Correlation Functions of the Critical Ising Model on a Torus

1988 Europhys. Lett. 5 95

(http://iopscience.iop.org/0295-5075/5/2/001)

EUROPHYSICS LETTERS 15 January 1988

Europhys. Lett., 5 (2), pp. 95-99 (1988)

Correlation Functions of the Critical Ising Model on a Torus.

P. DI FRANCESCO, H. SALEUR and J.-B. ZUBER

Service de Physique Théorique, CEN-Saclay - 91191 Gif-sur-Yvette Cedex, France

(received 30 September 1987; accepted 9 November 1987)

PACS. 02.40. – Geometry, differential geometry, and topology.
PACS. 64.60. – General studies of phase transitions.
PACS. 75.40D – Ising and other classical spin models.

Abstract. – Using different kinds of bosonization techniques, we derive expressions for the correlation functions of the critical Ising model on a torus. They are solutions of linear differential equations generalizing those written by Belavin, Polyakov and Zamolodchikov in the plane, involving derivatives with respect to the torus modular ratio. Two- and four-spin correlation functions provide an expression of the renormalized coupling constant M_4/M_2^2, in good agreement with numerical studies.

Much progress has been done recently in the study of 2D critical phenomena by using their conformal invariance [1-3]. For a very large class of systems, the so-called degenerate theories [1], all possible scaling dimensions are given by the Kac formula [4] in terms of a single number, the central charge c, and correlation functions on the plane satisfy linear differential equations which have been solved in some cases [5]. Study of 2D critical systems on a torus [6] is also of interest. For instance, the constraint of modular invariance of the partition function has allowed a systematic classification [7, 8] of all possible degenerate theories, determining their complete operator content.

In this letter, we address the question of the critical correlation functions on a torus. They must satisfy linear differential equations generalizing those of the plane [9], involving now derivatives with respect to the modular ratio. We consider the Ising model only, for which we determine *all* correlators with arbitrary numbers of spins, disorder and energy operators. This provides in principle the partition function in the whole critical domain [10]. As an application, we give an expression for the renormalized coupling constant M_4/M_2^2, in good agreement with lattice numerical studies [11]. A detailed version of this work will be published elsewhere [12].

We first recall that the Ising model is described in the vicinity of the critical temperature T_c by a Majorana fermion field with action [13]

$$\mathcal{A} = \frac{1}{2\pi} \int \mathrm{d}^2 x \, (\psi \partial_{\bar z} \psi + \bar\psi \, \partial_z \bar\psi + im \psi \bar\psi) \tag{1}$$

(where we use complex coordinates $z = x + iy$), the mass m being proportional to $T - T_c$. On a torus, periodic (P) or antiperiodic (A) boundary conditions (b.c.) must be assigned to $(\psi, \bar\psi)$

along the periods ω_1, ω_2, thus giving rise to four sectors labelled by $\nu = 1, \ldots, 4$ for PP, PA, AP, AA. The partition function has been evaluated [14, 10] and reads for $m = 0$

$$Z = \sum_{\nu=1}^{4} Z_\nu = \sum_{\nu=2}^{4} \frac{|\theta_\nu(0, \tau)|}{2\,|\eta(\tau)|}, \tag{2}$$

where the θ_ν are Jacobi theta-functions [15], η is Dedekind's function [14] and $\tau = \omega_2/\omega_1$ is the modular ratio (in the following we take $\omega_1 = 1$). The associated central charge is $c = 1/2$, and the fundamental (primary) operators [1] are the energy ε and the spin S, which describe the response of the model to a change of temperature or magnetic field. From (1), the energy operator is represented by $\varepsilon \equiv i\psi\bar{\psi}$, and thus energy correlation functions at T_c can be evaluated in a simple way. If $\nu \neq 1$, knowledge of the fermionic propagator $\langle\psi\psi\rangle_\nu$ [9, 14] gives

$$\langle \varepsilon(1)\,\varepsilon(2)\rangle_\nu = \left|\frac{\theta_1'(0)}{\theta_\nu(0)}\right|^2 \left|\frac{\theta_\nu(z_1 - z_2)}{\theta_1(z_1 - z_2)}\right|^2 \tag{3}$$

(where the τ-dependence of theta-functions is not written for brevity). Care must be taken with the first sector which has $Z_1 = 0$, but can nevertheless contribute to other quantities. For instance, in the case of $\langle\varepsilon\rangle$ one has $\langle\varepsilon\rangle_{\nu\neq 1} = 0$ by Wick's theorem, while the latter provides an undetermined form if $\nu = 1$. A calculation in the presence of a mass term gives [10] in the limit $m \to 0$

$$\langle\varepsilon\rangle = \frac{\sum_{\nu=1}^{4} Z_\nu \langle\varepsilon\rangle_\nu}{Z} = \frac{Z_1\langle\varepsilon\rangle_1}{Z} = \frac{2\pi|\eta|^3}{\sum_{\nu=2}^{4} |\theta_\nu(0)|}, \tag{4}$$

where *only* $\nu = 1$ contributes to the numerator. In the case of (3) one can use the short-distance expansion [1]

$$\varepsilon(1)\,\varepsilon(2) \sim |z_1 - z_2|^{-2} \sum \alpha_{N\bar{N}} A_{N\bar{N}}(z_1, \bar{z}_1)(z_1 - z_2)^N (\bar{z}_1 - \bar{z}_2)^{\bar{N}}, \tag{5}$$

where the operators $A_{N\bar{N}}$ belong to the conformal block of the identity. Since $Z_1\langle\mathbf{1}\rangle = Z_1 = 0$, repeated application of Ward's identity on the torus [9] gives $Z_1\langle A_{N\bar{N}}\rangle = 0$, and thus $Z_1\langle\varepsilon\varepsilon\rangle_1 = 0$. Then

$$\langle\varepsilon(1)\,\varepsilon(2)\rangle = \frac{\sum_{\nu=1}^{4} Z_\nu\langle\varepsilon\varepsilon\rangle_\nu}{Z} = \left|\frac{\theta_1'(0)}{\theta_1(z_1 - z_2)}\right|^2 \frac{\sum_{\nu=2}^{4} |\theta_\nu(z_1 - z_2)|^2/|\theta_\nu(0)|}{\sum_{\nu=2}^{4} |\theta_\nu(0)|}. \tag{6}$$

Correlation functions of $2n$ energies are evaluated in the same way, and give in each sector $\nu \neq 1$ the square modulus of a Pfaffian constructed out of fermionic propagators. $(2n+1)$-point functions are nonzero due to the $\nu = 1$ sector, as in (4). We discuss them later.

We turn now to spin correlation functions. The formulation (1) is not convenient here because, due to the nonlocality of the Jordan-Wigner transformation [13], S is not local in terms of ψ, $\bar{\psi}$. For calculations on the plane, various authors [16, 17] have considered instead a *duplicated* Ising model. The resulting $c = 2 \times 1/2 = 1$ theory involves a Dirac fermion field, and may be bosonized in terms of a Gaussian field ϕ [18]. *Squares* of spin correlation functions translate into correlation functions of exponentials of ϕ [17], and can be easily evaluated. An alternative (and perhaps more physical) method proceeds via the Ashkin-Teller (AT) model. The latter presents a critical line which renormalizes onto a Gaussian field theory [19], a special point of which corresponds simply to two noninteracting Ising models. The product $S_1 S_2$ is well known as the polarization operator [19], and represented by $\cos\phi$. At the decoupling point, squares of Ising functions will thus be obtained by considering correlators of $\cos\phi$.

Both approaches can be repeated on the torus where the b.c. must be treated carefully. The formal bosonization along the lines of ref. [16] supplemented by precise prescriptions on ϕ [20] gives results in *each* sector separately. For explicit computations we refer to [12]. It differs now from calculations within the AT approach. In the latter, the field turns out to be an angle identified with its opposite [21], *i.e.* lies on the orbifold $S^1/\mathbf{Z}_2$ of radius 1 (or 1/2 by duality). Accordingly, the functional integral splits into a sum of sectors in which the field ϕ undergoes a shift multiple of 2π across the torus: $\phi(z+1)=\phi(z)+2\pi m$, $\phi(z+\tau)=$ $=\phi(z)+2\pi m'$, and of three sectors where it changes sign along either or both periods: those are the twisted (T) sectors TP, PT, TT. The correlation function, therefore, reads

$$\langle S(1)\dots S(2n)\rangle^2 =$$

$$=\frac{\sum_{mm'} Z_{mm'}\langle\cos\phi(1)\dots\cos\phi(2n)\rangle_{mm'}+Z_{\mathrm{PT}}\langle\dots\rangle_{\mathrm{PT}}+Z_{\mathrm{TP}}\langle\dots\rangle_{\mathrm{TP}}+Z_{\mathrm{PP}}\langle\dots\rangle_{\mathrm{PP}}}{\sum_{mm'} Z_{mm'}+Z_{\mathrm{PT}}+Z_{\mathrm{TP}}+Z_{\mathrm{TT}}}. \tag{7}$$

The computation in a shifted sector (mm') is done by introducing the classical background satisfying the b.c. and using doubly periodic propagator [14] for the quantum fluctuations, whereas in each twisted sector one uses Wick's theorem with the appropriately antisymmetrized propagator. From (7) the *total* correlation function is thus obtained. The consistency of both approaches is instructive to check and involves various identities for the square modulus of theta-functions. We refer the reader to [12] for details and give only the first correlators here.

$$Z_\nu\langle S(1)S(2)\rangle_\nu=\frac{|\theta_1'(0)|^{1/4}}{2|\eta|}\,\frac{\left|\theta_\nu\left(\frac{z_1-z_2}{2}\right)\right|}{|\theta_1(z_1-z_2)|^{1/4}}, \tag{8}$$

$$Z_\nu\langle S(1)S(2)S(3)S(4)\rangle_\nu=\left\{\frac{|\theta_1'(0)|}{|\eta|^2}\left|\theta_\nu\left(\frac{z_1+z_2-z_3-z_4}{2}\right)\right|^2\cdot\right.$$

$$\left.\cdot\left|\frac{\theta_1(z_1-z_2)\,\theta_1(z_3-z_4)}{\theta_1(z_1-z_3)\,\theta_1(z_1-z_4)\,\theta_1(z_2-z_3)\,\theta_1(z_2-z_4)}\right|^{1/2}+\text{perm.}\right\}^{1/2}. \tag{9}$$

Contrary to (3), (8) is not periodic. Indeed, since S is not local in terms of ψ, $\tilde\psi$, shifting z_1-z_2 by 1, τ or $1+\tau$ modifies the b.c. and thus the sector. Only the total correlation function $\langle SS\rangle=\sum Z_\nu\langle SS\rangle_\nu/Z$ is periodic. The sector 1 contributes now, as expected from (4) and $SS\sim\varepsilon$.

One can as well consider disorder operators. Self-duality is broken on a torus, so $\langle\mu\mu\rangle$ is *a priori* different from $\langle SS\rangle$. There are, in fact, *four* disorder operator correlation functions, depending on the topology of the frustration line Γ. If Γ is homotopic to a point when $z_1\to z_2$ one has

$$\langle\mu\mu\rangle=\frac{-Z_1\langle SS\rangle_1+\sum_{\nu=2}^{4}Z_\nu\langle SS\rangle_\nu}{Z}. \tag{10}$$

The others are then deduced by modular transformations, and involve similar sums with one single minus sign. If Γ is homotopic to one period ω_1, ω_2 or $\omega_1+\omega_2$, as $z_1 \to z_2$, one has the short-distance behaviour $\langle \mu(1)\mu(2)\rangle \simeq |z_1-z_2|^{-1/4} Z^{\text{frust}}/Z$, where Z^{frust} is the frustrated Ising partition function [22] with antiperiodic b.c. for the spin along the corresponding period.

A remarkable property of eqs. (8), (9) is that they satisfy Eguchi and Ooguri's [9] differential equations expressing the degeneracy of S at level 2, and generalizing those written by BPZ in the plane [1]. One has, for instance [9],

$$\left\{\frac{4}{2}\partial_z^2 + [\zeta(z) - 2\eta_1 z]\,\partial_z - 2i\pi\,\partial_\tau - \frac{\eta_1}{4} - \frac{\mathscr{P}(z)}{16}\right\} Z_\nu \langle S(z,\bar z) S(0,0)\rangle_\nu = 0\,, \tag{11}$$

where η_1 is a standard constant of elliptic function theory, $\mathscr{P}$ and ζ are Weierstrass and zeta-functions [15].

Our method allows in fact to give the most general correlator. In the AT model at the decoupling point the product $\varepsilon_1\varepsilon_2$ of energies of the two Ising models translates into the marginal operator [19], *i.e.* $(\nabla\phi)^2$, while $\mu_1\mu_2$ corresponds to $\sin\phi$. Thus we have

$$\langle S(1)\ldots S(2n)\mu(2n+1)\ldots\mu(2n+2p)\,\varepsilon(2n+2p+1)\ldots\varepsilon(2n+2p+q)\rangle^2 =$$
$$= 2^{n+p}(-)^q\langle\cos\phi(1)\ldots\sin\phi(2n+1)\ldots[\nabla\phi(2n+2p+1)]^2\ldots\rangle\,, \tag{12}$$

where ϕ is a free-boson field with propagator behaving as $\langle\phi(1)\phi(2)\rangle = -1/2\log|z_1-z_2|$ at short distances. As above, the computation may be carried out either sector by sector, using the bosonization prescription, or for the full correlation function, as in eq. (7). The consistency between these various expressions, or with eq. (6) (for $n+p=0$) follows from identities between θ-functions. We refer the reader to [12] for details. For example, the explicit form of the 3-point function of the energy operator reads

$$\langle\varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\rangle = \frac{Z_1\langle\varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\rangle_1}{Z} = \frac{1}{Z}\left|\theta_1'''(0) - \theta_1'(0)\left[\left(\frac{\theta_1'}{\theta_1}\right)''(z_1-z_2) + \text{perm.}\right]\right|,$$

where, as argued above for $\langle\varepsilon\rangle$, only the first sector contributes.

Finally we present a numerical check of (8), (9). In [11] Burkhardt and Derrida have considered a lattice Ising model on squares $N\times N$, and calculated using a transfer matrix technique the first moments of the magnetization $M_{2n} = \langle(\sum S_i)^{2n}\rangle$. They have obtained, in particular, the renormalized coupling constants $V(N) = M_4/M_2^2$ for $N \leqslant 14$. In the limit $N\to\infty$, the values converge to a universal constant evaluated as $V = 1.1670 \pm 0.0015$. Now we can give an analytic expression of M_4 and M_2 using (8), (9), since

$$M_{2n} = \int_{\text{square}} d^2x_2\ldots d^2x_{2n}\,\langle S(1)\ldots S(2n)\rangle\,.$$

The integrals cannot be performed analytically, but we have estimated them using a Monte Carlo method. For 10 samples of 10^5 points we obtain $V = 1.168 \pm 0.005$ which is in good agreement with the above value. We have also checked the τ-dependence of M_2, M_4 against transfer matrix data.

REFERENCES

[1] BELAVIN A., POLYAKOV A. and ZAMOLODCHIKOV A., *Nucl. Phys. B*, **241** (1984) 333.
[2] FRIEDAN D., QIU Z. and SHENKER S., *Phys. Rev. Lett.*, **52** (1984) 1575.
[3] CARDY J., in *Conformal Invariance*, to appear in *Phase Transitions and Critical Phenomena*, edited by C. DOMB and J. L. LEBOWITZ (Academic Press, New York, N.Y.) 1987.
[4] KAC V. G., in *Group Theoretical Methods in Physics, Lecture Notes in Physics*, Vol. **94** (Springer-Verlag, New York, N.Y.) 1979.
[5] DOTSENKO V. S. and FATEEV V. A., *Nucl. Phys. B*, **240** [FS12] (1984) 312; **251** [FS13] (1985) 691.
[6] CARDY J., *Nucl. Phys. B*, **270** (1986) 186.
[7] CAPPELLI A., ITZYKSON C. and ZUBER J. B., *Nucl. Phys. B*, **280** (1987) 445.
[8] CAPPELLI A., ITZYKSON C. and ZUBER J. B., Preprint Saclay SPhT/87-059; KATO A., Tokyo preprint UT. 509.
[9] EGUCHI T. and OOGURI H., *Nucl. Phys. B*, **282** (1987) 127.
[10] SALEUR H. and ITZYKSON C., Preprint Saclay SPhT/87-001.
[11] BURKHARDT T. W. and DERRIDA B., *Phys. Rev. B*, **32** (1985) 7273.
[12] DI FRANCESCO P., SALEUR H. and ZUBER J. B., Preprint Saclay SPhT/87-097.
[13] SCHULTZ D., MATTIS D. C. and LIEB H., *Rev. Mod. Phys.*, **36** (1964) 856.
[14] ITZYKSON C. and ZUBER J. B., *Nucl. Phys. B*, **275** (1986) 580.
[15] BATEMAN PROJECT, *Higher Transcendental Functions*, Vol. **2** (Mac Graw Hill, New York, N. Y.) 1953.
[16] ZUBER J. B. and ITZYKSON C., *Phys. Rev. D*, **15** (1977) 2875.
[17] LUTHER A. and PESCHEL I., *Phys. Rev. B*, **12** (1975) 3908.
[18] COLEMAN S., *Phys. Rev. D*, **11** (1975) 2088.
[19] KADANOFF L. P. and BROWN A. C., *Ann. Phys.*, **121** (1979) 318.
[20] EGUCHI T. and OOGURI H., *Phys. Lett. B*, 187 (1987) 127.
[21] ELITZUR S., GROSS E., RABINOVICI E. and SEIBERG N., Weizmann preprint WIS.86/32; YANG S. K., Nordita preprint 87/3P; SALEUR H., Preprint Saclay SPhT/87-046.
[22] ZUBER J. B., *Phys. Lett.*, **176** (1986) 127; CARDY J., *Nucl. Phys. B*, **275** (1986) 200.