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MODULAR INVARIANCE IN NON-MINIMAL TWO-DIMENSIONAL CONFORMAL THEORIES

P. DI FRANCESCO, H. SALEUR and J.B. ZUBER

Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

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We construct modular invariants in non-minimal conformal theories of central charge $c<1$. We show these can also be considered as partition functions of $c=1$ theories, and describe a free field with defect lines on a torus. This is applied to the determination of partition functions for critical Q-state Potts and $O(n)$ models, with a special emphasis on the polymer ($n=0$) case.

1. Introduction

Conformal invariant theories in two dimensions [1] play a central role in string theory and in the study of critical phenomena. The analysis of their properties and their classification are actively pursued. After the introduction of minimal theories [1] that possess only a finite number of primary fields, and of its important subset of $c<1$ unitary theories [2], new families have been discovered using the conformal algebra [3].

The observation by Cardy [4] that the consistency of conformal theories defined on a torus restricts severely their operator content has opened a new route to their systematic classification [5, 6]. The idea is that the partition function of a conformal theory on a torus may be written as [4, 5]

$$Z=\mathrm{Tr}\left(q^{L_0-c/24}\,\bar{q}^{\bar{L}_0-c/24}\right), \tag{1.1}$$

where $q=\exp(2i\pi\tau)$, and $\tau=\omega_2/\omega_1$ is the modular ratio of the torus. This trace which runs over all states of the Hilbert space may be decomposed on the various irreducible representations of the two Virasoro algebras in the form

$$Z=\sum_{h,\bar{h}} N_{h\bar{h}}\chi_h(q)\chi_{\bar{h}}(\bar{q}). \tag{1.2}$$

The non-negative integer $N_{h,\bar{h}}$ represents the multiplicity of the operator of dimensions $(h,\bar{h})$ in the partition function. The characters χ_h read:

$$\chi_h=\mathrm{Tr}_h\left(q^{L_0-c/24}\right) \tag{1.3}$$

in each irreducible representation of highest weight h.

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Imposing that Z is a modular invariant function of τ then puts strong constraints on the N's, hence on the operator content of the theory, leading ultimately to a classification of the possible partition functions. This has been recently achieved [7] for the minimal theories. These have a central charge

$$c(p, p') = 1 - \frac{6(p-p')^2}{pp'}. \tag{1.4}$$

p, p' are two coprime positive integers, and the allowed values of h, $\bar{h}$ are given by the Kac formula [8]

$$h_{rs} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}, \tag{1.5}$$

with the constraint that the integers r, s satisfy the bounds:

$$\begin{aligned} &1 \leqslant r \leqslant p' - 1, \\ &1 \leqslant s \leqslant p - 1. \end{aligned} \tag{1.6}$$

Using modular invariance, a classification of such theories has been proposed. This has also been extended to superconformal theories [9] where a similar classification has been found [10].

In this paper, we want to consider non-minimal theories. We restrict ourselves to central charges less than one, in particular as in (1.4), but relax the conditions (1.6). This is of direct relevance in the study of 2D critical phenomena. For example, it is known [11] that there are lattice realizations of Ising-like systems which lead to a non-minimal $c = \frac{1}{2}$ conformal theory. Even more interestingly, the scaling limit of polymer systems seems to be described [12] by the $c(3,2) = 0$ theory, which can obviously not be minimal. Moreover, various works [12,13] indicate that the spectrum of conformal dimensions in these systems is reproduced by the Kac formula (1.5) partly with integer indices r, s, and partly with *half integer* ones. This seems a rather intriguing feature. We recall that the h_{rs} given by (1.5) with integral indices have an intrinsic meaning: they correspond to degenerate representations of the Virasoro algebra [8,14]. Fractional indices, on the other hand, do not seem to have any particular meaning. Is modular invariance of any relevance for non-minimal theories, and can it shed some light on this? These were the original motivations of this work.

We first construct modular invariant partition functions in a rather abstract way, using the same elliptic functions as in minimal theories [7]. These partition functions may be expanded on conformal characters of $c(p, p')$ theories: they involve an infinite number of primary fields, in particular the one with negative dimension

$h=\bar{h}=-(p-p')^2/4pp'$. This operator leads to a small q behaviour [15–16] described by an effective [17] central charge equal to one. Our partition functions may indeed be expanded also on $c=1$ characters. These non-minimal theories thus appear as gaussian theories in disguise. This is presented in sect. 2, while technical details on characters [18–20] are collected in the appendix.

In sect. 3 we give a reinterpretation of these partition functions in terms of a Coulomb [21–23] gas, or alternatively of a free field theory on a torus with defect lines.

Sect. 4 is devoted to applications to a few statistical mechanical models. We use the preceding construction to derive partition functions of critical Q-state Potts [24] or $O(n)$ [25] models on a torus for arbitrary Q, n. Special attention is payed to polymers, for which the above mentioned results are recovered.

Sect. 5 contains a few final comments.

A last word on notations. As we handle in this paper partition functions of various origins, we have introduced distinct notations. $Z^{(\alpha,\mu)}(q, M)$, as defined in eq. (2.5) denotes the functions constructed by the formal procedure of sect. 2. $Z_c[g, f]$ (eq. (3.12)), with the argument q made implicit, stands for the Coulomb partition function for coupling g and magnetic charges integer multiples of f. Finally $\mathfrak{Z}$ in sect. 4 represents the partition function of explicit statistical models, and $\tilde{\mathfrak{Z}}(\dots)$ suitably modified versions of it.

We hope the reader will not be bothered by these notations.

2. Construction of modular invariants

In this section, we construct families of modular invariants, relying on the analysis and results (and the conjecture) of ref. [7]. For any *even* positive integer M, we introduce the set of functions

$$K_\nu(q, M)=\frac{1}{\eta(q)}\sum_{n=-\infty}^{\infty} q^{(nM+\nu)^2/2M}, \tag{2.1}$$

where $\eta(q)$ is Dedekind's function (see appendix, eq. (A.15)). (The integers M and ν were denoted N and λ in ref. [7], but the reason for this change of notations will appear soon.) We recall the main steps of [7].

(i) These functions satisfy

$$K_\nu=K_{\nu+M}=K_{-\nu}. \tag{2.2}$$

There are therefore $\frac{1}{2}M+1$ independent functions, for $0\leqslant\nu\leqslant\frac{1}{2}M$, but it is more convenient to keep ν as a mod M variable.

(ii) Writing $q=\exp(2i\pi\tau)$, with τ in the upper half-plane, we let the modular group act on τ, and hence on these functions. One proves that they form a unitary representation of the modular group.

(iii) If one wants to construct modular invariants of the general form:

$$Z = \sum_{\nu, \bar{\nu}} \mathcal{N}_{\nu\bar{\nu}} K_\nu K_{\bar{\nu}}^* , \tag{2.3}$$

one finds that there is an independent invariant associated with each factorization of $\frac{1}{2}M$:

$$\tfrac{1}{2}M = \alpha^2 P \cdot P' , \tag{2.4}$$

where P and P' are coprimes. With this pair of coprimes, we associate a pair of integers R_0 and S_0 such that $R_0 P - S_0 P' = 1$ and construct the number $\mu = R_0 P + S_0 P'$ mod(M/α^2), which satisfies $\mu^2 = 1$ mod$(2M/\alpha^2)$. A modular invariant is then obtained by taking:

$$\mathcal{N}_{\nu, \bar{\nu}}^{(\alpha, \mu)} = \begin{cases} \sum_{\xi \in \mathbf{Z}/\alpha \mathbf{Z}} \delta_{\bar{\nu}, \mu\nu + \xi M/\alpha \bmod M} & \text{if } \alpha | \nu \text{ and } \alpha | \bar{\nu} \\ 0 & \text{otherwise}, \end{cases}$$

$$Z^{(\alpha, \mu)}(q, M) = \sum \mathcal{N}_{\nu\bar{\nu}}^{(\alpha, \mu)} K_\nu K_{\bar{\nu}}^* . \tag{2.5}$$

In ref. [7], it was conjectured that this construction exhausts the set of independent modular invariants of the form (2.3)*.

(iv) The resulting invariant $Z^{(\alpha, \mu)}(q, M)$ must then be reexpressed in terms of the $\frac{1}{2}M + 1$ functions K_ν, for $0 \leqslant \nu \leqslant \frac{1}{2}M$. This operation does not introduce any sign, since $K_\nu = K_{M-\nu}$. We thus conclude that each choice of (α, μ) produces a modular invariant (2.3) with positive coefficients, and that the number of such independent invariants is

$$\varphi\left(\tfrac{1}{2}M\right) = \tfrac{1}{2}\left[\prod_{i \geqslant 1} (1 + r_i) + \delta\right] \tag{2.6}$$

if $\frac{1}{2}M = \prod_i p_i^{r_i}$, p_i primes, and $\delta = 1$ if M is a square (all r_i even), $\delta = 0$ otherwise. Contrary to the case studied in [7], if $\frac{1}{2}M$ is a square $\frac{1}{2}M = \overline{M}^2$, the trivial factorization $M/2\overline{M}^2 = 1.1$ gives rise to a non-vanishing invariant. Among the possible invariants, let us mention two simple cases. For any M, choosing $\alpha = \mu = 1$ yields the diagonal invariant:

$$Z^{(1,1)}(q, M) = \sum_{\nu=0}^{M-1} |K_\nu(q, M)|^2 = |K_0|^2 + 2 \sum_{\nu=1}^{M/2-1} |K_\nu|^2 + |K_{M/2}|^2 . \tag{2.7}$$

On the other hand, if $M/2$ is a multiple of some square ℓ^2

$$M = \ell^2 N, \qquad N \text{ even}, \tag{2.8}$$

* *Note added in proof.* That (2.5) exhausts all invariants of the form (2.3) has now been proved [37].

we can take $\alpha = \ell$, $\mu = 1 \bmod N$, and the resulting invariant couples only values of ν multiples of ℓ, $\nu = \ell\lambda$:

$$\sum_{\xi=0}^{\alpha-1} K_{\ell\lambda+\xi M/2}(q, M) = K_\lambda(q, N), \tag{2.9}$$

$$Z^{(\ell,1)}(q, \ell^2 N) = \sum_{\lambda=0}^{N-1} |K_\lambda(q, N)|^2 = Z^{(1,1)}(q, N). \tag{2.10}$$

This is the diagonal invariant of the functions K_λ relative to the even integer N.

So far the discussion has been quite independent of conformal characters. In appendix A, we review the computation of characters for the $c \leqslant 1$ theories. The most important conclusion is that any expression of the form $q^x/\eta(q)$, $x \geqslant 0$ may be expressed as a (finite or infinite) sum of characters, with positive integer coefficients.

$$\frac{q^x}{\eta(q)} = \chi_{h=x+(c-1)/24} + \cdots .$$

In particular, for $c = 1$,

$$\begin{array}{llll} \text{either} & h = \frac{1}{4}n^2, & n \in \mathbb{N}, & \dfrac{q^h}{\eta} = \displaystyle\sum_{m=0}^{\infty} \chi_{(n+2m)^2/4}, \\ \text{or} & h \neq \frac{1}{4}n^2, & & \dfrac{q^h}{\eta} = \chi_h . \end{array} \tag{2.11}$$

This means that any of the previous invariants constructed in terms of the K-functions yields a modular invariant expression, sesquilinear in the characters, and with positive integer coefficients, i.e. an acceptable partition function for any of the $c \leqslant 1$ theories.

Let us examine more closely the "rational case" (1.4), where c is specified by two coprimes p and p'. We then define

$$N = 2pp' \tag{2.12}$$

and for ℓ an arbitrary integer, it is natural, as we shall see, to choose:

$$M = \ell^2 N = 2\ell^2 pp' . \tag{2.13}$$

The corresponding K functions may be expressed as linear combinations of characters. First, any K_ν with ν multiple of ℓ reads

$$K_{\ell\lambda}(q, \ell^2 N) = \frac{1}{\eta} \sum_{n=-\infty}^{\infty} q^{(\ell N n+\lambda)^2/2N} \tag{2.14}$$

TABLE 1

Expression of the K-functions (2.1) for $M = 2\ell^2 pp'$ in terms of characters of the $c(p, p')$ theory

$$K_0 = \sum_{k=0}^{\infty} \left(k + \frac{1+(-1)^k}{2} \right) \chi_{(2k+1)p',p}$$

$$K_{\ell N} = \sum_{k=0}^{\infty} \left(k + \frac{1-(-1)^k}{2} \right) \chi_{(2k+1)p',p}$$

$$K_{\ell N/2} = \sum_{k=1}^{\infty} k \chi_{2kp',p}$$

$$K_{\ell p' s_0} = \sum_{k=0}^{\infty} \left(k + \frac{1+(-1)^k}{2} \right) \chi_{(2k+1)p',p-s_0} + k \chi_{2kp',s_0}$$

$$K_{\ell (N-p' s_0)} = \sum_{k=0}^{\infty} \left(k + \frac{1-(-1)^k}{2} \right) \chi_{(2k+1)p',p-s_0} + k \chi_{2kp',s_0}$$

$$K_{\ell p r_0} = \sum_{k=0}^{\infty} \left(k + \frac{1+(-1)^k}{2} \right) \chi_{(2k+1)p'+r_0,p} + k \chi_{(2k+1)p'-r_0,p}$$

$$K_{\ell (N-p r_0)} = \sum_{k=0}^{\infty} \left(k + \frac{1-(-1)^k}{2} \right) \chi_{(2k+1)p'+r_0,p} + k \chi_{(2k+1)p'-r_0,p}$$

$$\begin{aligned} K_{\ell (p r_0 - p' s_0)} = \chi_{r_0,s_0} + \sum_{k=0}^{\infty} \left(k + \frac{1+(-1)^k}{2} \right) & [\chi_{(2k+1)p'+r_0,p-s_0} + \chi_{2(k+1)p'-r_0,s_0}] \\ & + k [\chi_{2kp'+r_0,s_0} + \chi_{(2k+1)p'-r_0,p-s_0}] \end{aligned}$$

$$\begin{aligned} K_{\ell (N-p r_0 + p' s_0)} = \sum_{k=0}^{\infty} \left(k + \frac{1-(-1)^k}{2} \right) & [\chi_{(2k+1)p'+r_0,p-s_0} + \chi_{2(k+1)p'-r_0,s_0}] \\ & + k [\chi_{2kp'+r_0,s_0} + \chi_{(2k+1)p'-r_0,p-s_0}] \end{aligned}$$

$$\begin{aligned} K_{\ell (p r_0 + p' s_0)} = \sum_{k=0}^{\infty} \left(k + \frac{1+(-1)^k}{2} \right) & \chi_{(2k+1)p'+r_0,p-s_0} + \left(k + \frac{1-(-1)^k}{2} \right) \chi_{2(k+1)p'-r_0,s_0} \\ & + k [\chi_{2kp'+r_0,s_0} + \chi_{(2k+1)p'-r_0,p-s_0}] \end{aligned}$$

and may therefore be expanded on the degenerate characters of appendix A (eqs. (A.19), (A.25)). The somewhat cumbersome formulae are displayed in table 1. On the other hand, if ν is *not* a multiple of ℓ, $\nu = \ell a + b$, K_ν reads:

$$K_\nu(q, \ell^2 N) = \frac{1}{\eta} \sum_{n=-\infty}^{\infty} q^{(n\ell N + a + b/\ell)^2/2N}$$

$$= \sum_{n=-\infty}^{\infty} \chi_{h_{\ell n}}, \tag{2.15}$$

with

$$h_{\ell n} = \frac{(\ell n N + a + b/\ell)^2 - (p - p')^2}{2N}. \tag{2.16}$$

There exist two numbers r_0 and s_0 such that

$$r_0 p - s_0 p' = 1 \tag{2.17}$$

expressing that p and p' are coprimes. We can thus rewrite $h_{\ell n}$ in (2.16) as

$$h_{\ell n} = \frac{(2\ell n p p' + (a + b/\ell) r_0 p - (a + b/\ell) s_0 p')^2 - (p - p')^2}{4pp'} \tag{2.18}$$

and interpret it as given by the Kac formula (1.5) with fractional indices r, s of denominator (at most) ℓ.

$$h = h_{r,s} = h_{2\ell n p' + (a+b/\ell) r_0, (a+b/\ell) s_0}. \tag{2.19}$$

We have thus constructed a finite dimensional representation of the modular group by means of a finite number of infinite sums of characters. We can then rewrite the modular invariants constructed in terms of K's as

$$Z = \sum N_{h\bar{h}} \chi_h \chi_{\bar{h}} \tag{2.20}$$

with non-negative integer coefficients $N_{h\bar{h}}$. In particular the invariant $Z^{(1,1)}(q, M)$ given in (2.7) involves all the characters either degenerate or of the form (2.15)–(2.16), whereas $Z^{(\ell,1)}(q, M) = Z^{(1,1)}(q, N)$ in (2.10) uses only the degenerate ones (A.19)–(A.25). Conversely we do not know if this construction exhausts all modular invariants of the form (2.20). It is interesting to observe that the modular invariants of the *minimal* theories may be recovered as linear combinations of the former invariants. We recall from ref. [7] that minimal characters can be simply represented by:

$$\chi_\lambda(q) = K_\lambda(q) - K_{\omega_0 \lambda}(q), \tag{2.21}$$

where $\omega_0 = r_0 p + s_0 p'$, r_0, s_0 the same integers as in (2.17). The class of minimal modular invariants considered in [7], may be written up to a factor, as:

$$
\begin{aligned}
4Z_{\min}^{(\alpha,\mu)} &= \sum_{\substack{\lambda \in \mathbf{Z}/N\mathbf{Z} \\ \lambda \text{ multiple of } \alpha}} \sum_{\xi \in \mathbf{Z}/\alpha\mathbf{Z}} \chi_\lambda \chi^*_{\mu\lambda+\xi N/\alpha} \\
&= \sum_\lambda \left(K_\lambda - K_{\omega_0\lambda}\right) \sum_\xi \left(K^*_{\mu\lambda+\xi N/\alpha} - K^*_{\omega_0(\mu\lambda+\xi N/\alpha)}\right) \\
&= 2\sum_\lambda \sum_\xi K_\lambda K^*_{\mu\lambda+\xi N/\alpha} - 2\sum_\lambda \sum_\xi K_\lambda K^*_{\omega_0\mu\lambda+\xi N/\alpha} \\
&= 2\left(Z^{(\alpha,\mu)} - Z^{(\alpha,\omega_0\mu)}\right).
\end{aligned}
\tag{2.22}
$$

An important observation is that each of the matrices (2.5) has

$$\mathcal{N}_{00}^{(\alpha,\mu)} = 1 \tag{2.23}$$

and that the corresponding invariant contains the contribution of the operator with the most negative conformal dimension

$$
\begin{aligned}
h = \bar{h} = h_0 &= -\frac{(p-p')^2}{4pp'}, \\
Z^{(\alpha,\mu)} &= |\chi_{p'p}|^2 + \cdots .
\end{aligned}
\tag{2.24}
$$

If $|p-p'| = 1$, this is the only spinless operator of negative dimension among those with integer indices. It seems that the only way to get rid of this operator by linear combinations of $Z^{(\alpha,\mu)}$ while preserving the positivity of the coefficient $N_{h\bar{h}}$ is to construct the minimal modular invariants (2.22). If moreover all the modular invariants are reached by our procedure, this would mean that any non-minimal theory with central charge $c < 1$ (but of the form (1.4)) would be affected by negative dimension operators.

What are the implications of these negative dimensions? In ref. [15,16], it has been shown that the value of the central charge c may be regarded as the coefficient of a finite size contribution to the free energy per unit length of a conformal theory on a strip, i.e.

$$Z \sim (q\bar{q})^{-c/24} \qquad \text{as } q \to 0. \tag{2.25}$$

This is, however, only justified when all the conformal dimensions appearing in the expansion of Z are non-negative, in particular in unitary theories. If some negative dimension is present, the small q behaviour of Z is dominated by the lowest

$h=\bar{h}=h_0$, and the "effective central charge" to be substituted in (2.25) is [17]:

$$c^{\text{eff}}=c-24h_0\,. \tag{2.26}$$

In the case at hand, eqs. (1.4) and (2.24) lead to

$$c^{\text{eff}}=1\,. \tag{2.27}$$

This suggests that the non-minimal $c<1$ conformal theories described by the invariant partition functions $Z^{(\alpha,\mu)}$ may be naturally given an alternative description in terms of a $c=1$ conformal theory, i.e. in terms of a free (gaussian) field.

This interpretation is supported by the fact that, as already mentioned, our modular invariant partition functions of the $c(p,p')$ theories may be expanded on $c=1$ characters.

$$\begin{aligned}
Z&=\sum_{h,\bar{h}}N_{h\bar{h}}\chi_h(q;c(p,p'))\chi_{\bar{h}}(\bar{q};c(p,p'))\\
&=\sum_{h',\bar{h}'}N_{h'\bar{h}'}\chi_{h'}(q;1)\chi_{\bar{h}'}(\bar{q};1)\\
&\sim(q\bar{q})^{-1/24}[1+0(q)]\qquad\text{for } q,\bar{q}\to 0\,.
\end{aligned}$$

Surprising as it may be, this equivalence originates in the presence of negative dimension operators. It corresponds to a redefinition of the ground state and of the generator which annihilates it. The "old" vacuum ($h=0$, $\bar{h}=0$) of the $c(p,p')$ theory is indeed not the state of minimal "energy" $L_0+\bar{L}_0$, and the true ground state is $|h=h_0,\ \bar{h}=h_0\rangle$.

In the $c=1$ theory, all the conformal dimensions are non negative. If some physical operator is assigned the dimension h in the $c(p,p')$ conformal theory, it is assigned the dimension $h'=h-h_0$ in the $c=1$ theory. These distinct conformal dimensions describe the power law behaviour of their correlation functions in the old and the new ground state, respectively.

This correspondence also implies a certain reshuffling of the primary fields (or heighest weight states). What used to be considered as a secondary field may be promoted to the status of primary field and vice versa. For example, in the $c(p,p')$ theory, the state $h=\bar{h}=0$ has no level one descendant. The contribution $h=\bar{h}=1$ to the partition function is regarded as coming from an independent primary field, whereas in the $c=1$ picture, it is interpreted as the contribution of a level-one descendant of $h=\bar{h}=-h_0$. This whole discussion parallels the construction of Dotsenko and Fateev [13].

In the next section, we show that the resulting $c=1$ partition functions have a simple and nice interpretation in terms of a Coulomb gas.

3. Free field or Coulomb gas representation

We now show that the modular invariants of the previous section may be interpreted as partition functions on a torus of Coulomb systems or free field with some kind of frustrations.

We first recall [21–23] that a two-dimensional Coulomb gas is a lattice system with electric and magnetic charges e_j, m_j (not necessarily integers) whose interaction is described by the action

$$A = \tfrac{1}{2}\mathrm{Re}\sum_{j\neq k}\left(\frac{e_j}{\sqrt{g}} + im_j\sqrt{g}\right)\mathscr{G}(\boldsymbol{R}_j - \boldsymbol{R}_k)\left(\frac{e_k}{\sqrt{g}} - im_k\sqrt{g}\right) + \sum_j \log X(e_j, m_j). \tag{3.1}$$

g is a coupling constant, $\mathscr{G}(\boldsymbol{r})$ is the lattice propagator $\mathscr{G}(\boldsymbol{R}) = \mathscr{G}_{\mathrm{R}}(\boldsymbol{R}) + i\mathscr{G}_{\mathrm{I}}(\boldsymbol{R})$ satisfying

$$\begin{cases} \Delta\mathscr{G}_{\mathrm{R}} = 2\pi\delta_{\boldsymbol{R},0} \\ \nabla_x\mathscr{G}_{\mathrm{I}} = \nabla_y\mathscr{G}_{\mathrm{R}}\,, \qquad \nabla_y\mathscr{G}_{\mathrm{R}} = -\nabla_x\mathscr{G}_{\mathrm{I}}\,, \end{cases} \tag{3.2}$$

which behaves at large separations as $\mathscr{G}(\boldsymbol{R}) \sim \ln(x+iy)$ and $X(e_j, m_j)$ are fugacities controlling the density of charges. Most two-dimensional statistical models are known [21–23] to renormalize at their critical point on the vacuum of (3.1) i.e. $X(e_j, m_j) = \delta_{e_j,0}\delta_{m_j,0}$. In this case, the various fields of charge (e, m) in the theory have scaling dimension and spin

$$\begin{cases} h + \bar{h} = \dfrac{e^2}{2g} + \dfrac{gm^2}{2} \\ h - \bar{h} = em\,, \end{cases} \tag{3.3}$$

i.e.

$$h = \frac{1}{4}\left(\frac{e}{\sqrt{g}} + m\sqrt{g}\right)^2, \qquad \bar{h} = \frac{1}{4}\left(\frac{e}{\sqrt{g}} - m\sqrt{g}\right)^2. \tag{3.4}$$

The "electric" fields $(e, m = 0)$ may be represented in a continuum theory as $\exp[ie\varphi(\boldsymbol{r})]$, φ being a gaussian free field with propagator $\langle\varphi(\boldsymbol{r})\varphi(\boldsymbol{r}')\rangle = -(1/g)\log|\boldsymbol{r} - \boldsymbol{r}'|$ derived from the action

$$A = \frac{-g}{4\pi}\int|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}y\,. \tag{3.5}$$

The "magnetic" fields $(e = 0, m)$ are disorder operators which create defects. Their correlation function is obtained [26] by imposing a discontinuity of $2\pi m$ of φ when one crosses a line connecting $\boldsymbol{r}$ to $\boldsymbol{r}'$. At any rate, the gaussian $c=1$ conformal theory is the natural framework for the Coulomb system.

Any modular invariant partition function $Z^{(\alpha,\mu)}(q, M)$ (cf. (2.3)–(2.5)) may be represented in terms of Coulomb gas conformal dimensions. First, we recall that if $\alpha \neq 1$:

$$Z^{(\alpha,\mu)}(q, M) = Z^{(1,\mu)}(q, M/\alpha^2), \tag{3.6}$$

with $\mu^2 = 1 \bmod(2M/\alpha^2)$. As at this stage, M is an arbitrary (even) integer, there is no loss of generality to assume that $\alpha = 1$. We then recall that the number μ stems from the factorization of $M/2$ into two coprimes P and P' (cf. eq. (2.4)). By definition, $Z^{(1,\mu)}$ is the sum:

$$Z^{(1,\mu)}(q, M) = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{\lambda=0}^{M-1} \sum_{n,\bar{n}=-\infty}^{\infty} q^{(Mn+\lambda)^2/2M}\, \bar{q}^{(M\bar{n}+\mu\lambda)^2/2M}. \tag{3.7}$$

One can then prove that

$$\begin{aligned} Z^{(1,\mu)}(q, M) &= \frac{1}{\eta(q)\eta(\bar{q})} \sum_{e,m=-\infty}^{\infty} q^{(Pe+P'm)^2/4PP'}\, \bar{q}^{(Pe-P'm)^2/4PP'} \\ &= \frac{1}{\eta(q)\eta(\bar{q})} \sum_{h,\bar{h}} q^h \bar{q}^{\bar{h}}, \end{aligned} \tag{3.8}$$

where the later sum runs over all h and $\bar{h}$ of the form

$$\begin{aligned} h &= \frac{1}{4}\left(\sqrt{\frac{P}{P'}}\, e + \sqrt{\frac{P'}{P}}\, m\right)^2, \\ \bar{h} &= \frac{1}{4}\left(\sqrt{\frac{P}{P'}}\, e - \sqrt{\frac{P'}{P}}\, m\right)^2, \end{aligned} \tag{3.9}$$

to be compared with (3.4). The derivation of (3.8) is straightforward. One shows that any term of (3.7) appears once in (3.8) and vice versa.

The $Z^{(1,\mu)}$ partition function may thus be interpreted (in a sense to be defined below) as the partition function of a Coulomb gas with coupling:

$$g = \frac{P'}{P} \qquad \text{or} \qquad g = \frac{P}{P'}. \tag{3.10}$$

Conversely, we can now introduce for g *arbitrary*

$$Z_c[g,1] = Z_c[1/g,1] = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{e,m\in\mathbf{Z}} q^{(e/\sqrt{g}+m\sqrt{g})^2/4} \bar{q}^{(e/\sqrt{g}-m\sqrt{g})^2/4}$$

$$= \frac{1}{\eta(q)\eta(\bar{q})} \sum_{e,m\in\mathbf{Z}} \exp\left(-2\pi\tau_I\left(\frac{e^2}{2g}+\frac{m^2 g}{2}\right)+2i\pi\tau_R em\right), \tag{3.11}$$

where we have written $q = e^{2i\pi\tau}$, $\tau = \tau_R + i\tau_I$. In particular for g rational $g = P/P'$, $Z_c[g,1] = Z^{(1,\mu)}(q, M)$. By means of a Poisson summation formula on both e and m it is easy to check that this expression is modular invariant; g being fixed one can also consider for arbitrary f

$$Z_c[g,f] = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{\substack{e\in\mathbf{Z}/f\\ m\in f\mathbf{Z}}} q^{(e/\sqrt{g}+m\sqrt{g})^2/4} \bar{q}^{(e/\sqrt{g}-m\sqrt{g})^2/4}, \tag{3.12}$$

which is equal to $Z_c[f^2 g, 1]$. Note that the *minimal* partition functions, which as noted in (2.22) are linear combinations of $Z^{(\alpha,\mu)}$, may be represented as combinations of coulombic partition functions. For example the diagonal modular invariant $((A_{p'-1}, A_{p-1})$ in the notations of [7]) which describes in particular for $p' = 3$, $p = 4$ the Ising model reads

$$2Z_{\min} = Z^{(1,1)}(q,N) - Z^{(1,\omega_0)}(q,N)$$

$$= Z_c\left[\frac{p}{p'}, p'\right] - Z_c\left[\frac{p}{p'}, 1\right], \tag{3.13}$$

the value $g = p/p'$ being naturally associated to $c(p, p')$ (see below sect. 4). We shall not pursue here the analysis of this queer relationship.

We want now to give a physical interpretation of (3.11). To this effect we perform again a Poisson summation, but only on the variable e. One finds

$$Z_c[g,1] = \sum_{m,m'\in\mathbf{Z}} \frac{\sqrt{g}}{\tau_I^{1/2}\eta(q)\eta(\bar{q})} \exp\left(-\pi g \frac{m'^2 + m^2(\tau_R^2+\tau_I^2) - 2\tau_R m m'}{\tau_I}\right). \tag{3.14}$$

The prefactor in front of the exponential is recognized as the partition function Z_1 of a free field [5] (3.5) and each term $Z_{m',m}$ in the sum corresponds to the introduction of two lines of defects. Indeed, suppose we demand that φ in (3.5) has a discontinuity of $2\pi m'$ $(2\pi m)$ when one crosses the geodesics ω_1, ω_2. For simplicity take $\omega_1 = L$, $\omega_2 = iT$. Then the classical solution which satisfies these constraints and $\Delta\varphi = 0$ is

$$\varphi_{\text{class}} = 2\pi\left(m'\frac{y}{T} + m\frac{x}{L}\right) \tag{3.15}$$

and the frustrated partition function

$$Z_{m',m} = \int[\mathrm{D}\varphi]\exp\left(-\frac{g}{4\pi}\int|\nabla(\varphi + \varphi_{\text{class}})|^2\,\mathrm{d}x\,\mathrm{d}y\right), \tag{3.16}$$

where φ is now doubly periodic, is easily obtained

$$Z_{m',m} = Z_1\exp\left(-\pi g TL\left(\frac{m'^2}{T^2} + \frac{m^2}{L^2}\right)\right) \tag{3.17}$$

in agreement with (3.14) for $\tau_{\mathrm{R}} = 0$, $\tau_{\mathrm{I}} = T/L$. The result remains of course valid for any τ. $Z_{m',m}$ is not modular invariant, with transformation laws

$$\begin{aligned} Z_{m',m}(-1/\tau) &= Z_{-m,m'}(\tau), \\ Z_{m',m}(1+\tau) &= Z_{m'+m,m}(\tau), \end{aligned} \tag{3.18}$$

which can be easily recovered from the above interpretation. Summation over m'; m gives $Z_{\mathrm{c}}[g,1]$, describing a free field with an arbitrary discontinuity multiple of 2π across ω_1, ω_2. It is modular invariant and thus remains unchanged under a redefinition of ω_1, ω_2. In the same way, $Z_{\mathrm{c}}[g,f]$ corresponds to discontinuities multiple of $2\pi f$. Returning to a Coulomb gas picture, the charge content of such a model on a torus is clear. From the definition of correlation functions of magnetic charges given above, it follows that they are quantized as integer multiples of f: $m \in f\mathbb{Z}$, and thus $e \in \mathbb{Z}/f$, in order to have integer spin.

4. Application to some statistical mechanics models

In this section, we interpret the partition functions Z_{c} (3.12) in terms of Q-state Potts or $\mathrm{O}(n)$ models.

Let us recall some basic results. The Q-state Potts model [24] is first defined for $Q \in \mathbb{N}^*$ by the action

$$A = \frac{1}{T}\sum_{\langle j,k\rangle}\delta_{\sigma_j\sigma_k}, \tag{4.1}$$

where T is the temperature, the sum is restricted to nearest neighbors of a regular lattice (we shall consider for definiteness the square lattice) and $\sigma_j = 1,\ldots,Q$. By

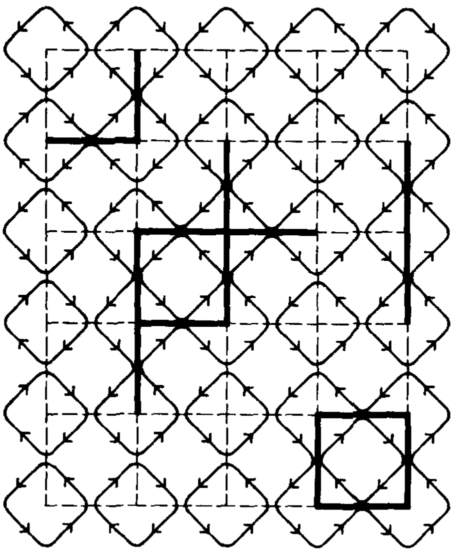

Fig. 1. A typical graph in the high temperature expansion of $\mathfrak{Z}_Q$ (4.2) and its alternative polygon representation. After arbitrary orientation, the polygons are considered as walls between regions of constant height in a solid-on-solid model.

high temperature expansion of the partition function one gets the Whitney polynomial [24]

$$\mathfrak{Z}_Q = \sum_{\{\sigma\}} e^{A} = \sum_{\text{graphs}} (e^{1/T} - 1)^{\mathcal{N}_B} Q^{\mathcal{N}_C} . \tag{4.2}$$

The sum is taken over all distributions of bonds on the edges of the lattice, $\mathcal{N}_B$ being the number of bonds and $\mathcal{N}_C$ the number of clusters (connected components) in a given graph. (4.2) defines now a model for $Q \in \mathbb{R}$; the case $Q \to 1$ corresponds to the percolation problem [27]. A graph in (4.2) can also be represented by a polygon decomposition [28] of the "surrounding lattice" (see fig. 1). Use of Euler's relation gives then

$$\mathfrak{Z}_Q = Q^{\mathcal{N}_s/2} \sum_{\text{graphs}} \left[(e^{1/T} - 1) Q^{-1/2} \right]^{\mathcal{N}_B} Q^{\mathcal{N}_P/2} , \tag{4.3}$$

where $\mathcal{N}_s$ is the total number of sites and $\mathcal{N}_p$ the number of polygons. If $Q \in [0, 4]$, there is a second order phase transition at the self-dual point [24]

$$(e^{1/T_c} - 1) Q^{-1/2} = 1 . \tag{4.4}$$

This can be studied by mapping onto the Coulomb gas, after $\mathfrak{Z}_Q$ is interpreted [23] as the partition function of a solid on solid (SOS) model. For this purpose one introduces discrete height variables φ on faces and vertices of the surrounding lattice and one considers a polygon in the representation (4.3) – after arbitrary orientation – as a wall between two regions of constant height. By convention, the corresponding φ difference is taken to be $\frac{1}{2}\pi$, the highest φ being on the left of each arrow. The Boltzmann weight of a given SOS configuration is obtained by a product of phase factors e^{iu} (e^{-iu}) for left (right) corner of a wall. Since the difference between the total number of left and right turns on a polygon is ± 4 for the square lattice, one has (at criticality) $\mathfrak{Z}_Q = Q^{\mathcal{N}_s/2}\mathfrak{Z}_{SOS}$ provided $Q^{1/2} = 2\cos 4u$. The SOS model renormalizes on (3.1) with a coupling constant given by

$$Q = 2 + 2\cos\tfrac{1}{2}\pi g, \qquad g \in [2,4]. \tag{4.5}$$

One can follow the same procedure for the O(n) model defined initially [25] for $n \in \mathbb{N}^*$ by

$$\mathfrak{Z}_n = \int \prod_i \mathrm{d}\boldsymbol{S}_i \prod_{\langle j,k\rangle} \left(1 + \frac{1}{T}\boldsymbol{S}_j \cdot \boldsymbol{S}_k\right), \tag{4.6}$$

where $\boldsymbol{S}$ is a n-component spin, $|\boldsymbol{S}|^2 = n$. By high temperature expansion one gets on the hexagonal lattice

$$\mathfrak{Z}_n = \sum_{\text{graphs}} (1/T)^{\mathcal{N}_B} n^{\mathcal{N}_P}. \tag{4.7}$$

The sum is over all configurations of non-intersecting self-avoiding polygons (fig. 2), $\mathcal{N}_P$ being the number of polygons and $\mathcal{N}_B$ the total number of bonds in a given graph. (4.7) can then be considered for $n \in \mathbb{R}$, the case $n \to 0$ corresponding to the polymer problem [29]. One transforms (4.7) into a SOS model [25] by introducing

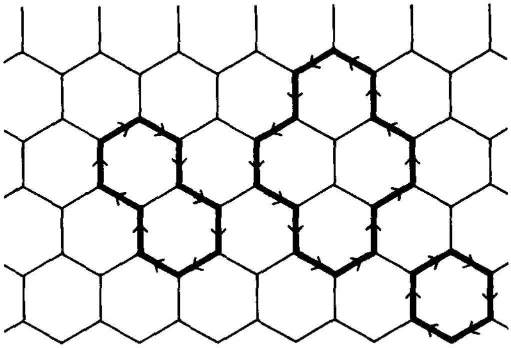

Fig. 2. A graph in the high temperature expansion of $\mathfrak{Z}_n$ (eq. (4.7)).

height variables φ on the centers of the hexagons, a loop being a wall of step π between two regions of constant height. The Boltzmann weight is obtained by a factor $1/T$ for each bond, times e^{iv} (e^{-iv}) for each left (right) turn. Then $\mathcal{Z}_n = \mathcal{Z}_{\mathrm{SOS}}$ provided $n = 2\cos 6v$. If $n \in [-2,2]$ there is a second order phase transition for [25]

$$\frac{1}{T_c} = \left[2 + (2-n)^{1/2}\right]^{-1/2}, \tag{4.8}$$

where the model renormalizes on (3.1) with coupling constant given by [25]

$$n = -2\cos \pi g, \qquad g \in [1,2]. \tag{4.9}$$

An important property is that (4.7) is also critical for $T < T_c$ (this may be regarded [30] as the two-dimensional analogue of Goldstone singularities), the corresponding g being another branch of (4.9), $g \in [0,1]$.

So far, we have assumed implicitly that we worked with free boundary conditions. Suppose now we consider (4.3) or (4.7) on a strip with transverse periodic boundary conditions. Then, in the associated SOS models defined above, a polygon which wraps around the cylinder has a weight 2 instead of $2\cos 4u$ ($2\cos 6v$) because the numbers of left and right turns are equal. The correspondence between $\mathcal{Z}_Q$ ($\mathcal{Z}_n$) and $\mathcal{Z}_{\mathrm{SOS}}$ is thus no more valid. As suggested in [13, 15] this can be repaired by adding a pair of electric charges $\pm e_0$ at $\pm\infty$. These contribute to the partition function by $\exp[ie_0(\varphi_\infty - \varphi_{-\infty})]$. Each polygon wrapping around the cylinder modifies $(\varphi_\infty - \varphi_{-\infty})$ by an amount $+\frac{1}{2}\pi(\pm\pi)$, having thus a new weight $2\cos\frac{1}{2}e_0\pi$ ($2\cos e_0\pi$) while polygons homotopic to a point have a weight unchanged. One gets then the desired result if $e_0 = 8u/\pi$ ($6v/\pi$). The SOS model which renormalizes onto the gaussian free field has $c = 1$ while after the introduction of these charges one finds the behaviour of the free energy at small q corresponds to [13, 15]

$$c = 1 - \frac{6e_0^2}{g}, \tag{4.10}$$

which gives

$$c_Q = 1 - \frac{3(g-4)^2}{2g}, \tag{4.11}$$

$$c_n = 1 - \frac{6(g-1)^2}{g}, \tag{4.12}$$

the latter formula being valid for both $T = T_c$ and $T < T_c$ in (4.7). For instance O(n) models at T_c and g rational p/p' are associated with $c(p, p')$ theories. Percolation and polymers have both $c = 0$ [12].

However, since our partition functions Z_c have $c^{\text{eff}}=1$, they do not involve charges at infinity. Working now on a torus we will thus define a modified Potts model by

$$\tilde{\mathfrak{Z}}_Q^{(1/2)} = Q^{\mathcal{N}_s/2} \sum_{\text{graphs}} \left[(\mathrm{e}^{1/T}-1)Q^{-1/2}\right]^{\mathcal{N}_B} Q^{\mathcal{N}_P/2} 2^{\tilde{\mathcal{N}}_P}. \tag{4.13}$$

The sum is the same as in (4.3) but the $\tilde{\mathcal{N}}_p$ polygons non-homotopic to a point in a given graph have a weight 2 instead of $Q^{1/2}$. In the same way we replace (4.7) by

$$\tilde{\mathfrak{Z}}_n^{(1/2)} = \sum_{\text{graphs}} (1/T)^{\mathcal{N}_B} n^{\mathcal{N}_P} 2^{\tilde{\mathcal{N}}_P}. \tag{4.14}$$

The weights are now correctly reproduced by the associated SOS models. In a free surface model however, the different heights must be compatible: by describing a closed path and computing φ differences for each wall crossed one must find a total variation $\delta\varphi = 0$. Since in (4.13)–(4.14) the polygons wrapping around the torus can be arbitrarily oriented in two possible ways, $\tilde{\mathfrak{Z}}_Q$ ($\tilde{\mathfrak{Z}}_n$) describe in fact SOS models with defects of the type (3.15). For the Potts case, the variation $\delta\varphi$ obtained by describing a geodesic of the torus is a multiple of π because each wall corresponds to a step $\frac{1}{2}\pi$ and $\tilde{\mathcal{N}}_p$ is always even. For the O(n) case it is of the same form, $\tilde{\mathcal{N}}_p$ being now arbitrary but walls corresponding to steps π. Since these topological defects remain unrenormalized [15] we conclude that at criticality the continuum limit of $\tilde{\mathfrak{Z}}_Q^{(1/2)}$ or $\tilde{\mathfrak{Z}}_n^{(1/2)}$ is precisely $Z_c[g, \frac{1}{2}]$. In the same way, one can define other models $\tilde{\mathfrak{Z}}^{(f)}$ by taking only configurations where the variations $\delta\varphi$ are multiple of $2\pi f$ with a continuum limit $Z_c[g, f]$.

Some examples are of special interest. First consider $n \to 0$, $T = T_c$ which describes the self-avoiding walk (polymer) problem [29]. So far, most studies concerned exponents. The asymptotic variation of the radius of gyration of a polymer graph with the number of monomers $\mathcal{N}_B$

$$\langle\ |\boldsymbol{R}_{\text{gyr}}|^2 \rangle \sim \mathcal{N}_B^{2\nu} \tag{4.15}$$

defines ν (which is also the standard correlation length exponent for $n \to 0$) known [25] to be $\frac{3}{4}$. In the same way, the number of configurations for $\mathbb{L}$ chains of common length $\mathcal{N}_B/\mathbb{L}$ which connect two fixed points behaves as

$$\Omega_{\mathbb{L}} \sim \mu^{\mathcal{N}_B} \mathcal{N}_B^{\gamma_{\mathbb{L}}-1}, \tag{4.16}$$

μ being a non-universal connectivity constant ($\mu = T_c$) and $\gamma_{\mathbb{L}}$ a critical exponent. $\gamma_{\mathbb{L}}$ can be expressed by [31]

$$\gamma_{\mathbb{L}} = (2 - 2x_{\mathbb{L}})\nu - \mathbb{L}, \tag{4.17}$$

a)

b)

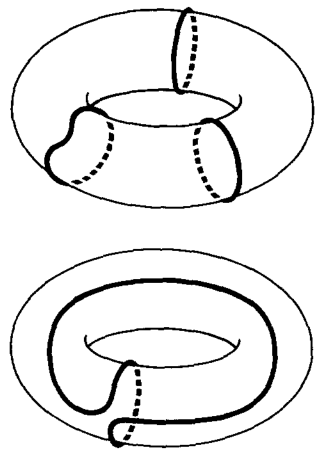

Fig. 3. The partition function (4.19) describes a grand canonical ensemble of non-intersecting self-avoiding loops on a torus which are non-homotopic to a point, such as in 3a or 3b. The weight of a configuration is $(1/\mu)^{\mathcal{N}_B} 2^{\tilde{\mathcal{N}}_P}$ where $\tilde{\mathcal{N}}_P$ is the number of these loops, $\mathcal{N}_B$ the total number of bonds (monomers) and μ the connectivity constant (4.16).

where $x_{\mathbb{L}}$ is the scaling dimension of some composite operator in the $O(n)$, $n \to 0$ model, and has been recently obtained [31]

$$x_{\mathbb{L}} = \tfrac{1}{48}(9\mathbb{L}^2 - 4). \tag{4.18}$$

The knowledge of the $x_{\mathbb{L}}$ gives then the gamma exponents of arbitrary networks [32] as well as contact exponents [33]. In addition to this we are now able to construct a non-trivial partition function for polymers on a torus in contrast to the standard $\mathfrak{Z}_{n=0} = 1$. Indeed

$$\tilde{\mathfrak{Z}}^{(1/2)}_{n=0} = \sum_{\text{graphs}} (1/\mu)^{\mathcal{N}_B} 2^{\tilde{\mathcal{N}}_P}, \tag{4.19}$$

which describes a grand canonical system of chains of variable length non-homotopic to a point (fig. 3) has the continuum limit

$$Z_c\left[\tfrac{3}{2}, \tfrac{1}{2}\right] = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{n_1, n_2} q^{(8n_1 + 3n_2)^2/96}\, \bar{q}^{(8n_1 - 3n_2)^2/96}, \tag{4.20}$$

which is also $Z^{(1,17)}(q, 48)$ in the notations of sect. 2. When developed on the characters of the $c = 0$ theory, (4.20) reproduces all the $x_{\mathbb{L}}$ in (4.18). It contains also new dimensions which describe the properties of chains with a fixed number of rotations around extremities [34]. Eq. (4.20) gives the spectrum of the transfer matrix numerically studied in [12].

In the same way we consider the case $Q \to 1$. It corresponds to the bond percolation problem in which bonds of the lattice are randomly occupied with a probability $p = 1 - e^{-1/T}$. At the threshold $p_c = 1 - e^{-1/T_c}$ an infinite cluster appears. Although $\mathfrak{Z}_{Q=1} = 1$ we can construct an interesting partition function

$$\tilde{\mathfrak{Z}}^{(1/2)}_{Q=1} = \sum_{\text{graphs}} \left(\frac{p_c}{1-p_c} \right)^{\mathcal{N}_B} 2^{\tilde{\mathcal{N}}_P}, \tag{4.21}$$

with continuum limit

$$Z_c\left[\tfrac{8}{3}, \tfrac{1}{2}\right] = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{n_1, n_2} q^{(3n_1+2n_2)^2/24}\, \bar{q}^{(3n_1-2n_2)^2/24}, \tag{4.22}$$

which is also $Z^{(1,5)}(q, 12)$. When developed on the characters for $c = 0$, it gives dimensions which describe various properties of perimeters (or "hull") of clusters [35].

Consider next the standard Ising model with action

$$A = \frac{1}{T} \sum_{\langle j,k \rangle} S_j S_k, \tag{4.23}$$

$S_j = \pm 1$. On the hexagonal lattice, the partition function $\mathfrak{Z}_{n=1}$ is the same as in (4.6) up to a change $\operatorname{th} 1/T \to 1/T$. With the usual boundary conditions for (4.23) on a torus one has at criticality $\mathfrak{Z}_{n=1} = Z_{\min}$ ($p = 4$, $p' = 3$) (3.13). If instead one modifies the weights for polygons of the high temperature expansion non-homotopic to a point as in (4.14) one can get for instance $\tilde{\mathfrak{Z}}^{(3)}_{n=1} = Z_c[\frac{4}{3}, 3] = Z^{(1,1)}(q, 24)$. Although these two partition functions give the same free energy in the thermodynamic limit, they have a very different expression on a torus, the expansion on $c = \frac{1}{2}$ characters involving only the minimal block (1.6) in the first case and the whole Kac table in the later one.

Finally, all the loops are treated in the same way in $\tilde{\mathfrak{Z}}^{(1/2)}_{n=2}$, namely with the weight $n = 2$. We thus expect $\mathfrak{Z}_{n=2} = \tilde{\mathfrak{Z}}^{(1/2)}_{n=2}$ to represent the Kosterlitz-Thouless point of the XY model:

$$\mathfrak{Z}_{\mathrm{KT}} = \mathfrak{Z}_{n=2} = Z_c[4, 1] = Z^{(1,1)}(q, 8),$$

in accordance with the well-known value $g = 4$ of the Coulomb coupling at this point [23].

On the other hand, the same kind of argument does not hold for the $Q = 4$ Potts model, since the transformation of (4.2) into (4.3) is not valid on a torus. The function $\tilde{\mathfrak{Z}}^{(1/2)}_{Q=4} = Z_c[1, 1]$ does *not* coïncide with $\mathfrak{Z}_{Q=4}$.

5. Conclusion

In this paper, we have constructed modular invariant partition functions for non-minimal $c < 1$ theories. They incorporate the contribution of an infinite number of primary fields, and may accommodate operators of dimension given by the Kac formula with non-integer indices. These partition functions have been seen to describe also $c = 1$ theories. This is naturally incorporated in their Coulomb or free field representation. The fact that minimal partition functions may also be represented as linear combinations of Coulomb partition functions is more surprising. Somehow, these results provide a relation between the conformal theory approach and the standard lore that most two-dimensional statistical mechanics models are expressed at criticality in terms of free fields. This connection seems different from the one mentioned in [5]. We do not know if our procedure exhausts all non-minimal $c \leqslant 1$ partition functions (for instance Q-state Potts or $O(n)$) models with loops weighted irrespective of their homotopy class).

Our whole discussion relies on the assumption that the partition function of the system is of the form (1.1). This is perfectly justified in unitary conformal theories that describe the continuum limit of some discrete statistical model, endowed with a positive transfer matrix. In non-unitary theories, and especially in cases where there is no natural definition of a transfer matrix, this assumption seems more questionable. One may imagine a different option, where the symbol "Tr" takes into account the signature of the states of the Hilbert space $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$

$$Z = \left(\mathrm{Tr}_{\mathscr{H}_+} - \mathrm{Tr}_{\mathscr{H}_-}\right)\left(q^{L_0 - c/24}\,\bar{q}^{\bar{L}_0 - c/24}\right).$$

For instance, the partition function of hamiltonian walks on the Manhattan lattice [36] which corresponds to the special $T = 0$ point in the low temperature critical phase of the $n = 0$ model [30] may be shown to have the continuum limit $Z_{\mathrm{Manh}} = Z_1^{-2}$. When expanded in power of $q, \bar{q}$, this leads to an expression with plus and minus signs, analogous to the formula above. This would lead to a totally different formalism, involving signed characters: clearly, more work is needed to explore this possibility.

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Appendix A

CHARACTERS OF THE VIRASORO ALGEBRA

Highest weight representations of the Virasoro algebra are constructed by repeated action of the generators L_n, $n < 0$ on a highest weight (h.w.) state $|h\rangle$ which satisfies by definition

$$L_0|h\rangle = h|h\rangle, \qquad L_n|h\rangle = 0 \qquad \forall n > 0. \tag{A.1}$$

The linear span of the states

$$|\psi\{\alpha\}\rangle = L_{-1}^{\alpha_1} \dots L_{-n}^{\alpha_n}|h\rangle \tag{A.2}$$

is called the Verma module of h.w. h and denoted V_h. All the states in V_h are eigenvectors of L_0 with integer spaced eigenvalues

$$L_0|\psi\{\alpha\}\rangle = (N+h)|\psi\{\alpha\}\rangle, \tag{A.3}$$

where $N = \Sigma k\alpha_k$ is the level of the state; the dimension of the subspace of level N is $p(N)$, the number of partitions of N.

Such a representation may not be irreducible, depending on the values of c and h. Reducibility (or "degeneracy") occurs when at some level N, there is a "singular" (or "null") vector $|\psi\rangle$, linear combination of $|\psi\{\alpha\}\rangle$, that satisfies (A.1). For c parametrized as

$$c = 1 - \frac{6}{x(x+1)} \tag{A.4}$$

(where x may be complex) a theorem [8, 14] states that this occurs whenever h may be written as

$$h = h_{rs} = \frac{[(x+1)r - xs]^2 - 1}{4x(x+1)}, \tag{A.5}$$

with r and s integers of the same sign. Moreover the smallest level N for which a singular vector $|\psi\rangle$ exists is the positive infimum of $r \cdot s$ over all pairs (r, s) satisfying (A.5). For example, in the case (1.4), the parameter x is rational, $x = p'/(p-p')$, and if there is a pair of integers (r, s), there is an infinite number of them since

$$h_{rs} = h_{r+p', s+p} = h_{-r,-s}. \tag{A.6}$$

The singular states and their descendants form one submodule V_{h_1}, or two distinct submodules $V_{h'_1}$ and $V_{h''_1}$, depending on the case [18]. The irreducible representation is obtained from V_h by factoring out all the singular states, i.e. by constructing the coset V_h/V_{h_1} or $V_h/(V_{h'_1} \oplus V_{h''_1})$. In the latter case, $V_{h'_1}$ and $V_{h''_1}$ have actually a non-empty intersection, which is again of the form $V_{h'_2} \oplus V_{h''_2}$, etc.

The character (1.3) of an irreducible representation of the Virasoro algebra is, up to the factor $q^{h-c/24}$, the generating function of the number of states at level N. When h is *not* of the form (A.5), this number is the number of partitions of N, hence

$$\chi_h(q) = \frac{q^{h-c/24}}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{A.7}$$

When h is a zero of Kac determinant (i.e. of the form (A.5)), and if all the singular states form a single submodule V_{h_1}, the irreducible character is simply:

$$\chi_h(q) = \frac{q^{h-c/24} - q^{h_1-c/24}}{\prod_1^{\infty}(1-q^n)}. \tag{A.8}$$

Finally, when there are two distinct submodules $V_{h_1'}$ and $V_{h_1''}$, the irreducible character reads:

$$\chi_h(q) = \left(q^h - q^{h_1'} - q^{h_1''} + q^{h_2'} + q^{h_2''} - \cdots\right)\frac{q^{h-c/24}}{\prod_1^{\infty}(1-q^n)}. \tag{A.9}$$

Let us first specialize to the case of representations of central charge

$$c = 1 - \frac{6(p-p')^2}{pp'}, \tag{A.10}$$

with p and p' two coprime integers. Then eq. (A.5) reads

$$h_{rs} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}. \tag{A.11}$$

If y is not integer, the representation of highest weight

$$h = \frac{y^2-(p-p')^2}{4pp'} \tag{A.12}$$

is certainly irreducible, and its character is of the form (A.7), or equivalently:

$$\chi_h(q) = \frac{q^{y^2/2N}}{\eta(q)}, \tag{A.13}$$

where

$$N = 2pp', \tag{A.14}$$

and

$$\eta(q) = q^{1/24}\prod_1^{\infty}(1-q^n) \tag{A.15}$$

is Dedekind's function.

On the other hand, consider values of h of the form (A.11). As already noticed in (A.6), there is some arbitrariness in (r, s). For any such pair of integers, define

$$\lambda = pr - p's\,. \tag{A.16}$$

Feigin and Fuchs [18] have shown that one is in the case (A.9) if $r \neq 0 \bmod p'$ and $s \neq 0 \bmod p$, in the case (A.8) otherwise.

Denoting r_0 and s_0 as two integers satisfying

$$1 \leqslant r_0 \leqslant p' - 1\,,$$
$$1 \leqslant s_0 \leqslant p - 1\,, \tag{A.17}$$

we may distinguish the following cases:

(i)
$$(r, s) = (kp', p)\,, \qquad k \geqslant 1 \text{ integer}\,,$$
$$\lambda = (k-1)pp'\,, \tag{A.18a}$$
$$\chi_h(q) = q^{\lambda^2/2N}\frac{1 - q^{kpp'}}{\eta(q)} = \frac{1}{\eta}\left\{q^{[(k-1)pp']^2/2N} - q^{[(k+1)pp']^2/2N}\right\},$$

(ii)
$$(r, s) = (r_0 + kp', p) = (p' - r_0, kp)\,, \qquad k \geqslant 0 \text{ integer}\,,$$
$$\lambda = (r_0 + (k-1)p')p\,, \tag{A.18b}$$
$$\chi_h(q) = q^{\lambda^2/2N}\frac{1 - q^{kp(p'-r_0)}}{\eta(q)} = \frac{1}{\eta}\left\{q^{[(k-1)pp' + pr_0]^2/2N} - q^{[(k+1)pp' - r_0 p]^2/2N}\right\},$$

(iii)
$$(r, s) = (kp', s_0)\,, \qquad k \geqslant 1 \text{ integer}\,,$$
$$\lambda = (kp - s_0)p'\,, \tag{A.18c}$$
$$\chi_h(q) = q^{\lambda^2/2N}\frac{1 - q^{kp's_0}}{\eta(q)} = \frac{1}{\eta}\left\{q^{[kpp' - p's_0]^2/2N} - q^{[kpp' + p's_0]^2/2N}\right\},$$

(iv)
$$(r, s) = (r_0 + kp', s_0)\,, \qquad k \geqslant 0 \text{ integer}\,,$$
$$\lambda = pr_0 - p's_0 + kpp'\,,$$
$$\tilde{\lambda} = pr_0 + p's_0 + kpp'\,, \tag{A.18d}$$
$$\chi_h(q) = \sum_{\substack{n \in \mathbb{Z} \\ n \notin [-k, -1]}} \frac{q^{(nN+\lambda)^2/2N} - q^{(nN+\tilde{\lambda})^2/2N}}{\eta(q)}\,.$$

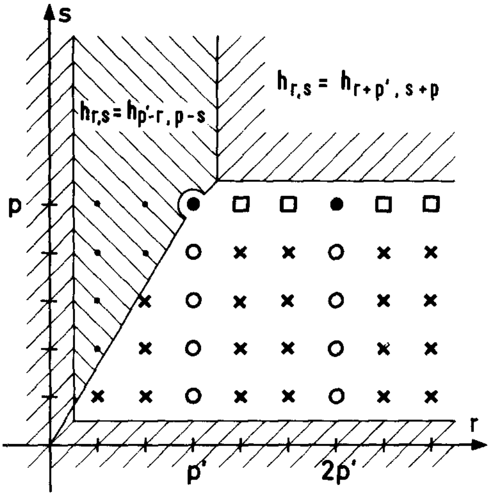

Fig. 4. Conformal grid: plot of the independent degenerate representations (here for $p'=3$, $p=5$). ● $(r,s)=(kp',p)$, □ $(r,s)=(r_0+kp',p)$, ○ $(r,s)=(kp',s_0)$, × $(r,s)=(r_0+kp',s_0)$, with $1\leqslant r_0\leqslant p'-1$, $1\leqslant s_0\leqslant p-1$, $k\geqslant 0$.

It is easy to see that these four cases exhaust all the values of the Kac table (A.11), thanks to the symmetries (A.6) (see fig. 4). The latter case includes of course the minimal characters χ_{r_0,s_0} [19,20]. Some of these formulas had already appeared [5].

For any value of the central charge $c<1$ which is not of the form (A.10), the coefficient x in (A.4) is irrational. Then if h is of the form (A.5), r and s are unique (up to a sign) and the character χ_h reads:

$$\chi_h = \frac{q^{[(x+1)r-xs]^2/4x(x+1)} - q^{[(x+1)r+xs]^2/4x(x+1)}}{\eta(q)}. \tag{A.19}$$

Next, we turn to the case $c=1$. This may be regarded as the $x\to\infty$ limit of the previous formulae, and accordingly, the only degenerate representations occur for

$$h=\tfrac{1}{4}n^2, \qquad n\in\mathbb{N}, \tag{A.20}$$

where n is related to the labels r,s of eq. (A.5) by $n=|r-s|$. The degeneracy occurs at level $\inf(r\cdot s)=n+1$ and the corresponding character reads:

$$\chi_h(q)=\frac{q^{n^2/4}-q^{(n+2)^2/4}}{\eta(q)}. \tag{A.21}$$

For completeness, we briefly discuss the case $1<c<25$. The parameter x in (A.4) has then a non-vanishing imaginary part. This gives rise in general to complex values of the Kac zeros. The only real zeros $h=h_{rs}$ occur for $r=s$ and are then negative:

$$h_{r,r}=\tfrac{1}{24}(1-c)(r^2-1)\leqslant 0\,, \tag{A.22}$$

and the corresponding character is

$$\begin{aligned}\chi_{r,r}&=\frac{q^{h_{r,r}-c/24}\left(1-q^{r^2}\right)}{\Pi(1-q^n)}\\&=\frac{1}{\eta(q)}\left[q^{r^2(1-c)/24}-q^{r^2(25-c)/24}\right].\end{aligned} \tag{A.23}$$

It is convenient to invert these equations for $c\leqslant 1$ and express $(1/\eta)q^{h-c/24}$ as a function of characters. This is trivial whenever h is not a zero of Kac determinant. When it is, one may check in all cases (A.18), (A.19), (A.21), that $(1/\eta)q^{h-(c-1)/24}$ may always be written as a (finite or infinite) sum of characters, with *positive integer* coefficients. For $c<1$, x rational, the results are as follows ($N=2pp'$):

$$n\geqslant 0 \qquad \frac{1}{\eta}q^{(nN)^2/2N}=\sum_{k=n}^{\infty}\chi_{(2k+1)p',p}\,, \tag{A.24a}$$

$$n\geqslant 0 \qquad \frac{1}{\eta}q^{(nN+pp')^2/2N}=\sum_{k=n}^{\infty}\chi_{2(k+1)p',p}\,, \tag{A.24b}$$

$$n>0 \qquad \frac{1}{\eta}q^{(nN-p's_0)^2/2N}=\sum_{k=n}^{\infty}\chi_{2kp',s_0}+\chi_{(2k+1)p',p-s_0}\,, \tag{A.24c}$$

$$n\geqslant 0 \qquad \frac{1}{\eta}q^{(nN+p's_0)^2/2N}=\sum_{k=n}^{\infty}\chi_{(2k+1)p',p-s_0}+\chi_{2(k+1)p',s_0}\,, \tag{A.24d}$$

$$n>0 \qquad \frac{1}{\eta}q^{(nN-pr_0)^2/2N}=\sum_{k=n}^{\infty}\chi_{(2k+1)p'-r_0,p}+\chi_{(2k+1)p'+r_0,p} \tag{A.24e}$$

$$n\geqslant 0 \qquad \frac{1}{\eta}q^{(nN+pr_0)^2/2N}=\sum_{k=n}^{\infty}\chi_{(2k+1)p'+r_0,p}+\chi_{(2k+3)p'-r_0,p}\,, \tag{A.24f}$$

$$\begin{aligned}n\geqslant 0 \qquad \frac{1}{\eta}q^{(nN-pr_0+p's_0)^2/2N}=&\sum_{k=n}^{\infty}\chi_{(2k+1)p'+r_0,s_0}+\chi_{(2k+1)p'-r_0,s_0}\\&+\chi_{(2k+1)p'+r_0,p-s_0}+\chi_{(2k+1)p'-r_0,p-s_0}\,,\end{aligned} \tag{A.24g}$$

$$\begin{aligned}n>0 \qquad \frac{1}{\eta}q^{(nN-pr_0-p's_0)^2/2N}=&\sum_{k=n}^{\infty}\chi_{2kp'+r_0,s_0}+\chi_{2kp'-r_0,s_0}\\&+\chi_{(2k+1)p'+r_0,p-s_0}+\chi_{(2k+1)p'-r_0,p-s_0}\,.\end{aligned} \tag{A.24h}$$

It is important to remember that in these expressions, r_0 and s_0 satisfy (A.17). In particular, notice that equation (A.24f) is *not* obtained from (A.24e) by changing r_0 into $-r_0$. Changing (r_0, s_0) into $(p'-r_0, p-s_0)$, however, is a licit operation. For $c<1$, x irrational, eq. (A.20) is easy to invert, because irrationality of x prevents $h+rs$ from being a zero of the Kac determinant:

$$\frac{q^{[(x+1)r-xs]^2/4x(x+1)}}{\eta} = \chi_{([(x+1)r-xs]^2-1)/4x(x+1)} + \chi_{([(x+1)r+xs]^2-1)/4x(x+1)} \,. \tag{A.25}$$

Finally, for $c=1$, inversion of (2.22) yields:

$$\frac{q^{n^2/4}}{\eta} = \sum_{m=0}^{\infty} \chi_{(n+2m)^2/4} \,. \tag{A.26}$$

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