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# Conjugation properties of tensor product and fusion coefficients

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**Abstract** We review some recent results on properties of tensor product and fusion coefficients under complex conjugation of one of the factors. Some of these results have been proven, some others are conjectures awaiting a proof, one of them involving hitherto unnoticed observations on ordinary representation theory of finite simple groups of Lie type.

**Keywords** Lie groups · Lie algebras · Fusion categories · Conformal field theories · Quantum symmetries · Drinfeld doubles

**Mathematics Subject Classification** 81R50 · 81T40 · 20C99 · 18D10

## 1 Notations and results

### 1.1 Notations

In the following $\lambda$, $\mu$, etc. label either finite dimensional irreps of a simple Lie algebra $\mathfrak{g}$ or of the corresponding simply connected compact Lie group $G$; or (for the sake of comparison) irreps of a finite group $\Gamma$; or integrable irreps of an affine algebra $\widehat{\mathfrak{g}}_k$ at a finite integral level $k$.

Following a standard abuse of notations, for the Lie groups and algebras, $\lambda$ denotes both the highest weight of the representation and the representation itself.

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$N_{\lambda\mu}^{\nu}$ denotes, respectively, the coefficients of decomposition of the tensor product $\lambda \otimes \mu$ into inequivalent irreps $\nu$ (Littlewood–Richardson coefficients) of $G$ or $\Gamma$; or the coefficients of decomposition of the fusion product denoted $\lambda \star \mu$ into irreps $\nu$ of $\widehat{\mathfrak{g}}_k$.

It is often convenient to regard this set of coefficients as elements of (possibly infinite) matrices; thus,

$$N_{\lambda\mu}^{\nu} = (N_\lambda)_{\mu}^{\nu}. \tag{1}$$

These coefficients satisfy the sum rule

$$\dim_\lambda \dim_\mu = \sum_\nu N_{\lambda\mu}^{\nu} \dim_\nu \tag{2}$$

where $\dim_\alpha$ denotes the dimension, resp. the *quantum dimension*, of the irrep $\alpha$ of $G$, $\mathfrak{g}$ or $\Gamma$, resp. of $\widehat{\mathfrak{g}}$. When $\lambda$ refers to a representation of complex type, we denote by $\bar{\lambda}$ the (equivalence class of its) complex conjugate. Recall that among the simple Lie algebras, only those of type $A_r$, any $r, r > 1$; $D_r$, $r$ odd; $E_6$ admit complex representations.

For a given pair $(\lambda, \mu)$, consider the moments of the $N$'s

$$m_r := \sum_\nu (N_{\lambda\mu}^{\nu})^r \qquad r \in \mathbb{N}.$$

In particular, $m_0$ counts the number of distinct (i.e., non equivalent) $\nu$'s appearing in the decomposition of $\lambda \otimes \mu$, resp. $\lambda \star \mu$.

For non-real $\lambda$ and $\mu$, we want to compare $m_r$ and $\bar{m}_r := \sum_\nu (N_{\lambda\bar{\mu}}^{\nu})^r$.

Call $\mathfrak{P}$ the property that the *multisets* $\{N_{\lambda\mu}^{\nu}\}$ and $\{N_{\lambda\bar{\mu}}^{\nu'}\}$ are identical. Since for given $\lambda$ and $\mu$ these multisets are finite, there is an equivalence

$$m_r = \bar{m}_r \quad \forall r \in \mathbb{N} \quad \Leftrightarrow \quad \mathfrak{P}. \tag{3}$$

### 1.2 A list of results and open questions

We start with a fairly obvious statement

**Proposition 1** [3] *For any Lie group, any finite group or any affine Lie algebra, $m_2 = \bar{m}_2$, i.e., $\sum_\nu (N_{\lambda\mu}^{\nu})^2 = \sum_{\nu'} (N_{\lambda\bar{\mu}}^{\nu'})^2$.*

See below the (easy) proof in Sect. 1.3. Much more surprising is the following.

**Proposition 2** [1] *For any simple Lie algebra, or any affine simple Lie algebra, $m_1 = \bar{m}_1$, i.e., $\sum_\nu N_{\lambda\mu}^{\nu} = \sum_{\nu'} N_{\lambda\bar{\mu}}^{\nu'}$. This is not generally true for finite groups $\Gamma$. This is not generally true for quantum doubles of finite groups either* [2].

**Problem 1** *For a given finite group $\Gamma$, find a criterion on $(\Gamma, \lambda, \mu)$ for Proposition 2 to hold.*

**Table 1** ✓ means that the property is true and proven; X that it is not true in general and there are counterexamples; ✓ ? that the property has been checked in many cases (see text) but that a general proof is still missing

| | $SU(3)$ or $su(3)$ or $\widehat{su}(3)$ | $SU(4)$ or $su(4)$ or $\widehat{su}(4)$ | $G$ or $\mathfrak{g}$ other simple Lie group or Lie algebra | $\widehat{\mathfrak{g}}$ other affine Lie algebra | $\Gamma$ finite simple group of Lie type | $\Gamma$ other finite group |
|---|---|---|---|---|---|---|
| $m_2 = \bar{m_2}$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| $m_1 = \bar{m_1}$ | ✓ | ✓ | ✓ | ✓ | ✓ ? | X |
| $m_0 = \bar{m_0}$ | ✓ | ✓ ? | X | X | X | X |
| $m_r = \bar{m}_r\ \forall r \Leftrightarrow \mathfrak{P}$ | ✓ | X | X | X | X | X |

**Proposition 3** [3] *For the Lie group $SU(3)$, $m_r = \bar{m}_r$ for all $r$,* i.e., *we have property $\mathfrak{P}$. Moreover, we know a (non-canonical and non-unique) piece-wise linear bijection $(\nu, \alpha) \leftrightarrow (\nu', \alpha')$, where $\alpha$ is a multiplicity index running over $N_{\lambda\mu}^{\nu}$ values. This property $\mathfrak{P}$ is not true in general for higher rank $SU(N)$ nor for other Lie groups.*

**Proposition 4** [3] *For the affine algebra $\widehat{su}(3)$ at finite level $k$, $m_r = \bar{m}_r$ for all $r$,* i.e., *we have property $\mathfrak{P}$. This is not true in general for higher rank $\widehat{su}(N)$ or other affine algebras.*

This is, however, satisfied by low-level representations.

**Problem 2** *For each $\widehat{\mathfrak{g}}_k$, find a criterion on $(\lambda, \mu, k)$ for Proposition 4 to hold.*

Also missing in $\widehat{su}(3)$ is a general mapping $\nu \leftrightarrow \nu'$ compatible with the level. Although we found one in a few particular cases, a general expression is still missing.

**Problem 3** *For each level in $\widehat{su}(3)_k$, find a piece-wise linear bijection $\nu \leftrightarrow \nu'$.*

A weaker property than property $\mathfrak{P}$, which follows from it, is that $m_0 = \bar{m}_0$.

**Proposition 5** [3] *For the affine algebra $\widehat{su}(3)$ at finite level $k$, $m_0 = \bar{m}_0$. This seems to be also true for $\widehat{su}(4)$, but this is not true in general for higher rank $\widehat{su}(N)$, $N \geq 5$, or other affine algebras.*

This is, however, satisfied by low-level representations.

**Problem 4** *For each $\widehat{\mathfrak{g}}_k$, find a criterion on $(\lambda, \mu, k)$ for Proposition 5 to hold.*

These results on the equality of various $m_k$ and $\bar{m}_k$ are summarized in Table 1.

### 1.3 Comments, remarks, examples and counterexamples

- The equality $m_2 = \bar{m}_2$ is the easiest to interpret and to prove. More explicitly it asserts that

$$\sum_{\nu} (N_{\lambda\mu}^{\nu})^2 = \sum_{\nu'} (N_{\lambda\bar{\mu}}^{\nu'})^2 \,. \tag{4}$$

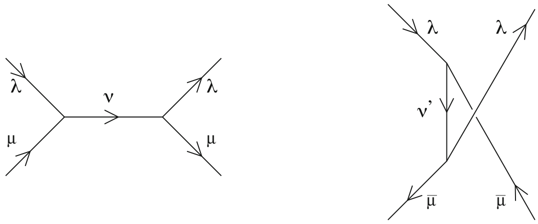


**Fig. 1** Graphical representation of $m_2 = \bar{m}_2$. Each $\lambda\mu\nu$ vertex carries the multiplicity $N_{\lambda\mu}^{\nu}$, and likewise for $\lambda\bar{\mu}\nu'$ on the *right*. Sums over $\nu$, respectively, $\nu'$ are equal

*Proof* The number of invariants $N^0_{\lambda\mu\bar{\lambda}\bar{\mu}}$ in $\lambda \otimes \mu \otimes \bar{\lambda} \otimes \bar{\mu}$ may be written as

$$
\begin{aligned}
N^0_{\lambda\mu\bar{\lambda}\bar{\mu}} &\overset{\text{(i)}}{=} \sum_{\nu,\nu'} N_{\lambda\mu}^{\nu} N_{\bar{\lambda}\bar{\mu}}^{\nu'} N^0_{\nu\nu'} \overset{\text{(ii)}}{=} \sum_{\nu,\nu'} N_{\lambda\mu}^{\nu} N_{\bar{\lambda}\bar{\mu}}^{\nu'} \delta_{\nu'\bar{\nu}} \overset{\text{(iii)}}{=} \sum_{\nu} N_{\lambda\mu}^{\nu} N_{\bar{\lambda}\bar{\mu}}^{\bar{\nu}} = \sum_{\nu} (N_{\lambda\mu}^{\nu})^2 \\
&\overset{\text{(iv)}}{=} N^0_{\lambda\bar{\mu}\bar{\lambda}\mu} = \sum_{\nu} N_{\lambda\bar{\mu}}^{\nu} N_{\bar{\lambda}\mu}^{\bar{\nu}} = \sum_{\nu} (N_{\lambda\bar{\mu}}^{\nu})^2
\end{aligned}
\tag{5}
$$

where we have made use of (i) associativity of the tensor or fusion product, (ii) $N^0_{\nu\nu'} = \delta_{\nu'\bar{\nu}}$, (iii) invariance under conjugation $N_{\bar{\lambda}\bar{\mu}}^{\bar{\nu}} = N_{\lambda\mu}^{\nu}$, and (iv) commutativity $N^0_{\lambda\mu\bar{\lambda}\bar{\mu}} = N^0_{\lambda\bar{\mu}\bar{\lambda}\mu}$.

Graphically, this may be represented as in Fig. 1. In physical terms, and in the context of particle physics, it expresses the fact that the numbers of independent amplitudes in the "$s$ channel" $\lambda \otimes \mu \to \lambda \otimes \mu$ and in the "crossed $u$ channel" $\lambda \otimes \bar{\mu} \to \lambda \otimes \bar{\mu}$ are the same.

- In contrast, the equality $m_1 = \bar{m}_1$ is neither natural nor general. While it is valid for all simple Lie algebras, either finite dimensional or affine, (see the discussion and elements of proofs in the next section), it is known not to be true for general finite groups. Counterexamples are provided by some finite subgroups of $SU(3)$, see below in Sect. 1.5, and also [1], and the detailed discussion in [2].
- Even more elusive and exceptional is the equality $m_0 = \bar{m}_0$, which happens to be true in $SU(3)$ or for the affine algebra $\widehat{su}(3)$, as a particular case of the more general property $\mathfrak{P}$ that they satisfy. Curiously we have found evidence (but no proof yet) that it also holds true for $SU(4)$ and $\widehat{su}(4)$ (this was tested in $\widehat{su}(4)_k$ up to level $k = 15$), but it fails in general for higher rank $SU(N)$ or $\widehat{su}(N)$.
- Finally the equality $m_r = \bar{m}_r$ for all $r$, or equivalently property $\mathfrak{P}$, is satisfied in $SU(3)$ [3] and in $\widehat{su}(3)$ at all levels [4].

*Example 1* In $SU(3)$, for the ten-dimensional representations,

$$
\begin{aligned}
(2,1) \otimes (2,1) &= 1(4,2) + 1(5,0) + 1(2,3) + 2(3,1) + 1(0,4) + 2(1,2) \\
&\quad +1(2,0) + 1(0,1) \\
(2,1) \otimes (1,2) &= 1(3,3) + 1(4,1) + 1(1,4) + 2(2,2) + 1(3,0) + 1(0,3) \\
&\quad +2(1,1) + 1(0,0)
\end{aligned}
\tag{6}
$$

on which we do observe all the above properties: $m_2 = \bar{m}_2 = 14$, $m_1 = \bar{m}_1 = 10$, $m_0 = \bar{m}_0 = 8$ and the multisets of multiplicities are both $\{1, 1, 1, 1, 1, 1, 2, 2\}$, or in short, $\{1^6 2^2\}$ (where we note the number $n$ of occurrences of multiplicity $m$ by $m^n$).

*Example 2* In $SU(4)$, with $\lambda = \mu = (1, 2, 2)$, we find for the multiplicities $N_{\lambda\mu}^{\nu}$ the multiset $\{1^{17} 2^8 3^9 4^8 5^1 6^3 7^3\}$ while for those for $\lambda \otimes \bar{\mu}$ it is $\{1^{16} 2^{12} 3^6 4^3 5^8 6^3 8^1\}$, whence $m_2 = \bar{m}_2 = 538$, $m_1 = \bar{m}_1 = 136$, $m_0 = \bar{m}_0 = 49$ but the multisets are clearly different.

*Example 3* In $SU(5)$, for $\lambda = (1, 1, 1, 0)$, $\mu = (1, 1, 0, 1)$, we find that the list of $N_{\lambda\mu}^{\nu}$ reads $\{1^{12} 2^6 3^3 4^3\}$ while that of $N_{\lambda\bar{\mu}}^{\nu'}$ reads $\{1^{15} 2^3 3^4 4^3\}$. We check that $m_2 = \bar{m}_2 = 111$ and $m_1 = \bar{m}_1 = 45$ but $m_0 = 24 \neq \bar{m}_0 = 25$.

*Example 4* In $SO(10)$ (Lie algebra $D_5$), with $\lambda = \mu = (1, 1, 0, 1, 0)$, the two multisets are, respectively, $\{1^{17} 2^{10} 3^3 4^8 5^6 6^3 7^2 8^2 12^1\}$ and $\{1^{15} 2^{11} 3^8 4^7 5^2 6^3 7^1 8^2 9^2 10^1\}$ from which we check that $m_2 = \bar{m}_2 = 840$, $m_1 = \bar{m}_1 = 168$, $m_0 = \bar{m}_0 = 52$ while the two multisets are manifestly different.

*Example 5* In $SO(10)$ (Lie algebra $D_5$), with $\lambda = \mu = (1, 1, 1, 1, 0)$, we find $m_1 = \bar{m}_1 = 4456$ and $m_2 = \bar{m}_2 = 184{,}216$ but $m_0 = 240$ and $\bar{m}_0 = 243$, hence a counterexample to the property of Proposition 5.

*Example 6* In $E_6$, likewise, we may find pairs of $\lambda$, $\mu$ which violate Propositions 4 and 5. Take $\lambda = \mu = (1, 1, 0, 0, 0; 1)$; $\bar{\mu} = (0, 0, 0, 1, 1; 1)$;[1] we find $m_1 = \bar{m}_1 = 947$, $m_2 = \bar{m}_2 = 14{,}163$ but $m_0 = 119$, $\bar{m}_0 = 123$. (Incidentally, the reducible representation encoded by $\lambda \otimes \mu$ in that case has dimension $63{,}631{,}071{,}504 = 252{,}252^2$.)

### 1.4 Related properties of the modular *S*-matrix

In the case of an affine algebra $\widehat{\mathfrak{g}}_k$, it is well known that the fusion coefficients are given by Verlinde formula [5]

$$N_{\lambda\mu}^{\nu} = \sum_{\kappa} \frac{S_{\lambda\kappa} S_{\mu\kappa} S_{\nu\kappa}^*}{S_{0\kappa}} . \tag{7}$$

**Proposition 6** [1] *For the affine algebra $\widehat{\mathfrak{g}}_k$ at finite level $k$, $\Sigma(\kappa) := \sum_{\nu} S_{\kappa\nu}$ vanishes if the irrep $\kappa$ is either of complex or of quaternionic type.*

For $\kappa$ complex, $\kappa \neq \bar{\kappa}$, this implies immediately Proposition 2, since, using the fact that $S_{\mu\bar{\kappa}} = S_{\bar{\mu}\kappa} = S_{\mu\kappa}^*$,

$$\sum_{\nu} N_{\lambda\mu}^{\nu} = \sum_{\kappa} \frac{S_{\lambda\kappa} S_{\mu\kappa} \sum_{\nu} S_{\nu\kappa}^*}{S_{0\kappa}} = \sum_{\nu, \kappa = \bar{\kappa}} \frac{S_{\lambda\kappa} S_{\mu\kappa} S_{\nu\kappa}^*}{S_{0\kappa}} = \sum_{\nu, \kappa = \bar{\kappa}} \frac{S_{\lambda\kappa} S_{\bar{\mu}\kappa} S_{\nu\kappa}^*}{S_{0\kappa}} = \sum_{\nu} N_{\lambda\bar{\mu}}^{\nu} . \tag{8}$$

[1] We use the common convention that the component of the vertex located on the short branch of the Dynkin diagram is written at the end.

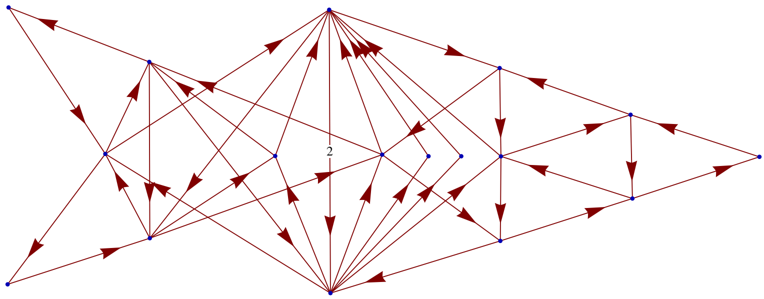


**Fig. 2** Tensor product graph for the subgroup $\Sigma(3 \times 360)$ of $SU(3)$. The middle vertical edge carries a multiplicity 2

But conversely, as shown in [1] by a fairly simple argument, Proposition 2 implies that $\Sigma(\kappa) = 0$ if $\kappa \neq \bar{\kappa}$.

The fact that the sum $\Sigma(\kappa)$ also vanishes for $\kappa$, a representation of quaternionic type, though of no direct relevance for the present discussion, is also a curious observation and was proved in [1] as the result of a case by case analysis.

### 1.5 The case of finite groups

Finite groups (admitting complex representations) do not generally satisfy Proposition 2. Consider for example the finite subgroup $\Gamma = \Sigma(3 \times 360)$ of $SU(3)$ [6,7]. A simple way to show that the equality of $m_1$ and $\bar{m}_1$ is not satisfied is to draw the oriented graph whose vertices are the irreps of $\Gamma$ and whose adjacency matrix is the matrix $N_{f\mu}^{\nu}$, where $f$ denotes one of the three-dimensional irreducible representations, see Fig. 2. In there, pairs of complex conjugate representations are images under a reflection through the horizontal axis. The sum $\sum_\nu N_{f\mu}^{\nu}$ counts the number of oriented edges exiting vertex $\mu$. It is clear that the sums relative to $\mu$ associated with the outmost upper and lower vertices are different.

Ultimately, we found among subgroups of $SU(3)$ the following counterexamples [1] to Proposition 2: $\Sigma(3\times 72)$, $\Sigma(3\times 360)$, and the subgroups of the type $F_{3m} = \mathbb{Z}_m \rtimes \mathbb{Z}_3$, where $m$ should be a prime of the type $6p + 1$.

Could the validity of Proposition 2 be related to the *modularity* of the tensor (or fusion) category, which holds true for Lie groups and affine algebras, but not generally for finite groups? In [2], we explored that possibility by constructing the Drinfeld doubles of subgroups of $SU(2)$ and $SU(3)$. While tensor product in Drinfeld doubles is known to be modular, we found again many counterexamples to Proposition 2, in particular for the double of the same group $\Sigma(3 \times 360)$. We conclude that the property encapsulated in Proposition 2 is not a modular property but rather seems to be a Lie theory property. See below in Sect. 2 a remark on the role of the Weyl group in the proof.

The validity of Proposition 2 is not directly related, either, to the simplicity of the group considered; indeed, the Mathieu groups $M_{11}$, $M_{12}$, $M_{21}$ $M_{22}$, $M_{23}$, $M_{24}$ are simple finite groups, but Proposition 2 is only valid for $M_{12}$ and $M_{21}$ (the latter, although simple, does not appear in the list of sporadic simple groups because it is isomorphic with $PSL(3,4)$).

We also considered those Chevalley groups that admit complex representations—otherwise Proposition 2 would be trivially verified. For small values of $n \geq 1$ and $q$ (a power of a prime) we looked at examples from the families $A_n(q) = SL(n+1,q)$, $B_n(q) = O(2n+1,q)$, $C_n(q) = Sp(2n,q)$, $D_n(q) = \Omega^+(2n,q)$, $G_2(q)$, and also from the families called $2A_n(q) = SU(n+1,q)$, $2B_n(q) = Sz(q)$ (Suzuki), $2D_n(q) = \Omega^-(2n,q)$, $2G_2(q)$ (Ree), $3D_4(q)$, with the notations used in MAGMA [8]. We could not explicitly study examples from the families $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, or $2F_4(q)$ (Ree), $2E_6(q)$, because of the size of their character table. The largest simple group of Lie type that we considered (and obeying Proposition 2) was $G_2(5)$, with 5,859,000,000 elements, 44 conjugacy classes (or irreps), and only four complex irreps. Altogether we tested about 70 Chevalley groups, 33 of them being simple, and 37 had complex irreps, so that testing the sum rule (Proposition 2) for them was meaningful. Among those 37 groups with complex irreps, 21 were simple and the sum rule was obeyed by all of them; among the $37 - 21 = 16$ non-simple groups with complex irreps, we found four cases for which the sum rule fails. In all cases where this sum rule failed for a non-simple Chevalley group, it turned out to hold for the corresponding projective group (a simple quotient of the latter): for instance the rule fails for the non-simple group $A_2(7) = SL(3,7)$ but it holds for the simple group $PSL(3,7)$ (and also holds for the non-isomorphic simple group $2A_2(7) = SU(3,7) = PSU(3,7)$). Although the obtained results may not be statistically significant they seem to indicate that Proposition 2 is valid for simple groups of Lie type. We did not try to prove this property but if it happens to be true, one may expect, for finite groups of Lie type, that the Weyl group could play a role in the proof, like in the case of simple Lie groups (see below).

## 2 A sketch of proofs

The proof of Proposition 1 has been given above. We shall content ourselves with sketches of proofs for the other propositions.

The proof of Proposition 2 may be split into two steps.

**Lemma 1** *Proposition 2 holds for $\lambda = \omega_p$, a fundamental weight.*

**Lemma 2** *Proposition 2 holds for any product of the fundamental representations.*

*Proof* The first lemma was established for the $A_r$, $D_{r\,\mathrm{odd}}$, $E_6$ simple Lie algebras (the others do not have complex irreps) making use of the Racah–Speiser formula, or of its affine extension. The latter expresses the tensor coefficient $N_{\lambda\mu}^{\nu}$ as a weighted sum over suitable elements of the (classical or affine) Weyl group, see [1] for details. Restricting $\lambda$ to be a fundamental weight makes the discussion amenable to a fairly simple analysis of a finite number of cases.

The second lemma follows simply from the associativity and commutativity of the tensor or fusion product, using the outcome of Lemma 1:

$$\sum_\nu (N_{\omega_p})_\mu^{\ \nu} = \sum_\nu (N_{\bar\omega_p})_\mu^{\ \nu} \ .$$

This, together with the commutativity of the $N$ matrices, entails that for any monomial $N_{\omega_{j_1}} \cdots N_{\omega_{j_q}}$

$$\begin{aligned}
\sum_\nu \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_q}}\right)_\mu^{\ \nu} &= \sum_{\nu'} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}}\right)_\mu^{\ \nu'} \sum_\nu \left(N_{\omega_{j_q}}\right)_{\nu'}^{\ \nu} \\
&\overset{\text{(Lemma 1)}}{=} \sum_{\nu'} \left(N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}}\right)_\mu^{\ \nu'} \sum_\nu \left(N_{\bar\omega_{j_q}}\right)_{\nu'}^{\ \nu} \\
&= \sum_\nu \left(N_{\bar\omega_{j_q}} N_{\omega_{j_1}} \cdots N_{\omega_{j_{q-1}}}\right)_\mu^{\ \nu} = \cdots \\
&= \sum_\nu \left(N_{\bar\omega_{j_1}} \cdots N_{\bar\omega_{j_q}}\right)_\mu^{\ \nu} \qquad (9)
\end{aligned}$$

This completes the proof of the two lemmas. As any $N_\lambda$ is a polynomial in the commuting $N_{\omega_p}$, $p = 1, \ldots, r$, $N_\lambda = P_\lambda(N_{\omega_1}, \ldots, N_{\omega_r})$ and $N_{\bar\lambda} = P_\lambda(N_{\bar\omega_1}, \ldots, N_{\bar\omega_r})$, this also establishes Proposition 2. □

The salient feature of this approach is the crucial role played by the (classical or affine) Weyl group.

Propositions 3 and 4, which deal with the explicit case of the classical or affine $su(3)$ algebra, have been established through a detailed and laborious analysis which will not be repeated here. We only mention that a variety of graphical representations of the determination of the $N_{\lambda\mu}^{\nu}$ coefficients has been used. We refer the reader to [3] and [4] for details.

## 3 Conclusion

In this letter, we have reviewed some recent results on conjugation properties of tensor product (or fusion) multiplicities. Although quite simple to state, it appears that these results were not previously known, and that some are fairly difficult to prove. In particular, we feel that our proofs of Propositions 2, 3 and 4 lack elegance and may miss some essential concept. Hopefully, some inspired reader will come with new insights into these matters.

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