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CRITICAL ISING CORRELATION FUNCTIONS IN THE PLANE AND ON THE TORUS

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Explicit expressions are given for all correlation functions of spin, disorder and energy operators of the critical Ising model in the plane or on the torus. Formulae for insertions of energy-momentum tensors may also be written, giving access to correlation functions of secondary fields. Two alternative methods are used throughout the paper: bosonization of the fermion representation of the Ising model or use of the free field (orbifold) interpretation of the Ashkin-Teller model. Many identities result from the consistency between these alternative approaches.

1. Introduction

This long paper is devoted to the calculation of correlation functions of the critical Ising model in the plane and on the torus. These correlation functions are interesting objects in statistical mechanics, and our expressions may be tested against numerical calculations [1]. They are also objects of theoretical importance, since they provide a natural way to study deviations from criticality. Moreover, they give access to information on the corresponding conformal field theory: operator product coefficients, matrix elements of operators. In this paper, we shall see that they are also instructive from a technical standpoint, as their calculation involves several ideas of current interest in conformal field theory and string theory: orbifolds, spin structures, theta function identities and bosonization.

The lattice action of the Ising model reads:

$$\mathscr{A} = -\frac{1}{T}\sum_{\langle i,j\rangle}\sigma_i\sigma_j\,, \qquad \sigma_i = \pm 1\,, \tag{1.1}$$

where T is the temperature and $\langle i, j\rangle$ denotes pairs of neighbouring sites on say the square lattice. The Jordan-Wigner transformation allows us to re-express this model in terms of anticommuting variables [2]. In the vicinity of the critical point T_c, these variables form the two components of a real (Majorana) free fermion field [3–4], of action

$$\mathscr{A}=\frac{1}{2\pi}\int \mathrm{d}^2x\,(\psi\,\bar\partial\psi+\tilde\psi\,\partial\tilde\psi+im\psi\tilde\psi) \tag{1.2}$$

(where we are using complex notations $z=x^1+ix^2$, $\partial=\partial/\partial z$ etc...). The mass m is proportional to $T-T_c$. Therefore at the critical point, to which we shall restrict our attention in this study, the two fields ψ and $\tilde\psi$ decouple in the action and have factorized correlation functions:

$$\langle\psi(z_1)\dots\tilde\psi(\bar w_1)\dots\rangle=\langle\psi(z_1)\dots\rangle\langle\tilde\psi(\bar w_1)\dots\rangle. \tag{1.3}$$

Moreover the ψ (resp. $\tilde\psi$) correlation functions are meromorphic functions of the arguments $z_1,\dots$ (resp. $\bar w_1,\dots$), and are nothing but Pfaffians of the two-point functions (or "propagators")

$$\langle\psi(z_1)\dots\psi(z_{2n})\rangle=\mathrm{Pf}\langle\psi(z_i)\psi(z_j)\rangle. \tag{1.4}$$

The short distance behaviour of this propagator is universal:

$$\langle\psi(z)\psi(w)\rangle\simeq\frac{1}{z-w}+\cdots, \tag{1.5}$$

but its explicit form depends of course on the boundary conditions imposed on ψ. On a torus of periods ω_1 and ω_2, for example, the two-component fermion field Ψ may and must be assigned periodic or anti-periodic boundary conditions along ω_1 or ω_2. This means that any functional integral over Ψ splits into four sectors (or "spin structures").

From (1.2) we learn that the energy operator of the Ising model, which controls its approach to criticality, is represented by the product

$$\varepsilon(z,\bar z)=i\psi(z)\tilde\psi(\bar z). \tag{1.6}$$

This representation reduces the calculation of energy correlation functions to a product of Pfaffians. We shall, however, present alternative expressions in terms of boson free fields, which will be shown to be equivalent thanks to non-trivial identities between rational or elliptic functions.

The expression of the original spin variable σ_i or rather of its continuous limit $\sigma(z,\bar z)$ in terms of the fermion field, on the other hand, is non-trivial, because the Jordan-Wigner transformation is non-local. To circumvent this difficulty, we adopt two alternative methods. The first one, introduced in ref. [5], used the technique of bosonization of a free *complex* fermion field to express the *square* of the spin correlation functions in the plane in terms of a bosonic free field. We shall argue that the same relation remains valid on a compact surface like the torus, with due attention paid to boundary conditions.

The second method, which may look more physical, proceeds via the Ashkin-Teller (AT) model, which consists of two Ising models on a square lattice coupled by a four-spin term

$$\mathscr{A}=\sum_{\langle i,j\rangle}-\frac{1}{T}\left(\sigma_i\sigma_j+\tau_i\tau_j\right)-\frac{1}{T'}\sigma_i\sigma_j\tau_i\tau_j,\qquad \sigma_i,\tau_i=\pm 1\,. \tag{1.7}$$

It presents a critical line given by the self-duality condition $\exp(-2/T')=\sinh(2/T)$ which terminates at $\operatorname{cotanh}(2/T)=2$, and for $T'=\infty$ it decouples into two independent Ising models. A reformulation of (1.7) as a solid-on-solid surface model shows [6] it renormalizes at criticality onto a free scalar field theory with action

$$\mathscr{A}=\frac{g}{4\pi}\int\left(\vec{\nabla}\phi\right)^2, \tag{1.8}$$

g being a coupling constant given by

$$g=\frac{8}{\pi}\sin^{-1}\left[\frac{\operatorname{cotanh}(2/T)}{2}\right] \tag{1.9}$$

and the various operators are identified as exponentials of ϕ, topological objects involving discontinuities or twists, and $(\vec{\nabla}\phi)^2$ for the marginal operator. This general formalism can then be used at the decoupling point ($g=2$) to obtain results for the Ising model. Such a gaussian field has a propagator which behaves at short distance, irrespective of boundary conditions on ϕ, as

$$\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle\sim-\frac{1}{g}\log|z-w|\,. \tag{1.10}$$

On a torus the functional integrals for (1.7) must be summed [7,8] over an infinite number of sectors. Three of these correspond simply to antiperiodic (twisted) boundary conditions for ϕ along one (or both) periods, and the others to shifted boundary conditions, where ϕ has a discontinuity multiple of 2π along the two periods. This means that the free field ϕ is to be considered as an *angle*, with moreover the identification $\phi=-\phi \bmod 2\pi$, i.e. ϕ belongs to the orbifold $S^1/\mathbb{Z}_2$. At the decoupling point, the relation with the fermionic sectors of (1.2) is not simple but has been clarified as far as the partition functions are concerned [9,8] (sect. 3).

One of the issues of this paper is to unravel the corresponding relation for the correlation functions.

Besides the fermion fields ψ and $\tilde{\psi}$ and the spin and energy operators σ and ε, there is an important object in the critical Ising model, as in any conformal theory, viz the energy-momentum tensor with its two components $T(z)$, $\bar{T}(\bar{z})$ which reads simply for the *Majorana* field

$$\begin{aligned} T(z) &= -\tfrac{1}{2}:\psi(z)\,\partial_z\psi(z): \\ &= -\tfrac{1}{2}\lim_{w\to z}\left[\frac{1}{2}\left(\psi(z)\,\partial_w\psi(w) - \partial_z\psi(z)\psi(w)\right) - \frac{1}{(z-w)^2}\right], \end{aligned} \quad (1.11)$$

with an analogous expression of $\bar{T}$ in terms of $\tilde{\psi}$.

Likewise, the energy momentum tensor of the gaussian field (1.8) reads

$$T(z) = -g:(\partial_z\phi)^2: = -\lim_{w\to z}\left\{g\,\partial_z\phi(z,\bar{z})\,\partial_w\phi(w,\bar{w}) + \frac{1}{2}\frac{1}{(z-w)^2}\right\}. \quad (1.12)$$

In the Ising model, we are mostly interested in correlation functions of the energy ε and of the spin σ operators. The correlation functions of their descendants result from the application of Virasoro generators on σ and ε, or alternatively from insertions of T's and $\bar{T}$'s into the correlation functions. We shall present a technique to derive explicit expressions for all these functions below. We recall that ε and σ are primary operators and as such, their correlation functions satisfy various identities.

(i) As in any conformal theory, conformal Ward identities relate correlation functions with various numbers of insertions of T (or $\bar{T}$). These identities encode the variation of the fields σ, ε (and $T, \bar{T}$) under changes of coordinates. Their explicit expression has first been written in the plane [4], then extended to a Riemann surface of arbitrary genus [10] (see below sect. 3).

(ii) The energy and spin operators have also the additional property of being "degenerate" at level 2. This means that the combination $L_{-2} - (3/2(2h+1))L_{-1}^2$ of Virasoro generators, with $h = \frac{1}{2}$ (resp. $\frac{1}{16}$) for ε, (resp. σ) cancels them. The action of L_{-1} is a simple differentiation with respect to z, while the one of L_{-2} may be derived from the insertion of T. Usually, the latter is eliminated using the Ward identity (i), to yield a partial differential equation for $\langle\sigma\ldots\varepsilon\ldots\rangle$ [4, 10]. In this work, as we have access to correlation function with arbitrary numbers of insertions of T's, we should be able to test separately (i) and (ii).

(iii) Finally, the operator product algebra [4] provides consistency checks between various correlation functions. At short distances:

$$\psi(z)\psi(w) = \frac{1}{(z-w)} + \cdots , \tag{1.13a}$$

$$\varepsilon(z,\bar{z})\varepsilon(w,\bar{w}) = \frac{1}{|z-w|^2} + \mathrm{O}(1), \tag{1.13b}$$

$$\sigma(z,\bar{z})\sigma(w,\bar{w}) = \frac{1}{|z-w|^{1/4}} + C_{\sigma\sigma\varepsilon}|z-w|^{3/4}\varepsilon(w,\bar{w}) + \cdots , \tag{1.13c}$$

$$\psi(z)\sigma(w,\bar{w}) = \frac{1}{(z-w)^{1/2}}\mu(w,\bar{w}) + \cdots , \tag{1.13d}$$

$$\tilde{\psi}(\bar{z})\sigma(w,\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^{1/2}}\mu(w,\bar{w}) + \cdots , \tag{1.13e}$$

$$\psi(z)\tilde{\psi}(\bar{w}) = -i\varepsilon(w,\bar{w}) + \cdots . \tag{1.13f}$$

The field μ denotes the disorder operator [11], dual to the spin σ, endowed with the same conformal weights and algebra. These relations may be summarized in a more compact way in terms of conformal blocks [4] as:

$$\begin{aligned}
&[\psi][\psi] = [1],\\
&[\varepsilon][\varepsilon] = [1],\\
&[\sigma][\sigma] = [1] + [\varepsilon],\\
&[\psi][\sigma] = [\mu],\\
&[\psi][\mu] = [\sigma],\\
&[\tilde{\psi}][\sigma] = [\mu],\\
&[\psi][\tilde{\psi}] = [\varepsilon].
\end{aligned}$$

For example, by a suitable short distance expansion, the $\langle \varepsilon\sigma\sigma \rangle$ function should be extracted from the 4-spin function, etc… .

To summarize, as the Ising model is represented in its critical regime by a free fermion field, the problem of computing its correlation functions might look trivial. It is not so for four main reasons:

(i) The non-local character of the spin variables in terms of fermions.

(ii) The fact that the original Ising fermions are real (Majorana) rather than complex (Dirac) makes their bosonization less direct. Actually, as we shall see, this suggests that duplicating the Ising model and hence computing *squares* of correlation functions may simplify matters. The resulting $c = 1$ theory is then amenable to bosonization [12–14].

(iii) The fact that our physical observables σ and ε are "non-chiral", i.e. are functions of both z and $\bar{z}$, prevents us from using existing techniques and formulae [15–16]. As a result, correlation functions appear as *sums* of modulus squares of analytic functions.

(iv) On a torus (or a higher genus surface), the role and contribution of the various sectors of boundary conditions of the free fermion or of the equivalent free boson have to be examined with care. Ultimately, we shall find that our correlation functions are *combinations* of chiral correlation functions computed in string theory [16]. Whether there is a more direct route to this result is not clear to us.

The paper is organized as follows. In sect. 2, we present expressions for the correlation functions in a plane, and introduce some of the themes which will recur in the following sections: bosonization versus Ashkin-Teller, identities of various origins.... Sect. 3 is a reminder of some results about the Ising and AT models on a torus: partition functions, sectors, expectation values of the energy operator. Sect. 4 is mainly devoted to the two-point function of the energy operator. The equivalence between the fermionization-bosonization and the AT approaches will result from a detailed analysis of the contribution of the various sectors, and involves a lot of identities between Jacobi θ-functions. We insist on carrying it through, at the possible price of some lengthiness, to establish the coherence of the different standpoints.

We turn to the case of the spin-spin correlation function in sect. 5, where we present three different methods of computation; one of them relies on the chiral bosonization which we recall there. Again the mutual equivalence of these methods, the role of the various sectors and some of the information that may be extracted from this spin-spin function are analysed in detail. Sect. 6 generalizes this to higher correlation functions on the torus and sect. 7 contains our final comments. Finally, an appendix gives some details on the equivalence between chiral and non-chiral bosonizations.

2. Ising correlation functions in the plane

In this section, we present various expressions for the correlation functions of the Ising model (at the critical point) in the plane*. This aims at preparing the reader for the discussion of the case of the torus.

* Many results presented in this section have been obtained in an earlier unpublished work by one of us with C. Itzykson.

2.1. As explained in the introduction, the energy correlation functions are simply a product of Pfaffians of ψ or $\tilde{\psi}$ propagators:

$$\langle \varepsilon(z_1, \bar{z}_1) \dots \varepsilon(z_{2n}, \bar{z}_{2n}) \rangle^I = \text{Pf}\big(\langle \psi(z_i)\psi(z_j) \rangle\big) \times \text{Pf}\big(\langle \tilde{\psi}(\bar{z}_i)\tilde{\psi}(\bar{z}_j) \rangle\big). \quad (2.1)$$

With our conventions, the propagators of the ψ and $\tilde{\psi}$ fields in the plane read:

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}, \qquad \langle \tilde{\psi}(\bar{z})\tilde{\psi}(\bar{w}) \rangle = \frac{1}{\bar{z}-\bar{w}}, \quad (2.2)$$

and it is understood that the antisymmetric matrix $\langle \psi(z_i)\psi(z_j) \rangle$ has no diagonal term.

Therefore, in the plane:

$$\langle \varepsilon(z_1, \bar{z}_1) \dots \varepsilon(z_{2n}, \bar{z}_{2n}) \rangle^I = \left| \text{Pf}\left(\frac{1}{z_i - z_j} \right) \right|^2. \quad (2.3)$$

Throughout this paper, we shall denote such a correlation function $\langle \varepsilon(1) \dots \varepsilon(2n) \rangle$, and keep $z, \bar{z}$ coordinates only when we want to stress the holomorphic (or antiholomorphic) properties of the correlator.

In the Ashkin-Teller model, the energy operator is known [6] to be represented by the cosine of the free boson field of eq. (1.4):

$$\varepsilon^{\text{AT}} \propto \cos 2\phi \quad (2.4)$$

up to a normalization. In a field theoretical language, this is expressed by the equivalence between the massive Thirring model and the sine-Gordon system [17–18]

$$\langle \varepsilon(1) \dots \varepsilon(2n) \rangle^{\text{AT}} \propto \langle \cos 2\phi(1) \dots \cos 2\phi(2n) \rangle$$
$$= \frac{1}{2^{2n}} \sum_{\substack{\varepsilon_i = \pm 1 \\ \sum \varepsilon_i = 0}} \prod_{1 \leqslant i < j \leqslant 2n} |z_i - z_j|^{4\varepsilon_i \varepsilon_j / g}. \quad (2.5)$$

The constraint $\sum \varepsilon_i = 0$ expresses the "electric neutrality", consequence of the translation invariance $\phi(x) \to \phi(x) + \text{const.}$

At the decoupling point $g = 2$, where the AT model reduces to two independent Ising models, ε is the sum of two Ising energy operators:

$$\varepsilon^{\text{AT}} = \varepsilon^{I_1} + \varepsilon^{I_2}. \quad (2.6)$$

In the plane, where the expectation value of an odd number of Ising energy operators vanishes (because of the self-duality of the model under which ε is odd), this gives:

$$\langle \varepsilon(1)\dots\varepsilon(2n)\rangle^{\mathrm{AT}}\big|_{g=2} = 2\langle\varepsilon(1)\dots\varepsilon(2n)\rangle^{I} + 2\sum_{p=1}^{n-1}\left[\langle\varepsilon(1)\dots\varepsilon(2p)\rangle^{I}\langle\varepsilon(2p+1)\dots\varepsilon(2n)\rangle^{I} + \text{perm.}\right], \tag{2.7}$$

and implies a series of identities equating sums of modulus squares of Pfaffians with the r.h.s. of eq. (2.5) computed at $g=2$. For example, for the four-point function:

$$\left|\frac{1}{z_{12}z_{34}} - \frac{1}{z_{13}z_{24}} + \frac{1}{z_{14}z_{23}}\right|^2 + \left|\frac{1}{z_{12}z_{34}}\right|^2 + \left|\frac{1}{z_{13}z_{24}}\right|^2 + \left|\frac{1}{z_{14}z_{23}}\right|^2$$
$$= \left|\frac{z_{12}z_{34}}{z_{13}z_{24}z_{14}z_{23}}\right|^2 + \left|\frac{z_{13}z_{24}}{z_{12}z_{34}z_{14}z_{23}}\right|^2 + \left|\frac{z_{14}z_{23}}{z_{12}z_{34}z_{13}z_{24}}\right|^2, \tag{2.8}$$

where $z_{ij} = z_i - z_j$.

2.2. For reasons discussed in the introduction, it seems more suitable to write expressions for the *squares* of the Ising correlation functions. We claim that for the energy correlation functions, up to a normalization:

$$\langle\varepsilon(1)\dots\varepsilon(2n)\rangle^2 \propto \left\langle \left(\vec{\nabla}\phi(1)\right)^2 \dots \left(\vec{\nabla}\phi(2n)\right)^2\right\rangle, \tag{2.9}$$

where ϕ is again a bosonic spinless free field of propagator

$$\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\tfrac{1}{4}\log(z-w)(\bar{z}-\bar{w}). \tag{2.10}$$

In eq. (2.9), $(\vec{\nabla}\phi)^2 = g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi = 4\,\partial_z\phi\,\partial_{\bar{z}}\phi$.

One may prove (2.9) directly from the previous expression (2.3). On the right-hand side, the contractions of $\partial\phi$ and $\bar{\partial}\phi$ decouple:

$$\langle\partial_z\phi(z,\bar{z})\,\partial_w\phi(w,\bar{w})\rangle = -\frac{1}{4(z-w)^2},$$
$$\langle\partial_{\bar{z}}\phi(z,\bar{z})\,\partial_{\bar{w}}\phi(w,\bar{w})\rangle = -\frac{1}{4(\bar{z}-\bar{w})^2}, \tag{2.11}$$

so that eq. (2.9) follows from the identity:

$$\det\left(\frac{1}{z_i - z_j}\right) = \mathrm{Pf}^2\left(\frac{1}{z_i - z_j}\right) = \mathrm{Hf}\left(\frac{1}{(z_i - z_j)^2}\right). \tag{2.12}$$

We recall that the Haffnian of a $2n \times 2n$ symmetric matrix A_{ij} is defined as

$$\mathrm{Hf}(A_{ij}) = \frac{1}{n!2^n} \sum_{\substack{\text{permutations}\\ (i_1,\dots,i_{2n})\\ \text{of } (1,\dots,2n)}} A_{i_1 i_2} \dots A_{i_{2n-1} i_{2n}} \tag{2.13}$$

and again here it is understood that our matrix A has no diagonal term. Identity (2.12) may be proved in a straightforward way, but for later purposes it is more useful to see it as a limiting case of the celebrated Cauchy determinant formula:

$$\det\left(\frac{1}{z_i - w_j}\right) = (-1)^{n(n-1)/2} \frac{\prod_{i<j}(z_i - z_j)(w_i - w_j)}{\prod_{i,j}(z_i - w_j)}. \tag{2.14}$$

Indeed, taking $w_j = z_j + \varepsilon_j$, letting all ε_i tend to zero and identifying the regular term yields (2.12). This completes the mathematical proof of (2.9).

It is also useful to develop a physicist's interpretation of this result, appealing to the Ashkin-Teller lore. The square of an Ising energy correlation function may be regarded as the product of two correlation functions pertaining to two decoupled Ising systems, or alternatively as the correlation function of the products of the two energy operators for the two independent Ising constituents of the Ashkin-Teller model at its decoupling point:

$$\begin{aligned}(\langle \varepsilon(1)\dots\varepsilon(2n)\rangle^I)^2 &= \langle \varepsilon_1(1)\dots\varepsilon_1(2n)\rangle^{I_1}\langle \varepsilon_2(1)\dots\varepsilon_2(2n)\rangle^{I_2}\\ &= \langle(\varepsilon_1\varepsilon_2)(1)\dots(\varepsilon_1\varepsilon_2)(2n)\rangle^{\mathrm{AT}}.\end{aligned} \tag{2.15}$$

Now the operator $\varepsilon_1\varepsilon_2$ in the Ashkin-Teller model has been identified [6] as the marginal operator coupled to the continuous parameter. In the coulombic or free boson reinterpretation of the AT model this is known [6] to map onto the operator $(\vec{\nabla}\phi)^2$. The value $g = 2$ at the decoupling point gives the propagator (2.10).

2.3. As explained in the introduction, the computation of spin correlation functions is less straightforward, because of the non-locality of σ in terms of ψ. In ref. [5], a solution was given, along the following steps:

(i) The spin-spin correlation function *of the lattice model* was expressed, through the Jordan-Wigner transformation, as the expectation value of a product of fermionic operators $\psi_- \equiv i(\tilde{\psi} - \psi)$ and $\psi_+ = \tilde{\psi} + \psi$:

$$\langle \sigma(0)\sigma(r)\rangle = \left\langle \psi_-(0)\left[\prod_{i=1}^{r-1} \psi_+(i)\psi_-(i)\right]\cdot \psi_+(r)\right\rangle \tag{2.16}$$

(where coordinates are measured in lattice units).

(ii) The Ising model was then duplicated: the square of the correlation function (2.16) was expressed in terms of the two copies of the Majorana field, ψ_1 and ψ_2 (and $\tilde{\psi}_1, \tilde{\psi}_2$), with which a Dirac field

$$D = \begin{pmatrix} D(z) \\ \tilde{D}(\bar{z}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_1 + i\psi_2 \\ \tilde{\psi}_1 + i\tilde{\psi}_2 \end{pmatrix} \tag{2.17}$$

was constructed.

(iii) In the continuum limit, this Dirac field was bosonized [18,19] in terms of a free boson field* ϕ such that the conserved current J_μ reads:

$$J^\mu = \overline{D}\gamma^\mu D = \varepsilon^{\mu\nu}\,\partial_\nu\phi\,. \tag{2.18}$$

In today's language and complex notations, we would say that each chirality component D and $\tilde{D}$ is written as:

$$D(z) = \exp i\varphi(z)\,,$$
$$\tilde{D}(\bar{z}) = \exp i\tilde{\varphi}(\bar{z}) \tag{2.19}$$

(up to factors, normal ordering prescriptions, etc...), that the two components of the current satisfy:

$$J_z(z) = J^{\bar{z}}(z) = 2\,\partial\phi = \partial\varphi\,,$$
$$J_{\bar{z}}(\bar{z}) = J^z(\bar{z}) = -2\,\bar{\partial}\phi = \bar{\partial}\tilde{\varphi}\,, \tag{2.20}$$

and the free field ϕ is nothing but

$$\phi(z,\bar{z}) = \tfrac{1}{2}(\varphi(z) - \tilde{\varphi}(\bar{z}))\,. \tag{2.21}$$

(iv) A careful analysis of short distance singularities in operator products led ultimately to the expression

$$\langle \sigma(z,\bar{z})\sigma(w,\bar{w})\rangle^2 = \mathcal{N}_1\langle \cos\phi(z,\bar{z})\cos\phi(w,\bar{w})\rangle \tag{2.22}$$

* Actually, the analysis of ref. [5] was performed in the critical regime at $T \neq T_c$; the fermion field was then massive and ϕ was a sine-Gordon field. Moreover, we use slightly different conventions in the bosonization which translate ϕ and change its normalization: $\sin\sqrt{\pi}\,\phi$ becomes $\cos\phi$.

up to a certain normalisation $\mathcal{N}_1$, to be adjusted by the short distance behaviour to $\mathcal{N}_1 = 2$. Of course, this relation is trivially true in the plane, since it gives nothing more than the power law: $\langle\sigma(1)\sigma(2)\rangle = 1/|z_1 - z_2|^{1/4}$. The point is that, as we shall argue in sect. 5, it holds also on a torus.

(v) Such an expression was finally generalized to the $2n$-point function "on-line", i.e. assuming the $2n$ spins to be aligned in the plane. It seems very reasonable to drop this restriction and to write:

$$\langle\sigma(z_1,\bar{z}_1)\dots\sigma(z_{2n},\bar{z}_{2n})\rangle^2 = \mathcal{N}_n\left\langle\prod_{i=1}^{2n}\cos\phi(z_i,\bar{z}_i)\right\rangle$$

$$= \frac{\mathcal{N}_n}{2^{2n}}\sum_{\substack{\varepsilon_i=\pm1\\ \Sigma\varepsilon_i=0}}\prod_{i<j}|z_i - z_j|^{\varepsilon_i\cdot\varepsilon_j/2} \qquad (2.23a)$$

and the short distance expansion show that $\mathcal{N}_n = 2^n$. The same formula was also obtained in ref. [17] by a different method.

Changing ϕ into $\frac{1}{2}\pi - \phi$ reverses the sign of the energy operator $\cos 2\phi$ and must be identified with the duality transformation. In (2.23) it changes the spin operator into its dual, the disorder operator μ, with the same correlation functions:

$$\langle\mu(1)\dots\mu(2n)\rangle^2 = \mathcal{N}_n\left\langle\prod_{i=1}^{2n}\sin\phi(i)\right\rangle = \langle\sigma(1)\dots\sigma(2n)\rangle^2. \qquad (2.23b)$$

This suggests a more general formula:

$$\langle\sigma(z_1,\bar{z}_1)\dots\sigma(z_{2n},\bar{z}_{2n})\mu(w_1,\bar{w}_1)\dots\mu(w_{2m},\bar{w}_{2m})\rangle^2$$

$$= \mathcal{N}_{n+m}\left\langle\prod_{i=1}^{2n}\cos\phi(z_i,\bar{z}_i)\prod_{j=1}^{m}\sin\phi(w_j,\bar{w}_j)\right\rangle$$

$$= \frac{(-1)^m}{2^{n+m}}\sum_{\substack{\varepsilon_i=\pm1\\ \varepsilon'_k=\pm1\\ \Sigma\varepsilon_i+\Sigma\varepsilon'_k=0}}\prod_k\varepsilon'_k\prod_{\substack{i<j\\ k<l}}|z_i - z_j|^{\varepsilon_i\varepsilon_j/2}|w_k - w_l|^{\varepsilon'_k\varepsilon'_l/2}\prod_{i,k}|z_i - w_k|^{\varepsilon_i\varepsilon'_k/2}. \qquad (2.23c)$$

For example, for $n = m = 1$:

$$\langle\sigma(1)\mu(2)\sigma(3)\mu(4)\rangle^2 = \frac{1}{2}\frac{1}{|(z_1 - z_3)(z_2 - z_4)|^{1/2}}\frac{1}{|x(1-x)|^{1/2}}[|x| + |1-x| - 1]$$

in terms of the cross-ratio $x = (z_1 - z_2)(z_3 - z_4)/(z_1 - z_3)(z_2 - z_4)$. This has the correct $z_2 \to z_4$ ($x \to \infty$) limit, and agrees up to a sign (a misprint?) with ref. [4]. The $x \to 0$ limit, on the other hand, is consistent with short distance expansion of σ and μ of the form:

$$\sigma(z,\bar{z})\mu(w,\bar{w}) = \frac{1}{\sqrt{2}|z-w|^{1/4}}\left\{e^{i\pi/4}(z-w)^{1/2}\psi(w) + e^{-i\pi/4}(\bar{z}-\bar{w})\tilde{\psi}(\bar{w})\right\} + \cdots \tag{2.24}$$

As in the case of energy correlations, these expressions may also be derived from considerations on the Ashkin-Teller model. The square of the Ising spin correlation may be regarded as the correlator

$$\langle \sigma(1)\tau(1)\sigma(2)\tau(2)\dots\sigma(2n)\tau(2n)\rangle$$

of the AT model at its decoupling point. The product $\sigma\tau$, however, is well known in the AT model [6]: it is the "polarization operator", of conformal dimensions $h = \bar{h} = \frac{1}{8}$ represented in terms of the free field ϕ as the "electric operator" $\exp i\phi$, or rather as $\cos\phi$.

It might seem that the bosonization of the Ising fermion or the considerations on the AT free field are just equivalent. It is so indeed in the plane. We shall see, however, in sects. 3–5, the subtleties associated with the boundary conditions on a torus say.

2.4. The merit of the bosonization approach sketched above is that it may yield explicit expressions for other correlators of interest, namely mixed correlators containing both fermions and spins. It is not difficult to repeat the steps (i) to (v) above in the presence of "spectator" fermions. The duplication of the model leads for example to:

$$\begin{aligned}
&2\langle \Psi(z,\bar{z})\Psi(w,\bar{w})\sigma(1)\dots\sigma(2n)\rangle\langle\sigma(1)\dots\sigma(2n)\rangle \\
&\quad= \langle \Psi_1(z,\bar{z})\Psi_1(w,\bar{w})\sigma_1(1)\dots\sigma_1(2n)\rangle\langle\sigma_2(1)\dots\sigma_2(2n)\rangle \\
&\qquad+ \langle\sigma_1(1)\dots\sigma_1(2n)\rangle\langle \Psi_2(z,\bar{z})\Psi_2(w,\bar{w})\sigma_2(1)\dots\sigma_2(2n)\rangle \\
&\quad= 2\mathcal{N}_n\langle D^*(z,\bar{z})D(w,\bar{w})\cos\phi(1)\dots\cos\phi(2n)\rangle
\end{aligned} \tag{2.25a}$$

and

$$\begin{aligned}
&2\langle \Psi(z,\bar{z})\Psi(w,\bar{w})\Psi(u,\bar{u})\Psi(v,\bar{v})\sigma(1)\dots\sigma(2n)\rangle\langle\sigma(1)\dots\sigma(2n)\rangle \\
&\quad+2\langle \Psi(z,\bar{z})\Psi(w,\bar{w})\sigma(1)\dots\sigma(2n)\rangle\langle \Psi(u,\bar{u})\Psi(v,\bar{v})\sigma(1)\dots\sigma(2n)\rangle \\
&\quad+2\langle \Psi(z,\bar{z})\Psi(u,\bar{u})\sigma(1)\dots\sigma(2n)\rangle\langle \Psi(w,\bar{w})\Psi(v,\bar{v})\sigma(1)\dots\sigma(2n)\rangle \\
&\quad+2\langle \Psi(z,\bar{z})\Psi(v,\bar{v})\sigma(1)\dots\sigma(2n)\rangle\langle \Psi(w,\bar{w})\Psi(u,\bar{u})\sigma(1)\dots\sigma(2n)\rangle \\
&\quad= 4\mathcal{N}_n\langle D^*(z,\bar{z})D(w,\bar{w})D^*(u,\bar{u})D(v,\bar{v})\cos\phi(1)\dots\cos\phi(2n)\rangle
\end{aligned} \tag{2.25b}$$

etc.... In these expressions Ψ, D, D^* denotes one of the two chiral components of the spinors:

$$\Psi = \begin{pmatrix} \psi(z) \\ \bar{\psi}(\bar{z}) \end{pmatrix}, \qquad D = \sqrt{\tfrac{1}{2}}\,(\Psi_1 + i\Psi_2), \qquad D^* = \sqrt{\tfrac{1}{2}}\,(\Psi_1 - i\Psi_2). \tag{2.26}$$

These expressions enable us to form other correlators, by suitable limiting procedures, in particular $\langle \varepsilon \ldots \sigma \ldots \rangle$ or $\langle T \ldots \sigma \ldots \rangle$. We have for example:

$$\begin{aligned} &\langle \varepsilon(z,\bar{z})\sigma(1)\ldots\sigma(2n)\rangle\langle\sigma(1)\ldots\sigma(2n)\rangle \\ &\quad = \mathcal{N}_n\langle \cos 2\phi(z,\bar{z})\cos\phi(1)\ldots\cos\phi(2n)\rangle, \end{aligned} \tag{2.27}$$

$$\begin{aligned} &\langle \varepsilon(z,\bar{z})\varepsilon(w,\bar{w})\sigma(1)\ldots\sigma(2n)\rangle\langle\sigma(1)\ldots\sigma(2n)\rangle \\ &\quad + \langle \varepsilon(z,\bar{z})\sigma(1)\ldots\sigma(2n)\rangle\langle \varepsilon(w,\bar{w})\sigma(1)\ldots\sigma(2n)\rangle \\ &\quad = 2\mathcal{N}_n\langle \cos 2\phi(z,\bar{z})\cos 2\phi(w,\bar{w})\cos\phi(1)\ldots\cos\phi(2n)\rangle, \end{aligned} \tag{2.28}$$

and

$$2\langle T(z)\sigma(1)\ldots\sigma(2n)\rangle\langle\sigma(1)\ldots\sigma(2n)\rangle = \mathcal{N}_n\langle \mathbb{T}(z)\cos\phi(1)\ldots\cos\phi(2n)\rangle, \tag{2.29}$$

$$\begin{aligned} &2\langle T(z)T(u)\sigma(1)\ldots\sigma(2n)\rangle\langle\sigma(1)\ldots\sigma(2n)\rangle \\ &\quad + 2\langle T(z)\sigma(1)\ldots\sigma(2n)\rangle\langle T(u)\sigma(1)\ldots\sigma(2n)\rangle \\ &\quad = \mathcal{N}_n\langle \mathbb{T}(z)\mathbb{T}(u)\cos\phi(1)\ldots\cos\phi(2n)\rangle. \end{aligned} \tag{2.30}$$

On the left-hand sides of these two latest equations, $T(z)$ denotes the stress-energy of the $c = \frac{1}{2}$ real fermion system (cf. eq. (1.11)), whereas $\mathbb{T}$ on the r.h.s. denotes the tensor pertaining to the $c = 1$ system:

$$\begin{aligned} \mathbb{T}(z) &= -\lim_{w\to z}\left[\tfrac{1}{2}\left(D^*(z)\,\partial_w D(w) - \partial_z D^*(z) D(w)\right) - \frac{1}{(z-w)^2}\right] \\ &= -\lim_{w\to z}\left[2\,\partial_z\phi(z,\bar{z})\,\partial_w\phi(w,\bar{w}) + \frac{1}{2}\frac{1}{(z-w)^2}\right]. \end{aligned} \tag{2.31}$$

Notice the change of the normalization by a factor 2 from (2.23) to (2.29). This factor 2, which comes from the duplication, will be important in a while.

Finally, it is very natural to unify expressions (2.9) and (2.23) into a single formula:

$$\langle\sigma(1)\dots\sigma(2m)\mu(2m+1)\dots\mu(2n)\varepsilon(2n+1)\dots\varepsilon(2n+p)\rangle^2$$
$$=\mathcal{N}_{np}\langle\cos\phi(1)\dots\cos\phi(2m)\sin\phi(2m+1)\dots$$
$$\sin\phi(2n)\big(\vec{\nabla}\phi(2n+1)\big)^2\dots\big(\vec{\nabla}\phi(2n+p)\big)^2\rangle \tag{2.32}$$

and

$$2\langle T(z)\sigma(1)\dots\mu(2m+1)\dots\varepsilon(2n+1)\dots)\rangle\langle\sigma(1)\dots\mu(2m+1)\dots\varepsilon(2n+1)\dots\rangle$$
$$=\mathcal{N}_{np}\langle\mathbb{T}(z)\cos\phi(1)\dots\sin\phi(2m+1)\dots\big(\vec{\nabla}\Phi(2n+1)\big)^2\dots\rangle, \tag{2.33}$$

etc...

Confrontation of (2.27)–(2.28) against (2.32) leads to consistency relations, of which we have only checked the first ones ($n=m=1$, $p=1$ and 2).

The short distance behaviour fixes the normalization:

$$\mathcal{N}_{np}=(-1)^p 2^n. \tag{2.34}$$

2.5. We are now ready to test the various identities and relations between correlators discussed at the end of the introduction.

(*i*) Ward identities in the plane for n primary fields A_i of conformal weights h_i, have a general form [4].

$$\langle T(z)A_1(1)\dots A_n(n)\rangle=\sum_{i=1}^{n}\left\{\frac{h_i}{(z-z_i)^2}+\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right\}\langle A_1(1)\dots A_n(n)\rangle, \tag{2.35}$$

whence

$$2\langle T(z)A_1(1)\dots A_n(n)\rangle\langle A_1(1)\dots A_n(n)\rangle$$
$$=\sum_{i=1}^{n}\left\{\frac{2h_i}{(z-z_i)^2}+\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right\}\langle A_1(1)\dots A_n(n)\rangle^2. \tag{2.36}$$

It is clear now that the Ward identity relating $\langle T\sigma\dots\mu\dots\varepsilon\dots\rangle$ and $\langle\sigma\dots\sigma\mu\dots\mu\varepsilon\dots\varepsilon\rangle$ (with $h_\sigma=\frac{1}{16}$ and $h_\varepsilon=\frac{1}{2}$) is satisfied if and only if $\langle\mathbb{T}\cos\phi\dots\sin\phi\dots(\vec{\nabla}\phi)^2\dots\rangle$ and $\langle\cos\phi\dots\sin\phi\dots(\vec{\nabla}\phi)^2\dots\rangle$ satisfy it (with $h_{\cos\phi}$

$= \frac{1}{8}$ and $h_{(\vec{\nabla}\phi)^2} = 1$). The latter, however, involves only free fields, and may be readily checked. The identity involving two T insertions may be handled in the same way.

(*ii*) *Degeneracy equations*. We want to check that

$$\left\langle \left[\left(L_{-2} - \tfrac{4}{3}L_{-1}^2\right)\sigma\right](1)\ldots\mu(2m+1)\ldots\varepsilon(2n+1)\ldots\right\rangle = 0, \tag{2.37}$$

$$\left\langle \left[\left(L_{-2} - \tfrac{3}{4}L_{-1}^2\right)\varepsilon\right](2n+1)\ldots\varepsilon(2n+p)\sigma(1)\ldots\mu(2m+1)\ldots\right\rangle = 0. \tag{2.38}$$

L_{-1} is just z differentiation, while the action of L_{-2} may be computed by a contour integral from eq. (2.29).

$$\begin{aligned}
&2\langle (L_{-2}\sigma)(1)\sigma(2)\ldots\sigma(2m)\mu(2m+1)\ldots\mu(2n)\varepsilon(2n+1)\ldots\varepsilon(2n+p)\rangle \\
&\quad\times\langle\sigma(1)\ldots\mu(2m+1)\ldots\varepsilon(2n+1)\ldots\rangle \\
&\quad= \mathcal{N}_{np}\oint\frac{\mathrm{d}z}{2\pi i(z-z_i)}\langle \mathbb{T}(z)\cos\phi(1)\ldots\sin\phi(2m+1)\ldots\left(\vec{\nabla}\phi(2n+1)\right)^2\ldots\rangle \\
&\quad= \mathcal{N}_{np}\langle (L_{-2}\cos\phi)(1)\ldots\sin\phi(2m+1)\ldots\left(\vec{\nabla}\phi(2n+1)\right)^2\ldots\rangle .
\end{aligned} \tag{2.39}$$

Therefore, eq. (2.37) is equivalent to the *non-linear* identity

$$\begin{aligned}
&\left\langle \left(\left(L_{-2} - \tfrac{4}{3}L_{-1}^2\right)\cos\phi\right)(1)\ldots\sin\phi(2m+1)\ldots\left(\vec{\nabla}\phi(2n+1)\right)^2\ldots\right\rangle \\
&\quad\times\langle\cos\phi(1)\ldots\sin\phi(2m+1)\ldots\left(\vec{\nabla}\phi(2n+1)\right)^2\ldots\rangle \\
&\quad+\tfrac{2}{3}\langle (L_{-1}\cos\phi)(1)\ldots\sin\phi(2m+1)\ldots\left(\vec{\nabla}\phi(2n+1)\right)^2\ldots\rangle^2 = 0,
\end{aligned} \tag{2.40}$$

with a similar expression for eq. (2.38). The merit of this method is that solving or checking complicated partial differential (linear) equations has been replaced by verifying a mere combinatorial identity. Still, the general expressions are cumbersome and difficult to handle. For example, for $n = m$, $p = 0$, eq. (2.40) is equivalent to:

$$\begin{aligned}
&\sum_{(\varepsilon,\varepsilon')}\prod_{i<j}|z_i - z_j|^{(\varepsilon_i\varepsilon_j + \varepsilon_i'\varepsilon_j')/2} \\
&\quad\times\left[\sum_{i>1}\frac{1}{(z_1-z_i)^2}\delta_{\varepsilon_i,1}\delta_{\varepsilon_i',1} + \sum_{j>i>1}\frac{\varepsilon_i\varepsilon_j}{(z_1-z_i)(z_1-z_j)}\delta_{\varepsilon_i,\varepsilon_i'}\delta_{\varepsilon_j,\varepsilon_j'}\right] = 0,
\end{aligned} \tag{2.41}$$

with $\sum\varepsilon = \sum\varepsilon' = 0$. We contented ourselves with checking such identities only in the first cases: eq. (2.37) for $n = m = 1$ (trivial) and $n = 2$, $p = 1$, eq. (2.38) for $n = m = 1$, $p = 2$, $n = 1$, $m = 2$, $p = 0$ and $n = m = 0$, $p = 4$.

(*iii*) Finally, we turn to the operator product expansions (1.13). Part of this information has already been implicitly incorporated in the bosonization formulae and their normalization. Less trivial is the determination of the coefficient $C_{\sigma\sigma\varepsilon}$ in the product $\sigma\sigma$. Starting from the 4-spin correlator written as in (2.23) and identifying the first two terms in the expansion as $z_1 \to z_2$, $z_3 \to z_4$, we find

$$C^2_{\sigma\sigma\varepsilon} = \tfrac{1}{4},$$

hence $C_{\sigma\sigma\varepsilon} = \pm\tfrac{1}{2}$. The plus sign may be chosen at the price of a possible redefinition of the energy operator ε.

This completes the tests of our general formula (2.32). Our task will now be to extend these results to a critical Ising model in a finite box with periodic boundary conditions, i.e. on a torus. We shall see that all the expressions for the correlation functions look very similar to those of the plane, with monomials $z_i - z_j$ replaced by appropriate Jacobi θ-functions (and with a little extra decoration). Before embarking on that discussion, however, we need to recall a few facts about the partition function, the stress energy tensor and related quantities for the Ising and AT models on a torus.

3. Generalities on the torus

We collect in this section various results which will be useful in the following. The torus is described by two complex periods ω_1, ω_2, $\tau = \omega_2/\omega_1 = \tau_R + i\tau_I$ is the modular ratio, and $q = e^{2i\tau\pi}$. With no loss of generality we take from now on $\omega_1 = 1$.

3.1. We first consider partition functions. For the Ising model, the successive steps of the solution impose the summation of the functional integral over four sectors [2], thus

$$Z^I = \int \mathscr{D}[\psi, \bar{\psi}] \exp(-\mathscr{A}) \sim \sum_{\alpha\beta} \left|\det(-\Delta_{\alpha\beta})\right|^{1/2}, \tag{3.1}$$

where $\alpha, \beta \in \{0, \tfrac{1}{2}\}$ and $\det(-\Delta_{\alpha\beta})$ is the determinant of the laplacian evaluated with the boundary conditions for the field

$$\phi(z+1) = e^{2i\pi\alpha}\phi(z),$$
$$\phi(z+\tau) = e^{2i\pi\beta}\phi(z). \tag{3.2}$$

Standard calculations [20] give $\det(-\Delta_{00}) = 0$ due to the zero mode, and the regularized values

$$\left(\det(-\Delta_{1/2,0})\right)^{1/2} = |\theta_4(0,\tau)/\eta(\tau)| = |\eta^2(\tau/2)/\eta^2(\tau)|,$$

$$\left(\det(-\Delta_{0,1/2})\right)^{1/2} = |\theta_2(0,\tau)/\eta(\tau)| = \left|\frac{2\eta^2(2\tau)}{\eta^2(\tau)}\right|,$$

$$\left(\det(-\Delta_{1/2,1/2})\right)^{1/2} = |\theta_3(0,\tau)/\eta(\tau)| = \left|\frac{\eta^4(\tau)}{\eta^2(2\tau)\eta^2(\tau/2)}\right|, \tag{3.3}$$

where η is Dedekind's function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and $\theta_\nu(z,\tau)$ are the usual Jacobi theta functions [21]. In the following we generally use the index ν to denote fermionic sectors or spin structures but sometimes, we characterize them by two numbers a and b taking the values 0 or $\frac{1}{2}$: $(a,b) = (\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},0)$, $(0,0)$ and $(0,\frac{1}{2})$ for $\nu = 1,2,3,4$ respectively. Notice that $a = \alpha + \frac{1}{2} \bmod 1$, $b = \beta + \frac{1}{2} \bmod 1$. With these conventions, $\theta_\nu = \theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ in Mumford's notations [21]. Z^I reads thus

$$Z^I = \frac{1}{2|\eta|}\left(|\theta_2(0)| + |\theta_3(0)| + |\theta_4(0)|\right), \tag{3.4}$$

with no contribution from the first sector (the normalization is chosen such that the identity operator is non-degenerate). Transformation formulae of the θ_ν's ensure it is modular invariant. The small q behaviour corresponds [22] to the central charge $c = \frac{1}{2}$, and the decomposition in characters of the Virasoro algebra [23, 20] to the associated minimal theory with only three primary fields: the identity, the energy and the spin. For the Ashkin-Teller model, the free field mapping generates an infinite number of sectors on the torus, as has been explained in ref. [8]. Those corresponding to shifts of the field ϕ appear because the introduction of surface variables is only local and cannot be done in a consistent way on a torus. The value of g given in (1.9) corresponds to a normalization of ϕ such that discontinuities are multiples of 2π. The functional integral

$$Z_{mm'}(g) = \int_{\substack{\phi(z+1)=\phi(z)+2\pi m \\ \phi(z+\tau)=\phi(z)+2\pi m'}} \mathscr{D}[\phi]\exp\left(-g/4\pi\int|\vec{\nabla}\phi|^2\right) \tag{3.5}$$

has been calculated [24, 25]

$$Z_{mm'}(g) = \frac{\sqrt{g}}{\tau_I^{1/2}|\eta|^2}\exp\left[-\pi g\frac{m'^2 + m^2(\tau_R^2 + \tau_I^2) - 2mm'\tau_R}{\tau_I}\right] \tag{3.6}$$

($\sqrt{g}$ appears here due to the subtraction of the zero mode). The sum over m, m'

gives then the first contribution to Z^{AT} as a "coulombic" partition function

$$Z_{\mathrm{c}}(g)=\sum_{mm'\in\mathbb{Z}} Z_{mm'}=\frac{1}{|\eta|^2}\sum_{\substack{e,m\\ \in\mathbb{Z}}} q^{(e/\sqrt{g}+m\sqrt{g})^2/4}\bar{q}^{(e/\sqrt{g}-m\sqrt{g})^2/4} \tag{3.7}$$

(this last equality being obtained by the Poisson formula) which is modular invariant. The small q behaviour corresponds to $c=1$ and the other terms to gaussian primary operators with dimension $x=h+\bar{h}=e^2/2g+\frac{1}{2}gm^2$ and spin $s=em$. To reproduce the order-disorder operators, one must add [7,8] to (3.7) three sectors with antiperiodic (twisted) boundary conditions for ϕ along at least one period, giving the other contribution (independent of g) as

$$\sum_{\alpha\beta}{}'\left|\det(-\Delta_{\alpha\beta})\right|^{-1/2} \tag{3.8}$$

(where the prime on the summation means that the doubly periodic sector is omitted). It is also modular invariant, and contains in particular the spin dimension $x_\sigma=\frac{1}{8}$, constant along the critical line. One gets finally

$$Z^{\mathrm{AT}}=\tfrac{1}{2}Z_{\mathrm{c}}(g)+\sum_{\nu}{}'\frac{|\eta|}{|\theta_\nu(0)|}, \tag{3.9}$$

the relative normalization being fixed by degeneracy arguments [7,8]. In the following we use (mm'), c, or $(\alpha\beta)$ to denote AT sectors.

At the decoupling point $g=2$, $Z^{\mathrm{AT}}=(Z^I)^2$. This is easily verified. Indeed the known identity

$$2\eta^3=\theta_2(0)\theta_3(0)\theta_4(0) \tag{3.10}$$

allows one to identify the crossed terms in the square of (3.4) with the contribution of the twisted sectors (3.9). The direct terms reproduce the coulombic contribution due to the identity

$$\sum{}'|\theta_\nu(0)|^2=\frac{2\sqrt{2}}{\tau_{\mathrm{I}}^{1/2}}\sum_{\substack{mm'\\ \in\mathbb{Z}}}\exp\left[-2\pi\frac{m'^2+m^2(\tau_{\mathrm{R}}^2+\tau_{\mathrm{I}}^2)-2mm'\tau_{\mathrm{R}}}{\tau_{\mathrm{I}}}\right], \tag{3.11}$$

which is a particular case of a result we establish later (4.19).

In other words [9], taking the square of Z^I amounts to computing the partition function of a system of two independent free Majorana fields Ψ_1 and Ψ_2. If these two fields have the same BC (direct terms in the expansion of $(Z^I)^2$), one may construct a Dirac field out of them, as in sect. 3, whose partition function is represented by Z_c. The cross-terms, on the contrary, lead to the twisted terms in (3.9). Thus there is no one-to-one correspondence between the fermionic sectors in the Ising model and the sectors of the AT model.

It is worth noticing that the last results could be recovered in a rather different approach [26]. For simplicity we consider only the decoupling point. Then it is known [27] that the transfer matrix of the AT model with periodic boundary conditions on a strip of width $\frac{1}{2}L$ corresponds in a hamiltonian language to the XX model on a chain of length L

$$\mathscr{H} = -\tfrac{1}{4}\sum_i \sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y , \tag{3.12}$$

where the σ are Pauli matrices, summed over the BC, (i) $(\sigma_{L+1}^x, \sigma_{L+1}^y) = (\sigma_1^x, \sigma_1^y)$, (ii) $(\sigma_{L+1}^x, \sigma_{L+1}^y) = -(\sigma_1^x, \sigma_1^y)$, (iii) $(\sigma_{L+1}^x, \sigma_{L+1}^y) = (\sigma_1^x, -\sigma_1^y)$. In the first two sectors, standard manipulations allow one to transform (3.12) into a diagonal [17] Luttinger model with periodic or antiperiodic BC. The spectrum is one of two decoupled Ising models whose number of excitations are of the same parity. This in turn is transformed into the hamiltonian of a free scalar field using the procedure of ref. [26]. This field is not periodic but presents discontinuities which are multiples of the number of excitations, or equivalently of the magnetization $\langle \Sigma\sigma_i^z \rangle$. Finally, the transfer matrix calculation is reformulated in a functional integral language to recover $Z_c(2)$. One recovers in particular the fact that the coulombic part of Z^{AT} $(g=2)$ comes from the direct terms in $(Z^I)^2$. A similar procedure can be used to treat (iii) and recover the twisted sectors.

3.2. The (doubly periodic) sector $\nu = 1$ does not contribute to the Ising partition function; however it cannot be always discarded. A simple example is provided by the calculation of the mean value of the energy operator at criticality on the torus. For $\nu = 2, 3, 4$, $Z_\nu \neq 0$ and a standard application of Wick's theorem gives $\langle \psi\bar{\psi} \rangle_\nu = 0$ while for $\nu = 1$, since $Z_1 = 0$ one gets an undetermined form. In ref. [28], Z_1 was calculated in presence of a mass term. Taking the derivative with respect to m one then finds

$$Z_1 \langle \varepsilon \rangle_1 (m) \to \pi |\eta|^2 , \qquad m \to 0 \tag{3.13}$$

(ε being normalized by (1.13b)). Thus the mean value of the energy is

$$\langle \varepsilon \rangle^I = \frac{\sum_{\nu=1}^{4} Z_\nu \langle \varepsilon \rangle_\nu}{Z^I} = 2\pi \frac{|\eta|^3}{\sum_\nu{}' |\theta_\nu(0)|} , \tag{3.14}$$

in agreement with lattice calculation of Ferdinand and Fisher [29]. Here the result comes from the $\nu = 1$ sector *only*.

This result admits a natural interpretation in the language of free fermions. Z_1 vanishes because in the grassmannian integral in the doubly periodic sector

$$Z_1 = \int \mathscr{D}[\psi, \tilde{\psi}] \mathrm{e}^{-\mathscr{A}}$$

the zero modes $\psi_0, \tilde{\psi}_0$ do not appear in the integrand and $\int \mathrm{d}\psi_0 \, \mathrm{d}\tilde{\psi}_0 = 0$. In the computation of $Z_1 \langle \varepsilon \rangle_1$ however one finds

$$\begin{aligned} Z_1 \langle \varepsilon \rangle_1 &\propto \int \mathscr{D}[\psi, \tilde{\psi}] \psi \tilde{\psi} \, \mathrm{e}^{-\mathscr{A}} \\ &= \int \mathrm{d}\psi_0 \, \mathrm{d}\tilde{\psi}_0 \, \psi_0 \tilde{\psi}_0 \int \prod \mathrm{d}\psi_k \, \mathrm{d}\tilde{\psi}_k \, \mathrm{e}^{-\mathscr{A}} \\ &\propto 1 \times |\eta|^2 . \end{aligned}$$

It is instructive to rederive (3.14) using the AT model at the decoupling point. As mentioned above (sect. 2) the operator energy ε translates into $\cos 2\phi$ in the scalar field formulation (1.8). A calculation using the naive (infinite) value for the coincident point propagator would give zero for $\langle \varepsilon \rangle^{\mathrm{AT}}$ since the Wick's theorem $\langle \cos 2\phi \rangle = \mathrm{e}^{-2\langle \phi^2 \rangle} = 0$. We shall instead use renormalized values which have been obtained in ref. [28] by adding a mass term $m^2\phi^2$ to (1.8), differentiating partition functions with respect to m^2 and letting m go to zero. We quote only the results here

$$\exp\left(-2\langle \phi^2 \rangle_{0,\frac{1}{2}}\right) = 4\pi \mathrm{e}^{-\gamma} \left| \frac{\eta^2(2\tau)}{\eta(\tau)} \right|^2 = \pi \mathrm{e}^{-\gamma} |\theta_2(0)|^2 ,$$

$$\exp\left(-2\langle \phi^2 \rangle_{\frac{1}{2},0}\right) = \pi \mathrm{e}^{-\gamma} \left| \frac{\eta^2(\tau/2)}{\eta(\tau)} \right|^2 = \pi \mathrm{e}^{-\gamma} |\theta_4(0)|^2 ,$$

$$\exp\left(-2\langle \phi^2 \rangle_{\frac{1}{2},\frac{1}{2}}\right) = \pi \mathrm{e}^{-\gamma} \left| \frac{\eta^5(\tau)}{\eta^2(2\tau)\eta^2(\tau/2)} \right|^2 = \pi \mathrm{e}^{-\gamma} |\theta_3(0)|^2 , \tag{3.15}$$

while $\langle \phi^2 \rangle_{00}$ is still infinite due to the zero mode. The numerical constant $\pi \mathrm{e}^{-\gamma}$ (γ = Euler's constant) depends on the regularization procedure (a zeta regularization was used in this calculation). Then we find

$$\langle \cos 2\phi \rangle^{\mathrm{AT}} = \frac{\sum'_{\alpha\beta} Z_{\alpha\beta} \mathrm{e}^{-2\langle \phi^2 \rangle_{\alpha\beta}}}{Z^{\mathrm{AT}}} = \pi \mathrm{e}^{-\gamma} \frac{|\eta|^3}{\sum'_{\nu} |\theta_\nu(0)|} \tag{3.16}$$

the universal (q-dependent) part of which agrees with $\langle \varepsilon \rangle^{f}$ in (3.14).

We can as well recover (3.14) in a third way, using the fact that at the decoupling point the square of Ising energy correlation functions is a correlation function of the marginal operator-$|\vec{\nabla}\phi|^2$ in the AT model (2.9). We show later that $\langle|\vec{\nabla}\phi|^2\rangle_{(\alpha\beta)\neq(00)} = 0$. In the coulombic sector $\langle|\vec{\nabla}\phi|^2\rangle$ is calculated by differentiating $Z_c(g)$ with respect to g

$$\langle|\vec{\nabla}\phi|^2\rangle_c = -\frac{4\pi}{\tau_I}\frac{\partial}{\partial g}\log Z_c(g). \tag{3.17}$$

One has

$$\frac{\partial}{\partial g}Z_c(g) = \frac{\pi\tau_I}{|\eta|^2}\sum_{\substack{em\\ \in\mathbb{Z}}}\left(\frac{e^2}{g^2} - m^2\right)(q\bar{q})^{e^2/4g+gm^2/4}\left(\frac{q}{\bar{q}}\right)^{em/2}. \tag{3.18}$$

For $g=2$, and using Jacobi's identity

$$\tfrac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^n(2n+1)q^{(n+1/2)^2/2} = \eta^3(q), \tag{3.19}$$

one gets [30]

$$\left.\frac{\partial}{\partial g}Z_c(g)\right|_{g=2} = \tfrac{1}{2}\pi\tau_I|\eta|^4. \tag{3.20}$$

Thus

$$\langle|\vec{\nabla}\phi|^2\rangle\big|_{g=2} = -\frac{2\pi^2|\eta|^4}{Z_c(2)} \tag{3.21}$$

and

$$\langle|\vec{\nabla}\phi|^2\rangle^{\mathrm{AT}} = -\langle\varepsilon\rangle^{I^2}, \tag{3.22}$$

in agreement with (3.13). The minus sign may appear surprising since both sides of the equality are squares of real quantities. One should not forget, however, that they are renormalized quantities.

3.3. Another quantity of interest is the mean value of the stress energy tensor. It is simply related to the partition functions by [10] $\langle T\rangle = 2i\pi\partial_\tau \log Z$. Using the differential equation satisfied by theta functions [21]

$$\partial_z^2\theta_\nu(z,q) = 4i\pi\partial_\tau\theta_\nu(z,q) \tag{3.23}$$

and

$$\theta_1'(0) = 2\pi\eta^3, \tag{3.24}$$

we get for the Ising model and $\nu \neq 1$

$$\langle T\rangle_\nu = i\pi\partial_\tau \log\frac{\theta_\nu(0)}{\eta} = \frac{1}{12}\left[\frac{3\theta_\nu''(0)}{\theta_\nu(0)} - \frac{\theta_1'''(0)}{\theta_1'(0)}\right] = -\frac{e_{\nu-1}}{4}, \qquad (3.25)$$

where e_ν's are standard constants of theta function theory [21]. To give a meaning to the stress energy tensor in the first sector, we use the same procedure as above by using calculations for the massive Ising model of ref. [28]. One finds

$$\langle T\rangle_1 = 2i\pi\partial_\tau \log Z_1(m) \to -\eta_1 + \frac{\pi}{\tau_I}, \qquad m \to 0, \qquad (3.26)$$

where

$$\eta_1 = -\frac{1}{6}\frac{\theta_1'''(0)}{\theta_1'(0)}. \qquad (3.27)$$

Thus $Z_1\langle T\rangle_1 = 0$ and

$$\langle T\rangle^I = \frac{\sum_\nu Z_\nu\langle T\rangle_\nu}{Z^I} = -\frac{1}{4}\frac{\sum_\nu{}'|\theta_\nu(0)|e_{\nu-1}}{\sum_\nu{}'|\theta_\nu(0)|}, \qquad (3.28)$$

which can also be recovered via the AT model.

Related to the stress energy tensor is the Ward identity on the torus. It has been derived in details in ref. [10] and we give only the result here

$$\begin{aligned}&\langle T(z)A_1(1)\dots A_n(n)\rangle - \langle T\rangle\langle A_1(1)\dots A_n(n)\rangle\\ &\quad= \sum_{i=1}^{n}\left\{h_i\left[\mathscr{P}(z-z_i)+2\eta_1\right] + \left[\zeta(z-z_i)+2\eta_1 z_i\right]\partial_{z_i}\right\}\langle A_1(1)\dots A_n(n)\rangle\\ &\qquad+2i\pi\partial_\tau\langle A_1(1)\dots A_n(n)\rangle, \end{aligned}\qquad (3.29)$$

where the ζ and Weierstrass $\mathscr{P}$ functions are related to the θ functions by

$$\zeta(z) = \frac{\theta_1'(z)}{\theta_1(z)} + 2\eta_1 z,$$

$$\mathscr{P}(z) = -\zeta'(z). \qquad (3.30)$$

Formula (3.29) is the natural doubly periodic extension of (2.35) taking into account that $\langle T\rangle \neq 0$ on the torus. The additional term involving the derivative of the correlation function with respect to τ follows from the deformation of the torus in a coordinate transformation.

The correlation functions involving degenerate operators satisfy then differential equations [4]. For an operator A degenerate at level 2 we have

$$\frac{3}{2(2h+1)}\partial_z^2\langle A(z,\bar z)A_1(1)\dots A_n(n)\rangle - \langle T\rangle\langle A_1(1)\dots A_n(n)\rangle$$
$$=\left\{2h\eta_1+2\eta_1 z\,\partial_z+\sum_{i=1}^n\left\{h_i\left[\mathscr{P}(z-z_i)+2\eta_1\right]+\left[\zeta(z-z_i)+2\eta_1 z_i\right]\partial_{z_i}\right\}\right\}$$
$$\times\langle A(z,\bar z)A_1(1)\dots A_n(n)\rangle+2i\pi\partial_\tau\langle A(z,\bar z)A_1(1)\dots A_n(n)\rangle, \qquad (3.31)$$

which can also be recast as

$$\left\{\frac{3}{2(2h+1)}\partial_z^2-2\eta_1 z\,\partial_z-\sum_{i=1}^n\left[\zeta(z-z_i)+2\eta_1 z_i\right]\partial_{z_i}-2i\pi\partial_\tau-2h\eta_1\right.$$
$$\left.-\sum_{i=1}^n h_i\left[\mathscr{P}(z-z_i)+2\eta_1\right]\right\}Z\langle A(z,\bar z)A_1(1)\dots A_n(n)\rangle=0, \quad (3.32)$$

where the differential operator acts on the product $Z\langle\quad\rangle$. Thus the differential equation will be satisfied for the total correlation function if it is in each sector separately. This of course is also expected since a coordinate transformation leaves a given sector unchanged.

4. Two-point energy correlation functions on the torus

4.1. Stress energy tensor. We first recall the free fermion propagators corresponding to (2.2) in the different sectors [10,20]

$$\langle\psi(z)\psi(w)\rangle_\nu=\mathscr{P}_\nu(z-w), \qquad \nu=2,3,4, \qquad (4.1)$$

where

$$\mathscr{P}_\nu(z)=(\mathscr{P}-e_{\nu-1})^{1/2}=\frac{\theta_1'(0)}{\theta_\nu(0)}\frac{\theta_\nu(z)}{\theta_1(z)}. \qquad (4.2)$$

Using these results it is easy to check the value of $\langle T\rangle_\nu$ given in (3.26) and to obtain [10]

$$\langle T(z)T(w)\rangle_\nu-\langle T\rangle_\nu^2=-\tfrac{1}{4}\left[(\mathscr{P}_\nu')^2-\mathscr{P}_\nu''\mathscr{P}_\nu\right]. \qquad (4.3)$$

This can be shown to agree with the Ward identity with two insertions of T and $n=0$

$$\langle T(z)T(w)\rangle_\nu - \langle T\rangle_\nu^2 = \tfrac{1}{12}c\mathscr{P}''(z-w) + 2[\mathscr{P}(z-w)+2\eta_1]\langle T\rangle_\nu + 2i\pi\partial_\tau\langle T\rangle_\nu \tag{4.4}$$

for $c=\frac{1}{2}$.

Note that the general structure of (4.4) is imposed by analyticity and periodicity arguments. Because of the short distance expansion

$$T(z)T(w) \simeq \frac{c/2}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial_w T}{z-w} + \text{regular terms}, \tag{4.5}$$

one has

$$\langle T(z)T(w)\rangle_\nu = \tfrac{1}{12}c\mathscr{P}''(z-w) + 2\langle T\rangle_\nu\mathscr{P}(z-w) + \text{const}, \tag{4.6}$$

$\mathscr{P}''$ (resp. $\mathscr{P}$) being up to a constant the unique elliptic functions singular as $z^{-4}(z^{-2})$ when $z\to 0$. The Ward identity allows in addition the determination of the constant term.

4.2. Energies. The two-point energy correlation function is then immediately obtained in each sector $\nu\neq 1$ as

$$\langle\varepsilon(z)\varepsilon(w)\rangle_\nu = |\mathscr{P}_\nu(z-w)|^2 = \left|\frac{\theta_1'(0)}{\theta_\nu(0)}\right|^2\left|\frac{\theta_\nu(z-w)}{\theta_1(z-w)}\right|^2, \tag{4.7}$$

i.e. the square modulus of an analytic function. It is periodic and satisfies the differential equation (3.32) as checked in [10]. It is then tempting to conclude that the total correlation function reads

$$\langle\varepsilon(z)\varepsilon(w)\rangle = \frac{\sum_\nu' Z_\nu\langle\varepsilon\varepsilon\rangle_\nu}{Z^I} = \frac{|\theta_1'(0)|^2}{|\theta_1(z-w)|^2}\,\frac{\sum_\nu'|\theta_\nu(z-w)|^2/|\theta_\nu(0)|}{\sum_\nu'|\theta_\nu(0)|}, \tag{4.8}$$

which enjoys the correct modular properties

$$\langle\varepsilon(z)\varepsilon(w)\rangle(\tau+1) = \langle\varepsilon(z)\varepsilon(w)\rangle(\tau),$$

$$\langle\varepsilon(z)\varepsilon(w)\rangle(-1/\tau) = |\tau|^2\langle\varepsilon(\tau z)\varepsilon(\tau w)\rangle(\tau). \tag{4.9}$$

This expression was already proposed in ref. [20]. To establish its correctness however we must prove the absence of any contribution from the first sector. For this, consider the short distance expansion

$$Z_\nu \varepsilon(z)\varepsilon(w) \simeq |z-w|^{-2}\sum \alpha_{N\bar{N}} Z_\nu A_{N\bar{N}}(z,\bar{z})(z-w)^N(\bar{z}-\bar{w})^{\bar{N}}, \qquad (4.10)$$

where $A_{N\bar{N}}$ are operators in the conformal block of the identity [$\mathbb{1}$]. Taking the mean value one finds on the r.h.s. of the equality terms as $Z_\nu\langle A_{N\bar{N}}\rangle_\nu$. If $\nu=1$, this vanishes for the identity and stress energy tensor and thus, by successive application of the Ward identity, for any descendant of $\mathbb{1}$. The l.h.s. must then be also zero, confirming (4.8). The specific heat, i.e. the integral of (4.8) on the torus is ultraviolet divergent. A regularized value has been obtained in ref. [28],

$$C = -\tfrac{1}{2}\pi\log(\pi e^{-\gamma}|\omega_1|) - \pi\frac{\sum'|\theta_\nu(0)|\log|\theta_\nu(0)|}{\sum'|\theta_\nu(0)|} - \tau_I|\omega_1|^2\langle\varepsilon\rangle^2, \qquad (4.11)$$

where we have reinstated $|\omega_1|\neq 1$ for a while to insist on the characteristic $\log|\omega_1|$ dependence.

4.2. We now rederive (4.8) using the AT model. First, we establish a useful identity for the square modulus of theta functions. Consider for convenience θ_3, defined by

$$\theta_3(z,\tau) = \sum_{N=-\infty}^{\infty} \exp(i\pi\tau N^2 - 2i\pi N z). \qquad (4.12)$$

One has

$$|\theta_3(z,\tau)|^2 = \sum_{N,M=-\infty}^{\infty} \exp\big(i\pi(\tau N^2 - \bar{\tau}M^2) - 2i\pi(Nz - M\bar{z})\big). \qquad (4.13)$$

We perform the summation over

$$P = N - M,$$
$$Q = N + M, \qquad (4.14)$$

where P, Q are now integers of the same parity. Then (4.13) reads

$$|\theta_3(z,\tau)|^2 = \Bigg(\sum_{\substack{P,Q\\ \text{even}}} + \sum_{\substack{P,Q\\ \text{odd}}}\Bigg)\exp\big[-\tfrac{1}{2}\pi\tau_I(P^2+Q^2) - i\pi PQ\tau_R$$
$$+ 2i\pi P\,\mathrm{Re}\,z + 2\pi Q\,\mathrm{Im}\,z\big]. \qquad (4.15)$$

In the first sum we set $Q = 2m$ and perform a Poisson transformation over m to get

$$\frac{1}{\sqrt{2\tau_I}} \sum_{\substack{P \text{ even} \\ P'}} \exp\left\{ -\frac{\pi}{2} P^2 \frac{\tau_R^2 + \tau_I^2}{\tau_I} - \frac{\pi P'^2}{2\tau_I} + \pi P P' \frac{\tau_R}{\tau_I} \right.$$

$$\left. + 2i\pi P \operatorname{Re} z - 2i\pi \frac{\operatorname{Im} z}{\tau_I} (P\tau_R - P') + 2\pi \frac{\operatorname{Im}^2 z}{\tau_I} \right\}$$

$$= \sum_{P \text{ even},\, P'} \Lambda_{PP'}(z, \tau), \tag{4.16}$$

which defines $\Lambda_{PP'}(z, \tau)$. In the second term we set in the same way $Q = 2m + 1$ and we obtain

$$\sum_{P,\, Q_{\text{odd}}} \exp\left[-\tfrac{1}{2}\pi\tau_I (P^2 + Q^2) - i\pi P Q \tau_R + 2i\pi P \operatorname{Re} z + 2\pi Q \operatorname{Im} z \right]$$

$$= \sum_{P_{\text{odd}},\, P'} (-1)^{P'} \Lambda_{PP'}(z, \tau). \tag{4.17}$$

Thus finally

$$|\theta_3(z, \tau)|^2 = \left(\sum_{\substack{P, P' \\ \text{even}}} + \sum_{\substack{P \text{ even} \\ P' \text{ odd}}} + \sum_{\substack{P \text{ odd} \\ P' \text{ even}}} - \sum_{\substack{P, P' \\ \text{odd}}} \right) \Lambda_{PP'}(z, \tau). \tag{4.18a}$$

Identities for other θ's follow by shifting z by half periods. Indeed $|\theta_1|^2$ is obtained multiplying (4.18) by $(-1)^{P+P'}$, $|\theta_2|^2$ by $(-1)^{P'}$ and $|\theta_4|^2$ by $(-1)^P$. In general, we write

$$|\theta_\nu(z, \tau)|^2 = \sum_{P, P' \in \mathbb{Z}} \varepsilon^{(\nu)}_{P/2, P'/2} \Lambda_{P, P'}(z, \tau), \tag{4.18b}$$

where the signs $\varepsilon^{(\nu)}_{P/2, P'/2}$ depend only on the class of P and P' modulo 2 and are given by table 1. This reproduces the sign assignments given in ref. [12] to the shifted (or "winding") sectors of a boson field in order to match the four fermion spin structures. As a first application one has

$$\sum_\nu |\theta_\nu(z, \tau)|^2 = \frac{2\sqrt{2}}{\sqrt{\tau_I}} \exp\left(\frac{2\pi \operatorname{Im}^2 z}{\tau_I} \right)$$

$$\times \sum_{mm' \in \mathbb{Z}} \exp\left[-2\pi \frac{m^2(\tau_R^2 + \tau_I^2) + m'^2 - 2mm'\tau_R}{\tau_I} \right.$$

$$\left. + 4i\pi m \operatorname{Re} z - 4i\pi \frac{m\tau_R - m'}{\tau_I} \operatorname{Im} z \right], \tag{4.19}$$

which establishes in particular (3.11) for $z = 0$.

TABLE 1
The signs $\varepsilon^{(\nu)}_{mm'}$ weighting the contributions of the (m, m')-sector in eqs. (4.18b), (5.18) or (A.3)

(m, m') mod 1 / $\begin{bmatrix} a \\ b \end{bmatrix} \simeq \nu$	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$
$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 1$	+	−	−	−
$\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = 2$	+	−	+	+
$\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = 4$	+	+	−	+
$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 3$	+	+	+	−

Here, $m, m' \in \frac{1}{2}\mathbb{Z}$, and $\varepsilon^{(\nu)}_{m,m'}$ depend only on their residue mod 1; ν or $\begin{bmatrix} a \\ b \end{bmatrix}$ are the two alternative descriptions of fermionic sectors introduced at the beginning of sect. 3.

The energy is represented by $\cos 2\phi$ in the AT model, and we calculate now its correlation functions sector by sector. First we consider the coulombic (shifted) sectors. The free field propagator for periodic BC corresponding to (1.8)–(1.10) is [20]

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{2g}\log\left[\frac{|\theta_1(z-w)|^2}{|\theta_1'(0)|^2}\exp\left(-2\pi\frac{\mathrm{Im}^2(z-w)}{\tau_I}\right)\right]$$
$$= -\frac{1}{2g}\log \Gamma(z-w,\tau). \tag{4.20}$$

If ϕ is constrained by (3.5), one can write $\phi = \hat{\phi} + \phi_{\text{class}}$ where $\hat{\phi}$ is now doubly periodic and ϕ_{class} is given by

$$\phi_{\text{class}} = i\pi\frac{m\bar{\tau} - m'}{\tau_I}z + \text{c.c.} = a_{mm'}z + \text{c.c.}, \tag{4.21}$$

with $\Delta\phi_{\text{class}} = 0$, or equivalently

$$\phi_{\text{class}} = 2\pi m \,\mathrm{Re}\, z - 2\pi\frac{m\tau_R - m'}{\tau_I}\mathrm{Im}\, z. \tag{4.22}$$

Then

$$\langle e^{ie\phi(z,\bar{z})} e^{-ie\phi(w,\bar{w})} \rangle_{mm'} = \left(\frac{|\theta_1'(0)|}{|\theta_1(z-w)|} \right)^{e^2/g} \times \exp\left[\frac{\pi e^2}{g} \frac{\mathrm{Im}^2(z-w)}{\tau_\mathrm{I}} + 2i\pi em \mathrm{Re}(z-w) - 2i\pi e \frac{m\tau_\mathrm{R} - m'}{\tau_\mathrm{I}} \mathrm{Im}(z-w) \right] \quad (4.23)$$

and

$$Z_\mathrm{c}(g) \langle e^{ie\phi(z,\bar{z})} e^{-ie\phi(w,\bar{w})} \rangle_\mathrm{c} = \frac{\sqrt{g}}{\tau_\mathrm{I}^{1/2}|\eta|^2} \left(\frac{|\theta_1'(0)|}{|\theta_1(z-w)|} \right)^{e^2/g} \exp\left(\frac{\pi e^2}{g} \frac{\mathrm{Im}^2(z-w)}{\tau_\mathrm{I}} \right) \times \sum_{mm' \in \mathbb{Z}} \exp\left\{ -\pi g \frac{m'^2 + m^2(\tau_\mathrm{R}^2 + \tau_\mathrm{I}^2) - 2mm'\tau_\mathrm{R}}{\tau_\mathrm{I}} + 2i\pi em \mathrm{Re}(z-w) - 2i\pi e \frac{m\tau_\mathrm{R} - m'}{\tau_\mathrm{I}} \mathrm{Im}(z-w) \right\}. \quad (4.24)$$

At the decoupling point $g=2$ one thus finds using (4.19)

$$Z_\mathrm{c} \langle \cos 2\phi(z,\bar{z}) \cos 2\phi(w,\bar{w}) \rangle_\mathrm{c} = \frac{1}{4|\eta|^2} \frac{|\theta_1'(0)|^2}{|\theta_1(z-w)|^2} \sum_\nu |\theta_\nu(z-w,\tau)|^2. \quad (4.25)$$

Using (3.13) and (4.7) we identify this

$$Z_\mathrm{c} \langle \cos 2\phi \cos 2\phi \rangle_\mathrm{c} = \sum_\nu{}' Z_\nu^2 \langle \varepsilon\varepsilon \rangle_\nu + (Z_1 \langle \varepsilon \rangle)^2. \quad (4.26)$$

We consider now the twisted sector $(\alpha\beta) = (\frac{1}{2}, 0)$. Then the propagator analogous to (4.20) reads

$$\langle \phi(z,\bar{z}) \phi(w,\bar{w}) \rangle_{\frac{1}{2},0} = -\frac{1}{2g} \log \frac{\Gamma[(z-w)/2, \tau/2]}{\Gamma[(z-w+1)/2, \tau/2]}, \quad (4.27)$$

where $\Gamma(z,\tau)$ is given in (4.20). To represent the energy correlation function, we cannot however consider $\exp[-(e=2)^2\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle_{\frac{1}{2},0}]$ alone since it is not periodic. We consider instead

$$\exp\left(-4\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle_{\frac{1}{2},0}\right)+\exp\left(4\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle_{\frac{1}{2},0}\right),$$

which gives at the decoupling point $g=2$

$$\frac{\left|\theta_1\left(\frac{z-w}{2},\frac{\tau}{2}\right)\right|^2}{\left|\theta_1\left(\frac{z-w+1}{2},\frac{\tau}{2}\right)\right|^2}+\frac{\left|\theta_1\left(\frac{z-w+1}{2},\frac{\tau}{2}\right)\right|^2}{\left|\theta_1\left(\frac{z-w}{2},\frac{\tau}{2}\right)\right|^2}.$$

After computing the residue at $z=w$ we find the correctly normalized result

$$2\langle\cos 2\phi(z,\bar{z})\cos 2\phi(w,\bar{w})\rangle_{\frac{1}{2},0}$$
$$=\frac{\pi^2}{4}\frac{|\theta_4(0)|^2}{|\theta_1(z-w)|^2}\left(\left|\theta_1\left(\frac{z-w}{2},\frac{\tau}{2}\right)\right|^4+\left|\theta_2\left(\frac{z-w}{2},\frac{\tau}{2}\right)\right|^4\right). \tag{4.28}$$

Eq. (4.28) amounts in fact to giving a meaning to all terms in the calculation of $\langle\cos 2\phi\cos 2\phi\rangle$ using Wick's theorem and renormalized expressions like (3.15). (We thus make the difference between $\cos 2\phi$ and $\sin 2\phi$.) Electric neutrality is indeed no more required since we are working in a twisted sector. On the other hand, using (4.7),

$$\langle\varepsilon\varepsilon\rangle_2+\langle\varepsilon\varepsilon\rangle_3=\frac{|\theta_1'(0)|^2}{|\theta_1(z-w)|^2}\left(\frac{|\theta_2(z-w)|^2}{|\theta_2(0)|^2}+\frac{|\theta_3(z-w)|^2}{|\theta_3(0)|^2}\right), \tag{4.29}$$

which can be rewritten using the duplication formula [21]

$$\theta_2(z)=\frac{\theta_2^2(z/2)\theta_3^2(z/2)-\theta_1^2(z/2)\theta_4^2(z/2)}{\theta_2(0)\theta_3^2(0)},$$
$$\theta_3(z)=\frac{\theta_2^2(z/2)\theta_3^2(z/2)+\theta_1^2(z/2)\theta_4^2(z/2)}{\theta_2^2(0)\theta_3(0)} \tag{4.30}$$

(all theta functions being evaluated with the same modular ratio τ)

$$\langle\varepsilon\varepsilon\rangle_2+\langle\varepsilon\varepsilon\rangle_3=2\pi^2\frac{|\theta_4(0)|^2}{|\theta_2(0)|^2|\theta_3(0)|^2}\frac{1}{|\theta_1(z-w)|^2}$$
$$\times\left(\left|\theta_2\left(\frac{z-w}{2}\right)\right|^4\left|\theta_3\left(\frac{z-w}{2}\right)\right|^4+\left|\theta_1\left(\frac{z-w}{2}\right)\right|^4\left|\theta_4\left(\frac{z-w}{2}\right)\right|^4\right). \tag{4.31}$$

Finally using the expressions of theta functions as infinite products one easily proves

$$\theta_2(z/2)\theta_3(z/2) = \left[\frac{\theta_2(0)\theta_3(0)}{2}\right]^{1/2} \theta_2(z/2, \tau/2),$$

$$\theta_1(z/2)\theta_4(z/2) = \left[\frac{\theta_2(0)\theta_3(0)}{2}\right]^{1/2} \theta_1(z/2, \tau/2), \tag{4.32}$$

and thus

$$2\langle\cos 2\phi(z,\bar{z})\cos 2\phi(w,\bar{w})\rangle_{\frac{1}{2},0} = \tfrac{1}{2}\left(\langle\varepsilon(z,\bar{z})\varepsilon(w,\bar{w})\rangle_2 + \langle\varepsilon(z,\bar{z})\varepsilon(w,\bar{w})\rangle_3\right), \tag{4.33}$$

or equivalently

$$2Z_{\frac{1}{2},0}\langle\cos 2\phi\cos 2\phi\rangle_{\frac{1}{2},0} = Z_2 Z_3\left(\langle\varepsilon\varepsilon\rangle_2 + \langle\varepsilon\varepsilon\rangle_3\right). \tag{4.34}$$

Similar identities follow then for the two other twisted sectors using modular transformations. For completeness we give forms similar to (4.28)

$$\begin{aligned} &2\langle\cos 2\phi(z,\bar{z})\cos 2\phi(w,\bar{w})\rangle_{0,\frac{1}{2}} \\ &\quad = 2|\tau|^{-2}\langle\cos 2\phi(z/\tau,\bar{z}/\bar{\tau})\cos 2\phi(w/\tau,\bar{w}/\bar{\tau})\rangle_{\frac{1}{2},0}\left(-\frac{1}{\tau}\right) \\ &\quad = \pi^2\frac{|\theta_2(0)|^2}{|\theta_1(z-w)|^2}\left(|\theta_1(z-w,2\tau)|^4 + |\theta_4(z-w),2\tau|^4\right) \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} &2\langle\cos 2\phi(z,\bar{z})\cos 2\phi(w,\bar{w})\rangle_{\frac{1}{2},\frac{1}{2}} \\ &\quad = 2\langle\cos 2\phi(z,\bar{z})\cos 2\phi(w,\bar{w})\rangle_{\frac{1}{2},0}(\tau+1) \\ &\quad = \pi^2\frac{|\theta_3(0,\tau)|^2}{|\theta_1(z-w)|^2|\theta_2(0)|^2|\theta_4(0)|^2} \\ &\qquad \times\left(|\theta_1(\tfrac{1}{2}(z-w),\tau)|^4|\theta_3(\tfrac{1}{2}(z-w),\tau)|^4\right. \\ &\qquad\quad \left.+|\theta_2(\tfrac{1}{2}(z-w),\tau)|^4|\theta_4(\tfrac{1}{2}(z-w),\tau)|^4\right). \end{aligned} \tag{4.36}$$

Collecting (4.26) and (4.33)–(4.36) we get finally

$$2\langle\cos 2\phi\cos 2\phi\rangle^{\mathrm{AT}} = \frac{\sum_\nu{}' Z_\nu\langle\varepsilon\varepsilon\rangle_\nu^I\sum_\nu{}' Z_\nu}{\left(\sum_\nu{}' Z_\nu\right)^2} + \left(\langle\varepsilon\rangle^I\right)^2 = \langle\varepsilon\varepsilon\rangle^I + \left(\langle\varepsilon\rangle^I\right)^2. \quad (4.37)$$

The additional term here comes, as in (2.7), from the fact that $\cos 2\phi \propto \varepsilon^{I_1} + \varepsilon^{I_2}$. Then, the correctly normalized expression for (4.37) is $2\langle\cos 2\phi\cos 2\phi\rangle = \frac{1}{2}\langle(\varepsilon^{I_1} + \varepsilon^{I_2})(\varepsilon^{I_1} + \varepsilon^{I_2})\rangle = \langle\varepsilon\varepsilon\rangle^I + (\langle\varepsilon\rangle^I)^2$.

4.4. It is also interesting to recover (4.8) using the operator $(\vec{\nabla}\phi)^2$ in the AT model. We first consider a shifted sector with indices mm'. Then using the notations of (4.21) $\partial_z\phi(z,\bar{z}) = \partial_z\hat{\phi} + a_{mm'}$ and

$$\begin{aligned}
&\tfrac{1}{16}\left\langle\left[\vec{\nabla}\phi(z,\bar{z})\right]^2\left[\vec{\nabla}\phi(w,\bar{w})\right]^2\right\rangle_{mm'} \\
&= \left\langle\left(\partial_z\hat{\phi} + a_{mm'}\right)\left(\partial_{\bar{z}}\hat{\phi} + \bar{a}_{mm'}\right)\left(\partial_w\hat{\phi} + a_{mm'}\right)\left(\partial_{\bar{w}}\hat{\phi} + \bar{a}_{mm'}\right)\right\rangle \\
&= \left(\left\langle\partial_z\hat{\phi}\,\partial_w\hat{\phi}\right\rangle + a_{mm'}^2\right)\times\text{c.c.} + \langle\partial_z\hat{\phi}\,\partial_{\bar{z}}\hat{\phi}\rangle^2 + \left\langle\partial_z\hat{\phi}\,\partial_{\bar{w}}\hat{\phi}\right\rangle\times\text{c.c.} \\
&\quad + 2|a_{mm'}^2|\left[\langle\partial_z\hat{\phi}\,\partial_{\bar{z}}\hat{\phi}\rangle + \langle\partial_z\hat{\phi}\,\partial_{\bar{w}}\hat{\phi}\rangle\right]. \qquad (4.38)
\end{aligned}$$

Using

$$\langle\phi(z,\bar{z})\hat{\phi}(w,\bar{w})\rangle = -\frac{1}{2g}\log\Gamma,$$

$$\partial_z^2\log\Gamma = -\mathscr{P} - 2\eta_1 + \frac{\pi}{\tau_{\mathrm{I}}}, \qquad (4.39)$$

we have

$$\langle\partial_z\hat{\phi}\,\partial_w\hat{\phi}\rangle + a_{mm'}^2 = -\frac{\mathscr{P}}{2g} + a_{mm'}^2 - \frac{\eta_1}{g} + \frac{\pi}{2g\tau_{\mathrm{I}}}. \qquad (4.40)$$

On the other hand

$$\sum a_{mm'}^2 Z_{mm'} = -\frac{2i\pi}{g}\partial_\tau Z_{\mathrm{c}}(g) + \frac{2i\pi}{g}Z_{\mathrm{c}}(g)\,\partial_\tau\log\left(\frac{1}{\tau_{\mathrm{I}}^{1/2}|\eta|^2}\right). \qquad (4.41)$$

But for $g = 2$

$$\tfrac{1}{2}Z_{\mathrm{c}}(2) = \sum_\nu{}' Z_\nu^2.$$

Thus

$$\partial_\tau Z_c(2) = 4\sum_\nu{}' Z_\nu\, \partial_\tau Z_\nu = -\frac{1}{4i\pi}\sum_\nu{}' Z_\nu^2 e_{\nu-1}\,, \tag{4.42}$$

where we have used (3.25). One has also

$$\partial_\tau \log\left(\frac{1}{\tau_{\mathrm{I}}^{1/2}|\eta|^2}\right) = \frac{\eta_1}{2i\pi} + \frac{i}{4\tau_{\mathrm{I}}} \tag{4.43}$$

and

$$\sum_{\substack{mm'\\ \in \mathbb{Z}}}\left(a_{mm'}^2 - \frac{\eta_1}{2} + \frac{\pi}{4\tau_{\mathrm{I}}}\right) Z_{mm'} = \tfrac{1}{2}\sum_\nu{}' Z_\nu^2 e_{\nu-1}\,. \tag{4.44}$$

In a very similar way by calculating $\partial_\tau\, \partial_{\bar\tau} Z_c(g)$ one finds

$$\sum_{\substack{mm'\\ \in \mathbb{Z}}}\left(a_{mm'}^2 - \eta_1 + \frac{\pi}{4\tau_{\mathrm{I}}}\right)\left(\bar a_{mm'}^2 - \bar\eta_1 + \frac{\pi}{4\tau_{\mathrm{I}}}\right) Z_{mm'} = \tfrac{1}{8}\sum_\nu{}' Z_\nu^2 e_{\nu-1}\bar e_{\nu-1} - \frac{\pi^2}{8\tau_{\mathrm{I}}^2}\,. \tag{4.45}$$

Since

$$\langle \partial_z\hat\phi\, \partial_{\bar z}\hat\phi\rangle = \frac{\pi}{2g\tau_{\mathrm{I}}} = -\langle \partial_z\hat\phi\, \partial_{\bar w}\hat\phi\rangle\,, \tag{4.46}$$

one has finally

$$\begin{aligned}
&\tfrac{1}{16} Z_c(2)\left\langle \left[\vec\nabla\phi(z,\bar z)\right]^2\left[\vec\nabla\phi(w,\bar w)\right]^2\right\rangle_c \\
&\quad = Z_c(2)\tfrac{1}{16}|\mathscr{P}(z-w)|^2 - \tfrac{1}{8}\mathscr{P}\sum_\nu{}' Z_\nu^2 \bar e_{\nu-1} \\
&\qquad - \tfrac{1}{8}\bar{\mathscr{P}}\sum_\nu{}' Z_\nu^2 e_{\nu-1} + \tfrac{1}{8}\sum_\nu{}' Z_\nu^2 e_{\nu-1}\bar e_{\nu-1}\,,
\end{aligned} \tag{4.47}$$

or

$$\begin{aligned}
&\tfrac{1}{2} Z_c(2)\left\langle \left[\vec\nabla\phi(z,\bar z)\right]^2\left[\vec\nabla\phi(w,\bar w)\right]^2\right\rangle_c \\
&\quad = \sum_\nu{}' Z_\nu^2 \langle\varepsilon\varepsilon\rangle_\nu^2\,.
\end{aligned} \tag{4.48}$$

We consider now the twisted sector $(\alpha, \beta) = (0, \frac{1}{2})$. The propagator reads

$$\langle \phi(z, \bar{z}) \phi(w, \bar{w}) \rangle_{0,\frac{1}{2}} = -\frac{1}{2g} \log \frac{\Gamma(z-w, 2\tau)}{\Gamma(z-w+\tau, 2\tau)} \tag{4.49}$$

and the only relevant terms are

$$\langle \partial_z \phi(z, \bar{z})\, \partial_w \phi(w, \bar{w}) \rangle_{0,\frac{1}{2}} = \frac{1}{2g} [\mathscr{P}(z-w, 2\tau) - \mathscr{P}(z-w+\tau, 2\tau)]\,; \tag{4.50}$$

thus for $g = 2$

$$\left\langle [\vec{\nabla}\phi(z, \bar{z})]^2 [\vec{\nabla}\phi(w, \bar{w})]^2 \right\rangle_{0,\frac{1}{2}} = |\mathscr{P}(z-w, 2\tau) - \mathscr{P}(z-w+\tau, 2\tau)|^2 . \tag{4.51}$$

Using (4.2) we can write

$$\mathscr{P}(z, 2\tau) - \mathscr{P}(z+\tau, 2\tau) = \left[\frac{\theta_1'(0, 2\tau)}{\theta_2(0, 2\tau)}\right]^2 \left\{ \left[\frac{\theta_2(z, 2\tau)}{\theta_1(z, 2\tau)}\right]^2 + \left[\frac{\theta_3(z, 2\tau)}{\theta_4(z, 2\tau)}\right]^2 \right\} . \tag{4.52}$$

The following relations between squares of theta functions [21]

$$\begin{aligned} \theta_1^2(z)\theta_4^2(0) &= \theta_3^2(z)\theta_2^2(0) - \theta_2^2(z)\theta_3^2(0), \\ \theta_4^2(z)\theta_4^2(0) &= \theta_3^2(z)\theta_3^2(0) - \theta_2^2(z)\theta_2^2(0), \end{aligned} \tag{4.53}$$

give

$$\begin{aligned} &[\theta_1^2(z, 2\tau)\theta_3^2(z, 2\tau) + \theta_2^2(z, 2\tau)\theta_4^2(z, 2\tau)]\, \theta_4^2(0, 2\tau) \\ &\quad = [\theta_3^4(z, 2\tau) - \theta_2^4(z, 2\tau)]\, \theta_2^2(0, 2\tau) \end{aligned} \tag{4.54}$$

while the duplication formula [21]

$$\theta_4(2z) = \frac{\theta_3^4(z) - \theta_2^4(z)}{\theta_4^3(0)} \tag{4.55}$$

gives then

$$\mathscr{P}(z, 2\tau) - \mathscr{P}(z+\tau, 2\tau) = [\theta_1'(0, 2\tau)]^2 \theta_4(0, 2\tau) \frac{\theta_4(2z, 2\tau)}{\theta_1^2(z, 2\tau)\theta_4^2(z, 2\tau)} . \tag{4.56}$$

Now we return to (z,τ) arguments. One has [21]

$$\theta_4(2z,2\tau)=\frac{\theta_3(z,\tau)\theta_4(z,\tau)}{\theta_4(0,2\tau)},$$

$$\theta_1(z,2\tau)\theta_4(z,2\tau)=\frac{1}{2}\frac{\theta_1(z,\tau)\theta_2\theta_3\theta_4(0,\tau)}{\theta_4^2(0,2\tau)},\tag{4.57}$$

which gives for the r.h.s. of (4.52)

$$\frac{4[\theta_1'(0,2\tau)]^2\theta_4^4(0,2\tau)}{\theta_2^2\theta_3^2\theta_4^2(0,\tau)}\frac{\theta_3(z,\tau)\theta_4(z,\tau)}{\theta_1^2(z,\tau)}.\tag{4.58}$$

From (4.57), $\theta_4^4(0,2\tau)=\theta_3^2(0,\tau)\theta_4^2(0,\tau)$ while [21]

$$\theta_1'(0,2\tau)=\frac{1}{2}\frac{\theta_2(0,\tau)\theta_1'(0,\tau)}{[\theta_3(0,\tau)\theta_4(0,\tau)]^{1/2}},\tag{4.59}$$

and finally

$$\mathscr{P}(z,2\tau)-\mathscr{P}(z+\tau,2\tau)=\frac{[\theta_1'(0,\tau)]^2}{\theta_3(0,\tau)\theta_4(0,\tau)}\frac{\theta_3(z,\tau)\theta_4(z,\tau)}{\theta_1^2(z,\tau)}.\tag{4.60}$$

Thus $\langle(\vec{\nabla}\phi)^2(\vec{\nabla}\phi)^2\rangle_{0,\frac{1}{2}}=\langle\varepsilon\varepsilon\rangle_3\langle\varepsilon\varepsilon\rangle_4$. Since $Z_{0,\frac{1}{2}}^{\mathrm{AT}}=2Z_3^IZ_4^I$ one has as well

$$Z_{0,\frac{1}{2}}\left\langle\left(\vec{\nabla}\phi\right)^2\left(\vec{\nabla}\phi\right)^2\right\rangle_{0,\frac{1}{2}}=2Z_3Z_4\langle\varepsilon\varepsilon\rangle_3\langle\varepsilon\varepsilon\rangle_4.\tag{4.61}$$

Similar identities for other sectors are then obtained by modular transformations.

Combining (4.48), (4.61) we obtain at the end

$$\left\langle\left(\vec{\nabla}\phi\right)^2\left(\vec{\nabla}\phi\right)^2\right\rangle^{\mathrm{AT}}=\left(\langle\varepsilon\varepsilon\rangle^I\right)^2,\tag{4.62}$$

as expected.

5. Spin-spin correlation function on the torus

Our purpose in this section is to obtain an analytic formula for the spin-spin correlation function of the Ising model on a torus.

We will proceed in three different ways. The first uses techniques introduced by Dixon, Friedan, Martinec and Shenker [15] and extensively employed by Atick and

Sen [16] in the slightly different context of "spin field" correlation functions of string theory on the torus. The extension of this method to higher correlation functions appears problematical.

The second uses arguments presented in sect. 2 and argued to extend to the torus, and bosonization techniques [13–14]. This method seems very powerful, does extend to all correlation functions (see sect. 6) and presumably to higher genus, but does not look as physical as the others, in view of our lack of understanding of the boundary conditions on the boson field.

Finally, as for the energy correlation functions, we present a third approach, based on the Ashkin-Teller model.

5.1. In the first approach, the operator product algebra of ψ or $\bar{\psi}$ with σ plays a crucial role. The $\frac{1}{2}$ exponent in (1.13) implies the monodromy property [31] of insertions of σ-spins in ψ or $\bar{\psi}$ correlators, namely that when the argument of σ describes a loop around the argument of a ψ or a $\bar{\psi}$, the correlator changes sign.

We consider the ψ propagator in the presence of two spins in the sector ν:

$$G_\nu(z, w, z_1, z_2) = \frac{\langle\psi(z)\psi(w)\sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\rangle_\nu}{\langle\sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\rangle_\nu}, \tag{5.1}$$

which is analytic in z, w except at the branch points and poles where it has the behaviour:

$$G_\nu \underset{z\to z_1}{\sim} \frac{1}{(z-z_1)^{1/2}},$$

$$G_\nu \underset{z\to z_2}{\sim} \frac{1}{(z-z_2)^{1/2}},$$

$$G_\nu \underset{w\to z_1}{\sim} \frac{1}{(w-z_1)^{1/2}},$$

$$G_\nu \underset{w\to z_2}{\sim} \frac{1}{(w-z_2)^{1/2}},$$

$$G_\nu = \frac{1}{z-w} + \text{reg. terms } (z\to w). \tag{5.2}$$

Using θ functions transformation properties, we find a candidate for G_ν, which is doubly periodic in z and w and has the requested behaviours (5.2)

$$G_\nu(z,w,z_1,z_2)=\frac{1}{2}\frac{\theta_1'(0)}{\theta_1(z-w)}\left[\frac{\theta_\nu\left(z-w+\frac{z_1-z_2}{2}\right)}{\theta_\nu\left(\frac{z_1-z_2}{2}\right)}\right.$$

$$\left.\times\left(\frac{\theta_1(z-z_1)\theta_1(w-z_2)}{\theta_1(z-z_2)\theta_1(w-z_1)}\right)^{1/2}+(2\leftrightarrow 1)\right]. \quad (5.3)$$

If the present case, analyticity arguments together with the antisymmetry of G_ν under the exchange $w\leftrightarrow z$ show that this function is unique [36].

Next step consists in evaluating the stress energy tensor insertion:

$$T(z)=\lim_{z\to w}\tfrac{1}{2}\left[\tfrac{1}{2}\left(\partial_z\psi(z)\,\psi(w)-\psi(z)\,\partial_w\psi(w)\right)+\frac{1}{(z-w)^2}\right], \quad (5.4)$$

$$\frac{\langle T(z)\sigma(z_1,\bar z_1)\sigma(z_2,\bar z_2)\rangle_\nu}{\langle\sigma(z_1,\bar z_1)\sigma(z_2,\bar z_2)\rangle_\nu}=\frac{1}{16}\left[\frac{\theta_1'}{\theta_1}(z-z_1)-\frac{\theta_1'}{\theta_1}(z-z_2)\right]^2$$

$$+\frac{1}{4}\frac{\theta_\nu'}{\theta_\nu}\left(\frac{z_1-z_2}{2}\right)\left[\frac{\theta_1'}{\theta_1}(z-z_1)-\frac{\theta_1'}{\theta_1}(z-z_2)\right]$$

$$+\text{reg. terms }(z\to z_1). \quad (5.5)$$

On the other hand, T is the generator of conformal transformations and we have in $z\to z_1$ limit:

$$\frac{\langle T(z)\sigma(z_1,\bar z_1)\sigma(z_2,\bar z_2)\rangle_\nu}{\langle\sigma(z_1,\bar z_1)\sigma(z_2,\bar z_2)\rangle_\nu}=\frac{h_\sigma}{(z-z_1)^2}+\frac{1}{(z-z_1)}\partial_{z_1}\log\langle\sigma(z_1,\bar z_1)\sigma(z_2,\bar z_2)\rangle_\nu$$

$$+\text{reg. terms }(z\to z_1). \quad (5.6)$$

Hence we recover the conformal dimension of the spin operator $h_\sigma=\frac{1}{16}$, and the (z_1,z_2) dependence of the two-point function:

$$\langle\sigma(z_1,\bar z_1)\sigma(z_2,\bar z_2)\rangle_\nu=C_\nu\frac{\left[\theta_\nu\left(\frac{z_1-z_2}{2}\right)\right]^{1/2}}{\left[\theta_1(z_1-z_2)\right]^{1/8}}. \quad (5.7)$$

In fact we could have considered the antianalytic part of this function, by insertion of $\tilde{\psi}(\bar{z})\tilde{\psi}(\bar{w})$, leading to the same $\bar{z}_1$, $\bar{z}_2$ dependence.

The spin-spin correlation function in a given sector ν is thus the square modulus of an analytic function (the reason of this remarkable property is not clear). We have for $\nu \neq 1$

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\rangle_\nu = \frac{|\theta_1'(0)|^{1/4}}{|\theta_\nu(0)|} \frac{\left|\theta_\nu\left(\frac{z_1 - z_2}{2}\right)\right|}{|\theta_1(z_1 - z_2)|^{1/4}}, \tag{5.8}$$

the normalization being chosen to give the short distance behaviour (1.13c). In the first sector, $\langle\sigma\sigma\rangle_1$ is infinite, since it appears in the short distance expansion a term proportional to the energy operator, and we have already argued that $\langle\varepsilon\rangle_1 = \infty$. Now we construct

$$\langle\sigma(1)\sigma(2)\rangle' = \frac{\sum_\nu Z_\nu\langle\sigma(1)\sigma(2)\rangle_\nu}{\sum' Z_\nu}. \tag{5.9}$$

An interesting property is that $\langle\sigma(1)\sigma(2)\rangle_\nu$ is not periodic by itself (contrary to what happened for the energies). This is because spin operators are not local in terms of fermions: translating $z_1 - z_2$ by 1, τ or $1+\tau$ amounts to creating a "frustration line" winding around the torus and changes the sign of the BC for ψ along ω_1, ω_2, or ω_1 and ω_2 [16]. Now the periodicity of the total correlation function (5.9) is ensured if the terms corresponding to different sectors in (5.9) exchange under these $z_1 - z_2$ shifts. This works clearly for $\nu \neq 1$, and fixes $Z_1\langle\sigma\sigma\rangle_1$ which is finite, thus

$$Z_\nu\langle\sigma\sigma\rangle_\nu = \tfrac{1}{2}(2\pi)^{1/3}|\theta_1'(0)|^{-1/12}\frac{\left|\theta_\nu\left(\frac{z_1 - z_2}{2}\right)\right|}{|\theta_1(z_1 - z_2)|^{1/4}}, \tag{5.10}$$

valid for any ν. One checks that (5.10) enjoys also modular covariance properties similar to (4.9)*.

In the present case of the two-point function, we can also justify the result (5.9)–(5.10) by proving that sector by sector, it satisfies the Ward identity and degeneracy equation (see below (5.19)–(5.20)).

We also notice that the *square* of this result, in each sector, is the *modulus square* of the Atick-Sen chiral spin correlation function. In other words:

$$\langle\sigma(z, \bar{z})\sigma(w, \bar{w})\rangle_\nu^2 = |\langle S^+(z)S^-(w)\rangle|^2 = \langle S^+(z)\tilde{S}^+(\bar{z})S^-(w)\tilde{S}^-(\bar{w})\rangle, \tag{5.11}$$

* The ansatz proposed in ref. [20] for the spin-spin correlation function is incorrect: it does not satisfy eq. (3.32).

where S^+ and S^- are their chiral spin fields, $\tilde{S}^+$ and $\tilde{S}^-$ spin fields of the opposite chirality. This is quite reminiscent of what we have found in the plane (2.22) and suggests applying the bosonization procedure to $\langle\sigma\sigma\rangle^2$.

5.2. The arguments which led to the representation (2.22) of the square of the spin-spin correlation function were based on: (i) a lattice regularization, and the explicit fermionization of the spins through the Jordan-Wigner transformation, followed by the duplication of the system: all these steps may be carried out on a torus; the duplication is done sector by sector, and leads to Dirac fermions with the same BC on their two components; (ii) explicit bosonization prescriptions for each chirality component of a free Dirac field: these have been developed recently [13–14] on a torus or a higher genus surface in a given field fermionic sector; (iii) short-distance expansions leading ultimately to the operator $\cos\phi$: these are universal and do not depend on the topology. We conclude that a natural ansatz for the square of the spin-spin correlation function is

$$\langle\sigma(z,\bar{z})\sigma(w,\bar{w})\rangle_\nu^2 = 2\langle\cos\phi(z,\bar{z})\cos\phi(w,\bar{w})\rangle_\nu . \tag{5.12}$$

We have, however, to explain how to compute with the field ϕ in the sector ν. We follow here the prescriptions of [14]. In each fermionic sector labelled by $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, there exist two fields $\varphi(z)$ and $\tilde{\varphi}(\bar{z})$, such that, as in (2.19):

$$D(z) = \exp i\varphi(z),$$

$$\tilde{D}(\bar{z}) = \exp i\tilde{\varphi}(\bar{z}). \tag{5.13}$$

In the calculation of expectation values involving φ, one imagines that φ is written as

$$\varphi(z) = 2\pi P z + \hat{\varphi}(z). \tag{5.14a}$$

P has eigenvalues $n+a$, $n\in\mathbb{Z}$, of relative weights

$$q^{(n+a)^2/2}\, e^{2\pi i(n+a)b}, \tag{5.14b}$$

while $\hat{\varphi}$ is subject to Wick theorem, with propagator

$$\langle\hat{\varphi}(z)\hat{\varphi}(w)\rangle = -\log\frac{\theta_1(z-w)}{\theta_1'(0)} . \tag{5.14c}$$

In ref. [14], this was supplemented by a normalization factor $[\theta_\nu(0)]^{-1}$. As we want to discuss also the contribution of the doubly periodic sector $\nu=1$, $a=b=\frac{1}{2}$, we rather compute $Z_\nu^\varphi\langle\ldots\rangle_\nu$ by the previous rules and add the overall factor $(2\eta)^{-1}(q)$

to the previous rules. For example:

$$Z_\nu^\varphi \langle e^{ie\varphi(z)} e^{-ie\varphi(w)} \rangle_\nu$$

$$= \frac{1}{2\eta(q)} \sum_{n=-\infty}^{\infty} q^{(n+a)^2/2} e^{2\pi i(n+a)(b+e(z-w))} \left(\frac{\theta_1'(0)}{\theta_1(z-w)} \right)^{e^2}$$

$$= \frac{1}{2\eta(q)} \theta_\nu(e(z-w)) \left(\frac{\theta_1'(0)}{\theta(z-w)} \right)^{e^2} . \tag{5.15}$$

We recall that $\theta_1'(0) = 2\pi\eta^3(q)$. Somehow, Z_ν^φ represents the analytic part (in τ or q) of the Dirac fermion partition function Z_ν^2, square of the Majorana partition function Z_ν.

$$Z_\nu^2 = Z_\nu^\varphi Z_\nu^{\tilde\varphi} = |Z_\nu^\varphi|^2$$

$$= \left| \frac{\theta_\nu(0)}{2\eta} \right|^2 ,$$

$$Z_\nu^\varphi = \frac{\theta_\nu(0)}{2\eta(q)} . \tag{5.16}$$

As in sect. 2, we finally set $\phi = \frac{1}{2}(\varphi - \tilde\varphi)$ in eq. (5.12): the computation of the r.h.s. can now be carried out:

$$Z_\nu^2 \langle \sigma(z,\bar z) \sigma(w,\bar w) \rangle_\nu^2 = \left| \frac{\theta_\nu\left(\frac{z-w}{2}\right)}{2\eta(q)} \left(\frac{\theta_1'(0)}{\theta_1(z-w)} \right)^{1/4} \right|^2 , \tag{5.17}$$

in agreement with the previous derivation (5.10).

These chiral bosonization prescription, which amount to extracting the analytic part of fermionic calculations [13], may look ad hoc. They enjoy, however, remarkable consistency conditions [14], that we shall examine in the next section. On the other hand, the physical meaning of these prescriptions remains obscure (to us), reflecting the absence of a functional integral over φ (or $\tilde\varphi$) with well defined boundary conditions.

The combined prescription on φ *and* $\tilde\varphi$, on the other hand, may be interpreted in a more perceptible way. Each fermionic sector by transformation similar to the calculation of eqs. (4.13)–(4.19), gives rise for correlation functions like $Z_\nu \langle \sigma\sigma \rangle_\nu^2$ to a sum of contributions with *signs* of the various shifted sectors of ϕ, corresponding to integer or half-integer winding numbers [12] (see appendix). Stated differently, one may suspect that in our computation of physical quantities which are non-chiral, we might avoid the use of chiral bosonization, and use rather the non-chiral one of ref. [12],

$$\tfrac{1}{2}Z_\nu^2\langle\sigma(z,\bar z)\sigma(w,\bar w)\rangle_\nu^2 = Z_\nu^2\langle\cos\phi(z,\bar z)\cos\phi(w,\bar w)\rangle_\nu$$
$$= \tfrac{1}{8}\sum_{m'\in\frac{1}{2}\mathbb{Z}} \varepsilon^{(\nu)}_{m,m'} Z_{m,m'}(g=2)\langle\cos\phi(z,\bar z)\cos\phi(w,\bar w)\rangle_{mm'}, \tag{5.18}$$

where the relevant expressions have been given in eqs. (3.6), (4.18b) and (4.23) and table 1.

5.3. We show here that our expression for $Z_\nu\langle\sigma\sigma\rangle_\nu$ satisfies the differential equation (3.32) in each fermionic sector ν, including the doubly periodic sector. After some algebra, one finds that this equation is equivalent to the identity:

$$2\left[\frac{\theta_\nu''}{\theta_\nu}\left(\frac{z}{2}\right) + \left(\frac{\theta_\nu'}{\theta_\nu}\left(\frac{z}{2}\right)\right)^2\right] + \frac{\theta_1''}{\theta_1}(z) - 4\frac{\theta_1'}{\theta_1}(z)\frac{\theta_\nu'}{\theta_\nu}\left(\frac{z}{2}\right) + 6\eta_1 = 0, \tag{5.19}$$

which is proved by the standard procedure. The left-hand side is shown to be doubly periodic and with vanishing residue at the possible poles $z=0$ and $\frac{1}{2}$, $(1+\tau)/2$, $\tau/2$ depending on whether $\nu = 2,3,4$: it is thus a constant which is computed at a special point.

As explained in sect. 2, we can actually test separately the Ward identity and the degeneracy equation which lead to (3.32). The insertion of $T(z)$ into the spin-spin correlation function is related to that of $\mathbb{T}(z)$ is $\langle\mathbb{T}(z)\cos\phi(1)\cos\phi(2)\rangle_\nu$

$$Z_\nu^2\langle\mathbb{T}(z)\cos\phi(1)\cos\phi(2)\rangle_\nu$$
$$= -Z_\nu^2\lim_{w\to z}\left\langle\left[2\,\partial_z\phi(z)\,\partial_w\phi(w) + \frac{1}{2}\frac{1}{(z-w)^2}\right]\cos\phi(1)\cos\phi(2)\right\rangle_\nu$$
$$= \left\{\frac{1}{2}\frac{\theta_\nu''}{\theta_\nu}\left(\frac{z_1-z_2}{2}\right) + \frac{1}{2}\frac{\theta_\nu'}{\theta_\nu}\left(\frac{z_1-z_2}{2}\right)\left[\frac{\theta_1'}{\theta_1}(z-z_1) - \frac{\theta_1'}{\theta_1}(z-z_2)\right]\right.$$
$$\left. + \frac{1}{8}\left[\frac{\theta_1'}{\theta_1}(z-z_1) - \frac{\theta_1'}{\theta_1}(z-z_2)\right]^2 + \eta_1\right\} Z_\nu^2\langle\cos\phi(1)\cos\phi(2)\rangle_\nu. \tag{5.20}$$

By the same kind of argument as in sect. 2, the Ward identity for $\langle\sigma\sigma\rangle$ is equivalent to the Ward identity for $\langle\cos\phi\cos\phi\rangle$. The latter which involves only a free field should be an identity which is rather a test of the computation rules with φ. One finds that it holds provided:

$$\frac{\theta_1''}{\theta_1}(z-z_1)+\frac{\theta_1''}{\theta_1}(z-z_2)+\frac{\theta_1''}{\theta_1}(z_1-z_2)+\frac{2\theta_1'}{\theta_1}(z_1-z_2)\left[\frac{\theta_1'}{\theta_1}(z-z_1)-\frac{\theta_1'}{\theta_1}(z-z_2)\right]$$

$$-2\frac{\theta_1'}{\theta_1}(z-z_1)\frac{\theta_1'}{\theta_1}(z-z_2)+6\eta_1=0, \tag{5.21}$$

which is again proved by the same method as (5.19) to which it reduces for $\nu=1$ and $z-z_2=2(z-z_1)$.

5.4. We now comment on some of the features of our result (5.9)–(5.10).

The presence of a term coming from the first sector is natural, due to $Z_1\langle\varepsilon\rangle_1\neq 0$. We have indeed [16]

$$Z_1\langle\sigma\sigma\rangle_1=\tfrac{1}{2}(2\pi)^{1/3}|\theta_1'(0)|^{-1/12}\frac{\left|\theta_1\left(\frac{z-w}{2}\right)\right|}{|\theta_1(z-w)|^{1/4}}$$

$$\simeq\tfrac{1}{2}|z-w|^{1-1/4}\pi|\eta|^2,\qquad z\to w, \tag{5.22}$$

in agreement with $\sigma(z)\sigma(w)\simeq c_{\sigma\sigma\varepsilon}|z-w|^{1-1/4}\varepsilon(z)$, $c_{\sigma\sigma\varepsilon}=\frac{1}{2}$ and $Z_1\langle\varepsilon\rangle_1=\pi|\eta|^2$. The singularity (5.22) being factorized all remaining terms in the expansion of (5.22) have even powers of $z-w$, $\bar{z}-\bar{w}$. This is easily explained writing the short distance expansion

$$Z_\nu\sigma(z)\sigma(w)\simeq|z-w|^{-1/4}\sum\alpha_{N\bar{N}}(z-w)^N(\bar{z}-\bar{w})^{\bar{N}}A_{N\bar{N}}(z,\bar{z})$$

$$+|z-w|^{1-1/4}\sum\beta_{N\bar{N}}(z-w)^N(\bar{z}-\bar{w})^{\bar{N}}B_{N\bar{N}}(z,\bar{z}), \tag{5.23}$$

where $A_{N\bar{N}}$ (resp. $B_{N\bar{N}}$) are operator belonging to $[\mathbb{1}]$ ($[\varepsilon]$).

Take now the mean value of both sides for $\nu=1$. First $Z_1\langle A_{N\bar{N}}\rangle_1$ is zero for any $N,\bar{N}$ as discussed after eq. (4.10). Similarly, since $\mathscr{P}(z-w)$ in (3.29) has an expansion with even powers of $z-w$, the only operators of $[\varepsilon]$ with non-zero $Z_1\langle B_{N\bar{N}}\rangle_1$ are those with $N,\bar{N}$ even. This explains the structure of (5.22).

In the sectors $\nu\neq 1$, after factorization of $|z-w|^{-1/4}$, one gets also an expansion in even powers of $z-w, \bar{z}-\bar{w}$. Here, all the expectation values for operators belonging to $[\varepsilon]$, as well as belonging to $[\mathbb{1}]$ with $N,\bar{N}$ odd, vanish. One has in particular

$$Z_\nu\langle\sigma\sigma\rangle_\nu\simeq\frac{Z_\nu}{|z-w|^{1/4}}\left|1+\frac{(z-w)^2}{48}\left[\frac{3\theta_\nu''(0)}{\theta_\nu(0)}-\frac{\theta_1'''(0)}{\theta_1'(0)}\right]\right|^2, \tag{5.24}$$

in agreement with (3.25) and $c_{\sigma\sigma T}=\frac{1}{4}$.

It is also interesting to consider the small q (strip) limit. Then one finds

$$Z\langle\sigma(z,\bar{z})\sigma(w,\bar{w})\rangle \simeq \frac{(\pi)^{1/4}(q\bar{q})^{-1/48}}{|\sin\pi(z-w)|^{1/4}}\left\{1+(q\bar{q})^{1/16}\left(\left|\cos\tfrac{1}{2}\pi(z-w)\right|+\left|\sin\tfrac{1}{2}\pi(z-w)\right|\right)+\cdots\right\}, \tag{5.25}$$

the first term is the usual result of the logarithmic mapping [23]. On the other hand, taking for simplicity $\tau_R=0$, we can write $\langle\sigma\sigma\rangle$ in the transfer matrix formalism [23] as

$$Z\langle\sigma(z,\bar{z})\sigma(w,\bar{w})\rangle=\sum|\langle NK|\hat{\sigma}|nk\rangle|^2\exp(-E_N\tau_I+iK)$$
$$\times\exp[(E_N-E_n)\mathrm{Im}(z-w)+i(k-K)\mathrm{Re}(z-w)]. \tag{5.26}$$

Thus in the limit $\tau_I\to\infty$, i.e. $q\to 0$ we have

$$Z\langle\sigma(z,\bar{z})\sigma(w,\bar{w})\rangle$$
$$=\mathrm{e}^{-E_0\tau_I}\sum|\langle 0|\hat{\sigma}|nk\rangle|^2\exp((E_0-E_n)\mathrm{Im}(z-w)+ik\,\mathrm{Re}(z-w))$$
$$+\mathrm{e}^{-E_1\tau_I}\sum|\langle 1|\hat{\sigma}|nk\rangle|^2\exp((E_1-E_n)\mathrm{Im}(z-w)+ik\mathrm{Re}(z-w))$$
$$+\ldots, \tag{5.27}$$

$|0\rangle$ and $|1\rangle$ being respectively the ground state ($E_0=\frac{1}{6}\pi c=\frac{1}{12}\pi$) and the first (scalar) excited state. The factor $(q\bar{q})^{1/16}$ appearing in (5.25) corresponds thus to $|1\rangle=|\sigma\rangle$ with $E_1-E_0=2\pi(h_\sigma+\bar{h}_\sigma)=\frac{1}{4}\pi$. Expanding now (5.25) in exponentials of $z-w$, $\bar{z}-\bar{w}$ gives then various sum rules for $\hat{\sigma}$ matrix elements. One has [23]

$$\frac{(\pi)^{1/4}}{|\sin\pi(z-w)|^{1/4}}=(2\pi)^{1/4}\sum_{n,\bar{n}=0}^{\infty}\frac{\Gamma(n+1/8)}{\Gamma(1/8)n!}\frac{\Gamma(\bar{n}+1/8)}{\Gamma(1/8)n!}$$
$$\times\exp\left[-2\pi\left(\tfrac{1}{8}+n+\bar{n}\right)\mathrm{Im}(z-w)+2i\pi(n-\bar{n})\mathrm{Re}(z-w)\right], \tag{5.28}$$

which gives [23] for instance

$$\sum_{\substack{\text{independent}\\ \text{operators at}\\ \text{level } N,N \text{ in}\\ [\sigma]}}\langle 0|\hat{\sigma}|\tfrac{1}{16}+N,\tfrac{1}{16}+N\rangle^2=(2\pi)^{1/4}\left(\frac{\Gamma(N+1/8)}{\Gamma(1/8)N!}\right)^2, \tag{5.29}$$

$\hat{\sigma}$ coupling $|0\rangle$ to states of $[\sigma]$ only. In the same way we find

$$\frac{(\pi)^{1/4}}{|\sin\pi(z-w)|^{1/4}}\left(|\cos\tfrac{1}{2}\pi(z-w)|+|\sin\tfrac{1}{2}\pi(z-w)|\right)$$

$$=(2\pi)^{1/4}\sum_{\substack{n,\bar{n}\\ m,\bar{m}\\ n+\bar{n}\text{ even}}}^{\infty}\frac{\Gamma(n-1/2)}{\Gamma(-1/2)n!}\frac{\Gamma(\bar{n}-1/2)}{\Gamma(-1/2)\bar{n}!}\frac{\Gamma(m+1/8)}{\Gamma(1/8)m!}\frac{\Gamma(\bar{m}+1/8)}{\Gamma(1/8)\bar{m}!}$$

$$\times\exp\left[-2\pi\left(-\tfrac{1}{8}+\tfrac{1}{2}(n+\bar{n})+m+\bar{m}\right)\mathrm{Im}(z-w)\right.$$

$$\left.+2i\pi\left(\tfrac{1}{2}(n-\bar{n})+m-\bar{m}\right)\mathrm{Re}(z-w)\right]. \qquad (5.30)$$

If $n=\bar{n}=m=\bar{m}=0$ we find $|\langle\sigma|\hat{\sigma}|0\rangle|^2=(2\pi)^{1/4}$, while $n=\bar{n}=1$, $m=\bar{m}=0$ gives $|\langle\sigma|\hat{\sigma}|\varepsilon\rangle|^2=(2\pi)^{1/4}\cdot\frac{1}{4}=(2\pi)^{1/4}c^2_{\sigma\sigma\varepsilon}$ as expected. Sum rules similar to (5.29) are then deduced, for instance

$$\sum_{\substack{\text{indep. op. at}\\ \text{level } NN \text{ in}\\ [\varepsilon]}}|\langle\sigma|\hat{\sigma}|\tfrac{1}{2}+N,\tfrac{1}{2}+N\rangle|^2=(2\pi)^{1/4}\sum_{m,\bar{m}=0}^{N}\frac{\Gamma(1/2+2N-2m)}{\Gamma(-1/2)(1+2N-2m)!}$$

$$\times\frac{\Gamma(1/2+2N-2\bar{m})}{\Gamma(-1/2)(1+2N-2\bar{m})!}\frac{\Gamma(m+1/8)}{\Gamma(1/8)m!}\frac{\Gamma(\bar{m}+1/8)}{\Gamma(1/8)\bar{m}!}. \qquad (5.31)$$

Conversely, the knowledge of all coefficients in operator product expansions should in principle allow one to recover (5.31), (5.29) or (5.25) via the logarithmic mapping although it seems a difficult task in practice.

5.5. We can once again verify (5.9), (5.10) using the AT model, for which the correlation function $\langle\cos\phi\cos\phi\rangle^{\mathrm{AT}}$ should equal $(\langle\sigma\sigma\rangle^{I})^2$. With the same steps as above we have first putting $e=1$ in (4.24)

$$Z_c(2)\langle\cos\phi(z,\bar{z})\cos\phi(w,\bar{w})\rangle_c=\frac{1}{4|\eta|^2}\left(\frac{|\theta_1'(0)|}{\theta_1(z-w)}\right)^{1/2}\sum_{\nu}\left|\theta_\nu\left(\frac{z-w}{2}\right)\right|^2, \qquad (5.32)$$

and thus

$$Z_c\langle\cos\phi\cos\phi\rangle_c=\sum_{\nu}Z_\nu^2\langle\sigma\sigma\rangle_\nu^2. \qquad (5.33)$$

Then we discuss for instance the $(\alpha,\beta)=(0,\frac{1}{2})$ sector. A straightforward repetition of the calculation with $e = 1$ instead of $e = 2$ gives

$$2\langle\cos\phi(z,\bar{z})\cos\phi(w,\bar{w})\rangle_{0,\frac{1}{2}}$$
$$= \pi^{1/2}\frac{|\theta_2(0)|^{1/2}}{|\theta_1(z-w)|^{1/2}}\left(|\theta_1(z-w,2\tau)| + |\theta_4(z-w,2\tau)|\right). \qquad (5.34)$$

On the other hand the identities

$$\theta_1(z,2\tau) = \frac{\theta_1(z/2,\tau)\theta_2(z/2,\tau)}{\theta_4(0,2\tau)},$$
$$\theta_4(z,2\tau) = \frac{\theta_3(z/2,\tau)\theta_4(z/2,\tau)}{\theta_4(0,2\tau)}, \qquad (5.35)$$

give

$$Z_1\langle\sigma\sigma\rangle_1 Z_2\langle\sigma\sigma\rangle_2 + Z_3\langle\sigma\sigma\rangle_3 Z_4\langle\sigma\sigma\rangle_4$$
$$= \tfrac{1}{4}(2\pi)^{2/3}\frac{|\theta_1'(0)|^{-1/6}}{|\theta_1(z-w)|^{1/2}}\left(\left|\theta_1\left(\frac{z-w}{2}\right)\theta_2\left(\frac{z-w}{2}\right)\right| + \left|\theta_3\left(\frac{z-w}{2}\right)\theta_4\left(\frac{z-w}{2}\right)\right|\right)$$
$$= \tfrac{1}{4}(2\pi)^{2/3}\frac{|\theta_1'(0)|^{-1/6}|\theta_4(0,2\tau)|}{|\theta_1(z-w)|^{1/2}}\left(|\theta_1(z-w,2\tau)| + |\theta_4(z-w,2\tau)|\right). \qquad (5.36)$$

Since $\theta_4(0,2\tau) = [\theta_3(0,\tau)\theta_4(0,\tau)]^{1/2}$, $Z_{0,\frac{1}{2}} = |\theta_3(0,\tau)||\theta_4(0,\tau)|/2|\eta|^2$, one finds then

$$2Z_{0,\frac{1}{2}}\langle\cos\phi\cos\phi\rangle_{0,\frac{1}{2}} = 2\left(Z_1 Z_2\langle\sigma\sigma\rangle_1\langle\sigma\sigma\rangle_2 + Z_3 Z_4\langle\sigma\sigma\rangle_3\langle\sigma\sigma\rangle_4\right) \qquad (5.37)$$

and similar relations obtained by modular transformations. Combining (5.33), (5.37) gives as expected

$$2\langle\cos\phi\cos\phi\rangle^{\mathrm{AT}} = \left(\langle\sigma\sigma\rangle^{I}\right)^2. \qquad (5.38)$$

5.6. The first method we used for determining the spin-spin function also yields the disorder-disorder correlation function(s) $\langle\mu(1)\mu(2)\rangle_\nu$. We start from (2.24) and rewrite it as

$$\psi(z)\sigma(w,\bar{w}) = \frac{e^{i\pi/4}}{\sqrt{2}}\frac{\mu(w,\bar{w})}{(z-w)^{1/2}} + \cdots. \qquad (5.39)$$

Then performing the $w \to z_1$ and $z \to z_2$ limits in (5.3) we get:

$$\frac{i}{2}\frac{\langle\mu(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2)\rangle_\nu}{\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_\nu} = -\frac{1}{2}\frac{\theta_1'(0)}{\theta_1(z_2-z_1)}\frac{\theta_\nu\left(\frac{z_2-z_1}{2}\right)}{\theta_\nu\left(\frac{z_1-z_2}{2}\right)}\frac{\left(-\theta_1^2(z_1-z_2)\right)^{1/2}}{\theta_1'(0)}$$

$$= \pm\frac{i}{2}\varepsilon_\nu, \tag{5.40}$$

where ε_ν is the parity of the θ_ν function: $\varepsilon_1 = -1$, $\varepsilon_\nu = +1$ for $\nu \neq 1$. The sign is fixed to be $+$ by the short distance behaviour

$$\mu(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2) = \frac{1}{|z_1-z_2|^{1/4}} - \frac{1}{2}|z_1-z_2|^{3/4}\varepsilon(z_2,\bar{z}_2) + \cdots \tag{5.41}$$

obtained from (1.13c) by the duality transformation.

The total disorder-disorder function reads then:

$$\langle\mu(z,\bar{z}_1)\mu(z_2,\bar{z}_2)\rangle = \frac{\sum_\nu \varepsilon_\nu Z_\nu\langle\sigma(1)\sigma(2)\rangle_\nu}{\sum' Z_\nu}. \tag{5.42}$$

Contrary to the spin-spin function, this expression is *not* periodic on the torus: changes of z_1 into z_1+1, $z_1+\tau$, $z_1+1+\tau$ permute the $Z_\nu\langle\sigma\sigma\rangle_\nu$ and generate three other disorder functions. The disorder-disorder correlator may be regarded as creating a fustration line joining z_1 and z_2. In working out the short-distance limit $z_1 \to z_2$ above, and determining the signs in (5.40), we have implicitly assumed that this frustration line shrinks to a point. The effect of translations of z_1 along ω_1, ω_2 or $\omega_1+\omega_2$, however, is to let this line wind around the torus along the corresponding direction. Letting then z_1 tend to z_2 leaves a closed non-contractible frustration line. More precisely, if one denotes by a superscript $\alpha = 1,\ldots,4$ the four functions obtained from (5.42) by translation of z_1 by 0, $1+\tau$, τ respectively, then in the limit $z_1 \to z_2$:

$$\langle\mu(1)\mu(2)\rangle^{(\alpha)} \approx \frac{\varepsilon_2^{(\alpha)}Z_2+\varepsilon_3^{(\alpha)}Z_3+\varepsilon_4^{(\alpha)}Z_4}{Z_2+Z_3+Z_4}\frac{1}{|z_1-z_2|^{1/4}}, \tag{5.43}$$

with $\varepsilon_\nu^{(\alpha)} = \pm 1$ and $\varepsilon_\nu^{(\alpha)} = -1$ iff $\alpha = \nu$.

The numerators in (5.43) for $\alpha = 2, 3, 4$ are nothing but the frustrated partition functions of the Ising model on a torus and match the expressions given in ref. [32].

6. Higher-correlation functions on a torus

This section is devoted to higher correlation functions. Candidates for the n-point correlation functions of the energy operator, or for the *squares* of correlators of spins and energies are now easy to write, and our main task here is to comment on the various consistency relations that these expressions must satisfy and that we should check.

In each fermionic sector different from the doubly periodic one, $\nu \neq 1$, we have three alternative formulae for the energy correlators:

$$\begin{aligned}\langle \varepsilon(1)\dots\varepsilon(2n)\rangle_\nu &= \left|\mathrm{Pf}\left(\langle\psi(z_i)\psi(z_j)\rangle_\nu\right)\right|^2 \\ &= \left|\mathrm{Pf}\left[\mathscr{P}_\nu(z_i - z_j)\right]\right|^2,\end{aligned} \tag{6.1a}$$

$$\begin{aligned}&\sum_{p=1}^{n}\left[\langle\varepsilon(1)\dots\varepsilon(2p)\rangle_\nu\langle\varepsilon(2p+1)\dots\varepsilon(2n)\rangle_\nu + \text{perm.}\right] \\ &\quad = 2^{3n-1}\langle\cos 2\phi(1)\dots\cos 2\phi(2n)\rangle_\nu \\ &\quad = 2^{n-1}\sum_{\substack{\varepsilon_i=\pm 1\\ \sum\varepsilon=0}}\left|\frac{\theta_\nu\left(\sum\varepsilon_i z_i\right)}{\theta_\nu(0)}\right|^2\prod_{i<j}\left|\frac{\theta_1(z_i - z_j)}{\theta_1'(0)}\right|^{2\varepsilon_i\varepsilon_j},\end{aligned} \tag{6.1b}$$

and

$$\begin{aligned}\langle\varepsilon(1)\dots\varepsilon(2n)\rangle_\nu^2 &= \left\langle\left(\vec{\nabla}\phi(1)\right)^2\dots\left(\vec{\nabla}\phi(2n)\right)^2\right\rangle_\nu \\ &= \left|\sum_{k=0}^{n}(-1)^k\frac{\theta_\nu^{(2n-2k)}(0)}{\theta_\nu(0)}\right. \\ &\quad\times\left.\left[\left(\frac{\theta_1'}{\theta_1}\right)'(z_1 - z_2)\dots\left(\frac{\theta_1'}{\theta_1}\right)'(z_{2k-1} - z_{2k}) + \text{perm.}\right]\right|^2,\end{aligned} \tag{6.1c}$$

which generalize eqs. (2.3), (2.5) and (2.9). The consistency of (6.1a) and (6.1c) follows from Fay's identity [21, 13, 14]. This identity, which generalizes the Cauchy determinant formula of eq. (2.14) to an arbitrary genus Riemann surface reads on a torus ($i, j = 1, \dots, m$)

$$\begin{aligned}\det\left[\frac{\theta_\nu(z_i - w_j)\theta_1'(0)}{\theta_1(z_i - w_j)\theta_\nu(0)}\right] &= (-1)^{m(m-1)/2}\left[\theta_1'(0)\right]^m \\ &\quad\times\frac{\theta_\nu(\Sigma(z_i - w_i))}{\theta_\nu(0)}\frac{\prod_{i<j}\theta_1(z_i - z_j)\theta_1(w_i - w_j)}{\prod_{i,j}\theta_1(z_i - w_j)}.\end{aligned} \tag{6.2}$$

As in sect. 2, we take $m=2n$, $w_i=z_i+\varepsilon_i$, let ε_i go to zero, and keep the regular term. This result is precisely

$$\det\left[\mathscr{P}_\nu(z_i-z_j)\right]$$
$$=\sum_{k=0}^{n}(-1)^k\frac{\theta_\nu^{(2n-2k)}(0)}{\theta_\nu(0)}\left[\left(\frac{\theta_1'}{\theta_1}\right)'(z_1-z_2)\dots\left(\frac{\theta_1'}{\theta_1}\right)'(z_{2k-1}-z_{2k})+\text{perm.}\right]. \tag{6.3}$$

Consistency of (6.1a) and (6.1b) has been checked only for $n=1$ and 2.

The correlation functions with an odd number of energy operators vanish identically in the sectors $\nu\neq 1$. On the contrary, only these do not vanish in the sector $\nu=1$. This is a consequence of the operator product expansion and of the fact that $Z_1\langle[1]\rangle_1=0$, $Z_1\langle[\varepsilon]\rangle_1\neq 0$, as already discussed. Their explicit calculation may be carried out in a way analogous to (6.1c)

$$Z_1^2\langle\varepsilon(1)\dots\varepsilon(2n+1)\rangle_1^2=-\left\langle\left(\vec\nabla\phi(1)\right)^2\dots\left(\vec\nabla\phi(2n+1)\right)^2\right\rangle_1$$
$$=\left|\sum_{k=0}^{n}(-1)^k\frac{\theta_1^{(2n-2k+1)}(0)}{2\eta}\right.$$
$$\left.\times\left[\left(\frac{\theta_1'}{\theta_1}\right)'(z_1-z_2)\dots\left(\frac{\theta_1'}{\theta_1}\right)'(z_{2k-1}-z_{2k})+\text{perm.}\right]\right|^2. \tag{6.4}$$

This leads to non-vanishing odd correlators:

$$\langle\varepsilon(1)\dots\varepsilon(2n+1)\rangle=\frac{Z_1\langle\varepsilon\dots\varepsilon\rangle_1}{Z}. \tag{6.5}$$

The $2n$-spin correlator is given by

$$Z_\nu^2\langle\sigma(1)\dots\sigma(2n)\rangle_\nu^2=2^nZ_\nu^2\langle\cos\phi(1)\dots\cos\phi(2n)\rangle_\nu$$
$$=\frac{1}{|\eta(q)|^2}\sum_{\substack{\varepsilon_i=\pm1\\ \Sigma\varepsilon=0}}\left|\theta_\nu\left(\frac{\sum\varepsilon_iz_i}{2}\right)\right|^2\prod_{i<j}\left|\frac{\theta_1(z_i-z_j)}{\theta_i'(0)}\right|^{\varepsilon_i\varepsilon_j/2}, \tag{6.6}$$

in each of the sectors $\nu=1$ to 4.

We notice above in eq. (5.11) that the two-point correlation function squared may be expressed in terms of the correlation function of the chiral "spin fields" of refs. [15,16]. For the $2n$-point function, this connection reads:

$$\langle\sigma(1)\dots\sigma(2n)\rangle_\nu^2 \propto \left\langle \prod_{i=1}^{2n} \left(S^+(z_i)\tilde{S}^+(\bar{z}_i) + S^-(z_i)\tilde{S}^-(\bar{z}_i)\right)\right\rangle_\nu \tag{6.7}$$

Whether there is a more direct connection between the spins of the Ising models and these spin fields, is not clear to us.

As in the plane, taking suitable short distance limits enables one to recover energy correlation functions, through various identities.

We now generalize eqs. (6.1), (6.6) to the most general correlator of spins and energies in the form:

$$Z_\nu^2\langle\sigma(1)\dots\sigma(2n)\varepsilon(2n+1)\dots\varepsilon((2n+p))\rangle_\nu^2$$

$$= 2^n(-1)^p Z_\nu^2\left\langle\cos\phi(1)\dots\cos\phi(2n)\left(\vec{\nabla}\phi(2n+1)\right)^2\dots\left(\vec{\nabla}\phi(2n+p)\right)^2\right\rangle_\nu, \tag{6.8}$$

where the r.h.s. has to be computed using either the prescription of chiral bosonization of eq. (5.14) or those of eq. (5.18). The consistency is established in appendix A. As in sect. 2, operator product expansions provide consistency relations between these expressions.

As in eq. (2.33), one may also write a similar ansatz for correlators with insertions of T. For example:

$$2Z_\nu^2\langle T(z)\sigma(1)\dots\varepsilon(2n+p)\rangle_\nu\langle\sigma(1)\dots\varepsilon(2n+p)\rangle_\nu$$

$$= 2^n(-1)^p Z_\nu^2\left\langle \mathbb{T}(z)\cos\phi(1)\dots\left(\vec{\nabla}\phi(2n+p)\right)^2\right\rangle_\nu, \tag{6.9}$$

which enable us to write the Ward identities and the degeneracy equations, and to compute correlators of secondary fields.

Let us restrict ourselves to $2n$-spin correlators. After some algebra, one finds that the Ward identity is satisfied in the sector ν iff:

$$\sum_{\substack{\varepsilon_i=\pm 1\\ \Sigma\varepsilon_i=0}} |f_\varepsilon|^2 g_\varepsilon(z, z_1, \dots, z_{2n}) = 0, \tag{6.10}$$

where

$$|f_\varepsilon|^2 = \left|\theta_\nu\left(\tfrac{1}{2}\sum\varepsilon_i z_i\right)\right|^2 \prod_{1\leqslant i<j\leqslant 2n} |\theta_1(z_i - z_j)|^{\varepsilon_i\varepsilon_j/2} \tag{6.11}$$

and

$$g_\varepsilon(z,\dots,z_{2n}) = \sum_{i<j}\varepsilon_i\varepsilon_j\frac{\theta_1''}{\theta_1}(z_i-z_j) - \sum_i\left(\frac{\theta_1'}{\theta_1}\right)'(z-z_i) - 6n\eta_1$$
$$-\left[\sum_i\varepsilon_i\frac{\theta_1'}{\theta_1}(z-z_i)\right]^2 + 2\sum_{i<j}\varepsilon_i\varepsilon_j\frac{\theta_1'}{\theta_1}(z_i-z_j)$$
$$\times\left[\frac{\theta_1'}{\theta_1}(z-z_i) - \frac{\theta_1'}{\theta_1}(z-z_j)\right]. \quad (6.12)$$

By the same method again, one may show that each g_ε actually vanishes.

Finally the degeneracy equation $(L_{-2}-\frac{4}{3}L_{-1}^2)\sigma = 0$ leads as in eq. (2.40) to the following identity

$$\frac{1}{2^{4n}}\sum_{\substack{\varepsilon_i=\pm1\\ \Sigma\varepsilon_i=0}}\sum_{\substack{\varepsilon_i'=\pm1\\ \Sigma\varepsilon_i'=0}}|f_{\varepsilon,\nu}|^2|f_{\varepsilon',\nu}|^2\left\{\frac{1}{6}\left[\frac{\theta_\nu''}{\theta_\nu}\left(\frac{\varepsilon_k z_k}{2}\right) + \frac{\theta_\nu'}{\theta_\nu}\left(\frac{\varepsilon_k z_k}{2}\right)\sum_{i>1}\varepsilon_i\frac{\theta_1'}{\theta_1}(z_1-z_i)\right]\right.$$
$$-\frac{\varepsilon_1}{12}\sum_{i>1}\varepsilon_i\left(\frac{\theta_1'}{\theta_1}\right)'(z_1-z_i) + \frac{1}{24}\left[\sum_{i>1}\varepsilon_i\frac{\theta_1'}{\theta_1}(z_1-z_i)\right]^2 + \frac{\eta_1}{2}$$
$$+\frac{\varepsilon_1\varepsilon_1'}{6}\left[\frac{1}{2}\sum_{i>1}\varepsilon_1\frac{\theta_1'}{\theta_1}(z_1-z_i) + \frac{\theta_\nu'}{\theta_\nu}\left(\frac{\varepsilon_k z_k}{2}\right)\right]$$
$$\left.\times\left[\frac{1}{2}\sum_{i>1}\varepsilon_1'\frac{\theta_1'}{\theta_1}(z_1-z_i) + \frac{\theta_\nu'}{\theta_\nu}\left(\frac{\varepsilon_k' z_k}{2}\right)\right]\right\} = 0, \quad (6.13)$$

where in the expression $\varepsilon_k z_k$, the summation over k is implicit. We have already checked this identity for $n=1$, and recover in the limit $z_i \to z_j$, $i \neq j$ the plane identity (2.41). It is a remarkable feature that our bosonization process gives non-analytic identities between θ-functions.

7. Conclusion and perspectives

The reader who has followed us that far is hopefully convinced that we can write any correlation function on the plane or on the torus involving an arbitrary number of spin and/or energy operators. At the price of some combinatorics, we may even manufacture correlators involving arbitrary numbers of energy momentum tensors, hence generate also correlators for arbitrary secondary fields. Admittedly, we have not checked *all* Ward identities, degeneracy equations and consistency relations of various kinds, which amount to non-trivial identities between rational functions in the plane, theta functions on the torus. Our formulae, however, have a high degree of plausibility.

Let us stress the salient features of our analysis.

(i) By a systematic comparison between different methods, we have demonstrated the consistency of the chiral bosonization and of the orbifold nature of the critical Ashkin-Teller model. Let us emphasize that these provide two distinct bosonizations of the Ising model. Chiral bosonization enables one to compute (squares of) correlation functions in each fermionic sector, whereas the orbifold bosonization of the AT model yields directly the (square of the) full correlator.

(ii) The role of the various fermionic sectors has been elucidated. In particular, the doubly periodic sector has been shown to contribute to a variety of observables.

(iii) As was already apparent on the form of the four-spin correlation function on the plane, the correlators are *not* in general the squared modulus of some analytic function.

One can think of extending these results in various directions. The most obvious one is the computation of correlation functions of the $c=\frac{1}{2}$ unitary conformal theory on a higher genus surface. (Its interpretation as the critical limit of a microscopic Ising model becomes however delicate.) It is likely that consistency relations between correlation functions will generate hosts of interesting identities.

One may also try to generalize this work to other minimal or non-minimal conformal theories. Although these models do not have the simplicity of the Ising model with its underlying free fermion field, their partition functions admits a free boson representation [30, 33]. Repeating our work using this representation and extending it to other models would be very interesting. We recall that integral representations for correlators (in the plane) of minimal theories have been given by Dotsenko and Fateev [34], and that the four-spin correlation function of the AT model has been studied by Zamolodchikov [35].

Finally, one may ponder about the huge information stored in this infinite collection of Ising correlation functions at the critical point. *In principle*, this information should enable us to explore the critical region around the critical point. Perturbative calculations in that direction have already been attempted [28]. Using our correlators in a non-perturbative way remains a challenge.

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Appendix A

In this appendix, we prove the equivalence between the prescriptions of eqs. (5.14), (5.15) and those of eq. (5.18) namely between chiral and non-chiral bosonization, for a class of correlation functions:

$$\langle X\rangle = \left\langle \sum_k \partial_{v_k}\phi(v_k,\bar{v}_k)\prod_l \partial_{\bar{w}_l}\phi(w_l,\bar{w}_l)\prod_j e^{ie_j\phi(z_j,\bar{z}_j)}\right\rangle , \tag{A.1}$$

satisfying the condition of "electric charge conservation" (or momentum conservation, in the context of vertex operators of string theory).

$$\sum e_j = 0 . \tag{A.2}$$

In each fermionic sector, labelled by $\nu = 1,\dots,4$ or by $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, the claim is that:

$$Z_\nu^2 \langle X \rangle_\nu = \tfrac{1}{8} \sum_{m,\, m' \in \frac{1}{2}\mathbb{Z}} \varepsilon^{(\nu)}_{m,\,m'} Z_{mm'}(2) \langle X \rangle_{mm'} . \tag{A.3}$$

The l.h.s. is evaluated using the prescriptions of eqs. (5.14)–(5.15), whereas the computation of the r.h.s. is carried out by the same method as in eq. (4.23) and following, shifting the field ϕ by the classical field of winding numbers m, m' and weighting the contributions by $Z_{mm'}$, the signs $\varepsilon^{(\nu)}_{mm'}$ are given in eq. (3.6) and table 1. Although such a property is stated in refs. [12, 13], we find it useful to make its proof explicit here.

It is convenient to first generalize slightly the computation of (4.12)–(4.18) to

$$\begin{aligned}
\theta\begin{bmatrix} a \\ b \end{bmatrix}(0)\theta^*\begin{bmatrix} a \\ \bar{b} \end{bmatrix}(0) &= \theta\begin{bmatrix} a \\ 0 \end{bmatrix}(b)\theta^*\begin{bmatrix} a \\ 0 \end{bmatrix}(\bar{b}) \\
&= \sum_{n,\bar{n}} \exp\Big\{ i\pi\left[\tau(n+a)^2 - \bar{\tau}(\bar{n}+a)^2\right] \\
&\qquad\qquad + 2i\pi\left[(n+a)b - (\bar{n}+a)\bar{b}\right]\Big\} \\
&= \frac{1}{(2\tau_I)^{1/2}} \sum_{p,\,p' \in \mathbb{Z}} (-1)^{p'(p+2a)} \\
&\qquad \times \exp\left\{ -\frac{\pi}{2\tau_I}\left(p\tau_R - p' + b - \bar{b}\right)^2 - \frac{\pi\tau_I}{2} p^2 + i\pi p(b+\bar{b}) \right\} .
\end{aligned} \tag{A.4}$$

In this computation, a is integer or half-integer, but b and $\bar{b}$ are kept for a while arbitrary. This enables one to compute the l.h.s. of (A.3) by differentiations with respect to b or $\bar{b}$. For any functional $\mathscr{F}$ of the field $\phi = \frac{1}{2}(\varphi - \tilde{\varphi})$ we write:

$$\begin{aligned}
Z_\nu^2 \Big\langle \mathscr{F}\{\phi(z_j, \bar{z}_j)\} \Big\rangle_\nu &= \frac{1}{|2\eta|^2} \sum_{n,\bar{n}} \theta\begin{bmatrix} a \\ b \end{bmatrix}(0)\theta^*\begin{bmatrix} a \\ \bar{b} \end{bmatrix}(0) \\
&\quad \times \Big\langle\Big\langle \mathscr{F}\left\{ \pi(n+a)z_j - \pi(\bar{n}+a)\bar{z}_j + \tfrac{1}{2}\hat{\varphi}(z_j) - \tfrac{1}{2}\hat{\tilde{\varphi}}(\bar{z}_j) \right\} \Big\rangle\Big\rangle \\
&= \left\langle\left\langle \mathscr{F}\left\{ \frac{1}{2i}\left(z_j \frac{\partial}{\partial b} + \bar{z}_j \frac{\partial}{\partial \bar{b}} \right) + \frac{1}{2}\hat{\varphi}(z_j) - \frac{1}{2}\hat{\tilde{\varphi}}(\bar{z}_j) \right\} \right\rangle\right\rangle \left. \frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}\theta^*\begin{bmatrix} a \\ \bar{b} \end{bmatrix}}{|2\eta|^2} \right|_{b=\bar{b}} ,
\end{aligned} \tag{A.5}$$

where the symbol $\langle\langle\ \rangle\rangle$ means that contractions are done according to eq. (5.14a). The sum of the $\hat{\varphi}$ and $\hat{\bar{\varphi}}$ contractions, however, differs from the propagator (4.20) by the anholomorphic term $\pi\,\mathrm{Im}^2(z-w)/2\tau_{\mathrm{I}}$. It is a mere exercise of combinatorics to reshuffle (A.4) to build up this propagator, and we only sketch the steps. According to (A.5), we write

$$Z_\nu^2\langle X\rangle_\nu=\left\langle\left\langle\prod_k\left[\frac{1}{2i}\frac{\partial}{\partial b}+\frac{1}{2}\,\partial\hat{\varphi}(v_k)\right]\prod_l\left[\frac{1}{2i}\frac{\partial}{\partial\bar{b}}-\frac{1}{2}\,\bar{\partial}\hat{\bar{\varphi}}(w_l)\right]\right.\right.$$
$$\left.\left.\times\prod_j\exp\left\{\frac{1}{2}e_j\left(z_j\frac{\partial}{\partial b}+\bar{z}_j\frac{\partial}{\partial\bar{b}}\right)+\frac{1}{2}ie_j\left(\hat{\varphi}(z_j)-\hat{\bar{\varphi}}(\bar{z}_j)\right)\right\}\right\rangle\right\rangle$$
$$\times\left.\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}\theta*\begin{bmatrix}a\\\bar{b}\end{bmatrix}}{|2\eta|^2}\right|_{b=\bar{b}}. \tag{A.6}$$

The product $\Pi_j\exp(\frac{1}{2}e_j(z_j\,\partial/\partial b+\bar{z}_j\,\partial/\partial\bar{b}))$ is the operator which translates b by $\frac{1}{2}\Sigma e_jz_j$, $\bar{b}$ by $\frac{1}{2}\Sigma e_j\bar{z}_j$. Therefore

$$\prod_j\exp\left\{\frac{1}{2}e_j\left(z_j\frac{\partial}{\partial b}+\bar{z}_j\frac{\partial}{\partial\bar{b}}\right)+\frac{1}{2}ie_j\left(\hat{\varphi}(z_j)-\hat{\bar{\varphi}}(\bar{z}_j)\right)\right\}\theta\begin{bmatrix}a\\b\end{bmatrix}\theta*\begin{bmatrix}a\\\bar{b}\end{bmatrix}$$
$$=\frac{1}{(2\tau_{\mathrm{I}})^{1/2}}\sum_{p,p'}(-1)^{p'(p+2a)}\exp\left\{-\frac{\pi}{2\tau_{\mathrm{I}}}\left(p\tau_{\mathrm{R}}-p'+b-\bar{b}+\sum\tfrac{1}{2}e_j(z_j-\bar{z}_j)\right)^2\right\}$$
$$\times\prod_j\exp\left(\tfrac{1}{2}ie_j\left(\hat{\varphi}(z_j)-\hat{\bar{\varphi}}(\bar{z}_j)\right)\right)$$
$$=\frac{1}{(2\tau_{\mathrm{I}})^{1/2}}\sum_{p,p'}(-1)^{p'(p+2a)}\exp\left\{-\frac{\pi}{2\tau_{\mathrm{I}}}\left(p\tau_{\mathrm{R}}-p'+b-\bar{b}\right)^2-\frac{\pi\tau_{\mathrm{I}}}{2}p^2+i\pi p(b+\bar{b})\right\}$$
$$\times\exp\left\{-\frac{i\pi}{\tau_{\mathrm{I}}}(b-\bar{b})\sum e_j\,\mathrm{Im}\,z_j\right\}\prod_j\exp ie_j\left\{\phi_{\mathrm{class}}(z_j,\bar{z}_j)+\tfrac{1}{2}\hat{\varphi}(z_j)-\tfrac{1}{2}\hat{\bar{\varphi}}(z_j)\right\}$$
$$\times\exp\frac{\pi}{2\tau_{\mathrm{I}}}\left(\sum e_j\,\mathrm{Im}\,z_j\right)^2, \tag{A.7}$$

where ϕ_{class} is the classical field of eqs. (4.21)–(4.22), pertaining to winding numbers $m=\frac{1}{2}p$, $m'=\frac{1}{2}p'$. Then we let the other $\partial/\partial b$, $\partial/\partial\bar{b}$ derivatives of (A.6) operate.

One finds:

$$Z_\nu^2\langle X\rangle_\nu = \frac{1}{|2\eta|^2}\frac{1}{(2\tau_I)^{1/2}}\sum_{p,p'}\varepsilon^{(\nu)}_{\frac12 p,\frac12 p'}\exp\left\{-\frac{\pi}{2\tau_I}(p\tau_R-p')^2-\frac{\pi\tau_I}{2}p^2\right\}$$

$$\times\left\langle\left\langle\left[\prod_k\left\{\partial\phi(v_k,\bar v_k)-\frac{\pi}{2\tau_I}\sum e_j\,\mathrm{Im}\,z_j\right\}\right.\right.\right.$$

$$\left.\times\prod_l\left\{\bar\partial\phi(w_l,\bar w_l)+\frac{\pi}{2\tau_I}\sum e_j\,\mathrm{Im}\,z_j\right\}+\text{contracted terms}\right]$$

$$\left.\left.\times\prod_j\exp ie_j\phi(z_j,\bar z_j)\right\rangle\right\rangle\exp\frac{\pi}{2\tau_I}\left(\sum e_j\,\mathrm{Im}\,z_j\right)^2, \tag{A.8}$$

where ϕ denotes the combination $\phi_{\mathrm{class}}+\frac12(\hat\phi-\hat{\hat\phi})$ and the "contracted terms" stand for terms where pairs of curly brackets relative to (v_a, v_b), $(v_a,\bar v_b)$ or $(\bar v_a,\bar v_b)$ are removed and replaced respectively by $\pi/4\tau_I$, $-\pi/4\tau_I$ or $\pi/4\tau_I$.

Finally, the terms $\pm(\pi/2\tau_I)\Sigma e_j\,\mathrm{Im}\,z_j$ in the curly brackets are rewritten as $-(\pi/2\tau_I)\Sigma_j e_j\,\mathrm{Im}(v_k-z_j)$ or $(\pi/2\tau_I)\Sigma_j e_j\,\mathrm{Im}(w_l-z_j)$, thanks to the condition (A.2), and are nothing but the contribution of the anholomorphic term to the contraction of $\partial\phi$ or $\bar\partial\phi$ with $\Pi\exp(ie_j\phi(z_j,\bar z_j))$. Likewise the "contracted terms" build up this contribution in contractions of $\partial\phi_a$ with $\partial\phi_b$, etc..., and the last factor in (A.8) does the same for self-contractions within the exponentials:

$$\left\langle\left\langle\prod_j\exp\left(ie_j\phi(z_j,\bar z_j)\right)\right\rangle\right\rangle\left(\frac{\pi}{2\tau_I}\left(\sum e_j\,\mathrm{Im}\,z_j\right)^2\right)=\left\langle\prod_j\exp\left(ie_j\phi(z_j,\bar z_j)\right)\right\rangle_{\frac12 p,\frac12 p'} \tag{A.9}$$

because of the identity

$$\left(\sum e_j\,\mathrm{Im}\,z_j\right)^2=-\sum_{j<k}e_je_k\,\mathrm{Im}^2(z_j-z_k), \tag{A.10}$$

a consequence of (A.2).

Putting everything together, we obtain the announced result (A.3). As already noticed in (4.19), in the sum over ν, only integer m and m' contribute:

$$\sum_{\nu=1}^{4}Z_\nu^2\langle X\rangle_\nu=\tfrac12\sum_{m,m'\in\mathbb{Z}}Z_{m,m'}(2)\langle X\rangle_{mm'} \tag{A.11}$$

and the factor $\frac12$ matches the one of (3.9). This is the first step towards establishing in general the consistency between the bosonization and the AT approaches, namely between eq. (6.8) and:

$$\left(\sum Z_\nu \langle \sigma(1) \ldots \sigma(2n) \varepsilon(2n+1) \ldots \varepsilon(2n+p) \rangle_\nu \right)^2$$

$$= 2^n (-1)^p (Z^I)^2 \langle \cos\phi(1) \ldots \left(\vec{\nabla}\phi(2n+p)\right)^2 \rangle^{\mathrm{AT}}$$

$$= 2^n (-1)^p \left\{ \tfrac{1}{2} \sum Z_{mm'} \langle \cos\phi \ldots \left(\vec{\nabla}\phi\right)^2 \rangle_{mm'} + \sum_{\alpha\beta}{}' Z_{\alpha\beta} \langle \ldots \rangle_{\alpha\beta} \right\}. \quad \text{(A.12)}$$

Generalizing what we have found after the laborious computations of subsects. (4.3) and (5.5), we may conjecture that in general:

$$2\left(Z_1 Z_2 \langle \sigma \ldots \varepsilon \rangle_1 \langle \sigma \ldots \varepsilon \rangle_2 + Z_3 Z_4 \langle \sigma \ldots \varepsilon \rangle_3 \langle \sigma \ldots \varepsilon \rangle_4 \right)$$

$$= 2^n (-1)^p Z_{0,\frac{1}{2}} \langle \cos\phi \ldots \left(\vec{\nabla}\phi\right)^2 \rangle_{0,\frac{1}{2}}, \quad \text{(A.13)}$$

and similar other identities obtained by modular transformations. Together with (A.11), these identities would established (A.12).

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