Nuclear Physics B300 [FS22] (1988) 393–432
North-Holland, Amsterdam

GENERALIZED COULOMB-GAS FORMALISM FOR TWO DIMENSIONAL CRITICAL MODELS BASED ON SU(2) COSET CONSTRUCTION

P. DI FRANCESCO, H. SALEUR and J.-B. ZUBER

Institut de Recherche Fondamentale, Service de Physique Théorique, CEA–CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 29 February 1988

Conformal theories associated with the coset construction $SU(2)_k \times SU(2)_l / SU(2)_{k+l}$ are described by a generalized Coulomb-gas formalism. An integrable lattice model obtained by k fusions of the 6-vertex model has a continuous critical line containing the level-k Wess–Zumino–Witten model. Its continuum limit is described by tensor products of the parafermionic and free bosonic sectors. Modification of its boundary conditions by the introduction of floating charges, yields the coset models. The discussion extends to non-unitary models. Special attention is paid to the case $k=2$, corresponding to $N=1$ supersymmetry.

1. Introduction

The recent period has seen an intense activity in the construction of conformally invariant theories (for a review, see ref. [1]) and their classification. Modular invariance on the torus has proved to be an effective constraint for this purpose.

On the other hand, according to ideas [2] prior to modern developments in conformal invariance, most two-dimensional critical models are expected to derive from free bosonic theories. Steps have been taken to establish links between these two approaches [3–5]. In ref. [5] we have shown that all minimal partition functions classified in refs. [6, 7] are linear combinations of Gaussian partition functions [8–10]. A derivation of these expressions from the underlying lattice models was proposed in refs. [5, 11].

The purpose of this paper is to extend these considerations to theories based on the coset construction on the affine $\widehat{SU}(2)$ algebra [12–14]. This includes, in particular, $N=1$ supersymmetric theories.

In all cases, the underlying microscopic model is a spin $S=\frac{1}{2}k$ vertex model [15]. The latter presents a critical line with central charge $c=3k/(k+2)$. As discussed in this paper, its self-dual point turns out to be described by the SU(2) level-k Wess–Zumino–Witten (WZW) model [16, 17]. Along the critical line, the partition function on the torus can be reexpressed in terms of a free bosonic field and a Z_k

0169-6823/88/$03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

parafermionic theory [18,19] coupled through boundary conditions. For $k=1$, parafermions are absent and it is known [5,11] that the $c<1$ partition functions are obtained, starting from the $S=\frac{1}{2}$, 6-vertex model, by introducing "floating electric charges" related to the exponents of the classifying simply laced algebra. Likewise, for $k>1$ a similar modification of the bosonic contribution leads to the other discrete series based on the SU(2) coset construction. The interpretation of these theories, in terms of tensor products of parafermion and of (modified) free boson sectors, looks quite natural, in view of the previous observations of ref. [14]. What is less obvious is the way these sectors are coupled through boundary conditions, as discussed below.

In sect. 2, we review the case of $c<1$ minimal models, their filiation with the 6-vertex model and the corresponding $c=1$ continuous theories. Floating charges proportional to the exponents of some simply laced Lie algebra reduce the central charge from $c=1$ to $c<1$ and generate all the minimal models: this is summarized in a compact way in eq. (2.25), which was originally proposed by Kostov [20], the microscopic interpretation [5,11] of which is recalled. All these considerations are equally valid for non-unitary minimal models, as exemplified by the $c=-\frac{22}{5}$ theory describing the Lee–Yang edge singularity [21].

Sect. 3 extends these ideas to the next non-trivial case, $k=2$, which through the coset construction leads to the $N=1$ superconformal minimal theories. Here the basic model is a spin-1, 19-vertex model. It has a critical line, with $c=\frac{3}{2}$, originating from the $k=2$ WZW model, as tested by a numerical calculation of the transfer matrix spectrum. Inclusion of floating charges, as before, enables one to recover all the $c<\frac{3}{2}$ superminimal models classified by Cappelli [22] but the exceptional series $(\mathrm{D},\mathrm{E}_6)$, for which a special construction remains to be found.

Generalization to arbitrary k is then straightforward; the relevant formulae are collected in sect. 4.

Sect. 5 shows that similar considerations apply to the $N=2$ superconformal models [23]. We recall that the corresponding representation may be obtained by the coset construction applied to $\mathrm{SU}(2)\times\mathrm{U}(1)/\mathrm{U}(1)_{\mathrm{diag}}$.

Our final comments are presented in sect. 6, while three appendices contain some technical details and proofs.

2. Minimal conformal theories

2.1. We first discuss the case of minimal models, i.e. conformally invariant models for which the operator algebra closes with only a finite number of primary fields. As shown in ref. [24] this occurs for central charges

$$c(p,p')=1-\frac{6(p-p')^2}{pp'}<1, \tag{2.1}$$

where p, p' are two coprime positive integers. In this case the allowed values of h, $\bar{h}$ are given by the Kac formula [25]

$$h_{rs} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'}, \tag{2.2}$$

with the constraint that the integers r, s satisfy the bounds

$$\begin{aligned} 1 &\leqslant r \leqslant p' - 1, \\ 1 &\leqslant s \leqslant p - 1. \end{aligned} \tag{2.3}$$

Unitary theories [26] correspond to $|p - p'| = 1$. Using the requirement of modular invariance [27], all possible partition functions of minimal models on a torus

$$Z = \sum_{h,\bar{h}} N_{h\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}), \tag{2.4}$$

have been classified [6,7]. In eq. (2.4), $q = \exp 2i\pi\tau$, $\tau = \omega_2/\omega_1 = \tau_R + i\tau_I$ is the modular ratio, $N_{h\bar{h}}$ is the number of primary fields of dimensions $(h, \bar{h})$, χ_h is the character of the Virasoro algebra in the irreducible representation of highest weight h.

2.2. As suggested by various approaches [5,11,28], the underlying model in all the theories (2.1) is the 6-vertex model. It is obtained by putting arrows on each bond of the square lattice, in such a way that the current is conserved at each node, thus giving rise to 6 possible vertex configurations (fig. 1). (More generally, we will be led to consider models with $2S + 1$ states per bond in the following; this simple case appears thus as a spin $S = \frac{1}{2}$ vertex model). Imposing invariance under reversal of all arrows [29], one is left with three free parameters, the Boltzmann weights a, b, c (fig. 1). If Δ denotes

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}, \tag{2.5}$$

one may show that transfer matrices, on a strip of width L with the same value of Δ, commute. In the very anisotropic limit, the model is equivalent to a spin-$\frac{1}{2}$ XXZ

Fig. 1. Vertices of the 8-vertex model.

antiferromagnetic quantum chain with hamiltonian [29]

$$H \propto \sum_{i=1}^{L} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y - \Delta S_i^z S_{i+1}^z . \tag{2.6}$$

The critical region corresponds to $|\Delta| \leqslant 1$. In this case, setting

$$\Delta = -\cos \pi\lambda , \qquad 0 \leqslant \lambda \leqslant 1 , \tag{2.7}$$

a convenient parametrization of the weights is (up to a global normalization)

$$\begin{cases} a = \sin \frac{1}{2}\lambda(\pi - \alpha) , \\ b = \sin \frac{1}{2}\lambda(\pi + \alpha) , \\ c = \sin \lambda\pi , \end{cases} \tag{2.8}$$

where α is a spectral parameter $|\alpha| \leqslant \pi$. The continuum limit of the model can then be derived in various ways [30, 31]. For instance, one can reformulate it as a solid-on-solid (SOS) interface model [32]; height variables φ are introduced on the dual lattice in such a way that neighbouring φ differ by $\pm\pi$ depending on the orientation of the arrow which separates them. It is then argued [31] that renormalization group trajectories flow to the Gaussian fixed point, which we shall describe by the action

$$\mathscr{A} = \frac{g}{4\pi} \int |\nabla\varphi|^2 \, \mathrm{d}^2 x . \tag{2.9}$$

The value of the renormalized coupling constant is then easily obtained using Baxter's solution of the 8-vertex model [29]. Indeed, introduction of vertices with a non-conserved current (fig. 1), which correspond to vortices of charge $m = (2\pi)^{-1} \oint \nabla\varphi \, \mathbf{d}l = 2$, gives rise to a singularity of the free energy

$$f_s \sim |d|^{2/y} , \qquad y = 2\lambda . \tag{2.10}$$

On the other hand the scaling dimension of a vortex operator in eq. (2.9) is easily calculated [33] to be $x = \frac{1}{2} g m^2$. We have thus

$$g = 1 - \lambda , \qquad 0 \leqslant g \leqslant 1 ; \tag{2.11}$$

$g = 0$ corresponds to $\Delta = 1$, $c = 0$. If $\Delta > 1$ the model is completely frozen with ferroelectric order. The other limit, $g = 1$, gives rational weights in eq. (2.8). In this case $\Delta = -1$, $c = a + b$, and the model is isotropic and self-dual, due to the symmetry property of the partition function (in the plane) [29]

$$Z(a, b, c) = Z\left(\tfrac{1}{2}(a - b + c), \tfrac{1}{2}(-a + b + c), \tfrac{1}{2}(a + b + c)\right) . \tag{2.12}$$

The phase $\Delta < -1$ is antiferroelectric, while an infinite-order phase transition takes place at $\Delta = -1$. We note, finally, that the 6-vertex model can be mapped on a Thirring model [30], the bosonization of which yields eq. (2.9). If $\Delta = 0$ ($g = \lambda = \frac{1}{2}$) the four-fermion coupling vanishes and one gets a free (Dirac) fermion theory.

The partition function of the critical 6-vertex model on a torus can now be calculated. We recall, first, the regularized expression [34] of the partition function of the Gaussian theory (2.9)

$$Z_0 = \int_{\substack{\varphi \text{ doubly} \\ \text{periodic}}} [D\varphi] \mathrm{e}^{-\mathscr{A}} = \sqrt{\frac{g}{\tau_{\mathrm{I}}}} \frac{1}{|\eta(q)|^2}, \tag{2.13}$$

where η is Dedekind's function, i.e.

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.14}$$

In the mapping onto eq. (2.9), special care must be paid to boundary conditions. The correspondence between arrows and heights being only local, the latter cannot be defined consistently on the torus. Following a closed path along ω_1, ω_2, there can be [9] shifts of the variable φ: $\delta_1\varphi = 2\pi M$, $\delta_2\varphi = 2\pi M'$. In such a "sector", the continuum limit of the partition function reads

$$Z_{M,M'}(g) = \int_{\substack{\delta_1\varphi = 2\pi M \\ \delta_2\varphi = 2\pi M'}} [D\varphi] \, \mathrm{e}^{-\mathscr{A}} = Z_0 \exp\left(-\pi g \frac{|M' - M\tau|^2}{\tau_{\mathrm{I}}} \right). \tag{2.15}$$

It enjoys the modular covariance properties expected from its definition

$$Z_{MM'}(\tau) = Z_{cM' + dM, aM' + bM}\left(\frac{a\tau + b}{c\tau + d} \right). \tag{2.16}$$

If the model is considered on a lattice $L \times L'$ with L, L' even, M and M' are integers. In this case, summing over M, M' gives the Coulombic (or Gaussian) partition function

$$\begin{aligned} Z_{\mathrm{C}}(g) &= \sum_{M, M' \in Z} Z_{MM'}(g), \\ &= \frac{1}{|\eta|^2|} \sum_{E, M \in Z} q^{(E/\sqrt{g} + M\sqrt{g})^2/4} \bar{q}^{(E/\sqrt{g} - M\sqrt{g})^2/4}, \\ &= \frac{1}{|\eta|^2} \sum_{E, M \in Z} q^{\Delta_{\mathrm{EM}}} \bar{q}^{\bar{\Delta}_{\mathrm{EM}}}. \end{aligned} \tag{2.17}$$

The second equality, obtained after a Poisson transformation (2.17), displays the $q \to 0$ behaviour, $Z_C \sim (q\bar{q})^{-c/24}$ with $c = 1$. The conformal weights are given by

$$\Delta_{EM} + \bar{\Delta}_{EM} = x_{EM} = E^2/2g + gM^2/2,$$
$$\Delta_{EM} - \bar{\Delta}_{EM} = s_{EM} = EM, \tag{2.18}$$

they correspond to electromagnetic operators O_{EM}, i.e. combinations of spin-wave (vertex operators $\exp(iE\varphi)$) and vortex operators [33]. Eq. (2.17) appears also in string theory as the partition function of a free field compactified on a circle [8, 35] of radius $R = \sqrt{\frac{1}{2}g}$. It satisfies the symmetry

$$Z_C(g) = Z_C(1/g), \tag{2.19}$$

the fixed point of which is the self-dual point, $g = 1$, mentioned above. It is noteworthy that $Z_C(g = 1)$ is precisely equal to the SU(2), level $k = 1$ WZW partition function (see appendix A). We shall comment on this point later.

If the model is studied on a lattice with L or L' odd, the corresponding frustrations can be half an odd integer. A new modular invariant can then be constructed by considering

$$\left(\sum_{M \in Z + \frac{1}{2},\, M' \in Z} + \sum_{M \in Z,\, M' \in Z + \frac{1}{2}} + \sum_{M,\, M' \in Z + \frac{1}{2}} \right) Z_{MM'}(g). \tag{2.20}$$

Adding eqs. (2.17) and (2.20) gives $2Z_C(g/4)$. In the following we shall restrict ourselves to even $\times$ even lattices.

2.3. It has been observed [5] that *all* minimal partition functions [6, 7] can be reexpressed as linear combinations of Coulombic partition functions with various couplings (see formulae (A.5)–(A.9) in ref. [5]). We want to give here an a posteriori justification of this result. First, we recall that to get a central charge $c < 1$ starting from a free field with $c = 1$, it is necessary to add (in the plane) at infinity a charge e such that [3]

$$c = 1 - 6e^2/g. \tag{2.21}$$

Eq. (2.1) can then be realized with, e.g.

$$g = p/p', \qquad e = (p' - p)/p'. \tag{2.22}$$

Now, the dimensions h_{EM} of electromagnetic operators in this decorated free field read

$$h_{EM} = \Delta_{EM} - e^2/4g. \tag{2.23}$$

This enables us to rewrite the Kac formula (2.2) as

$$h_{rs}=\frac{(rp-sp')^2-(p-p')^2}{4pp'}$$

$$=\frac{\Lambda^2}{4pp'}-\frac{e^2}{4g}. \tag{2.24}$$

As shown in ref. [5], minimal partition functions are classified by a pair of simply laced Lie algebras (A,G) (where G can be of A, D or E type) and A and G have respective Coxeter numbers p and p'. Moreover, the spinless $(h=\bar{h})$ operators have conformal dimensions given by eq. (2.24) with r (respectively, s) taking its value among the Coxeter exponents of G (respectively, A). As p and p' are coprimes, the residual class of $\Lambda=rp-sp'$ modulo p' is itself one of the N exponents, n, of G. This suggests that one should supplement the free field by a set of $(N-1)$ fractional electric charges $e_n=n/p'$.

Let us now return to the torus. As shown in ref. [5], a way of adding an electric floating charge to eq. (2.17), respecting modular invariance, is to include an interaction term between the shifts $\cos(2\pi eM\wedge M')$ where $\wedge$ denotes the greatest common divisor. This suggests finally that the (A,G) partition functions may be reproduced by the expression

$$Z=\tfrac{1}{2}\sum_{M,M'\in Z} Z_{MM'}\left(\frac{p}{p'}\right)\sum_n \cos\left(2\pi\frac{n}{p'}M\wedge M'\right), \tag{2.25}$$

where n runs over the exponents of G. Performing the sum over n and decomposing onto various congruence classes of M and M', indeed reproduces formulae of ref. [5]. The expression (2.25) was first proposed by Kostov [20].

2.4. It seems rather remarkable from a conceptual point of view that the minimal partition functions can be reexpressed in terms of a decorated free field, thus justifying ideas which originated in works of Kadanoff [2]. Of course, this can be inferred from the presence of an underlying 6-vertex model. The precise correspondence has been studied [11] for unitary models, realized as restricted solid-on-solid models, and attached to the Dynkin diagram of the associated G algebra [36]. Fig. 2 shows the assignments of heights to each node of a Dynkin diagram. We restrict ourselves here to the case of A_N $(N=p'-1=p-2)$ models [28] $(N\geqslant 2)$.

As shown originally by Baxter [37], the 8-vertex model can be reformulated as a solid-on-solid model on the dual lattice, neighbouring heights differing by ± 1, with

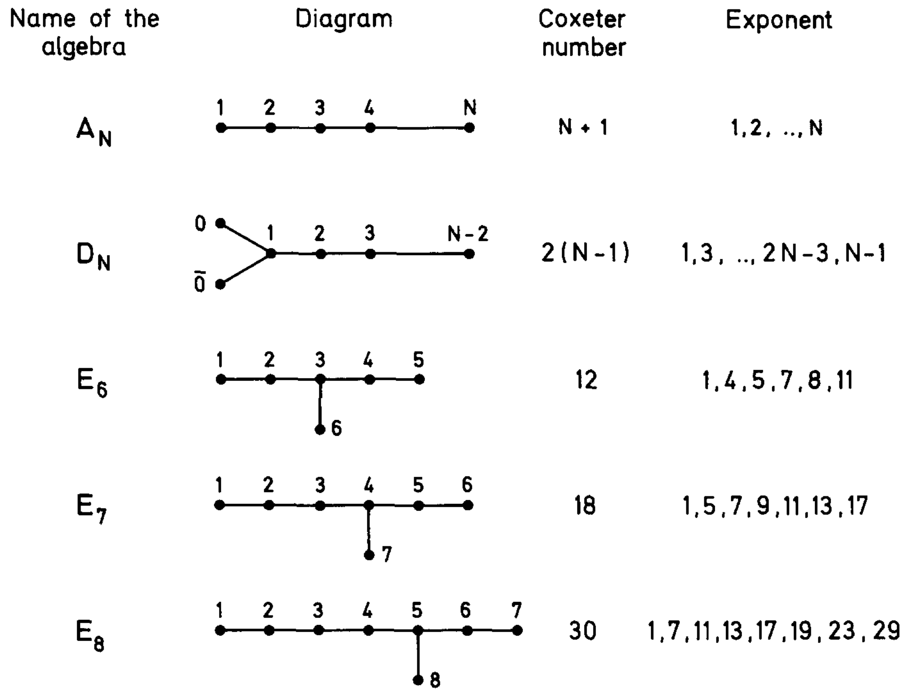

Fig. 2. The Dynkin diagrams (with a possible assignment of height variables to their nodes), Coxeter numbers and exponents of the simply laced Lie algebras.

interaction round face (IRF) type Boltzmann weights (fig. 3)

$$\begin{cases} w(l, l+1|l-1, l) = w(l, l-1|l+1, l) = \alpha_l, \\ w(l+1, l|l, l-1) = w(l-1, l|l, l+1) = \beta_l, \\ w(l+1, l|l, l+1) = \gamma_l, \\ w(l-1, l|l, l-1) = \delta_l, \end{cases} \tag{2.26}$$

where, up to a normalization factor

$$\begin{cases} \alpha_l = h\left(\frac{1}{2}\lambda(\pi+\alpha)\right), \\ \beta_l = h\left(\frac{1}{2}\lambda(\pi-\alpha)\right)\left[h(w_{l-1})h(w_{l+1})\right]^{1/2}/h(w_l), \\ \gamma_l = h(\pi\lambda)h\left(w_l + \frac{1}{2}\lambda(\alpha-\pi)\right)/h(w_l), \\ \delta_l = h(\pi\lambda)h\left(w_l + \frac{1}{2}\lambda(\pi-\alpha)\right)/h(w_l), \end{cases} \tag{2.27}$$

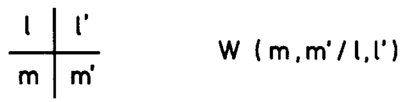

Fig. 3. Height configurations of the SOS model associated with the 8-vertex model.

$w_l = \pi(w + \lambda l)$, $h(x) = \theta_1(x/\pi)\theta_4(x/\pi)$, and θ_1, θ_4 are the usual Jacobi theta functions of (real) nome $\hat{q}$. The general relation between λ, α, $\hat{q}$ and the original 8-vertex model parameters is given in ref. [28]. When $w = 0$ and λ takes the value $1/N+1$, the heights l, due to $h(w_0) = h(w_{N+1}) = 0$, can be consistently restricted [28] to the set $l = 1, \dots, N$ and thus considered as attached to the Dynkin diagram of the A_N algebra (fig. 2) of Coxeter number $H = N + 1$.

The model (2.27) is critical when $\hat{q}$ is equal to zero; two cases are then possible. We consider here the transition from the so-called regime III to regime IV, characterized by $|\alpha| \leqslant \pi$. In this case, the corresponding vertex model becomes, precisely, the 6-vertex model with weights (2.8). (Note, however, that the SOS reformulation of the latter we used in eq. (2.9) is *not* directly related to the RSOS model (2.27)). It renormalizes onto the Gaussian model (2.9) with coupling constant $g = 1 - \lambda = (H-1)/H$ and comparison with eq. (2.25) gives $p = H - 1$, $p' = H$. As noticed in refs. [11, 38], one could also consider a dilute RSOS model which in turn corresponds to a dilute vertex model. It renormalizes onto eq. (2.9), with g given by another branch $g = 1 + \lambda = (H+1)/H$; in this case $p = H + 1$, $p' = H$.

The correspondence between the vertex and the RSOS model becomes more complicated on a torus; it requires the introduction of a boundary operator [39], which should translate in the continuum limit into the cosine interaction of eq. (2.25). Partition functions can nevertheless be calculated using a rather indirect procedure.

One performs, for this purpose, a graphical expansion drawing clusters which connect sites of the same height [40]. These clusters can, in turn, be represented using a polygon decomposition of the lattice [41]. The weight for contractible polygons is given by the largest eigenvalue of the incidence matrix G of the diagram i.e. $2\cos(\pi/H)$; it is the same as for the Potts model with $Q = 4\cos^2(\pi/H)$. Non-contractible loops in number $\mathcal{N}$ must be given a weight $\operatorname{Tr} G^{\mathcal{N}}$ [11]. This model, after arbitrary orientation of the loops, can be reinterpreted as a new SOS model, loops being considered as steps $\pm\pi$ between regions of constant height [42]. Its action maps onto (2.9) with $g = (H-1)/H$ (or $(H+1)/H$ on the dilute branch). On the other hand $\operatorname{Tr} G^{\mathcal{N}}$ decomposes as

$$\sum_n (2\cos(n\pi/H))^{\mathcal{N}}, \tag{2.28}$$

which, in the continuum limit, translates precisely into the sum of cosines of eq. (2.25) [5]. In summary, the diagram on which heights are living determines $p' = H$ in eq. (2.25). Then, depending on the branch, $p = H \pm 1$ and one finds two unitary modular invariants as expected. In fact, this derivation generalizes to the cases of D and E algebras as well [11].

2.5. It appears in ref. [28] that heights in the model (2.26) can also be restricted to the same set $l = 1, \dots, N$, choosing $\lambda = R/(N+1)$ with R and $N+1 = H$ coprimes. If $R > 1$, however, Boltzmann weights are no longer positive so this should correspond to the case of non-unitary models [36]. Indeed, the derivation of the partition function is the same, the weight of contractible loops becoming $2\cos(\pi R/H)$, while the one of non-contractible loops is unchanged. In the continuum limit one gets eq. (2.25) with $p' = H$ and, depending on the branch, $p = H \pm R$. This generalizes also to D and E algebras. Note, however, that by construction $|p - p'| < p'$, while this restriction does not appear in the classification of refs. [6, 7]. This construction thus excludes modular invariants which should be obtained by considering other determinations of g.

The simplest example of a non-unitary model is the Lee–Yang singularity [43] which occurs generally for magnetic models in an imaginary field. It has not yet been directly mapped onto a free field (2.9), although the continuum partition function is known to be given by eq. (2.25), with $p' = 5$, $p = 2$. The latter can, in fact, be derived using a slightly different formulation. Indeed, as shown by various graph expansions [44], the Lee–Yang singularity is expected to be the generic singularity of hard objects with a negative fugacity. Now, consider model (2.26) with $\lambda = \frac{2}{5}$. Define two square sublattices X and Y [28], and occupation variables related to the height variables l by $\sigma_i = \frac{1}{2}(3 - l_i)$ on X and $\sigma_i = \frac{1}{2}(l_i - 2)$ on Y. Then, with the above height restrictions, one has $\sigma_i = 0$ or 1, and $\sigma_i \sigma_j = 0$ if i, j are nearest neighbors. Thus, this translates eq. (2.26) into a hard square model with diagonal interaction [45], the Boltzmann weights of which read, for σ variables on a given face, as

$$\begin{cases} \omega_1 = w_{\mathrm{H}}(00|00) = h(\pi\lambda)h\left(\frac{1}{2}\lambda(5\pi - \alpha)\right)/h(2\pi\lambda), \\ \omega_2 = w_{\mathrm{H}}(10|00) = w_{\mathrm{H}}(00|01) = h\left(\frac{1}{2}\lambda(\pi - \alpha)\right)\left[\dfrac{h(\pi\lambda)}{h(2\pi\lambda)}\right]^{1/2}, \\ \omega_3 = w_{\mathrm{H}}(01|00) = w_{\mathrm{H}}(00|10) = h\left(\frac{1}{2}\lambda(\pi + \alpha)\right), \\ \omega_4 = w_{\mathrm{H}}(10|01) = h(\pi\lambda)h\left(\frac{1}{2}\lambda(3\pi + \alpha)\right)/h(2\pi\lambda), \\ \omega_5 = w_{\mathrm{H}}(01|10) = h\left(\frac{1}{2}\lambda(3\pi - \alpha)\right). \end{cases} \tag{2.29}$$

The model is critical for $\hat{q} \to 0$, when h becomes a sine function. Then, as shown by

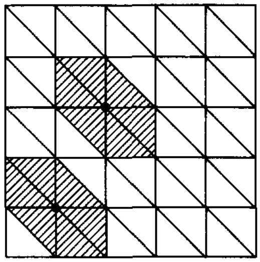

Fig. 4. Hard hexagons in the hard square model.

Baxter, critical properties depend only on

$$\Delta = \frac{\omega_1^2 - \omega_4\omega_5}{\omega_2\omega_3}, \tag{2.30}$$

which reads here as

$$\Delta = 2(2\cos\pi\lambda)^{1/2} - (2\cos\pi\lambda)^{3/2} - (2\cos\pi\lambda)^{-1/2}. \tag{2.31}$$

If $\omega_5 = 0$, this describes, as well, a free hard hexagon model [45] (fig. 4) of fugacity $z = \Delta^{-2}$, and weights

$$\begin{cases} \omega_1 = 1, \\ \omega_2 = \omega_3 = z^{1/4}, \\ \omega_4 = z^{1/2}; \end{cases} \tag{2.32}$$

a given hexagon being shared by four squares. The standard choice $\lambda = \frac{1}{5}$ gives $z = \frac{1}{2}(11 + 5\sqrt{5})$. On the other hand, $\lambda = \frac{2}{5}$ gives

$$z = \tfrac{1}{2}(11 - 5\sqrt{5}) < 0. \tag{2.33}$$

This singularity which appears as non-physical [46] in the hard hexagon point of view is expected to be in the universality class of the Lee–Yang problem. On the other hand, the above arguments establish that it corresponds to eq. (2.25) with $p' = 5$, $p = 2$ [21].

2.6. Finally, we return briefly to the $c = 1$ case. In addition to eq. (2.17), another line of models parametrized by g is obtained by adding the contribution of sectors where the field φ is antiperiodic along one or both periods of the torus, giving the

Z_2-orbifold partition functions [10,47]. The partition functions

$$Z_{rs} = \int_{\substack{\varphi(z+1) = e^{i\pi r}\varphi(z) \\ \varphi(z+\tau) = e^{i\pi s}\varphi(z)}} [D\varphi]\, e^{-\mathscr{A}}, \qquad r, s = 0, 1, \tag{2.34}$$

have been calculated in ref. [34] as

$$\begin{aligned} Z_{10} &= \left| \frac{\eta}{\theta_4(0)} \right|, \\ Z_{01} &= \left| \frac{\eta}{\theta_2(0)} \right|, \\ Z_{11} &= \left| \frac{\eta}{\theta_3(0)} \right|, \end{aligned} \tag{2.35}$$

(here θ_ν denote the Jacobi theta functions of argument τ). Summing over the different sectors, one gets

$$Z_{\text{orb}}(g) = \tfrac{1}{2} Z_{\text{C}}(g) + \sum_{\nu=2}^{4} \left| \frac{\eta}{\theta_\nu(0)} \right|, \tag{2.36}$$

which describes [48] the critical Ashkin–Teller model [29]. If $g = \frac{1}{2}$ the latter decouples into two independent Ising models and thus

$$Z_{\text{orb}}(g = \tfrac{1}{2}) = Z_{\text{Ising}}^2, \tag{2.37}$$

where

$$Z_{\text{Ising}} = \sum_\nu \frac{|\theta_\nu(0)|}{2|\eta|} = \sum_\nu Z_\nu, \tag{2.38}$$

and Z_ν are the partition functions of a free (Majorana) fermion with various boundary conditions [34]. One has, in particular,

$$Z_{\text{C}}(g = \tfrac{1}{2}) = 2 \sum_\nu Z_\nu^2, \tag{2.39}$$

while crossed terms in eq. (2.37) reproduce eq. (2.35) due to

$$2\eta^3 = \theta_2 \theta_3 \theta_4(0). \tag{2.40}$$

Models at $c = 1$ can also be obtained by repeating construction (2.25) with extended Lie algebras, since then there is an exponent $n = 0$. For $\hat{\text{A}}$, $\hat{\text{D}}$ one simply gets special points on lines (2.17), (2.36). For Ê one gets new, isolated points [11, 49]. The above models have been conjectured [50] to exhaust all possible $c=1$ theories. Let us point out, however, that combinations of the form

$$\tfrac{1}{2}[Z_{\mathrm{C}}(g)+Z_{\mathrm{C}}(g')] \tag{2.41}$$

are also modular invariant and, due to the symmetry $E, M \to -E, -M$ in eq. (2.17), expand on powers of $q, \bar{q}$ with positive integer coefficients. All the explicit lattice realizations of this remark known so far, however, are nonunitary and have a true central charge less than one.

3. Minimal superconformal theories

3.1. We consider now models which combine conformal invariance with $N=1$ supersymmetry (SUSY); they can be minimal with respect to the $N=1$ superconformal algebra. This occurs for central charges [51]

$$c=\tfrac{3}{2}-3(p-p')^2/pp'<\tfrac{3}{2}, \tag{3.1}$$

where either $p, p' \in 2N-1$ and $p \wedge p'=1$, or $p, p' \in 2N$, $\frac{1}{2}(p-p') \in 2N-1$ and $\frac{1}{2}p \wedge \frac{1}{2}p'=1$. In this case the allowed values of $h, \bar{h}$ are given by the formula [25]

$$h_{rs}=\frac{(rp-sp')^2-(p-p')^2}{8pp'}+\frac{t(2-t)}{16}, \tag{3.2a}$$

with the constraints

$$\begin{gathered} 1 \leqslant r \leqslant p'-1, \\ 1 \leqslant s \leqslant p-1, \\ t=|r-s| \bmod 2, \end{gathered} \tag{3.2b}$$

t equals 0 (respectively, 1) in the Neveu–Schwarz (Ramond) sector. Unitary theories correspond to $|p-p'|=2$, the simplest example describes the tricritical Ising model [52]. Using modular invariance, all possible partition functions of superconformal minimal models on a torus

$$Z=\sum_{h,\bar{h}}\left[N_{h\bar{h}}^{\mathrm{NS}}\chi_h^{\mathrm{NS}}\chi_{\bar{h}}^{\mathrm{NS}}+N_{h\bar{h}}^{\widetilde{\mathrm{NS}}}\chi_h^{\widetilde{\mathrm{NS}}}\chi_{\bar{h}}^{\widetilde{\mathrm{NS}}}+N_{h\bar{h}}^{\mathrm{R}}\chi_h^{\mathrm{R}}\chi_{\bar{h}}^{\mathrm{R}}\right], \tag{3.3}$$

have been classified [22]*. In eq. (3.3), NS, $\widetilde{\mathrm{NS}}$ and R denote various boundary

* Actually, only unitary superconformal minimal theories have been classified in [22] but it is easy to see that it extends to non-unitary ones with no additional cases. Incidentally, the case $p'=5$, $p=9$ leads to $c=\frac{13}{30}$; hence it realizes a non-unitary, non-minimal theory with $c<1$.

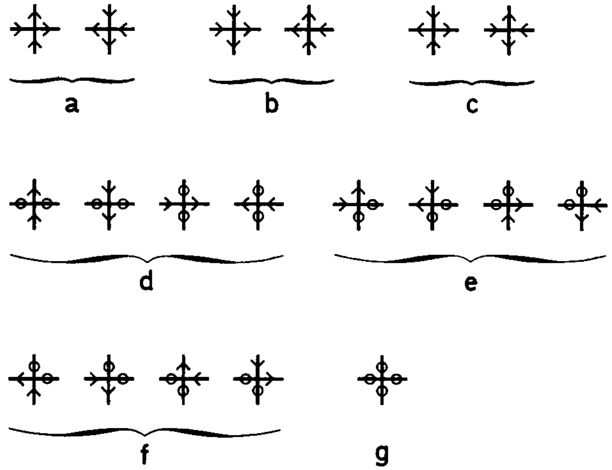

Fig. 5. Vertices of the 19-vertex model.

conditions on the fermion, and the χ's are the associated superconformal characters [53]. We have omitted in eq. (3.3) the $\tilde{\mathrm{R}}$ sector (doubly periodic) which decouples in modular transformations; its contribution is a pure number.

3.2. Recently Date, Jimbo, Kuniba, Miwa and Okado [54] have proposed generalized RSOS models obtained by a fusion procedure [15], the central charges of which are given by the unitary subset of eq. (3.1). This suggests that the underlying model in all the $N=1$ SUSY theories is a spin $S=1$ vertex model; the latter can be obtained by putting either an arrow or a dot (0-current) on each bond of the square lattice. Conservation of the current at a node gives then rise to 19 possible vertices (fig. 5). Commuting transfer matrices can be obtained for special sets of values of the weights.

Imposing *CPT* invariance [55] one is left with

$$\begin{cases} a=\sin\lambda(\pi-\alpha)\sin\lambda(\pi+\alpha)\,, \\ b=\sin\lambda\alpha\sin\lambda(2\pi-\alpha)\,, \\ c=\sin\lambda\pi\sin 2\lambda\pi\,, \\ d=\sin\lambda\alpha\sin\lambda(\pi-\alpha)\,, \\ e=\sin 2\lambda\pi\sin\lambda(\pi-\alpha)\,, \\ f=\sin 2\lambda\pi\sin\lambda\alpha\,, \\ g=\sin\lambda\pi\sin 2\lambda\pi-\sin\lambda\alpha\sin\lambda(\pi-\alpha)\,, \end{cases} \tag{3.4}$$

where λ can be real or purely imaginary and α is a spectral parameter. In a very anisotropic limit one gets a spin-1 antiferromagnetic XXZ quantum chain with hamiltonian

$$
\begin{aligned}
H \propto \sum_{i=1}^{L} & \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} - (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 \\
& - 2(\cos(\pi\lambda) - 1)\left[(S_i^z S_{i+1}^z)(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)(S_i^z S_{i+1}^z)\right] \\
& + 2\sin^2\pi\lambda\left[S_i^z S_{i+1}^z - (S_i^z S_{i+1}^z)^2 + 2(S_i^z)^2\right]. \qquad (3.5)
\end{aligned}
$$

To our knowledge, critical properties of eq. (3.4) have not been fully investigated. To derive them, we remark that the case $\lambda = 0$ (for which weights become rational functions) is expected, through non-abelian bosonization arguments [56], to describe the SU(2) level $k = 2$ WZW model. It is then instructive to notice that the corresponding partition function, obtained by using modular invariance, can be written as

$$
Z_{2-\mathrm{WZW}} = \sum_{r,s=0,1} \mathscr{Z}_2(r,s) \sum_{\substack{M=r[2] \\ M'=s[2]}} Z_{MM'}\left(g = \tfrac{1}{2}\right), \qquad (3.6)
$$

(see appendix A) where $\mathscr{Z}_2(r,s)$ denotes the partition function of an Ising ($\boldsymbol{Z}_2$) model with twisted boundary conditions $e^{i\pi r}(e^{i\pi s})$ on the spin variable. Here, and in the following, $[k]$ stands for "modulo k". This result can be explained in the following way. First, one can associate to the 19-vertex model a SOS model with height variables φ on the dual lattice, with neighbouring φ differing by 0, $\pm 2\pi$ depending on the current carried by the bond which separates them. This appears quite similar to the argument in sect. 2, except that, now, two neighbouring φ can be equal. On the other hand, if one considers a configuration of the 19-vertex model and puts heavy bonds on each 0-current bond, forgetting the remaining arrows, one gets loops which are precisely graphs in a low-temperature expansion of the Ising-model partition function (fig. 6). Eq. (3.6) suggests that in the continuum limit and for $\lambda = 0$, these two aspects decouple, the SOS model renormalizing onto the Gaussian free field (2.9) as for the 6-vertex case, the new degrees of freedom due to the possibility of a 0-step contributing a supplementary Ising factor. The only coupling in eq. (3.6) is due to boundary conditions and is also explained with these arguments. Indeed, if we consider the system on a lattice $L \times L'$ with L, L' even, the algebraic number of arrows crossed along the periods, which gives (M, M'), and the number of loops of heavy bonds crossed, which determines the spin flips (r, s), are of the same parity (fig. 6).

These arguments suggest, thus, that on an even $\times$ even torus, the 19-vertex model with weight (3.4) and λ real has the continuum-limit partition function

$$
Z_{\mathrm{SC}}(g) = \sum_{r,s=0,1} \mathscr{Z}_2(r,s) \sum_{\substack{M=r[2] \\ M'=s[2]}} Z_{MM'}(g), \qquad (3.7)
$$

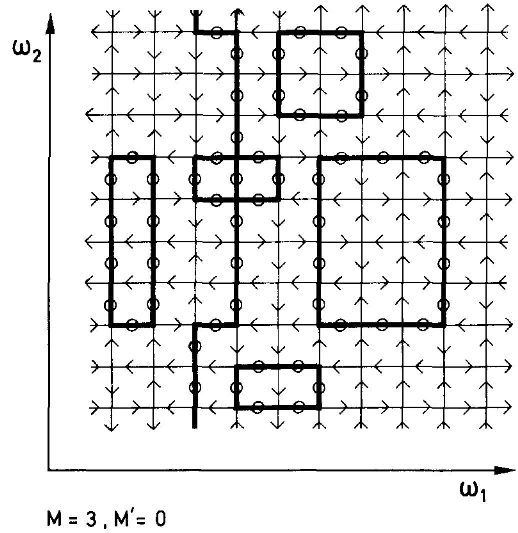

Fig. 6. A configuration of the 19-vertex model on the torus. Lines passing through bonds with a 0-current draw low-temperature graphs in the Ising model.

which satisfies the symmetry

$$Z_{\mathrm{SC}}(g) = Z_{\mathrm{SC}}\left(\frac{1}{4g}\right), \tag{3.8}$$

the fixed point of which is $g = \frac{1}{2}$, associated with $\lambda = 0$ as in eq. (3.6). Also, if $\lambda = \frac{1}{2}$, some weights vanish as for $\lambda = 1$ in eq. (2.8). This suggests that one should generalize eq. (2.11) to

$$g = \tfrac{1}{2} - \lambda, \qquad 0 \leqslant \lambda \leqslant \tfrac{1}{2}. \tag{3.9}$$

The point $\lambda = \frac{1}{2}$ should correspond, then, to a first-order transition towards ferroelectric (frozen) order, and $\lambda = 0$ an infinite-order transition towards antiferroelectric order.

If the model is studied on a lattice with L or L' odd, frustrations and flips can be of opposite parities. A new modular invariant is then obtained by considering

$$\sum_{r,s=0,1} \mathscr{Z}_2(r,s)\left(\sum_{\substack{M=r+1[2]\\ M'=s[2]}} + \sum_{\substack{M=r[2]\\ M'=s+1[2]}} + \sum_{\substack{M=r+1[2]\\ M'=s+1[2]}} \right) Z_{MM'}(g). \tag{3.10}$$

Adding eqs. (3.7) and (3.10) gives, due to the self-duality of the Ising model

(appendix A)

$$\mathscr{Z}_2 = \mathscr{Z}_2(0,0) = \tfrac{1}{2} \sum_{r,s=0,1} \mathscr{Z}_2(r,s), \tag{3.11}$$

a completely decoupled object

$$\tilde{Z}_{\mathrm{SC}} = \mathscr{Z}_2 \times Z_c(g). \tag{3.12}$$

3.3. The above arguments leading to eq. (3.7), although plausible, are not very rigorous. A first natural check is that Z_{SC} decomposes into powers of q and $\bar{q}$ with positive integer coefficients. This is indeed the case, since one can rewrite it after Poisson transformation as

$$Z_{\mathrm{SC}}(g) = \frac{1}{|\eta|^2}\Bigg[\left(|\chi_0|^2 + |\chi_{1/2}|^2\right) \sum_{E,\,M\,\mathrm{even}} + \left(\chi_0\chi_{1/2}^* + \chi_0^*\chi_{1/2}\right) \sum_{E,\,M\,\mathrm{odd}} + |\chi_{1/16}|^2 \sum_{E-M\,\mathrm{odd}}\Bigg] q^{(E/\sqrt{2g}+M\sqrt{2g})^2/8}\bar{q}^{(E/\sqrt{2g}-M\sqrt{2g})^2/8}, \tag{3.13}$$

where the χ's are characters of the Ising model. To verify our identifications more properly we have studied the 19-vertex model by numerical means, calculating the transfer matrix spectrum on strips of width $L \leqslant 10$, in the case $\alpha = \frac{1}{2}\pi$. This spectrum splits into sectors labelled by the value of the conserved total current in the ω_2 direction (fig. 6), which is expected to determine M. A first interesting quantity is the free energy per unit length on strips with L even ($M = 0$), which is expected to scale as [57]

$$f_L = f + \pi c/6L^2, \qquad c = \tfrac{3}{2}. \tag{3.14}$$

Estimates of c obtained by comparing two successive widths for some values of $\lambda \in [0, \frac{1}{2}]$ are given in table 1. They are in good agreement with $c = \frac{3}{2}$. On strips with L odd, the ground state is shifted as in eq. (3.10) and this occurs either with $M = 1$, $r = 0$ or $M = 0$, $r = 1$. Comparing numerical values deduced from eq. (3.9), one finds that the dimension $x = \frac{1}{8}$ of the Ising spin always gives the leading term, and thus

$$f_L = f + \pi\tilde{c}/6L^2, \qquad \tilde{c} = c - 12x = 0. \tag{3.15}$$

Numerical estimates of $\tilde{c}$ are also in good agreement with this value. We now restrict ourselves to L even. The scaling of gaps obtained by considering the ground state in the M sector (compared to the ground state in $M = 0$) depends on the

TABLE 1
Estimates of c obtained by comparing the ground states of the transfer matrix spectrum for two successive widths.

L	$\lambda = 0$	$\lambda = \pi/5$	$\lambda = 3\pi/10$
4	1.297	1.287	1.247
6	1.441	1.429	1.399
8	1.481	1.469	1.447
10	1.490	1.481	1.465

parity of M from eq. (3.13). For M odd they should behave like

$$m_L^{(M)} = \frac{L}{2\pi}\left(\tfrac{1}{8} + \tfrac{1}{2}gM^2\right), \tag{3.16}$$

while

$$m_L^{(M)} = \frac{L}{2\pi}\left(1 + \tfrac{1}{2}gM^2\right), \tag{3.17}$$

for M even. In particular, the case $M = 1$ gives access to the coupling constant g. Estimates of the latter, obtained using $m_L^{(1)}$ and eq. (3.16) are given in fig. 7a. They agree nicely with the conjecture (3.9). The general structure of eqs. (3.16) and (3.17) can also be checked. An interesting property emerging from eq. (3.13) is also the existence of an operator with dimension $x = 1$ (originating from $|\chi_{1/2}|^2$) constant along the critical line; it should be observed in the first gap of the $M = 0$ sector. Measurements of this gap give the estimates of fig. 7b, which agree reasonably well

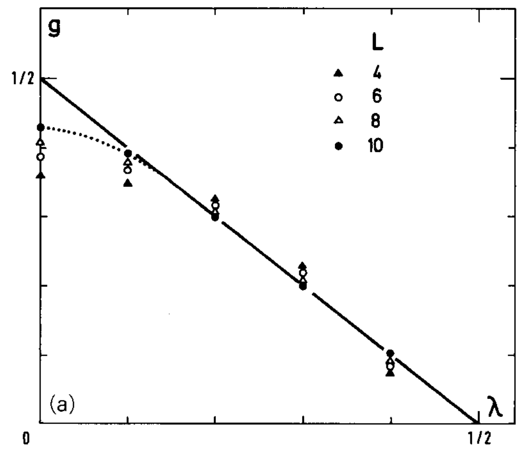

Fig. 7a. Numerical estimates of g (3.17). The solid line is the conjecture (3.9).

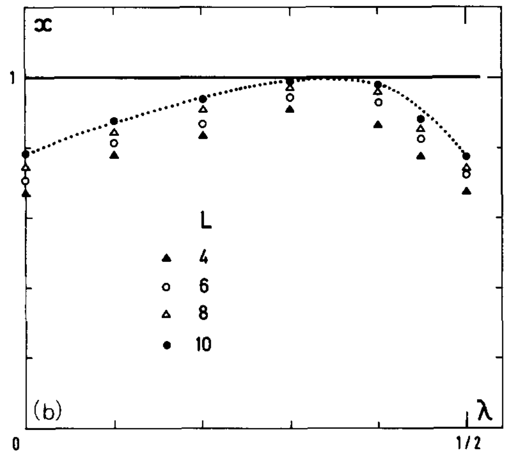

Fig. 7b. Numerical estimate of the lowest dimension in the $M=0$ sector. One expects $x=1$ along the critical line.

with this prediction. In all cases, the convergence becomes poor around $\lambda=0$. This is related to the presence of new marginal operators ($E=\pm 2$, $M=0$ or $E=0$, $M=\pm 2$ for instance), known [58] to induce additional logarithmic corrections.

We do not comment any further on this numerical study. It could be refined using various technical tricks. Also, following methods developed for the 6-vertex model [59], corrections to the Bethe Ansatz could be obtained, and the expression (3.13) analytically verified. We think, however, the above arguments are sufficient to support the conjecture (3.13).

A remarkable property of the 19-vertex model with weights (3.4) is that it exhibits $N=1$ supersymmetry (SUSY) in the continuum limit. This can be observed in formula (3.7) which appears as the partition function of a super free field with action

$$\mathscr{A}_{\mathrm{SC}}=\frac{g}{2\pi}\int\left[\partial_z\varphi\,\partial_{\bar z}\varphi-\psi\,\partial_{\bar z}\psi-\bar\psi\,\partial_z\bar\psi\right]\mathrm{d}z\,\mathrm{d}\bar z. \tag{3.18}$$

Indeed, the partition functions with twisted spins boundary conditions $\mathscr{Z}_2(r,s)$ are related to the corresponding Majorana fermion partition functions by

$$\begin{aligned}\mathscr{Z}_2(0,0)&=Z_{\mathrm{Ising}}=Z_2+Z_3+Z_4,\\ \mathscr{Z}_2(0,1)&=-Z_2+Z_3+Z_4,\\ \mathscr{Z}_2(1,0)&=Z_2-Z_3+Z_4,\\ \mathscr{Z}_2(1,1)&=Z_2+Z_3-Z_4.\end{aligned} \tag{3.19}$$

So eq. (3.7) can be rewritten as

$$Z_{\mathrm{SC}}(g) = Z_2\left(\sum_{M,M'\in 2Z} - \sum_{\substack{M\in 2Z\\ M'\in 2Z+1}} + \sum_{\substack{M\in 2Z+1\\ M'\in 2Z}} + \sum_{M,M'\in 2Z+1} \right) Z_{MM'}\left(\tfrac{1}{2}g\right)$$

$$+ Z_3\left(\sum_{M,M'\in 2Z} + \sum_{\substack{M\in 2Z\\ M'\in 2Z+1}} + \sum_{\substack{M\in 2Z+1\\ M'\in 2Z}} - \sum_{M,M'\in 2Z+1} \right) Z_{MM'}\left(\tfrac{1}{2}g\right)$$

$$+ Z_4\left(\sum_{M,M\in 2Z} + \sum_{\substack{M\in 2Z\\ M'\in 2Z+1}} - \sum_{\substack{M\in 2Z+1\\ M'\in 2Z}} + \sum_{M,M'\in 2Z+1} \right) Z_{MM'}\left(\tfrac{1}{2}g\right), \quad (3.20)$$

$$Z_{\mathrm{SC}}(g) = \sum_{r,s=0,1} \sum_{M,M'\in Z} \varepsilon^{rs}_{MM'} \int_{\substack{\psi(z+1)=\mathrm{e}^{i\pi r}\psi(z)\\ \psi(z+\tau)=\mathrm{e}^{i\pi s}\psi(z)\\ \text{same for } \tilde{\psi}}} \mathrm{d}[\psi\tilde{\psi}] \int_{\substack{\varphi(z+1)=\varphi(z)+2\pi M\\ \varphi(z+\tau)=\varphi(z)+2\pi M'}} [\mathrm{d}\varphi]\mathrm{e}^{-\mathscr{A}_{\mathrm{SC}}} \quad (3.21)$$

where

$$\varepsilon^{rs}_{MM'} = 1 - 2\delta_{r,\,M \bmod 2}\delta_{s,\,M' \bmod 2}\,.$$

The precise coupling between ψ and φ, induced by boundary conditions through eq. (3.7), is rather intriguing. One can check explicitly that eq. (3.13) can be expanded on $c=\frac{3}{2}$ characters (appendix B). Finally, we note that there is another formulation of eq. (3.18) using the $N=3$ Gross–Neveu model which plays here the role of the Thirring model in sect. 2 [60].

3.4. Using the "super Coulombic" partition functions (3.7), one expects to reproduce the superminimal partition functions of ref. [22] by the same modification as in sect. 2 for the ordinary minimal ones.

Starting from a free superfield and adding a charge at infinity [51b, 61] decreases c to

$$c = \tfrac{3}{2} - 6e^2/g, \quad (3.22)$$

suggesting (for instance)

$$g = p/2p', \qquad e = (p'-p)/2p'. \quad (3.23)$$

Now, dimensions of operators read

$$h_{\mathrm{EM}} = \Delta_{\mathrm{EM}} - e^2/4g + \delta, \quad (3.24)$$

the last term δ, which is equal to 0 or $\frac{1}{16}$, being the contribution of the Ising spin operator. This has to be compared with the Kac formula

$$h_{rs} = \frac{\Lambda^2}{8pp'} - \frac{(p'-p)^2}{8pp'} + \frac{t(2-t)}{16}\,. \tag{3.25}$$

As shown in ref. [22], minimal superconformal partition functions are still classified by a pair of simply laced algebras (G, G') of Coxeter numbers p and p'. In the case where $G = A$, one checks that modular invariants involve dimensions (3.25) with $\Lambda = 2n$, n running over the N exponents of G'.

This suggests to introduce the same $(N-1)$ additional electric charges as in sect. 2, $e_n = n/p'$, and to build partition functions, with the SUSY generalization of eq. (2.25), as

$$Z = \sum_{r,s=0,1} \mathscr{X}_2(r,s) \sum_{\substack{M=r[2]\\ M'=s[2]}} Z_{MM'}\left(\frac{p}{2p'}\right) \sum_n \cos\left(2\pi \frac{n}{p'} M \wedge M'\right). \tag{3.26}$$

Using similar techniques as used for minimal models, we show in appendix B that this indeed reproduces the results of ref. [22]. Performing the summation over n and decomposing into various congruence classes of M and M', eq. (3.26) can be reexpressed as a linear combination of Z_{SC}, $\tilde{Z}_{\mathrm{SC}}$ for various couplings.

In the classification of ref. [22] there appears, also, invariants labelled by the pair $(\mathrm{D}_{p/2+1}, \mathrm{E}_6)$ ($p' = 12$). They can be reexpressed as

$$\begin{aligned} Z = {} & \tilde{Z}_{\mathrm{SC}}\left(\tfrac{1}{2}pp'\right) + \tilde{Z}_{\mathrm{SC}}(p/2p') - \tilde{Z}_{\mathrm{SC}}\left(\tfrac{1}{32}pp'\right) - \tilde{Z}_{\mathrm{SC}}(8p/p') \\ & + Z_{\mathrm{SC}}(2p/p') + Z_{\mathrm{SC}}\left(\tfrac{1}{32}pp'\right) - Z_{\mathrm{SC}}\left(\tfrac{1}{8}pp'\right) - Z_{\mathrm{SC}}(p/2p'), \end{aligned} \tag{3.27}$$

Z_{SC} and $\tilde{Z}_{\mathrm{SC}}$ being defined respectively in (3.7), (3.12). But we have not been able to find for them a formula similar to eq. (3.26).

3.5. Now, eqs. (3.23) and (3.24) can be related to the construction of fused RSOS models in ref. [54]. We notice, first, that the weights (3.4), although this was not observed in their derivation [55], can be obtained by a fusion of the 6-vertex model (2.8). Similarly, fusion of the eight-vertex model defines a 21-vertex model [62] which can, in turn, be reformulated as a solid-on-solid model on the dual lattice, neighbouring heights differing by 0, ± 2, with IRF type Boltzmann weights. Their expression is similar to eq. (2.27) and cumbersome. We do not use it in the following and refer the reader to ref. [54]. These weights involve parameters λ, α, w and a (real) nome $\hat{q}$. When $w = 0$ and $\lambda = 1/(N+1)$, heights can be consistently restricted to the set $l = 1, \dots, N$ and thus considered as being attached to the Dynkin diagram of the A_N algebra of Coxeter number $H=N+1$. Note that because the steps are 0, ± 2, heights may be restricted to be even or odd, giving rise to two distinct models.

Criticality occurs when $\hat{q}=0$. In the regime III to regime IV transition, the 21-vertex model reduces then precisely to the 19-vertex model with weights (3.4). It renormalizes onto the super Gaussian model (3.18) with coupling constant $g=\frac{1}{2}-\lambda=(H-2)/2H$ and comparison with eq. (3.23) gives $p=H-2,\ p'=H$. One can also define a dilute model renormalizing onto another branch $g=\frac{1}{2}+\lambda=(H+2)/2H$; in this case $p=H+2,\ p'=H$.

Correspondence between the vertex and RSOS model becomes more complicated on the torus, and we leave to a further study the precise microscopic justification of topological terms in eq. (3.24). The procedure should, as well, generalize to D or E algebras, and to nonunitary cases choosing $\lambda=R/(N+1)$. In particular, we believe that the fusion of E_6 involves some subtleties [63], which could explain the appearance of the new type of invariant (3.27).

In this formalism, the tricritical Ising model occurs for $p=3,\ p'=5,\ c=\frac{7}{10}$. Indeed, in this case the only modular invariant is associated to the algebra A_4. Steps take the values 0, ± 2, and the two distinct models corresponding to even or odd heights are identical. In each case, height variables take two possible values and can be expressed as Ising spins with four spin interactions. Supersymmetry appears, thus, as very natural once it is traced back to the underlying 19-vertex model.

3.6. Finally, we return to the $c=\frac{3}{2}$ case. Various critical lines can then be constructed in a way similar to the $c=1$ case. In addition to $Z_{\mathrm{SC}}(g)$, $\tilde{Z}_{\mathrm{SC}}(g)=\mathscr{Z}_2\times Z_c(g)$, $\mathscr{Z}_2\times Z_{\mathrm{orb}}(g)$, one has, in particular, a generalized orbifold

$$Z_{\mathrm{S\,orb}}(g)=\tfrac{1}{4}Z_{\mathrm{SC}}(g)+\tfrac{3}{2}\sum_{\substack{r,s=0,1\\(r,s)\neq(0,0)}}\mathscr{Z}_2(r,s)Z_{rs}+\tfrac{15}{4}, \tag{3.28}$$

where Z_{rs} has been defined in eq. (2.34). If $g=\frac{1}{2}$

$$Z_{\mathrm{SC}}\left(g=\tfrac{1}{2}\right)=4\sum_{\nu}Z_{\nu}^{3}, \tag{3.29}$$

which agrees with the construction of SU(2) $k=2$ WZW model as a theory with three free Majorana fermions [64]. Combining eqs. (3.29) and (3.19)

$$Z_{\mathrm{S\,orb}}\left(g=\tfrac{1}{2}\right)=Z_{\mathrm{Ising}}^{3}, \tag{3.30}$$

so eq. (3.28) should describe the SUSY Ashkin–Teller model [60]. Of course, construction (3.26) can also be applied to extended Lie algebras, giving additional $c=\frac{3}{2}$ models. The complete classification of $c=\frac{3}{2}$ theories remains an open question (see, however, refs. [65, 66]).

4. General SU(2) ⊗ SU(2) / SU(2) theories

4.1. It is known that all the previous models (in the unitary case) can be obtained through the Goddard–Kent–Olive "coset" construction [12]

$$\frac{\mathrm{SU}(2)_k \otimes \mathrm{SU}(2)_{m-2}}{\mathrm{SU}(2)_{k+m-2}}. \tag{4.1}$$

The case $k=1$ corresponds to the series (2.1) and $k=2$ to the SUSY series (2.2). Recently, theories for a general value of k have been studied [14] with central charges deduced from eq. (4.1) to be

$$c=\frac{3k}{k+2}-\frac{6k}{m(m+k)}, \qquad m \geqslant 2. \tag{4.2}$$

These are likely to be minimal with respect to some extended conformal algebra [14b], the primary fields of which have the conformal weights

$$h_{rs}=\frac{[r(m+k)-sm]^2-k^2}{4km(m+k)}+\frac{t(k-t)}{2k(k+2)}, \tag{4.3}$$

where

$$\begin{gathered} 1 \leqslant r \leqslant m-1, \\ 1 \leqslant s \leqslant m+k-1, \\ t=|r-s \bmod 2k|, \qquad 0 \leqslant t \leqslant k. \end{gathered} \tag{4.4}$$

4.2. Associated lattice models have been proposed in ref. [54]. These are basically generalized RSOS models obtained by k fusions of the fundamental $k=1$ ones. Also, results (4.2) and (4.3) have been discussed using a Feigin–Fuchs construction involving a bosonic field and a parafermion of statistics Z_k [14]. This suggests immediately, generalizing the results of sects. 2 and 3, that the underlying model in all theories (4.2) is a spin $S=\frac{1}{2}k$ vertex model, obtained by assigning one of the $2S+1$ states on each bond of the square lattice. Conservation of the current at each node then gives rise to Γ_k possible vertices where

$$\Gamma_k = \frac{1}{(2k)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2k}\left(\frac{x^{k+1}-1}{x-1}\right)^4\Bigg|_{x=0}, \tag{4.5}$$

$\Gamma_1 = 6$, $\Gamma_2 = 19$, $\Gamma_3 = 44, \ldots$. As discussed in refs. [15, 54b] commuting transfer matrices can be obtained for a special set of weights which are precisely those derived from the fusion of the 6-vertex model (2.8). These weights involve a constant λ and a spectral parameter α. In the very anisotropic limit one gets a spin-$\frac{1}{2}k$ XXZ quantum chain [67].

The point $\lambda = 0$ (for which weights become rational functions) is expected through non-abelian bosonization arguments to describe the SU(2) level $-k$ WZW model [56]. A distinction appears here depending on the parity of k; we first discuss the case of k odd. There is then only one possible modular invariant [6, 7] which can be written as

$$Z_{k-\mathrm{WZW}} = \sum_{r,s=0,\ldots,k-1} \mathscr{Z}_k(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}(g=1/k), \tag{4.6}$$

where $\mathscr{Z}_k(r,s)$ denotes the partition function of the "$\boldsymbol{Z}_k$ Ising model" [18, 19] with twisted boundary conditions $\mathrm{e}^{2i\pi r/k}$ ($\mathrm{e}^{2i\pi s/k}$) on the spin variable (see appendix A for more details). Eq. (4.6) suggests a formula similar to (3.7) describing the critical line of the Γ_k-vertex model

$$Z_{k-\mathrm{C}}(g) = \sum_{r,s=0,\ldots,k-1} \mathscr{Z}_k(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}(g). \tag{4.7}$$

For $k = 1, 2$ this reduces of course to Z_{C}, Z_{SC}.

One can check that eq. (4.7) decomposes into powers of $q, \bar{q}$ with integer coefficients. The associated central charge is

$$c = 3k/(k+2). \tag{4.8}$$

Expression (4.7) has the duality symmetry

$$Z_{k-\mathrm{C}}(g) = Z_{k-\mathrm{C}}(1/k^2 g), \tag{4.9}$$

the fixed point of which is eq. (4.6), associated with $\lambda = 0$. On the other hand $\lambda = 1/k$ corresponds [54] to the vanishing of some weights as in eqs. (2.8) and (3.4);

this suggests the relation

$$g = 1/k - \lambda, \qquad 0 \leqslant \lambda \leqslant 1/k. \tag{4.10}$$

The appearance of parafermions in eq. (4.7) can be traced back to the vertex configurations, as given in sect. 3. Indeed, one can introduce, first, a SOS model on the dual lattice, where neighbouring heights φ differ by amounts depending on the state of the bond which separates them. Also, the $2S-1=k-1$ bonds with current $J<S$ can be associated with twists $\exp 2i\pi(2J/k)$, hence, giving graphs in the low-temperature expansion of a $\boldsymbol{Z}_k$ model. This makes sense since the total current is conserved at each node, so $\Sigma 2J = 0[k]$. Eq. (4.7) suggests that these two aspects decouple in the continuum limit and the model renormalizes onto a Gaussian free field (2.9), while the additional degrees of freedom are described by a parafermionic $\boldsymbol{Z}_k$ theory.

4.3. The construction of ref. [14] suggests to add a charge at infinity to decrease c to

$$c = 3k/(k+2) - 6(e^2/g). \tag{4.11}$$

Hence (for instance)

$$g = m/k(m+k), \qquad e = 1/(m+k). \tag{4.12}$$

Dimensions of operators now read

$$h_{\mathrm{EM}} = \Delta_{\mathrm{EM}} - e^2/4g + \delta_t \tag{4.13}$$

the last term being the dimension of the spin operator σ_t in the $\boldsymbol{Z}_k$ model $\delta_t = t(k-t)/2k(k+2)$ depending on the sector. On the other hand, the Kac formula (4.3) reads

$$h_{rs} = \frac{\Lambda^2}{4km(m+k)} - \frac{k}{4m(m+k)} + \frac{t(k-t)}{2k(k+2)}. \tag{4.14}$$

Modular invariants have been obtained in ref. [14a]. They are still classified by a pair of Lie algebras (A, G) and associated Λ's read $\Lambda = nk$ where n runs over the exponents of G.

This suggests that one should introduce

$$Z = \sum_{r,s=0,\ldots,k-1} \mathscr{Z}_k(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}\left[\frac{p}{kp'}\right] \sum_{\substack{n\in \text{exponents}\\ \text{of G}}} \cos\left(\frac{2\pi n}{p'} M \wedge M'\right). \tag{4.15}$$

We show in appendix C that eq. (4.15) with G = A and $p = m$, $p' = m + k$ or $p = m + k$, $p' = m$ indeed reproduces the invariants of ref. [14b]. It may be generalized to non-unitary cases by taking p and $(p - p')/k$ as coprimes. In this case, the central charge reads

$$c = \frac{3k}{k+2} - \frac{6(p-p')^2}{kpp'}. \tag{4.16}$$

Finally, eqs. (4.13) and (4.14) can be interpreted in physical terms. The Γ_k vertex model with $\lambda = 1/(N+1)$ is equivalent to the fused SOS models of ref. [54] at criticality, heights of which are attached to Dynkin diagram of A_N (note that since k is odd all heights are coupled). Using eq. (4.10) one finds the coupling constant $g = 1/k - \lambda = (H-k)/kH$ and comparison with eq. (4.13) gives $p = m = H - k$, $p' = m + k$. The dilute branch gives, similarly, $g = 1/k + \lambda = (H+k)/kH$, hence $p = m + k = H + k$, $p' = m$. On the torus, boundary conditions imply the introduction of the additional topological terms in eq. (4.15).

4.4. The case of k being even is more delicate. There exists, then, several modular invariants for the SU(2) WZW model, labelled by a Lie algebra G with Coxeter number $k+2$ (eq. (4.6) corresponds to G = A). With each of them is associated a twisted $\mathbf{Z}_k$ partition function $\mathscr{Z}_k^{(G)}$. One checks that

$$Z_{k-\mathrm{WZW}}^{(G)} = \sum_{r,s=0,\dots,k-1} \mathscr{Z}_k^{(G)}(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}(g=1/k). \tag{4.17}$$

Moreover, for G = D one has

$$\mathscr{Z}_k^{(D)}(r,s) = \tfrac{1}{2}\big[\mathscr{Z}_k(r,s) + (-)^r \mathscr{Z}_k(r, s+\tfrac{1}{2}k) + (-)^s \mathscr{Z}_k(r+\tfrac{1}{2}k, s) + (-)^{r+s} \mathscr{Z}_k(r+\tfrac{1}{2}k, s+\tfrac{1}{2}k)\big], \tag{4.18}$$

where $\mathscr{Z}_k \equiv \mathscr{Z}_k^{(A)}$. This suggests that there are several choices of boundary conditions (or weights) for the vertex model, leading to different critical partition functions

$$Z_{k-\mathrm{C}}^{(G)}(g) = \sum_{r,s=0,\dots,k-1} \mathscr{Z}_k^{(G)}(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}(g). \tag{4.19}$$

Eq. (4.19) satisfies the symmetry of eq. (4.9), and can be expanded into powers of q and $\bar{q}$ with positive integral coefficients. We expect eq. (4.10) to remain valid.

Modular invariants have been written [14a]. They are generally classified by a triplet of algebras $(\mathrm{G},\mathrm{G}')_{\mathrm{G}''}$. If G is of the A type, we expect partition functions to be given by the natural generalization of eqs. (2.25), (3.26) and (4.13), i.e.

$$Z = \sum_{r,s=0,\dots,k-1} \mathscr{Z}_k^{(\mathrm{G}'')}(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}\left[\frac{p}{kp'}\right] \sum_{\substack{n\in\text{exponents}\\ \text{of G}'}} \cos 2\pi \frac{n}{p'}(M\wedge M'), \tag{4.20}$$

where $p(p')$ are the Coxeter numbers of A(G′).

The symmetry between G and G″ noticed in ref. [14a] is not obvious on this formula.

5. $N=2$ superconformal minimal theories

5.1. In this section, we consider conformal invariant theories which are $N=2$ supersymmetric, i.e. contain two spin-$\frac{3}{2}$ superpartners of the energy–momentum tensor and a U(1) current.

They have been proved to be unitary and minimal for a discrete set of central charges $c<3$

$$c = 3k/(k+2) = 3 - 6/(k+2), \tag{5.1}$$

where k is a positive integer.

Representations of the $N=2$ superconformal algebra fall into three sectors (P), (A), (T) according to the moding of the four generators $T(z)$, $J(z)$, $G(z)$, $\tilde{G}(z)$ of respective conformal spins $2, 1, \frac{3}{2}, \frac{3}{2}$. These sectors correspond to: periodicity around the origin in the plane for all generators (P); antiperiodicity for G, $\tilde{G}$, (A), and finally twists, i.e. antiperiodicity for $\tilde{G}$ and J (T).

Modular invariants in the (A, P) sector have been constructed in ref. [23], and turn out to be classified by a simply laced algebra G. Using the same techniques as in the previous sections, the partition functions can be written in a way similar to eq. (4.7). One finds, for k even, that

$$Z_{k;\,N=2}^{(\mathrm{G})} = \tfrac{1}{2} \sum_{r,s\in \mathbf{Z}_k} \mathscr{Z}_k^{(\mathrm{G})}(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}\left(g=\frac{k+2}{2k}\right)$$

$$+ \sum_{r,s\in Z_{k/2}} \mathscr{Z}_k^{(\mathrm{G})}(2r,2s) \sum_{\substack{M=2r[k]\\ M'=2s[k]}} Z_{MM'}\left(g=\frac{k+2}{2k}\right), \tag{5.2}$$

and for k odd

$$Z^{(G)}_{k;\,N=2} = \tfrac{1}{2} \sum_{r,s\in Z_k} \mathscr{Z}^{(G)}_k(r,s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}\left(g = \frac{k+2}{2k}\right)$$

$$+ \tfrac{1}{2} \sum_{r,s\in Z_k} \mathscr{Z}^{(G)}_k(2r,2s) \sum_{\substack{M=r[k]\\ M'=s[k]}} Z_{MM'}\left(g = \frac{2(k+2)}{k}\right). \tag{5.3}$$

Let us give the expressions for $Z^{(G)}_{k;\,N=2}$ in the first cases $k = 1, 2, 3$

$$Z^{(A)}_{k=1,\,N=2} = \tfrac{1}{2}\left[Z_C(g=6) + Z_C(g=\tfrac{3}{2})\right]. \tag{5.4}$$

By Euler's identity, one can easily check that

$$Z^{(A)}_{k=1,\,N=2} = Z_C(6) - 1 = Z_C(\tfrac{3}{2}) + 1,$$

and $g = 6$ and $g = \frac{3}{2}$ are $N = 2$ supersymmetric [47, 68, 69] points of the Gaussian line

$$Z^{(A)}_{k=2,\,N=2} = \tfrac{1}{2}\left(Z_{SC}(g=1) + \tilde{Z}_{SC}(g=4)\right), \tag{5.5}$$

$$Z^{(A)}_{k=3,\,N=2} = \tfrac{1}{2}\left(Z_{3-C}(g=\tfrac{5}{6}) + Z_{3-C}(g=\tfrac{10}{3})\right). \tag{5.6}$$

These expressions are other realizations of the form (2.41). For the closure of the operator algebra, however, one must project the P-sector on states of even or odd fermion number. Expressions of the same nature can be obtained for the twisted sector [23b] as well.

6. Conclusion

In this paper, we have shown that conformal theories associated with the coset construction $SU(2)_k \times SU(2)_{m-2}/SU(2)_{k+m-2}$ may be realized by a common procedure. An integrable lattice model, obtained by k fusions of the 6-vertex model has a critical regime described by a continuous line of central charge $c = 3k/(k+2)$ containing the WZW model of level k. Modification of its boundary conditions, through the introduction of floating charges related to the exponents of a simply laced Lie algebra of Coxeter number m or $k+m$, yields the desired models. As the original vertex model can be described in terms of tensor products of parafermionic and bosonic sectors, so do the modified ones, with an appropriate alteration due to the floating charges.

This procedure is an alternative to the direct construction of lattice realizations of the coset models by the Kyoto group [54], by fusions of the ($k=1$) minimal models.

As in previous works [5,6], unitarity plays a minor role in the discussion. We think that generalizing the coset construction to non-unitary representations of the Virasoro or of Neveu–Schwarz–Ramond algebras would be a useful task. This presumably involves considering fractional level representations of the $\widehat{\mathrm{SU}}(2)$ Kac–Moody algebra.

Our method fails to reproduce some models involving exceptional Lie algebras; we believe that additional possibilities in the fusion and restriction of these models exist and remain to be discovered.

As mentioned at the end of sect. 4, the coset construction shows the symmetric roles played by the levels k and $m-2$. This symmetry, however, is not explicit in eq. (4.20). More generally, one might expect some permutation symmetry within the triplet of algebras $(\mathrm{G},\mathrm{G}',\mathrm{G}'')$ classifying the models. This may be made explicit for the ordinary minimal models, for which a formula equivalent to eq. (2.25) reads as

$$Z^{(\mathrm{A}_{p-1},\mathrm{G}_{p'-1})}=\tfrac{1}{4}\sum_{M,M'\in\mathbf{Z}} Z_{MM'}\left(\frac{1}{pp'}\right)\sum_{\substack{n\in\text{exponents}\\ \text{of }\mathrm{A}_{p-1}}}\cos\left(\frac{2\pi n}{p}M\wedge M'\right)$$

$$\times\sum_{\substack{n'\in\text{exponents}\\ \text{of }\mathrm{G}_{p'-1}}}\cos\left(\frac{2\pi n'}{p'}M\wedge M'\right).$$

For the other series of models, however, no such symmetric formula is known to us.

Our construction does not shed any light on the intriguing question of the "extended chiral algebras", discussed in refs. [14c, 70], for which the models under study are supposed to be minimal.

It is most likely that the same kind of considerations applies to models based on the coset construction with higher rank algebras [71]. Integrable vertex models have been identified [72], as well as generalized parafermions [73]. In the Coulomb interpretation, the free field lives on a higher dimensional torus [74].

Finally, as the parafermionic sectors seem to be an essential building block in our construction, it might be worth looking more closely at their properties. In particular, one may wonder if they cannot, in turn, be bosonized. It is likely that this may be achieved using a $(k-1)$-component free field. This was already apparent in ref. [75] where a vertex operator construction of the parafermion field itself was given. Also, it has been noticed that the parafermionic central charge $c_k=2(k-1)/(k+2)$ coincides with that of the models obtained by the $\mathrm{SU}(k)_1\times\mathrm{SU}(k)_1/\mathrm{SU}(k)_2$ coset construction. The latter may be bosonized [74] but it remains to identify the two theories. An alternative bosonization of these parafermionic theories, more economical and natural in view of their central charge $c_k < 2$, would appeal to only two boson fields.

It is a pleasure to thank I. Kostov and V. Pasquier for discussions. H.S. acknowledges early discussions with P. Ginsparg and D. Kastor and hospitality extended to him by the Enrico Fermi Institute at Chicago and by SLAC.

Appendix A

A.1. We first recall the expression of the affine SU(2) characters for representations of level k and isospin $\frac{1}{2}l$, and hence central charge $c = 3k/(k+2)$

$$\chi_l(q) = \frac{1}{\eta^3(q)} \sum_{t=-\infty}^{\infty} [2(k+2)t + l + 1]\, q^{[2(k+2)t+l+1]^2/4(k+2)}, \tag{A.1}$$

where l is integer and satisfies $0 \leqslant l \leqslant k$.

Corresponding partition functions of the SU(2) WZW model are classified by a single simply laced Lie algebra G with Coxeter number $k+2$

$$Z^{(\mathrm{G})}_{k-\mathrm{WZW}} = \sum_{l,\bar{l}=0}^{k} N^{(\mathrm{G})}_{l\bar{l}} \chi_l(q) \chi_{\bar{l}}(\bar{q}). \tag{A.2}$$

On the other hand the latter theories may be obtained, following Zamolodchikov and Fateev [18], by combining a free boson with a non-local (parafermionic) current algebra with $\boldsymbol{Z}_k \times \bar{\boldsymbol{Z}}_k$ symmetry. This decomposition is manifest in the following character formula

$$\chi_l(q) = \sum_{m=-k+1}^{k} \eta(q) C^l_m(q) \frac{\theta_m(q)}{\eta(q)}, \tag{A.3}$$

where

$$\theta_m(q) = \sum_{n=-\infty}^{\infty} q^{k(n+m/2k)^2}. \tag{A.4}$$

θ_m/η represents the bosonic contribution and C^l_m are the level-k SU(2) string functions [76]; ηC^l_m appears in eq. (A.3) as a branching coefficient of the coset construction for SU(2)/U(1). This leads to the parafermionic theories with central charges $c = 2(k-1)/(k+2)$ and partition functions

$$\mathscr{Z}^{(\mathrm{G})}_k = \tfrac{1}{2}|\eta|^2 \sum_{m=-k+1}^{k} \sum_{l,\bar{l}=0}^{k} N^{(\mathrm{G})}_{l\bar{l}} C^l_m(q) C^{\bar{l}}_m(\bar{q}). \tag{A.5}$$

One can also consider twisted boundary conditions $e^{2i\pi r/k}(e^{2i\pi s/k})$ for the Z_k-Ising spin, with the corresponding partition functions [19]

$$\mathscr{Z}_k^{(G)}(r,s) = \tfrac{1}{2}|\eta|^2 e^{-2i\pi rs/k} \sum_{m=-k+1}^{k} \sum_{l,\bar{l}=0}^{k} e^{2i\pi ms/k} N_{l\bar{l}}^{(G)} C_m^l(q) C_{m-2r}^{\bar{l}}(\bar{q}). \quad \text{(A.6)}$$

They enjoy modular covariance properties expected from their definition

$$\begin{aligned} S: &\quad \mathscr{Z}_k^{(G)}(r,s) = \mathscr{Z}_k^{(G)}(-s,r), \\ T: &\quad \mathscr{Z}_k^{(G)}(r,s) = \mathscr{Z}_k^{(G)}(r,r+s), \end{aligned} \quad \text{(A.7)}$$

and satisfy sum rules

$$\mathscr{Z}_k^{(G)}(r,s) = \frac{1}{k} \sum_{r's'} e^{2i\pi(rs'-sr')/k} \mathscr{Z}_k^{(G)}(r',s'), \quad \text{(A.8)}$$

reflecting self-duality; this identity may be interpreted as expressing the partition functions where the twist affects the spin variable in terms of those where it affects the dual disorder variable.

Other modular invariants may be obtained by summing (A.6) over the different orbits of the modular group. Also in the case of k even, it is interesting to notice that

$$\begin{aligned} \mathscr{Z}_k^{(D_{k/2+2})}(r,s) = \tfrac{1}{2}\Big[&\mathscr{Z}_k^{(A_{k+1})}(r,s) + (-)^r \mathscr{Z}_k^{(A_{k+1})}\big(r, s+\tfrac{1}{2}k\big) \\ &+ (-)^s \mathscr{Z}_k^{(A_{k+1})}\big(r+\tfrac{1}{2}k, s\big) + (-)^{r+s} \mathscr{Z}_k^{(A_{k+1})}\big(r+\tfrac{1}{2}k, s+\tfrac{1}{2}k\big). \end{aligned} \quad \text{(A.9)}$$

This has the simple consequence that $\mathscr{Z}_k^{(D)}$ is in fact obtained from $\mathscr{Z}_k^{(A)}$ by summing eq. (A.6) over even r, s.

A.2. We now introduce a bosonic partition function similar to eq. (A.6)

$$\hat{Z}(r,s) = \frac{1}{k|\eta|^2} e^{2i\pi rs/k} \sum_{m=-k+1}^{k} e^{-2i\pi ms/k} \theta_m \theta_{m-2r}^*, \quad \text{(A.10)}$$

and consider

$$\sum_{r,s} \mathscr{Z}_k^{(G)}(r,s)\hat{Z}(r,s) = \frac{1}{2k} \sum_{\substack{m,\bar{m},l,\bar{l} \\ r,s}} N_{l\bar{l}}^{(G)} e^{2i\pi(m-\bar{m})s/k} C_m^l \theta_{\bar{m}} C_{m-2r}^{\bar{l}*} \theta_{\bar{m}-2r}^*$$

$$= \tfrac{1}{2} \sum_{m,\bar{m},l,\bar{l}} N_{l\bar{l}}^{(G)} \left(C_m^l \theta_m C_{\bar{m}}^{\bar{l}*} \theta_{\bar{m}}^* + C_m^l \theta_{m+k} C_{\bar{m}}^{\bar{l}*} \theta_{\bar{m}+k}^* \right). \quad \text{(A.11)}$$

Using symmetry properties [76] of string functions and of $N_{l\bar{l}}$: $C_{k-m}^{k-l} = C_m^l = C_{-m}^l$, $N_{l\bar{l}} = N_{k-l,k-\bar{l}}$, one checks that eq. (A.11) reproduces eq. (A.2). By decomposing the θ-functions, $\hat{Z}(r,s)$ reads

$$\hat{Z}(r,s) = \frac{1}{k|\eta|^2} \sum_{m=-k+1}^{k} \sum_{n,\bar{n}\in Z} e^{2i\pi(r-m)s/k} q^{k(n+m/2k)^2} \bar{q}^{k(\bar{n}+(m-2r)/2k)^2}. \quad \text{(A.12)}$$

To reexpress it in terms of Coulombic partition functions (2.17), one has to solve

$$\tfrac{1}{2}\left(\frac{E}{\sqrt{g}} + M\sqrt{g}\right) = \sqrt{k}\left(n + \frac{m}{2k}\right),$$

$$\tfrac{1}{2}\left(\frac{E}{\sqrt{g}} - M\sqrt{g}\right) = \sqrt{k}\left(\bar{n} + \frac{m-2r}{2k}\right). \quad \text{(A.13)}$$

Setting $g = 1/k$ one gets

$$E = n + \bar{n} + (m-r)/k,$$

$$M = k(n - \bar{n} + r/k), \quad \text{(A.14)}$$

and then

$$\hat{Z}(r,s) = \frac{1}{k|\eta|^2} \sum_m \sum_{\substack{p-q \\ \text{even}}} e^{2i\pi(r-m)s/k} q^{\Delta_{p+(m-r)/k,\,kq+r}} \bar{q}^{\bar{\Delta}_{p+(m-r)/k,\,kq+r}}. \quad \text{(A.15)}$$

A Poisson transformation over $\bar{p}$ where $(p,q) = (2\bar{p}, 2\bar{q})$ or $(2\bar{p}+1, 2\bar{q}+1)$, yields

$$\hat{Z}(r,s) = \frac{1}{2k} \sum_{\bar{q},\bar{q}'} \sum_m e^{2i\pi(s/k)(r-m)} e^{2i\pi(m-r)/2k\cdot\bar{q}'}$$

$$\times \left[Z_{2k\bar{q}+r,\bar{q}'/2}(g=1/k) + (-1)^{\bar{q}'} Z_{k(2\bar{q}+1)+r,(\bar{q}'/2)}(g=1/k) \right]. \quad \text{(A.16)}$$

Summing over m mod $2k$ constrains $\bar{q}' = 2s$ $[2k]$ in both terms, and

$$\hat{Z}(r,s) = \sum_{\bar{q}\bar{q}'} Z_{k\bar{q}+r,\, k\bar{q}'+s}\left(g = \frac{1}{k}\right) = \sum_{\substack{M=r[k] \\ M'=s[k]}} Z_{M,M'}(g = 1/k). \qquad \text{(A.17)}$$

Together with eq. (A.10) we finally obtain

$$Z^{(\mathrm{G})}_{k-\mathrm{WZW}} = \sum_{r,s=0,\dots,k-1} \mathscr{Z}^{(\mathrm{G})}_{k}(r,s) \sum_{\substack{M=r[k] \\ M'=s[k]}} Z_{MM'}(g = 1/k), \qquad \text{(A.18)}$$

which is the result announced in eqs. (3.6), (4.6) and (4.17).

Appendix B

B.1. $N = 1$ superconformal characters for theories (3.1) read in the different sectors

$$\chi^{\mathrm{NS}}_{\lambda} = \frac{\eta^2(\tau)}{\eta(\tau/2)\eta(2\tau)}\left[K_{\lambda}(\tau) - K_{\tilde{\lambda}}(\tau)\right],$$

$$\chi^{\widetilde{\mathrm{NS}}}_{\lambda} = \frac{\eta(\tau/2)}{\eta(\tau)}\left[K_{\lambda}(\tau+1) - K_{\tilde{\lambda}}(\tau+1)\right],$$

$$\lambda^{\mathrm{R}}_{\lambda} = \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)}\left[K_{\lambda}(\tau) - K_{\tilde{\lambda}}(\tau)\right], \qquad \text{(B.1)}$$

where

$$K_{\lambda}(\tau) = \frac{1}{\eta(\tau)} \sum_{n=-\infty}^{\infty} q^{(nN+\lambda)^2/4N} \qquad \text{(B.2)}$$

and $N = 2pp'$, $\lambda = pr - p's$, $\tilde{\lambda} = pr + p's$. We are interested also in the limiting case $c = \frac{3}{2}$. Then, the highest weight NS (respectively R) representations are reducible iff $h = \frac{1}{8}(r-s)^2 + \delta$, $\delta = 0$ (respectively $\frac{1}{16}$) and the corresponding characters read [25]

$$\chi^{\mathrm{NS}}_{\lambda} = \frac{\eta(\tau)}{\eta(\tau/2)\eta(2\tau)}\left[q^{(r-s)^2/8} - q^{(r-s+2)^2/8}\right],$$

$$\chi^{\widetilde{\mathrm{NS}}}_{\lambda} = \frac{\eta(\tau/2)}{\eta(\tau)^2}\left[q^{(r-s)^2/8} - q^{(r-s+2)^2/8}\right],$$

$$\chi^{\mathrm{R}}_{\lambda} = \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)^2}\,q^{1/16}\left[q^{(r-s)^2/8} - q^{(r-s+2)^2/8}\right]. \qquad \text{(B.3)}$$

Otherwise

$$\chi_\lambda^{\mathrm{NS}} = \frac{\eta(\tau)}{\eta(\tau/2)\eta(2\tau)} q^h,$$

$$\lambda_\lambda^{\widetilde{\mathrm{NS}}} = \frac{\eta(\tau/2)}{\eta(\tau)^2} q^h,$$

$$\chi_\lambda^{\mathrm{R}} = \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)^2} q^h. \tag{B.4}$$

B.2. It is straightforward, though tedious, to reexpress all partition functions of ref. [22] in terms of "superCoulombic" partition functions eq. (3.7) and (3.12). We only sketch here the procedure followed: recognize first, the square moduli of prefactors in eq. (B.1) as fermionic partition functions $Z_\nu = |\theta_\nu(0|\tau)/2\eta(\tau)|$ $\nu = 2, 3, 4$ (cf. eq. (2.38)). The second step consists in matching electric and magnetic charges E and M and coupling constant g to reproduce each term in $|K_\lambda - K_{\tilde{\lambda}}|^2$ as $(1/|\eta|^2)\ q^{\Delta_{E,M}}\bar{q}^{\bar{\Delta}_{E,M}}$. We give the results, labelled by two simply laced algebras, of Coxeter numbers p, p':

p and p' odd;

$$Z_{\mathrm{A}_{p-1},\mathrm{A}_{p'-1}} = \tfrac{1}{2}\left(Z_{\mathrm{SC}}\left(\tfrac{1}{2}pp'\right) - Z_{\mathrm{SC}}\left(\frac{p}{2p'}\right)\right), \tag{B.5}$$

p and p' even;

$$Z_{\mathrm{A}_{p-1},\mathrm{A}_{p'-1}} = \tfrac{1}{2}\left(\tilde{Z}_{\mathrm{SC}}\left(\tfrac{1}{2}pp'\right) - Z_{\mathrm{SC}}\left(\frac{p}{2p'}\right)\right), \tag{B.6}$$

$p' = 2$ [4];

$$Z_{\mathrm{A}_{p-1},\mathrm{D}_{p'/2+1}} = \tfrac{1}{2}\left(\tilde{Z}_{\mathrm{SC}}\left(\tfrac{1}{2}pp'\right) + \tilde{Z}_{\mathrm{SC}}\left(\frac{2p}{p'}\right) - Z_{\mathrm{SC}}\left(\tfrac{1}{8}pp'\right) - Z_{\mathrm{SC}}\left(\frac{p}{2p'}\right)\right), \tag{B.7}$$

$p' = 0$ [4];

$$Z_{\mathrm{A}_{p-1},\mathrm{D}_{p'/2+1}} = \tfrac{1}{2}\left(\tilde{Z}_{\mathrm{SC}}\left(\tfrac{1}{2}pp'\right) + \tilde{Z}_{\mathrm{SC}}\left(\frac{2p}{p'}\right) - \tilde{Z}_{\mathrm{SC}}\left(\tfrac{1}{8}pp'\right) - Z_{\mathrm{SC}}\left(\frac{p}{2p'}\right)\right), \tag{B.8}$$

$p' = 12$;

$$Z_{\mathrm{A}_{p-1},\mathrm{E}_6} = \tfrac{1}{2}\bigg(\tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{2}pp'\big) + \tilde{Z}_{\mathrm{SC}}\bigg(\frac{2p}{p'}\bigg) - \tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{8}pp'\big) - \tilde{Z}_{\mathrm{SC}}\bigg(\frac{8p}{p'}\bigg) + Z_{\mathrm{SC}}\big(\tfrac{1}{32}pp'\big) - Z_{\mathrm{SC}}\bigg(\frac{p}{2p'}\bigg)\bigg), \tag{B.9}$$

$$Z_{\mathrm{D}_{p/2+1},\mathrm{E}_6} = \tfrac{1}{2}\bigg[\tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{2}pp'\big) + \tilde{Z}_{\mathrm{SC}}\bigg(\frac{2p}{p'}\bigg) - \tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{32}pp'\big) - \tilde{Z}_{\mathrm{SC}}\bigg(\frac{8p}{p'}\bigg) + Z_{\mathrm{SC}}\bigg(\frac{2p}{p'}\bigg) + Z_{\mathrm{SC}}\big(\tfrac{1}{32}pp'\big) - Z_{\mathrm{SC}}\big(\tfrac{1}{2}pp'\big) - Z_{\mathrm{SC}}\bigg(\frac{p}{2p'}\bigg)\bigg], \tag{B.10}$$

$p' = 18$;

$$Z_{\mathrm{A}_{p-1},\mathrm{E}_7} = \tfrac{1}{2}\bigg(\tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{2}pp'\big) + \tilde{Z}_{\mathrm{SC}}\bigg(\frac{2p}{p'}\bigg) - \tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{18}pp'\big) + Z_{\mathrm{SC}}\big(\tfrac{1}{72}pp'\big) - Z_{\mathrm{SC}}\big(\tfrac{1}{8}pp'\big) - Z_{\mathrm{SC}}\bigg(\frac{p}{2p'}\bigg)\bigg), \tag{B.11}$$

$p' = 30$;

$$Z_{\mathrm{A}_{p-1},\mathrm{E}_8} = \tfrac{1}{2}\bigg[\tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{2}pp'\big) + \tilde{Z}_{\mathrm{SC}}\bigg(\frac{2p}{p'}\bigg) - \tilde{Z}_{\mathrm{SC}}\big(\tfrac{1}{18}pp'\big) - \tilde{Z}_{\mathrm{SC}}\bigg(\frac{18p}{p'}\bigg) + Z_{\mathrm{SC}}\big(\tfrac{1}{72}pp'\big) + Z_{\mathrm{SC}}\bigg(\frac{9p}{2p'}\bigg) - Z_{\mathrm{SC}}\big(\tfrac{1}{8}pp'\big) - Z_{\mathrm{SC}}\bigg(\frac{p}{2p'}\bigg)\bigg]. \tag{B.12}$$

where Z_c, $\tilde{Z}_c$ are defined in eqs. (3.7)–(3.12). Except for eq. (B.10), all these expressions are reproduced by eq. (3.26).

Appendix C

C.1. Characters of theories given in eqs. (4.1) and (4.2) are obtained by the factorization formula [12, 14]

$$\chi_l^{(k)}\chi_{p-1}^{(m-2)} = \sum_q \chi_{lpq}\chi_{q-1}^{(m+k-2)}, \tag{C.1}$$

where the branching coefficients χ_{lpq} are expressed in terms of level-k string functions C^l_m

$$\chi_{lpq}=\sum_{\overline{m}=-k+1}^{k} C^l_{\overline{m}}\left(\sum_{\substack{t\in Z\\ m_{p,q}(t)=\overline{m}}} q^{\alpha_{pq}(t)}-\sum_{\substack{t\in Z\\ m_{p,-q}(t)=\overline{m}}} q^{\beta_{p,q}(t)}\right), \tag{C.2}$$

where

$$m_{p,q}(t)=p-q+2mt(\operatorname{mod}2k), \tag{C.3}$$

$$\alpha_{p,q}(t)=\beta_{p,-q}(t)=\frac{(2m(m+k)t+(m+k)p-mq)^2-k^2}{4km(m+k)}. \tag{C.4}$$

The most general invariant has been conjectured [14a] to be labelled by three simply laced algebras corresponding to the three SU(2) spaces of the coset construction

$$Z_{(\mathrm{G},\mathrm{G}')_{\mathrm{G}''}}=\sum_{\substack{p,q,l\\ \bar{p},\bar{q},\bar{l}}} N^{(\mathrm{G})}_{p+1,\bar{p}+1}N^{(\mathrm{G}')}_{q+1,\bar{q}+1}N^{(\mathrm{G}'')}_{l,\bar{l}}\chi_{lpq}\chi^*_{\bar{l}\bar{p}\bar{q}}, \tag{C.5}$$

where $N^{(\mathrm{G})}_{p\bar{p}}$ denotes the $\mathrm{A}_1^{(1)}$ solution for algebra G, and G, G′, G″ have respective Coxeter numbers $m, m+k, k+2$. In eq. (C.5) l must have the same parity as $p-q$; χ_{lpq} is zero otherwise.

C.2. Our purpose is now to show in a simple case that eq. (C.5) may indeed be reexpressed into eq. (4.20).

First, notice that expression (C.2) may be written as

$$\chi_{lpq}=\sum_{\overline{m}=-k+1}^{k} C^l_{\overline{m}}\theta_{\overline{m},p,q}, \tag{C.6}$$

where

$$\theta_{\overline{m},p,q}=\sum_{\substack{t\in Z\\ m_{p,q}(t)=\overline{m}}} q^{\alpha_{p,q}(t)}-\sum_{\substack{t\in Z\\ m_{p,-q}(t)=\overline{m}}} q^{\alpha_{p,-q}(t)}. \tag{C.7}$$

Introduce, now, a twisted bosonic partition function similar to eq. (A.10)

$$\hat{Z}^{p,q,\bar{p},\bar{q}}(r,s)=\frac{1}{k}\frac{e^{2i\pi rs/k}}{|\eta|^2}\sum_{\overline{m}=-k+1}^{k} e^{-2i\pi(\overline{m}s/k)}\theta_{\overline{m},p,q}\theta^*_{\overline{m}-2r,\bar{p},\bar{q}}. \tag{C.8}$$

Then, it is straightforward to see, following eqs. (A.10)–(A.14), that

$$\sum_{r,s} \mathscr{Z}_k^{(\mathrm{G}'')}(r,s)\hat{Z}^{pq\bar{p}\bar{q}}(r,s) = \sum_{l\bar{l}} N_{l,\bar{l}}^{(\mathrm{G}'')} \chi_{lpq}\chi^*_{\bar{l}\bar{p}\bar{q}}. \tag{C.9}$$

Defining

$$\hat{Z}^{(\mathrm{G},\mathrm{G}')}(r,s) = \sum_{p,q,\bar{p},\bar{q}} N_{p+1,\bar{p}+1}^{(\mathrm{G})} N_{q+1,\bar{q}+1}^{(\mathrm{G}')} \hat{Z}^{pq\bar{p}\bar{q}}(r,s), \tag{C.10}$$

one finds that

$$Z_{(\mathrm{G},\mathrm{G}')_{\mathrm{G}''}} = \sum_{r,s} \mathscr{Z}_k^{(\mathrm{G}'')}(r,s)\hat{Z}^{(\mathrm{G},\mathrm{G}')}(r,s). \tag{C.11}$$

C.3. The last step would consist in reexpressing $\hat{Z}^{(\mathrm{G},\mathrm{G}')}(r,s)$ in terms of $Z_{k-\mathrm{C}}$. We here restrict ourselves to the case $(\mathrm{G},\mathrm{G}') = (\mathrm{A}_{m-1}, \mathrm{A}_{m+k-1})$ and $m = 1 \bmod k$ which enables one to invert eq. (C.3) as

$$t = \frac{m_{pq} - p + q}{2} \bmod k. \tag{C.12}$$

Then eq. (C.10) reads for $r = 0$, $s = 1$ as

$$\hat{Z}^{(\mathrm{AA})}(0,1) = \sum_{\substack{p,q\\ \bar{m}}} \sum_{n=-k+1} \frac{\left(1 + (-1)^{\bar{m}+p-q}\right)}{2k|\eta|^2} \mathrm{e}^{-2i\pi\bar{m}/k} \left| \sum_{t\in Z} q^{\alpha_{pq}(kt + (\bar{m}-p+q)/2)} \right.$$
$$\left. - \sum_{t\in Z} q^{\alpha_{p-q}(kt+(\bar{m}-p+q)/2)} \right|^2. \tag{C.13}$$

Decomposing the square modulus gives rise to two types of terms, which we match against appropriate electromagnetic charges and coupling constant g direct terms: $g = km(m+k)$ and

$$\begin{aligned} E &= (m+k)p - mq + m(m+k)[k(t+\bar{t}) + \bar{m} - pq] \equiv \bar{m} \bmod k, \\ M &= (\bar{t} - t), \end{aligned} \tag{C.14}$$

cross terms: $g = km/(m+k)$ and

$$\begin{aligned} E &= p + m[k(t+\bar{t}) + \bar{m} - p] \equiv \bar{m} \bmod k, \\ M &= (m+k)[(\bar{t} - t)] + \frac{q - (m+k)q}{k}. \end{aligned} \tag{C.15}$$

After a Poisson transformation over E and some algebra, one is left with

$$\hat{Z}^{(\mathrm{A},\mathrm{A})}(0,1)=\tfrac{1}{2}\sum_{\substack{\overline{M}=0[k]\\ \overline{M}'=1[k]}}\left[Z_{\overline{M}\overline{M}'}\left(\frac{m(m+k)}{k}\right)-Z_{\overline{M}\overline{M}'}\left(\frac{m}{k(m+k)}\right)\right], \quad \text{(C.16)}$$

which extends to any r, s first by considering the $(0, s)$ case, then by using modular transformations, so that:

$$Z_{(\mathrm{AA})_{\mathrm{G}''}}=\tfrac{1}{2}\sum_{r,s}\mathscr{X}_k^{(\mathrm{G}'')}(r,s)\sum_{\substack{M=r[k]\\ M'=s[k]}}\left[Z_{MM'}\left(\frac{m(m+k)}{k}\right)-Z_{MM'}\left(\frac{m}{k(m+k)}\right)\right], \quad \text{(C.17)}$$

in agreement with eq. (4.15) and (4.20) for $\mathrm{G}'=\mathrm{A}$.

References

[1] C. Itzykson, H. Saleur and J.-B. Zuber, Conformal Invariance and Applications to Statistical Mechanics (World Scientific, Singapore, to appear)
[2] L.P. Kadanoff, J. Phys., A11 (1978) 1399
[3] Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 [FS12] (1984) 312; B251 [FS13] (1985) 691
[4] J.-L. Gervais and A. Neveu, Commun. Math. Phys. 100 (1985) 15; Nucl. Phys. B257 [FS14] (1985) 59; B264 (1986) 557
[5] P. Di Francesco, H. Saleur and J.-B. Zuber, J. Stat. Phys. 49 (1987) 57
[6] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 [FS18] (1987) 445
[7] A. Cappelli, C. Itzykson and J.-B. Zuber, Commun. Math. Phys. 113 (1987) 1;
A. Kato, Mod. Phys. Lett. A2 (1987) 585
[8] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253; B267 (1986) 75
[9] P. Di Francesco, H. Saleur and J.-B. Zuber, Nucl. Phys. B285 [FS19] (1987) 454
[10] S.K. Yang, Nucl. Phys. B285 [FS19] (1987) 183
[11] V. Pasquier, J. Phys. A20 (1987) L1229
[12] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103 (1986) 105
[13] P. Bowcock and P. Goddard, Nucl. Phys. B285 [FS19] (1987) 651;
P. Bouwknegt, Nucl. Phys. B290 [FS20] (1988) 507
[14] (a) F. Ravanini, Mod. Phys. Lett. A3 (1988) 271, 397;
(b) D. Kastor, E. Martinec and Z. Qiu, Phys. Lett. B200 (1988) 434;
(c) J. Bagger, D. Nemeschansky and S. Yankielowicz, Phys. Rev. Lett. 60 (1988) 389
[15] P.P. Kullish, N. Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. 5 (1981) 393
[16] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83
[17] D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493
[18] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 62 (1985) 215 [Zh. E.T.F. 89 (1985) 380]
[19] D. Gepner and Z. Qiu, Nucl. Phys. B285 [FS19] (1987) 423
[20] I. Kostov, private communication
[21] J.L. Cardy, Phys. Rev. Lett. 54 (1985) 1354
[22] A. Cappelli, Phys. Lett. 185B (1987) 82

[23] (a) F. Ravanini and S.K. Yang, Phys. Lett. B195 (1987) 202
(b) Z. Qiu, Phys. Lett. 198B (1987) 497; Nucl. Phys. B295 [FS21] (1988) 171;
(c) D. Gepner, Nucl. Phys. B296 (1987) 757;
(d) V.K. Dobrev and A. Ch. Ganchev, Mod. Phys. Lett. A3 (1988) 127
[24] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
[25] V.G. Kac, *in* Lecture notes in physics 94 (1979) p. 441 (Springer, Berlin)
[26] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; Commun. Math. Phys. 107 (1986) 535
[27] J.L. Cardy, Nucl. Phys. B270 (1986) 186
[28] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193
[29] R.J. Baxter, Exactly Solved Models in Statistical Mechanics, (Academic Press, New York, 1982)
[30] M.P.M. den Nijs, Phys. Rev. B23 (1981) 6111;
A. Luther and I. Peschel, Phys. Rev. B12 (1975) 3908
[31] H.J.F. Knops, Ann. Phys. 128 (1980) 448
[32] H. van Beijeren, Phys. Rev. Lett. 38 (1977) 993
[33] J.V. José, L.P. Kadanoff, S. Kirkpatrick and D.R. Nelson, Phys. Rev. B16 (1977) 1217
[34] C. Itzykson and J.-B. Zuber, Nucl. Phys. B275 [FS17] (1986) 580
[35] P. Ginsparg and C. Vafa, Nucl. Phys. B289 (1987) 414
[36] V. Pasquier, Nucl. Phys. B285 (1987) 162
[37] R.J. Baxter, Ann. Phys. 76 (1973) 1, 25, 48
[38] B. Nienhuis, J. Phys. A15 (1982) 199
[39] V.V. Bazhakov and N. Yu. Reshetikhin, Serpukhov preprint 87-102
[40] V. Pasquier, J. Phys. A20 (1987) 5707
[41] R.J. Baxter, S.B. Kelland and F.Y. Wu, J. Phys. A9 (1976) 397
[42] M.P.M. den Nijs, Phys. Rev. B27 (1983) 1674
[43] M.E. Fisher, Phys. Rev. Lett. 40 (1978) 1610
[44] D.A. Kurze and M.E. Fisher, Phys. Rev. B20 (1979) 2785
[45] R.J. Baxter, J. Phys. A13 (1980) L61
[46] D.S. Gaunt, J. Chem. Phys. 46 (1967) 3237
[47] D. Friedan and S. Shenker, unpublished, to appear in ref. [1].
[48] H. Saleur, J. Phys. A20 (1987) L1127
[49] P. Ginsparg, Nucl. Phys. B295 [FS21] (1988) 153
[50] R. Dijkgraaf, E. Verlinde and H. Verlinde, Proc. 1987 Cargèse Summer School, Commun. Math. Phys. 115 (1988) 649
[51] (a) H. Eichenherr, Phys. Lett. B151 (1985) 26;
(b) M.A. Bershadsky, V.G. Knizhnik and M.G. Teitelman, Phys. Lett. B151 (1985) 31
[52] D. Friedan, Z. Qiu and S. Shenker, Phys. Lett. B151 (1985) 37
[53] A. Meurman and A. Rocha-Caridi, Commun. Math. Phys. 107 (1986) 263;
A. Rocha-Caridi, Infinite lie algebras and conformal invariance in condensed matter and particle physics, ed. K. Dietz and V. Rittenberg (World Scientific, Singapore)
[54] (a) E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl. Phys. B290 [FS20] (1987) 231
(b) E. Date, M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 12 (1986) 209
[55] (a) A.B. Zamolodchikov and V.A. Fateev, Sov. J. Nucl. Phys. 32 (1980) 298
(b) T. Sogo, Y. Akutsu and T. Abe, Prog. Theor. Phys. 70 (1983) 730, 739
[56] I. Affleck, Nucl. Phys. B265 [FS15] (1985) 409;
I. Affleck and F.D.M. Haldane, Phys. Rev. B36 (1987) 5291
[57] (a) H.W.J. Blöte, J. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742;
(b) I. Affleck, Phys. Rev. Lett. 56 (1986) 746
[58] J.L. Cardy, J. Phys. A19 (1987) L1093; (C) A20 (1987) 5039
[59] (a) H. de Vega and M. Karowski, Nucl. Phys. B285 [FS19] (1987) 619;
(b) H. de Vega and F. Woynarovich, Nucl. Phys. B251 (1985) 439;
(c) F. Woynarovich and H.P. Eckle, J. Phys. A20 (1987) L97
[60] R. Shankar, Phys. Rev. Lett. 55 (1985) 453;
Y. Goldschmidt, Phys. Rev. Lett. 56 (1986) 1627

[61] G. Mussardo, G. Sotkov and M. Stanishkov, Trieste preprints 17/87/EP and 30/87/EP
[62] V.A. Fateev, Sov. J. Nucl. Phys. 33 (1981) 761
[63] V. Pasquier, in preparation
[64] A.B. Zamolodchikov and V.A. Fateev, Sov. J. Nucl. Phys. 43 (1986) 657 [Yad. Fiz. 43 (1986) 1031]
[65] O. Foda, Utrecht preprint THU-87-25
[66] L. Dixon, P. Ginsparg and J. Harvey, preprint HUTP 87/A085
[67] H.M. Babujian, Nucl. Phys. B215 [FS7] (1983) 317
[68] S.K. Yang and H.B. Zheng, Nucl. Phys. B285 [FS19] (1987) 410
[69] G. von Gehlen and V. Rittenberg, J. Phys. A20 (1987) 227;
V. Rittenberg, *in* String theory-quantum cosmology and quantum gravity – integrable and conformal invariant theories, ed. H. de Vega and N. Sanchez (World Scientific, Singapore)
[70] A.B. Zamolodchikov, Theor. Math. Phys. 63 (1985) 1205 [Teo. Mat. Fiz. 65 (1985) 347]
[71] (a) V.A. Fateev and A.B. Zamolodchikov, Nucl. Phys. B280 [FS18] (1987) 644
(b) F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Amsterdam–Utrecht preprints ITFA 87-12, THU 87-18 and ITFA 87-18, THU 87-21
(c) V.A. Fateev and S.L. Lykyanov, Int. J. Mod. Phys. A3 (1988) 507
(d) M.R. Douglas, Caltech prep. 68-1453, to appear in Nucl. Phys.
[72] M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 14 (1987) 123; Mod. Phys. Lett. B1 (1987) 73; Comm. Math. Phys. 116 (1988) 507
V. Pasquier, Nucl. Phys. B29S [FS21] (1978) 491
[73] D. Gepner, Nucl. Phys. B290 [FS20] (1987) 10
[74] I. Kostov, Saclay preprint SPLT/88-035
[75] T. Eguchi and K. Higashijima, *in* Recent developments in quantum field theory, ed. J. Ambjørn, B. Durrhus, J.L. Petersen (North-Holland, Amsterdam)
[76] V.G. Kac and D.H. Peterson, Adv. Math. 53 (1984) 125