temperature expansions of $3d$ Ising models on various lattices, that the nearest singularity occurs in many cases at a complex value of β (see figs. 4, 5).

These unphysical nearby singular points have received some attention a dozen of years ago, because they may spoil the analysis of the physical critical behavior. In particular it was noticed [5] that the Bethe approximation gives a qualitative account of them. We recall that this approximation results from truncating to second order the irreducible cluster expansion. Alternatively, it solves the Ising model on a Cayley-tree lattice, and gives a good approximation of the actual model at low temperature. In table 1, we display the series expansion of the full model together with the Bethe series; they disagree to order eight in $u = t^2 = e^{-4\beta}$. The approximation relies on the observation that a given spin interacts with the magnetic field h and with its q neighbours (q = coordination number), whose interaction with the rest of the lattice may be described by an effective field h_1. The self-consistency equation for $\rho = e^{-2h_1}$ reads

$$y = \rho\left(\frac{1+t\rho}{t+\rho}\right)^{q-1}, \tag{3.1}$$

and the free energy per site is

$$F(t) = \frac{\ln Z}{V} = (q-1)\ln(1+\rho t) + \left(1-\tfrac{1}{2}q\right)\ln\left(1+2\rho t+\rho^2\right). \tag{3.2}$$

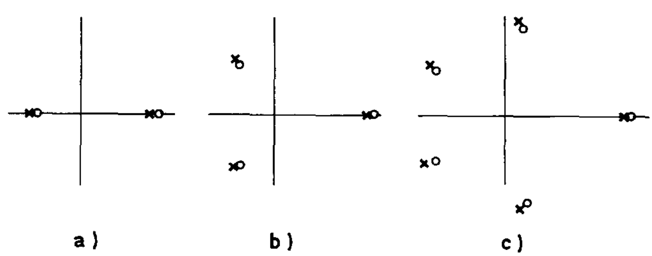


Fig. 4. Nearby complex singularities in the complex u plane for: (a) simple cubic lattice; (b) body centered cubic lattice; (c) face centered cubic lattice. Crosses indicate the singularities obtained from series expansions, open circles result from the Bethe approximation.

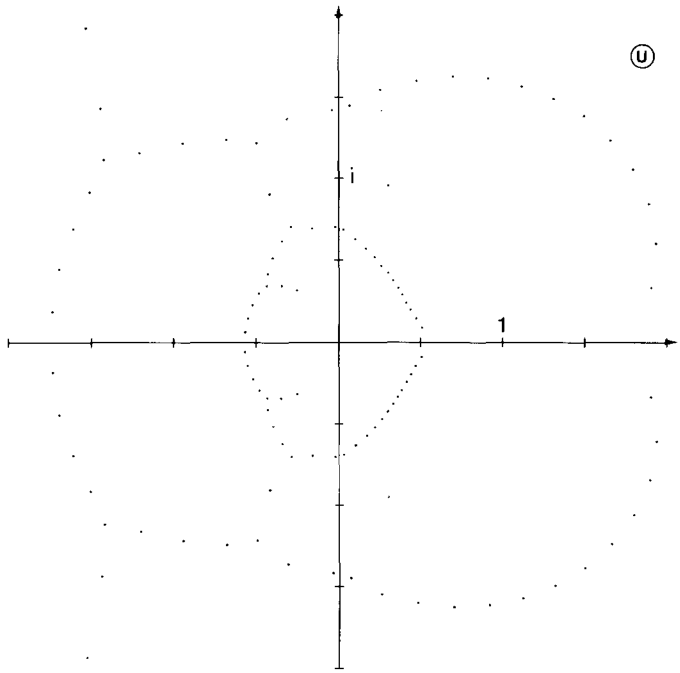


Fig. 5. Zeros of a body-centered cubic lattice ($3 \times 3 \times 3$ lattice).

If one introduces

$$p = \frac{t+\rho}{1+t\rho}, \tag{3.3}$$

it satisfies

$$p - t - y(1-pt)p^{q-1} = 0. \tag{3.4}$$

Singularities of F arise from singularities of $p(t, y)$. If $f(p, t, y)$ denotes the left-hand side of (3.4), they are obtained through elimination of p between $f = 0$, $\partial f/\partial p = 0$, and satisfy

$$y + \frac{1}{y} - 2 = -\frac{q^q}{(q-1)^{q-1}t^{q-2}}\left(t^2 - t_c^2\right)^3 \prod_{i=1}^{q-4/2}\left(t^2 - t_i^2\right)^2. \tag{3.5}$$

In a zero magnetic field, the critical point is at $t_c = (q-2)/q$, and there are $\frac{1}{2}(q-4)$ additional complex singularities, the location of which approximates well the patterns of fig. 4.

TABLE 1

Series expansion for the Ising three-dimensional model on a cubic lattice for $\frac{1}{6}tF'(t)$

Order n in $u = t^2$	Exact	Bethe	Improved Bethe
3	1	1	1
4	0	0	0
5	5	5	5
6	−7	−7	−7
7	35	35	35
8	−88	−96	−88
9	313	349	313
10	−935		−935
11	3113		3047
12	−9847		
13	32435		
14	−105161		
15	346966		
16	−1140024		
17	3778250		
18	12520405		
19	41694493		
20	−139015755		
t_c	0.6418	$\frac{2}{3} = 0.6667$	0.6570
t_i	$\pm i\, 0.5345$	$\pm i\, 0.5$	$\pm i\, 0.5158$

Column 1 exact results [11], column 2 Bethe approximation, column 3 improved Bethe approximation. The latter incorporates also clusters of overturned spins on squares, and is defined by:

$$F = 3\ln\left(1 + 4\rho t^2 + \rho^2(4t^2 + 2t^4) + 4\rho^3 t^2 + \rho^4\right) - 2\ln\left(1 + 2\rho' t + \rho'^2\right) - 7\ln\left(1 + \rho' t\right),$$

with:

$$\rho'^{11} = \rho^{12}\left(\frac{\rho' + t}{1 + \rho' t}\right)^7 \quad \text{and} \quad \rho'\frac{t + \rho'}{1 + \rho' t} = \rho\frac{t^2 + \rho(2t^2 + t^4) + 3\rho^2 t^2 + \rho^3}{t^2\rho^3 + \rho^2(2t^2 + t^4) + 3\rho t^2 + 1}.$$

In the presence of a magnetic field, the Bethe approximation satisfies the Lee-Yang theorem [12]. For real $t > t_c$, the r.h.s. of (3.5) is negative, and the Lee-Yang singularity lies on the unit circle:

$$y + \frac{1}{y} - 2 < 0 \Rightarrow |y| = 1.$$

Moreover, the approximation suggests that these Lee-Yang edge singularities, the critical point and the unphysical singularities all lie on a single manifold. It is not known whether such a property is shared by the actual model.

As h departs from zero, $y + 1/y - 2 \simeq 4h^2$ and the critical and unphysical singularities split in a different way:

$$t_c^2(h) \sim \left(\frac{q-2}{q}\right)^2 + Ah^{2/3}\exp\left[i\tfrac{1}{3}\pi(1+2k)\right], \qquad (k=0,1,2),$$

$$t_i^2(h) \sim t_i^2(0) \pm Bh. \tag{3.6}$$

The phases and exponents in h reflect the (classical) critical exponents which are different at the critical and unphysical points. In the Bethe approximation, the specific heat, magnetic susceptibility and correlation length diverge at t_i^2 with exponents $\alpha' = \frac{1}{2}$, $\gamma' = \frac{1}{2}$, $\nu' = \frac{1}{4}$, whereas at t_c^2 the ordinary mean-field exponents are found ($\alpha = 0$, $\gamma = 1$, $\nu = \frac{1}{2}$). At t_i^2, the spontaneous magnetization does not vanish:

$$M \sim M_{\text{reg}} + C\left(t^2 - t_i^2\right)^{1/2}.$$

Finally, within the Bethe approximation it is easy to draw the line to be identified with the curve of zeros, using its characterization of sect. 2. This is done on fig. 6 for the simple cubic lattice. It is made of two branches. One originates from the real negative singularity and corresponds to a cut of $p(t)$. The other one starts from t_c^2, at the (classical) angle of $\frac{1}{4}\pi$.

As mentioned above, the Bethe approximation may be seen as a member of a sequence of successive approximations. Any finite-order approximation leads to similar algebraic equations, and thus fails to reproduce the details of the actual critical behavior that we describe now. In table 1 we show some data for such an improved approximation.

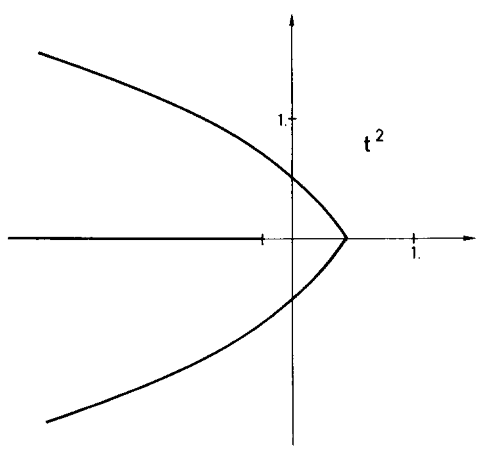


Fig. 6. The lines of discontinuity of Im F, in the Bethe approximation.

3.2. SERIES ANALYSIS

Rather long low-temperature series exist [11] for the specific heat, the spontaneous magnetization and the susceptibility on the three lattices of interest (s.c., b.c.c., f.c.c.), and a shorter one for the s.c. correlation length [13]. It is possible to analyse these series and to try to determine the critical exponents at the complex singularities. We have used various methods, ratio methods, D-log Padé approximants and a generalized Padé-approximation [14]. The results are summarized in table 2 where we also recall the classical exponents. In spite of the rather large dispersion of the estimates, there is some evidence for a universal critical behaviour which is neither classical (exponents disagree with Bethe's lattice predictions), nor identical to the ordinary critical behaviour. Especially striking is the case of the f.c.c. lattice where the two doublets of conjugate singularities exhibit approximately the same behaviour. This universal behaviour at the complex singularities had not been emphasized before, to the best of our knowledge. Using ratio methods, Thompson, Guttmann and Ninham [5] had given the s.c. exponents

$$\alpha' = 1.07, \qquad \beta' = -0.08, \qquad \gamma' = 1.12 - 1.20.$$

Notice that the previous estimate of $\alpha' > 1$ points to a weak divergence of the internal energy $U \sim (t^2 - t_i^2)^{1-\alpha'}$. Also the estimates for α', β', γ' and ν' are consistent with hyperscaling relations:

$$\alpha' + 2\beta' + \gamma' \simeq 2, \qquad 2 - \alpha' \simeq \nu' d (= 3\nu').$$

If this singular universal behavior is non-classical, what is the continuum field theory describing it? A possible candidate could be a ϕ^3 theory with an imaginary coupling: this would indeed be the continuous theory at a critical point where the original order parameter does not vanish. In particular it is the critical theory of

TABLE 2
Exponents at the complex singularities, determined from series analysis

	s.c.	b.c.c.	f.c.c.	Bethe
α'	1.25 ± 0.15	1.30 ± 0.1	1.35 ± 0.15 1.55 ± 0.15	$\frac{1}{2}$
β'	-0.05	~ -0.05	~ -0.05	$\frac{1}{2}$
γ'	1.1 ± 0.1	1.05 ± 0.1	1.05 ± 0.1	$\frac{1}{2}$
ν'	0.4 ± 0.1			$\frac{1}{4}$

For the f.c.c. lattice, the two figures refer to the two doublets of conjugate singularities. The estimates for γ' seem much more stable than those for α'.

the Lee-Yang edge singularity [9]. Then, the upper critical dimension d_c where the exponents become classical would be 6, as attested by $\frac{1}{2} = \alpha' = 2 - \nu' d_c = 2 - \frac{6}{4}$. However the $3d$ critical behaviour of the free energy at the Lee-Yang edge singularity as a function of the relevant parameter, namely h, has been determined by Kurtze and Fisher from the dimer series [9]

$$F \sim (h - h_c)^{\sigma+1}, \tag{3.7}$$

$1 - \sigma = 0.91$ should be compared with α' in table 2. The lack of agreement may be due to a loose estimate of α' (the series are badly behaved) or may signal that we have not properly identified the critical theory. Another possibility is a relation with similar singularities in lattice gauge theories to which we turn now. In this case we are dealing with expansions in terms of surfaces and these singularities might signal the existence of some well defined model for random surfaces.

3.3. HIGHER DIMENSIONS

The $3d$ Ising model at low temperature is equivalent to a Z_2 gauge theory at high temperature by Kramers-Wannier duality. This suggests to look at the possible complex singularities of the latter (or of other lattice gauge theories) in higher dimensions. On a four-dimensional hypercubic lattice Wilson [15] has derived a series for the free energy to 11th order in $u = t^2$; t being the first character expansion parameter:

$$t = \tanh\beta, \qquad I_1(\beta)/I_0(\beta), \qquad I_2(\beta)/I_1(\beta),$$

for $Z_2, U(1), SU(2)$ respectively. The coefficients of these series have alternating signs, which points to a closest singularity on the real *negative* axis in u. There is a real positive singularity, which depending on the case, signals the deconfining transition of U(1), the end of the metastable phase of Z_2, or might split into a pair of nearby complex singularities [16] (see fig. 7).

The Bethe approximation may be generalized to a d-dimensional gauge theory. One replaces the lattice by a Cayley tree of cubes. Such a lattice is defined

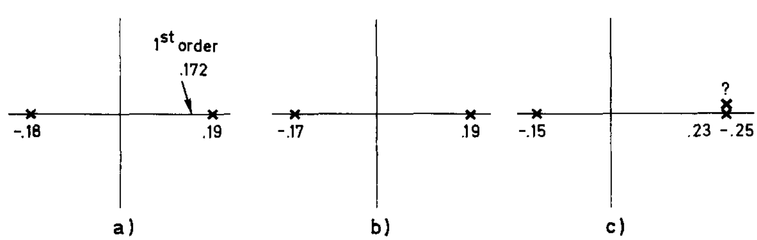


Fig. 7. Nearby complex singularities of $4D$ lattice gauge theories (a) Z_2; (b) U(1); (c): SU(2).

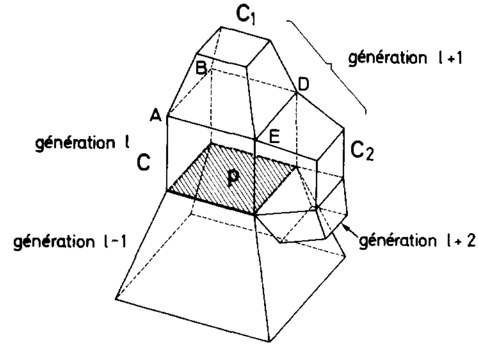


Fig. 8. Constructing a Cayley tree of cubes: each face like ABDE has $2(d-2)$ adjacent cubes (only two are represented here); cubes built on the faces of C_1 and C_2 will never meet.

recursively (fig. 8). Given a cube of generation l, we add $(2d-5)$ cubes of generation $l+1$ on each of its five free faces. On the resulting lattice, $2(d-2)$ cubes are incident on each face, and there is no cluster nor cycle of cubes. Accordingly, the Bethe approximation resums strong coupling diagrams of the original model which may be considered as trees of cubes, but misses or miscounts diagrams as those depicted on fig. 9.

On the other hand, as every link is shared by an infinite number of plaquettes, there is no weak-coupling expansion. To solve the model in its strong-coupling phase for the simplest case of a Z_2 gauge group, consider a cube C of generation l, and the plaquette p separating it from its ancestor (fig. 8). We call x_l the sum of closed diagrams made of C and its descendants not containing p, ty_l the sum of such diagrams containing p.

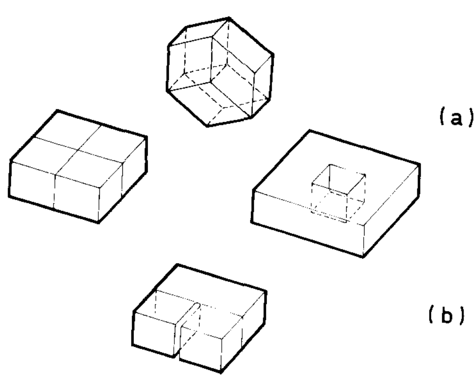


Fig. 9. Examples of diagrams omitted (a), or overcounted (b) in the Bethe approximation.

The following recursion relations hold in dimension d

$$x_l = \left[(x_{l+1} + t y_{l+1})^{\text{“}2d-5\text{”}} \right]^5,$$

$$y_l = \left[(t x_{l+1} + y_{l+1})^{\text{“}2d-5\text{”}} \right]^5, \tag{3.8}$$

where quotation marks mean that in the binomial expansion we set $t^{2k} \equiv 1$, $t^{2k+1} \equiv t$. Assuming that $\rho_l = y_l / x_l$ goes to a limit ρ as $l \to \infty$, we get the self-consistent equation

$$\rho = \left(\frac{(t+\rho)^{\text{“}2d-5\text{”}}}{(1+\rho t)^{\text{“}2d-5\text{”}}} \right)^5 . \tag{3.9}$$

When $d=3$, we recover eq. (3.1), in zero field, while for $d=4$, the following parametrization of the free energy follows:

$$F - (6 \ln \cosh \beta + 4 \ln 2) = 20 \ln(1 + 3\rho t + 3\rho^2 + \rho^3 t)$$

$$- 14 \ln(1 + 4\rho t + 6\rho^2 + 4\rho^3 t + \rho^4),$$

$$\rho = p^5, \qquad p = t + (1 - pt) p^5 \frac{3 + p^{10}}{1 + 3p^{10}} . \tag{3.10}$$

This has two singular points at $t^2 = -0.151$ and $t^2 = 0.190$. Moreover the transition from strong-coupling to the trivial weak-coupling phase $p \equiv 1$ is discontinuous (first order) as in the ordinary mean field picture. Comparing F_2 given above to $F_{\text{weak}} = 6\beta + \ln 2$ leads to a first-order transition at $t = 0.178$. All these numbers agree roughly with those given on fig. 7.

As d goes to infinity we have to rescale t so as to make dt^4 finite to have a sensible limit. The previous equation for p reduces to

$$p = t + 2dp^5,$$

which has two singular points at

$$t^2 = \pm \frac{16}{25} \frac{1}{\sqrt{10d}} .$$

This is the limit considered by Drouffe, Parisi and Sourlas [6].

Finally, information about the critical behaviour at the complex singularity in four dimensions is rather scarce. Padé analysis of the Z_2, U(1) and SU(2) short series gives an exponent α' ranging between 0.45 and 0.6, hence consistent with its classical value $\frac{1}{2}$. It is certainly preliminary to conclude that the upper critical dimension is four. Clearly, these nearby complex singularities deserve more attention.

We thank R. Balian, P. Moussa, G. Parisi, and N. Sourlas for useful discussions.

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