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Maps, immersions and permutations

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ABSTRACT

We consider the problem of counting and of listing topologically inequivalent "planar" 4-valent maps with a single component and a given number n of vertices. This enables us to count and to tabulate immersions of a circle in a sphere (spherical curves), extending results by Arnold and followers. Different options, where the circle and/or the sphere are/is oriented are considered in turn, following Arnold's classification of the different types of symmetries. We also consider the case of *bicolorable* and bicolored maps or immersions, where *faces* are bicolored. Our method extends to immersions of a circle in a higher genus Riemann surface. There the bicolorability is no longer automatic and has to be assumed. We thus have two separate countings in nonzero genus, that of bicolorable maps and that of general maps. We use a classical method of encoding maps in terms of permutations, on which the constraints of "one-componentness" and of a given genus may be applied. Depending on the orientation issue and on the bicolorability assumption, permutations for a map with n vertices live in S_{4n} or in S_{2n}. In a nutshell, our method reduces to the counting (or listing) of orbits of certain subset of S_{4n} (respectively, S_{2n}), under the action of the centralizer of a certain element of S_{4n} (respectively, S_{2n}). This is achieved either by appealing to a formula by Frobenius or by a direct enumeration of these orbits. Applications to knot theory are briefly mentioned.

Keywords: Embedded graphs; knot diagrams; immersed curves; closed curves; topological maps.

Mathematics Subject Classification 2010: 05C10, 05C30, 57M25, 83C47

1. Introduction

In the present paper, we are interested in the problem of enumerating the (equivalence classes of) curves with n double points, that one can draw on the sphere or on an orientable surface of genus g. Such curves may be regarded as the images of immersions of a circle in that surface. In the present paper, we shall be mainly dealing with immersions in a compact surface. See Fig. 1 for an illustration of the difference between immersions in the plane or in the sphere. The problem is tightly connected with the census of *knots* and *virtual knots*, but we shall content ourselves with brief comments about knots. Recall that for nonzero genus, (virtual) knot diagrams or drawings of curves exhibit *virtual crossings* in addition to their regular crossings: the former may be regarded as artifacts, due to the projection on the plane of the figure, see [1] and the example of Fig. 2.

Since the curves (or the knot diagrams), like the surfaces themselves, can be oriented, the discussion and the results will naturally split into four cases (OO, UO, OU, UU), that we define now: a genus g curve is the image of the circle, under an immersion $S^1 \to \Sigma$, the latter being an orientable surface of genus g, and, if the curve is not simple (if it crosses itself), the multiple points of the immersion should be double points with distinct tangents (in other words, we consider generic closed curves). When $g = 0$, this is called a generic spherical curve. Both the circle and the surface can be oriented (see Figs. 3 and 4). If S^1 is not oriented, one may consider the sets UU and UO of $\mathrm{Diff}(\Sigma)$-equivalent and $\mathrm{Diff}^+(\Sigma)$-equivalent unoriented curves. If S^1 is oriented, one considers the sets OU and OO of $\mathrm{Diff}(\Sigma)$-equivalent and $\mathrm{Diff}^+(\Sigma)$-equivalent oriented curves. $\mathrm{Diff}^+(\Sigma)$ denotes the group of orientation-preserving diffeomorphisms of the oriented surface Σ. For spherical curves, these four types of immersions have been considered by previous authors [2–4]. Correspondingly, in knot theory, one may consider knots up to mirror symmetry, and oriented or unoriented.

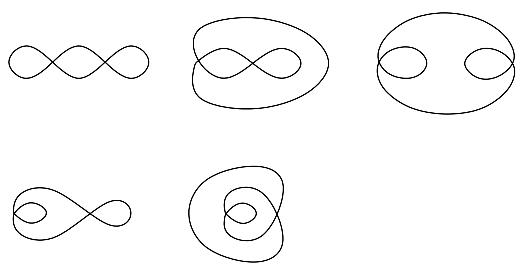


Fig. 1. Immersions of an unoriented circle with two double points. The five immersions in the plane give rise to two distinct immersions on the sphere, for instance the two lying on the left.

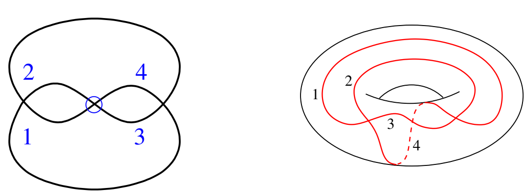


Fig. 2. The diagram on the left describes a genus 1 immersion and is *not* bicolorable. The little blue circle encircles a *virtual* crossing. On the right, the same is immersed in a torus.

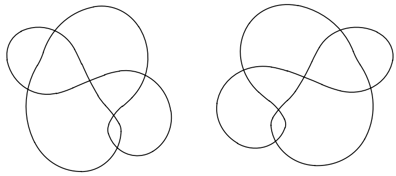


Fig. 3. Two immersions of an unoriented circle with $n = 6$ double points. Distinct on an oriented sphere, but equivalent on an unoriented sphere.

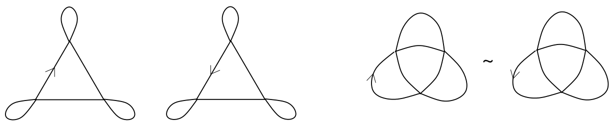


Fig. 4. Immersions of an oriented circle. Left: an $n = 3$ immersion not equivalent to its reverse; in contrast, the trefoil *is* equivalent to its reverse.

Following Carter [5], who coined this adjective in the UU case, one says that two immersed curves are OO, UO, OU or UU geotopic, if they are equivalent in the previous sense. Now it is clear that the operation of adding handles to a surface, in which a circle is immersed defines immersed curves in higher genus surfaces. It is therefore natural to consider the following definition [5]: two immersed curves are stably geotopic, if and only if there is a collection of handles, that can be added to either surface, or both, in such a way, that the curves become geotopic on the resulting surfaces. In this paper, we assume that the studied immersions are cellular, in the sense, that the complement of each associated immersed curve is homeomorphic to a collection of open disks, so that the classifications obtained in this paper, when $g > 0$ for the different kinds of immersions should always be understood up to stable geotopy (although this will not be in general repeated in the text). In other words, the genus given in our tables for an immersed curve with a given number of crossings is such that the chosen curve cannot be immersed in a surface of smaller genus. One could then use surgeries to obtain a classification for all generic immersed curves (see [6]).

On top of the question of orientation, we introduce the issue of *bicolorability* of the curve. By definition, a curve is bicolorable, if one can assign opposite colors to adjacent faces. While in genus 0, any self intersecting curve may be bicolored (with adjacent faces of opposite colors), it is no longer true for higher genus, see Fig. 2 for an example. Moreover, when a curve is bicolored, the two possible colorings may be or not (topologically) equivalent, see Fig. 5. This bicoloring is quite natural in the context of knot theory, where it amounts to considering the curve as an *alternating* knot, see Fig. 9 below; there are two ways of doing that, which may or not lead to equivalent knots. We shall thus append a suffix c, b or no suffix at all to the symbols OO, OU, etc.: OOc will refer to (inequivalent) bicolorings of immersions of an oriented circle in an oriented surface, OOb to bicolorable (but not bicolored)

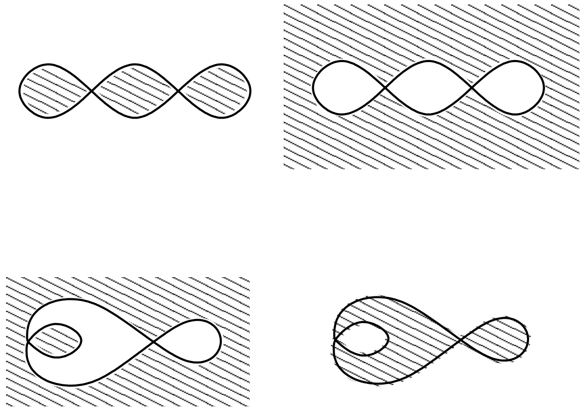


Fig. 5. Swapping colors: the two diagrams on the top are not equivalent, while the two diagrams on the bottom are (on the sphere, of course). The first two contribute 2 to |UUc| and 1 to |UUb|, the last two contribute 1 to both.

immersions, and OO alone to general, bicolorable or not, immersions. Likewise for immersions of type UO, OU and UU. This results in $3 \times 4 = 12$ different types of immersions, and the reader who is eager to see numbers may jump to Tables 8 and 9 to see their cardinals tabulated up to 10 crossings for all genera. The reader who prefers figures to numbers is directed to Figs. 15–17 for a complete list of indecomposable irreducible spherical curves of UU type, with respectively $n = 8$ and 9 crossings.

For $g > 0$ immersions they are few explicit results made available in the literature, see however[a] [7] and [8].

We shall regard curves with simple crossings (images of immersions) as 4-*valent maps*. Our immersions, being cellular, indeed define maps[b]: recall that a map is a graph embedded in a surface with its 2-cells (aka faces) homeomorphic to open disks. The fact that faces do not contain handles will be used repeatedly in this paper, in particular, when using the Euler formula to determine the genus of embedded curves. We should insist on the fact that, in this work, we consider circle immersions/maps, bicolorable or not: all the curves, that we consider have a single connected component (in the language of knot theory, we are interested in *knots*, not in *links*).

Matrix integrals in the large size limit, which are quite effective for the counting of maps of a given genus fail to distinguish maps with different numbers of components. We thus, use an alternative method regarding maps as *combinatorial*

[a]The latter reference (that we discovered at a later stage of our work) contains, for the UU case, a table of isomorphism classes of immersions, with given genus and a given number of crossings (up to five), obtained using Gauss diagrams and a method described in [5].
[b]They also define cellular embeddings of particular graphs called "simple assembly graphs without endpoints" in [9], see also [10].

maps, i.e. maps described by pairs of permutations, following an old idea by Walsh and Lehman [12], or some variants. The constraints of "one-componentness" and of fixed genus may be easily enforced in that description. Depending on the orientation issue and on the bicolorability assumption, permutations for a map with n vertices live in S_{4n} or in S_{2n}.

This method, however, yields labeled maps. To obtain unlabeled maps and immersions, a quotient by a relabeling group has to be performed. This is achieved by considering *orbits* of the combinatorial maps, under the action of some subgroup of the permutation group.

The set-up of the paper is as follows. In Sec. 2, we present the simplest version of the previous idea, where the two permutations encoding general immersions live in S_{4n}. The rapid growth of $(4n)!$ limits its practical use beyond $n = 6$. In Sec. 3, we consider bicolorable maps and introduce a better coding by pairs of permutations of S_{2n}. Orbits of these pairs under the action of the hyperoctahedral group yield immersions of type OOc. Section 4 is devoted to a study of the various types of bicolored or bicolorable immersions, that may be derived from the OO type. We derive some relations between the numbers of these different types (Theorem 4). In Sec. 5, we remove the assumption of bicolorability and encode the general maps and immersions by another choice for the pair of permutations of S_{2n}. Sections 6 and 7 gather results, comments on the asymptotia and on the application to knot theory and our conclusions. Appendix A gives some details on the algorithms used for counting orbits, Appendix B contains several tables of interest, that will be described later, and Appendix C reviews the connection between maps and Feynman diagrams of matrix or scalar integrals.

A notational comment: in the following, we make use of two notations for the cardinal of a set X, either $|X|$ or $\#X$.

2. UO Immersions, First Method using Permutations of S_{4n}

2.1. *The subset $X = [2^{2n}]$ of S_{4n} and its orbits ("X method")*

In the present section, we obtain the number of circle immersions of type UO, with n crossings, by counting the number of orbits of solutions for a particular set of equations written in the group S_{4n}, under the action of a particular subgroup. We shall actually recover part of these results later, with other methods, which are faster (see Secs. 3 and 5), but the technique presented here has an interest of its own.

Method: description of a curve by a permutation belonging to a particular conjugacy class X of S_{4n}

In a first stage, we consider a labeling of *half-edges* of the maps. For a 4-valent map with n vertices, there are $4n$ such half-edges, and we consider the symmetric group

S_{4n} acting on these labels. We choose $\sigma \in [4^n]$[c] to describe the clockwise linking pattern of half-edges at the vertices, and consider all possible pairings of half-edges (propagators in physicists' parlance) encoded in permutations $\tau \in [2^{2n}]$. Note that, this method of labeling half-edges is not original, it has been used by Walsh and Lehman [12] and rediscovered later by Drouffe, as quoted in [13].

Example of encoding
See below in Fig. 6, the map encoded by

$$\sigma = (1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)$$
$$\tau = (1,13)(2,5)(3,6)(4,16)(7,8)(9,12)(10,15)(11,14)$$

in cycle notation.

Orbits of $X = [2^{2n}]$ for the adjoint action of the centralizer of an element of $[4^n]$

Theorem 1. *Call* $\sigma = (1,2,3,4)(5,6,7,8)\ldots(4n-3,4n-2,4n-1,4n) \in [4^n] \subset S_{4n}$, *using cycle notation, and* $\mathcal{C}_\sigma = C(S_{4n},\sigma)$, *the centralizer of* σ *in* S_{4n}. *Let* $X = [2^{2n}]$ *denote the conjugacy class of* S_{4n}, *whose elements are products of* $2n$ *transpositions. Then, we have*:

Circle immersions of type UO, i.e. *immersions of the unoriented circle in an orientable and oriented surface of genus* g, *are in bijection with the orbits of* $\mathcal{C}_\sigma$ *acting by conjugation on the set of permutations* τ, *that belong to* X *and solve the simultaneous equations*:

$$\begin{array}{lll} \sigma^2\tau \in [(2n)^2] & \text{(I)} & \text{one-componentness} \\ c(\sigma\tau) = n+2-2g & \text{(II)}_g & \text{genus condition,} \end{array}$$

where $c(x)$ *is the function giving the number of cycles* (*including singletons*) *of the permutation* x.

Proof. That labeled maps are in one-to-one correspondance with pairs (σ,τ) has been known for long [12]. The sequence of labels as one goes across the crossings is described by the permutation $\sigma^2\tau$, and imposing condition (I) ensures that the curve

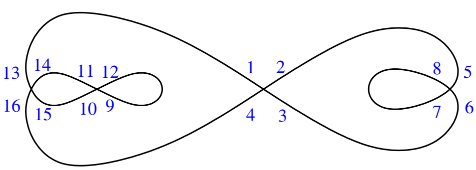


Fig. 6. The diagram encoded by $\tau = (1,13)(2,5)(3,6)(4,16)(7,8)(9,12)(10,15)(11,14)$.

[c]The notation $[\cdot]$ refers to the conjugacy classes of the permutation group.

has a single component (hence also that the graph is connected). Condition $(\mathrm{II})_g$ follows from Euler relation, if one realizes, that the number of faces of the map is just the number of cycles $c(\sigma\tau)$, (another observation made by many previous authors...). A change of labels by $\gamma \in S_{4n}$ acts on σ and τ by conjugation: $(\sigma, \tau) \to (\sigma^\gamma, \tau^\gamma)$, with $\alpha^\gamma := \gamma\alpha\gamma^{-1}$. The form of σ as well as conditions on permutations τ of the type (I), $(\mathrm{II})_g$, $(\mathrm{I}) \cap (\mathrm{II})_g$, are invariant under the action of any γ in the centralizer $\mathcal{C}_\sigma$ of σ, i.e.

$$\tau \text{ satisfies (I) and/or } (\mathrm{II})_g,\ \gamma \in \mathcal{C}_\sigma \Rightarrow \tau^\gamma \text{ satisfies it too.} \qquad \square$$

Heuristically, $\mathcal{C}_\sigma$ is the group of reparametrizations (relabelings) of the edges of the diagram, that leave the pattern of edges around each vertex unchanged. Quotienting by that group, i.e. considering its orbits for the adjoint action, thus enables one to go from labeled maps to unlabeled, topologically distinct maps. Finally, note that the definition of σ as describing the, say, *clockwise* linking pattern at vertices has singled out an orientation of the surface, while no information about the orientation of the circuit described by $\sigma^2\tau$ is provided: the maps are naturally asssociated with immersions of an *unoriented* circle in an *oriented* surface, hence of type UO in our nomenclature.

We shall use the following notations for the relevant subsets of $X = [2^{2n}]$:

$$X' = \{\tau \in [2^{2n}] \,|\, \sigma^2\,\tau \in [(2n)^2]\}$$
$$X'_g = \{\tau \in [2^{2n}] \,|\, \sigma^2\,\tau \in [(2n)^2] \text{ and } \sigma\tau \text{ has } n+2-2g \text{ cycles}\}.$$

In particular, we denote $X'' = X'_0$, corresponding to *planar* (in fact spherical) maps. The family of sets X'_g is a partition of X', and the set of orbits of the latter (identified with circle immersions), for the adjoint action of the subgroup $\mathcal{C}_\sigma$, is also partitioned into orbits corresponding to the various circle immersions of genus g.

Remarks. (i) In the group S_{4n}, the function c is related to the (n-independent) length function ℓ by $\ell(x) = 4n - c(x)$. Therefore equation $(\mathrm{II})_g$ also reads $\ell(\sigma\tau) = 3n - 2 + 2g$.

(ii) In the wording of the theorem, we made a convenient choice for σ; this is actually irrelevant since the choice amounts to labeling the half-edges in a specific way.

(iii) An arbitrary curve has $c(\sigma\tau) = n + 2 - 2g \geq 1$, therefore the possible values of g are such that $2g \leq n + 1$.

Examples. Let us choose $n = 4$, then $\sigma = (1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)$ in cycle notation, equivalently $\sigma = [2,3,4,1,6,7,8,5,10,11,12,9,14,15,16,13]$ in list notation.

First example: $\tau = (1,13)(2,5)(3,6)(4,16)(7,8)(9,12)(10,15)(11,14)$.
One checks that $\sigma^2\tau = (1,15,12,11,16,2,7,6)(3,8,5,4,14,9,10,13) \in [8^2]$, so τ obeys condition (I) and therefore encodes an immersion (possibly nonspherical).

One then evaluates $\sigma\tau = (1, 14, 12, 10, 16)(2, 6, 4, 13)(3, 7, 5)(11, 15)$; its number of cycles is 6 (there are two unwritten[d] singletons: (8) and (9), so that the genus is 0, and the immersion is actually spherical. The immersion encoded by permutation τ is given in Fig. 6. Notice that 1-cycles give rise (or come from) *kinks*, also known as *simple loops*.

Second example: $\tau = (1, 8)(2, 3)(4, 16)(5, 13)(6, 12)(7, 14)(9, 15)(10, 11)$.
One checks that $\sigma^2\tau = (1, 6, 10, 9, 13, 7, 16, 2)(3, 4, 14, 5, 15, 11, 12, 8) \in [8^2]$, but this time $\sigma\tau = (1, 5, 14, 8, 2, 4, 13, 6, 9, 16)(7, 15, 10, 12)$, which has two 1-cycles (3) and (11), and therefore a number of cycles equal to 4, so the permutation τ describes an immersion in a surface of genus 1. The encoding is made explicit in Fig. 7. Notice that, this circle immersion has four real crossings, as expected, but also one virtual one.

2.1.1. *Counting orbits*

One would like to count the orbits for the $\mathcal{C}_\sigma$ action on the sets X, X', X'_g and in particular on $X'' = X'_0$. How to find *a priori*, the number and the *lengths* of the $\mathcal{C}_\sigma$ orbits?

Burnside's lemma asserts that the number of orbits in, say, X' is related to the total number $\sum_\kappa |X'^\kappa|$ of fixed points in the action of $\kappa \in \mathcal{C}_\sigma$ acting in X', i.e. the number of pairs (κ, ξ), such that $\kappa\xi = \xi\kappa$, by

$$|X'/\mathcal{C}_\sigma| = \#\,\mathcal{C}_\sigma\text{−orbits in } X' = \frac{\sum_\kappa |X'^\kappa|}{|\mathcal{C}_\sigma|}. \tag{2.1}$$

This implies, however, the computation of $|X'| \times |\mathcal{C}_\sigma|$ pairs of products $(\kappa\xi, \xi\kappa)$, which becomes prohibitively large for $n \geq 6$.

Orbits, double classes and a formula by Frobenius
Let us first state a simple but useful theorem (that belongs to the folklore)

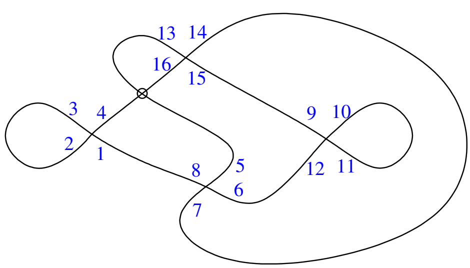


Fig. 7. The diagram encoded by $\tau = (1, 8)(2, 3)(4, 16)(5, 13)(6, 12)(7, 14)(9, 15)(10, 11)$. The virtual crossing is indicated by an open circle.

[d]Since n is fixed, it is unnecessary to write explicitly the 1-cycles, when using the cycle notation, but one should remember that the function $c(x)$ should count the total number of cycles.

Theorem 2. *Let G be a finite group and H be a subgroup of G. Take $x \in G$ and call* $\mathrm{Cl}(x)$ *its conjugacy class. Then the orbits for the adjoint action of H on* $\mathrm{Cl}(x)$ *are in one-to-one correspondence with double cosets $H\backslash G/K$, where $K = C(G, x)$ is the centralizer of x in G.*

Proof. Let $x \in G$. Then $y, y' \in \mathrm{Cl}(x)$ belongs to the same H-orbit iff $\exists\, h \in H$: $y' = hyh^{-1}$, but $y = gxg^{-1}$ and $y' = g'xg'^{-1}$, hence $g'xg'^{-1} = hgxg^{-1}h^{-1}$ or $g'^{-1}hgx = xg'^{-1}hg$, from which it follows that $k := g'^{-1}hg \in K := C(G, x)$ and $g' = hgk^{-1} \in HgK$. □

The counting of H-orbits in $\mathrm{Cl}(x)$ thus amounts to the counting of these double cosets. Frobenius [14] has given a formula for the number of double cosets $H\backslash G/K$. In essence, his method consists in computing in two different ways, the number of solutions of equation $hgk = g$ with $g \in G$, $h \in H$ and $k \in K$, with the result that

$$|H\backslash G/K| = \frac{|G|}{|H|\,|K|} \sum_{\mu} \frac{|H_\mu|\,|K_\mu|}{|G_\mu|}, \tag{2.2}$$

where the sum runs over conjugacy classes G_μ of G, $H_\mu = H \cap G_\mu$ and $K_\mu = K \cap G_\mu$.

We are going to make repeated use of this connection between orbits and double classes and of Frobenius' formula. In the problem at hand, $G = S_{4n}$, there is an one-to-one correspondence between the orbits of $\tau \in \mathrm{Cl} = X = [2^{2n}]$, under the action of $H = \mathcal{C}_\sigma = C(S_{4n}, \sigma)$ and double cosets of S_{4n} of the form $\mathcal{C}_\sigma\backslash S_{4n}/\mathcal{C}_\tau$ with $\mathcal{C}_\tau = C(S_{4n}, \tau)$. In this particular case of permutations, Frobenius' formula is easy to apply without calculating explicitly the intersections: classes G_μ are indexed by partitions, and H_μ, K_μ can be simply deduced from the knowledge of the cyclic structure of their own conjugacy classes. Note that, this method allows one to count the $\mathcal{C}_\sigma$-orbits in X, but not those in X' or in X''.

Poor man's method

In the latter case, the same group theoretical considerations are not applicable, and we resort to a direct construction of these orbits on a computer. This "brute force" method has the merit of giving not only the number and lengths of orbits, but also an explicit representative of each of them, thus providing a *catalogue* of the corresponding maps or immersions. In practice, this method may be used up to $n = 6$. For higher values, we resort to alternative methods presented in the next sections.

2.1.2. *Orbits lengths*

The length of orbits, for the $\mathcal{C}_\sigma$ action on any of the sets X, X' or X'_g, is not constant. For instance, when $n = 4$ there are 121 orbits for the $\mathcal{C}_\sigma$ action on X' (21 of genus 0, 64 of genus 1, and 36 of genus 2), but they are not all of the same length: 92 are of maximal size (namely 6144, the order of $\mathcal{C}_\sigma$), 23 are of size $|\mathcal{C}_\sigma|/2$, and 6 are of size $|\mathcal{C}_\sigma|/4$). This corresponds to the fact that the centralizer $C(\mathcal{C}_\sigma, \tau)$, in $\mathcal{C}_\sigma$, of a permutation τ describing some specific immersion is not necessarily trivial, and $\#\mathrm{Orb}(\tau) = |\mathcal{C}_\sigma|/|C(\mathcal{C}_\sigma, \tau)|$. Existence of "symmetries", for a specific immersion, is measured, or actually defined, by $C(\mathcal{C}_\sigma, \tau)$. The order ω of this group will not be given in our tables, but it is easy to obtain for every particular case. For large n, almost all orbits have trivial stabilizers [15], and an estimate for the total number of immersions, including all values of g, is asymptotically given by $|X'|/|\mathcal{C}_\sigma|$, equal to $\frac{(4n-2)!!}{4^n n!}$, when $n > 2$ (see below, Appendix C and Table 1).

2.2. *Results*

The group $\mathcal{C}_\sigma$. Given $\sigma \in [4^n]$, i.e. a product of n cyclic permutations on n disjoint sets of four objects, its centralizer $\mathcal{C}_\sigma$ is made of cycles operating on the n same sets of four objects, times any permutation of these n cycles. Whence the order

$$|\mathcal{C}_\sigma| = 4^n n!$$

i.e. $|\mathcal{C}_\sigma| = 4, 32, 384, 6144, 122880, 2949120\dots$ for $n = 1, 2, 3, 4, 5, 6$, see Table 2.

The set $X = [2^{2n}]$ and its $\mathcal{C}_\sigma$-orbits
A standard result is that $|X| = (4n-1)!!$.

How many orbits are there, when $\mathcal{C}_\sigma$ acts by conjugation on the class $X = [2^{2n}]$? By use of Frobenius' formula for double cosets (2.2), we find that for $n = 1, 2, \dots, 9$, there are

$$\#\mathcal{C}_\sigma\text{−orbits in } X = 2, 10, 54, 491, 6430, 119475, 2775582, 76733201, 2439149685. \tag{2.3}$$

One is of length 1: the orbit of σ^2.

The set X' and its $\mathcal{C}_\sigma$-orbits
We prove in Appendices C.2 and C.3, using a simple integral calculation or a purely combinatorial argument, that $|X'| = (4n-2)!!$. Acting on that X', $\mathcal{C}_\sigma$ has a number of orbits given by

$$\#\mathcal{C}_\sigma\text{−orbits in } X' = 1, 3, 13, 121, 1538, 28010, \dots \tag{2.4}$$

Taking $n = 4$ for example, using the orbit stabilizer theorem and denoting as above by ω the order of the centralizer $C(\mathcal{C}_\sigma, \tau)$, one finds that there are 92 orbits in X' with $\omega = 1$, 23 orbits with $\omega = 2$ and 6 orbits with $\omega = 4$, a total of 121. One checks that $92|\mathcal{C}_\sigma| + 23|\mathcal{C}_\sigma|/2 + 6|\mathcal{C}_\sigma|/4 = 645120 = |X'|$, as it should. Moreover $\#(\mathcal{C}_\sigma \backslash S_{4n}/\mathcal{C}_\tau) = 491$ corresponding to the $\mathcal{C}_\sigma$-orbits of X, but only 121 correspond to orbits of X'.

The number of $\mathcal{C}_\sigma$-orbits in X''
Among the $\mathcal{C}_\sigma$ orbits in X', we pick those that are such that $\sigma\tau$ has $n+2$ cycles (condition $(\mathrm{II})_0$ for genus 0). We find a number of relevant orbits equal

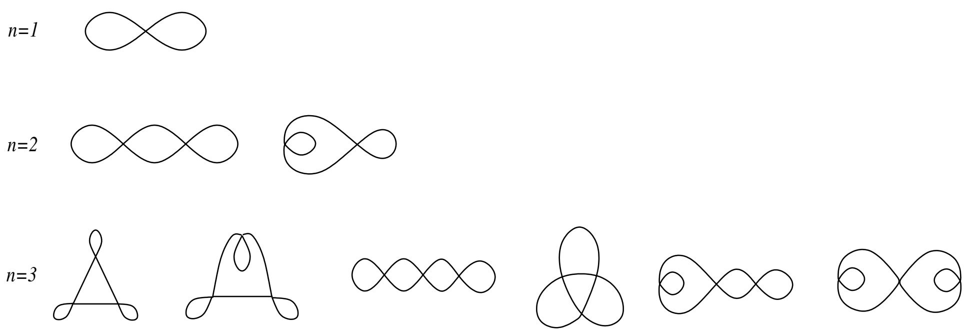


Fig. 8. Immersions of an unoriented circle in the oriented sphere with n double points, for $n = 1, 2, 3$.

to $1, 2, 6, 21, 99, 588, \ldots$. As discussed above, those are the numbers of immersions with n double points of an unoriented circle in the oriented sphere, see Fig. 8, [2, 11] and OEIS sequence A008987.

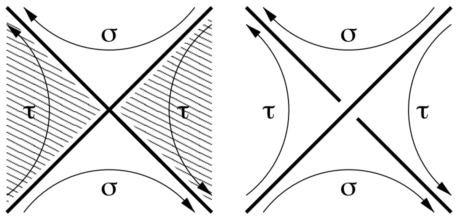
$$\#\text{UO spherical immersions} = 1, 2, 6, 21, 99, 588, 3829, \ldots. \tag{2.5}$$

In fact, the number 3829 (for $n = 7$) and further terms will be obtained below through a different method.

The following Table 1 summarizes, for $n = 1, \ldots, 6$, most of the results obtained using the above technique. The number of immersions in surfaces of specific genus $g > 0$ can be obtained in the same way (only the spherical ones appear in Table 1), but the corresponding values are gathered in Table 9, because we shall later recover and extend the results obtained in the present section. The last line of Table 1 refers to a quantity (free energy) defined in Appendix C.

Table 1. Orbits of X subsets. $X = [2^{2n}]$. The numbers in blue give the asymptotic estimate of the number of orbits. The numbers of spherical UO immersions are given by the line # $\mathcal{C}_\sigma$-orbits in X''. The total numbers of UO immersions (all genera) are given by the line # $\mathcal{C}_\sigma$-orbits in X'. Last entry $F_n^{(0,1)}$ of the table is defined in Appendix C.

n	1	2	3	4	5	6
$\lvert\mathcal{C}_\sigma\rvert = 4^n n!$	4	32	384	6144	122880	2949120
$\lvert X\rvert = (4n-1)!!$	3	105	10395	2027025	654729075	316234143225
# $\mathcal{C}_\sigma$-orbits in X	2	10	54	491	6430	119475
$\lvert X'\rvert = (4n-2)!!$ (I)	2	48	3840	645120	185794560	81749606400
# $\mathcal{C}_\sigma$-orbits in X'	1	3	13	121	1538	28010
$\lvert X'\rvert/\lvert\mathcal{C}_\sigma\rvert$	$\frac{1}{2}$	$\frac{3}{2}$	10	105	1512	27720
$\lvert X''\rvert$ (I)$\cap$(II)$_0$	2	32	1344	99840	11034624	1646100480
# $\mathcal{C}_\sigma$-orbits in X''	1	2	6	21	99	588
$\lvert X''\rvert/\lvert\mathcal{C}_\sigma\rvert$	$\frac{1}{2}$	1	3.5	16.25	89.8	558.17
$F_n^{(0,1)} = \frac{\lvert X''\rvert}{4^n n!}$	$\frac{1}{2}$	1	$\frac{7}{2}$	$\frac{65}{4}$	$\frac{449}{5}$	$\frac{3349}{6}$



(a) (b)

Fig. 9. White and shaded faces $\leftrightarrow$ over/under-crossings.

3. Bicolorable and Bicolored Maps of Types UO and OO using S_{2n}

In a nutshell: here, we shall get UOc, the bicolored immersions of type UO, then, forgetting the color assignment, we shall get UOb, the bicolorable immersions of type UO, which turns out to be identical to the immersions UO for genus 0 maps (spherical curves). This will be explained below.

3.1. *The set $Y = S_{2n}$, its orbits, and immersions of type* UOc ("*Y method*")

In the present section, we shall study the orbits of solutions for a particular set of equations written in a set Y, defined as the symmetric group S_{2n} itself, under the action of a particular subgroup that turns out to be its hyperoctahedral subgroup.

Method: description of a bicolored map by a pair of permutations of $Y = S_{2n}$

It is a well-known fact, that *planar* maps with vertices of even valency may have their faces bicolored. This applies of course to our 4-valent planar maps. For nonplanar (i.e. of genus $g > 0$) maps, this is no longer guaranteed, (as already discussed in the Introduction and examplified in Fig. 2) and we have to assume that the map is bicolorable, see below. We then turn to a more efficient encoding of such *bicolored* maps by permutations [16].

For a bicolored map with n vertices and $2n$ labeled edges, we deal with permutations of S_{2n}, instead of S_{4n} as above. A map is encoded into a *pair* of permutations $\sigma, \tau \in S_{2n}$: σ describes the sequence of edges as white faces are traveled clockwise, while τ describes the counterclockwise sequence of edges on shaded faces, see Fig. 9(a). When considering the map as (the plane projection of) an *alternating* knot, one uses the convention for overcrossings/undercrossings shown on Fig. 9(b). Define $\rho = \sigma^{-1}\tau$ and $\tilde{\rho} = \sigma\tau^{-1}$; it is clear that ρ describes the pairings of edges at overcrossings, and $\tilde{\rho}$ at undercrossings, and they are both a product of n disjoint transpositions, $\rho, \tilde{\rho} \in [2^n]$. The chain of edges as one follows a thread of the knot/link is thus described by $\rho\tilde{\rho} = \sigma^{-1}\tau\sigma\tau^{-1}$, and white, respectively, shaded, faces correspond to cycles of σ, respectively, τ.

Just as in Sec. 2.1, the two conditions of one-componentness and genus g amount to imposing

$$\rho\tilde{\rho} = \sigma^{-1}\tau\sigma\tau^{-1} \text{ has 2 equal cycles, i.e. } \rho\tilde{\rho} \in [n^2]. \quad (\mathrm{I}') \quad \text{one-componentness}$$
$$c(\sigma) + c(\tau) = n + 2 - 2g \quad (\mathrm{II}')_g \quad \text{genus } g.$$

We want to count all σ and τ subject to the above conditions. Actually, it is convenient to fix ρ in $[2^n]$, defining it for example, by $\rho = \rho_0$, $\rho_0(2i-1) := 2i$, $\rho_0(2i) = 2i-1, i = 1, \ldots, n$ (it is only a relabeling of the edges). This choice being made, a map is then described (up to the conjugate action of the centralizer of ρ, see below) by a single permutation σ, since $\tau = \sigma\rho$. Notice that $\tilde{\rho} = \sigma\rho\sigma^{-1}$. With this choice for ρ, the two conditions (I') and $(\mathrm{II}')_g$ can be written in terms of equations for σ (see Theorem 3, below).

As in Sec. 2, it is natural to define the subsets Y' and Y'_g of Y, made of those permutations σ that respectively obey the conditions (I') and $(\mathrm{I}') \cap (\mathrm{II}')_g$. The sets Y'_g constitute a partition of Y'.

Ultimately, in order to count the number of curves, one decomposes the previous subsets Y'_g, in particular, $Y'' = Y'_0$ for spherical curves, into orbits for the conjugate action of $\mathcal{C}_\rho$, the centralizer of ρ_0 in S_{2n}.

Finally, we observe that the convention that σ describes the clockwise sequence of labels on white faces (and τ the counterclockwise one on shaded faces) assumes that the sphere or the higher genus surface is oriented, while nothing specifies the orientation of the curve. Our orbits, in this section, are thus of type UO.

Bicolored versus bicolorable curves

One could think that the orbits of Y'_g should determine the various UO circle immersions of genus g. This is not so for two reasons, already mentioned in the Introduction. First, the curves obtained in this way, correspond to a bicoloring of a curve. For lack of a better name we call "bicolored immersions" the bicolored curves associated with the orbits of Y'_g (recall that in the language of knot theory, they describe alternating knots), and denote their set by UOc. Depending on whether the two alternative colorings (i.e. the two choices of alternating over- and under-crossings) are or not topologically equivalent, they will contribute differently to the counting of ordinary, uncolored immersions; this will be spelled out in Sec. 4.2. Secondly, for genus $g > 0$, not all curves are bicolorable, see Fig. 2 for an example. We shall call "*bicolorable* curves" or "bicolorable immersions" (not to be confused with the *bicolored* ones previously described) the curves obtained by this technique, *after* erasing the colors, and denote their set by UOb. Finally, we recall that UO refers to immersions studied in the previous section, with no assumption of bicolorability.

In the Tables 9 and 8, the reader can find the cardinals of these various sets of immersions, and check that $|\mathrm{UO}| = |\mathrm{UOb}|$ in genus 0, while for $g > 0$, $|\mathrm{UO}| > |\mathrm{UOb}|$, as expected since bicolorable curves, do not exhaust all possible genus g curves.

Example of encoding

See in Fig. 10, the example of the bicolored diagram described by $\sigma = [3,5,7,1,2,6,4,8] = (1,3,7,4)(2,5)(6)(8)$ and $\tau = [5,3,1,7,6,2,8,4] = (1,5,6,2,3)(4,7,8)$, hence $\rho = [2,1,4,3,6,5,8,7]$.

We summarize the above method as follows:

Theorem 3. *Call $\rho = (1,2)(3,4)\ldots(2n-3,2n-2)(2n-1,2n) \in [2^n] \subset S_{2n}$, using cycle notation, and $\mathcal{C}_\rho = C(S_{2n},\rho)$, the centralizer of ρ in S_{2n}. Bicolored circle immersions of the unoriented circle in an oriented surface of genus g, or* UOc *immersions for short, are in bijection with the orbits of $\mathcal{C}_\rho$ acting by conjugacy on S_{2n}, whose representatives σ solve*

$$\rho\sigma\rho\sigma^{-1} \text{ has 2 equal cycles, i.e. } \rho\tilde{\rho} \in [n^2] \text{ with } \tilde{\rho} = \sigma\rho\sigma^{-1} \qquad (\mathrm{I}')$$

$$c(\sigma) + c(\sigma\rho) = n + 2 - 2g \qquad (\mathrm{II}')_g$$

$c(x)$ being the function that gives the number of cycles (including singletons) of the permutation x.

Remarks. (i) In the wording of this theorem, we chose a particular value of ρ in the conjugacy class $[2^n]$, namely $\rho = \rho_0$, because it is simple and convenient, but we could have made any other choice in the same class since this just corresponds to a relabeling of some edge labels. We shall see in Sec. 3.2 how to further restrict the choice of σ.

(ii) It is useful to remember that $\rho^2 = \widetilde{\rho}^2 = 1$, that $\widetilde{\rho} = \rho^\sigma$ since, by definition, $\rho^\sigma = \sigma\rho\sigma^{-1}$, and that $\tau = \sigma\rho = \widetilde{\rho}\sigma$.

(iii) The set Y' defined by condition (I$'$) alone can also be written $Y' = \{\sigma \in S_{2n} : \sigma^\rho\sigma^{-1} \in [n^2]\}$.

Example. As an example, we give in Fig. 10, the diagram of Fig. 6 in this new description.

Structure of the centralizer

The centralizer $\mathcal{C}_\rho$ of ρ_0 is generated by transpositions of $(1,2)$, $(3,4), \ldots, (2n-1,2n)$, times a permutation of these n pairs. Its order is thus $2^n n!$. This group is called the hyperoctahedral group BC_n, as it is the group of symmetries of

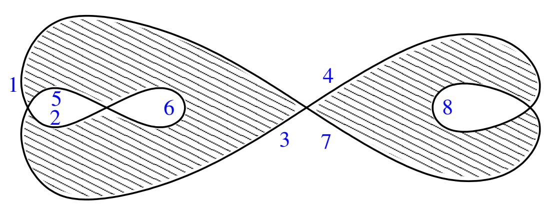


Fig. 10. The bicolored diagram encoded by $\sigma = [3,5,7,1,2,6,4,8] = (1,3,7,4)(2,5)(6)(8)$.

the n-cube. It admits several different geometric and algebraic presentations. One construction is as follows (see, for example [17]). The symmetric group S_{2n} acts on $\{1, 2, \ldots, 2n\}$ and therefore also on the set of partitions of the latter consisting of two-element subsets. Fix an element in this set (we shall choose $\{\{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{2n-1, 2n\}\}$) and denote by $\mathcal{C}_\rho$ its stabilizer. Clearly, $\mathcal{C}_\rho$ permutes the n two-element subsets among themselves and it is equal to the centralizer, in S_{2n} of the permutation $\rho = (1,2)(3,4)\ldots(2n-1, 2n)$, whence the notation. The subgroup $\mathcal{C}_\rho$ of S_{2n}, then appears as the semidirect product of $S_2 \times \cdots \times S_2$ (n times) and S_n, the latter acting by permuting the factors of the former (wreath product).

In order to put in perspective, what will be done in Sec. 3.2, let us make a few additional remarks. Call ϖ the map $S_{2n} \mapsto S_{2n}$ defined by $\varpi(x) = \rho x \rho^{-1}$. Clearly ϖ is a group homomorphism and an involution, moreover the subgroup $\mathcal{C}_\rho$ is the set of fixed points of this involution: $\mathcal{C}_\rho = \{x : x \in S_{2n} \mid \varpi(x) = x\}$. Define the map (not a group morphism) $\varphi : S_{2n} \mapsto S_{2n}$ by $\varphi(x) = \varpi(x^{-1})x$. Notice that $\varphi(x^{-1})$ and $\varphi(x)^{-1}$ belong to the same S_{2n}-conjugacy class since they are conjugated in $S_{2n} : \varphi(x)^{-1} = x^{-1}\varphi(x^{-1})x$. We have also $\varphi(x)^{-1} = \rho x \rho x^{-1}$, so that ρ being fixed, the condition characterizing the one-componentness of the permutation x encoding a curve reads simply $\varphi(x) \in [n^2]$. The reader will easily notice (see also [18]) that, for any k in $\mathcal{C}_\rho$, $\varphi(kx) = \varphi(x)$ and $\varphi(xk) = k\varphi(x)k^{-1}$. Therefore, φ induces a map from the space of double cosets[e] $\mathcal{C}_\rho \backslash S_{2n} / \mathcal{C}_\rho$ to the set of conjugacy classes in S_{2n}. Actually, we shall see in Sec. 3.2, that the counter image $Y' = \varphi^{-1}([n^2])$, considered as a subset of $\mathcal{C}_\rho \backslash S_{2n} / \mathcal{C}_\rho$, contains only one element: the double coset $\mathcal{C}_\rho \backslash \beta / \mathcal{C}_\rho$, where $\beta = (1, 2, 3, \ldots, 2n)$. As a double coset, Y' is then a disjoint union of left cosets $\sigma\mathcal{C}_\rho$, with $\sigma \in S_{2n}$ (they will be identified with the sets $V(r)$ of Sec. 3.2), parametrized by the homogenous space $R = \mathcal{C}_\rho / \mathcal{C}_\rho \cap \mathcal{C}_\rho^\beta = \mathcal{C}_\rho / D_n$, where D_n is the dihedral group. The space R, with $|\mathcal{C}_\rho|/|D|$ elements, will be described in Sec. 3.2 as parametrizing the "gauge condition" (in physicist's parlance).

Orbits of Y'

One finds $|Y'| = 2^{2n-1}(n-1)!n!$, see Appendix C.4 for a proof based on a simple integral. In practice, the orbits of Y' are obtained by methods similar to those of Sec. 2, (see also Appendix A).

Orbits of $Y'' = Y'_0$ and of Y'_g

Once the orbits of Y' are known, filtering according to their genus yields the orbits of each Y'_g, in particular of $Y'' = Y'_0$. For $n = 10$, we had to rely on a random sampling method (see Appendix A), but as, we have no *a priori* knowledge of $|Y''|$, we have no way to check the correctness of the result. The figures entered in red in Table 2 below, for $n = 10$, are thus likely estimates, awaiting an independent confirmation.

[e] A graphical way to encode these double cosets is described in [19, p. 401] see also [18].

Orbit lengths

Lengths of orbits of Y' may be read off the following table, with the notation $k^{\#\text{orbits of length } |\mathcal{C}_\rho|/k}$

$n = 1$	2^2					
$n = 2$	1^1	2^2				
$n = 3$	1^4	2^6	3^2	6^2		
$n = 4$	1^{44}	2^6	4^4			
$n = 5$	1^{352}	2^{62}	5^4	10^2		
$n = 6$	1^{3803}	2^{62}	3^{15}	6^6		
$n = 7$	1^{45696}	2^{766}	7^6	14^2		
$n = 8$	1^{644736}	2^{752}	4^{28}	8^8		
$n = 9$	$1^{10315716}$	2^{12264}	3^{202}	6^{22}	9^8	18^2

Results

The numbers of orbits for $g = 0$ are given in Table 2; for higher values of g, they are gathered in Table 9, under the entry UOc.

3.2. *The left coset $U = \beta\,\mathcal{C}_\rho$ and immersions of type* UOc *and* OOc *("U method").*

In a nutshell: we shall see in this section that, in order to determine the number of immersions of type UOc, we can replace the set Y' studied in the previous section by a particular subset U (a particular left coset of $\mathcal{C}_\rho$) and the adjoint action of $\mathcal{C}_\rho$ by its restriction to the dihedral subgroup D_n, which is much smaller. Moreover, by replacing the adjoint action of D_n by the adjoint action of $\mathbb{Z}_n$ (a particular cyclic subgroup of the latter), one obtains the number of immersions of type OOc. In Sec. 4, we shall see how, from this study, and by introducing several involutions, one can obtain the various types of immersions. As a side result, we shall also see how the stratification of U into subsets of genus g allows us to recover (in genus 0) the classification of the so-called "long curves" and to obtain new classifications when $g > 0$.

3.2.1. *The set R*

With $\rho = (1, 2)(3, 4) \ldots (2n - 1, 2n)$ fixed as before, what can be said about the values of $\tilde{\rho} = \sigma\rho\sigma^{-1}$ as $\sigma \in Y'$? Consider the sets

$$R := \{\tilde{\rho} \,|\, \tilde{\rho}\rho \in [n^2]\} \tag{3.1}$$

and for $r \in R$,

$$V(r) := \{\sigma \,|\, \sigma\rho\sigma^{-1} = r\}. \tag{3.2}$$

Table 2. Orbits for Y subsets.

n	1	2	3	4	5	6	7	8	9	10
$\vert\mathcal{C}_\rho\vert = 2^n n!$	2	8	48	384	3840	46080	645120	10321920	185794560	3715891200
$\vert Y\vert = \vert S_{2n}\vert = (2n)!$	2	24	720	40320	3628800	479001600	87178291200	20922789888000	6402373705728000	2432902008176640000
$\#\mathcal{C}_\rho$-orbits in $Y = S_{2n}$	2	8	34	182	1300	12634	153598	2231004	37250236	699699968
$\vert Y'\vert = 2^{2n-1}(n-1)!n!$	2	16	384	18432	1474560	176947200	29727129600	6658877030400	1917756584755200	690392370511872000
$\#\ \mathcal{C}_\rho$-orbits in Y' (I')	2	3	14	54	420	3886	46470	645524	10328214	
$\vert Y'\vert/\vert\mathcal{C}_\rho\vert = 2^{n-1}(n-1)!$	1	2	8	48	384	3840	46080	645120	10321920	185794560
$\vert Y''\vert$	2	16	336	12480	689664	51440640	4870932480	561752432640	76597275525120	120774980822263040
$\#\ \mathcal{C}_\rho$-orbits in Y'' (I')∩(II')	2	3	12	37	198	1143	7658	54559	413086	3251240
$\vert Y''\vert/\vert\mathcal{C}_\rho\vert$	1.	2.	7.	32.5	179.6	1116.33	7550.43	54423.3	412268.66	3250229.2
$F_n^{(0,1)} = \frac{\vert Y''\vert}{2(2n)!!}$	$\frac{1}{2}$	1	$\frac{7}{2}$	$\frac{65}{4}$	$\frac{449}{5}$	$\frac{3349}{6}$	$\frac{52853}{14}$	$\frac{217693}{8}$	$\frac{618403}{3}$	$\frac{8125573}{5}$

Note: The numbers in blue give the asymptotic estimate of the number of orbits.
Numbers of spherical UO bicolored immersions appear on the line $\#\,\mathcal{C}_\rho$-orbits in Y''.
Total numbers of UO bicolored immersions (all genera): line $\#\,\mathcal{C}_\rho$-orbits in Y'.
Last entry $F_n^{(0,1)}$ of the table is defined in Appendix C.
Here and below in this paper, figures in red are still awaiting confirmation, see above and Appendix A for explanations.

Table 3. Orbits for Z subsets.

n	1	2	3	4	5	6	7	8	9	10
$\vert\mathcal{C}'_\rho\vert = n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$\vert Z'\vert = (2n-1)!$	1	6	120	5040	362880	39916800	6227020800	1307674368000	355687428096000	121645100408832000
$\#\mathcal{C}'_\rho$-orbits in Z'	1	4	22	218	3028	55540	1235526	32434108	980179566	33522177088
$\vert Z'\vert/\vert\mathcal{C}'_\rho\vert = \frac{(2n-1)!}{n!}$	1	3	20	210	3024	55440	1235520	32432400	980179200	33522128640
$\vert Z''\vert$	1	4	42	780	21552	803760	38054160	2194345440	149604053760	11794431720960
$\#\ \mathcal{C}'_\rho$- orbits in Z''	1	3	9	37	182	1143	7553	54559	412306	3251240
$\vert Z''\vert/\vert\mathcal{C}'_\rho\vert = 2F_n^{(0,1)}$	1	2	7	$\frac{65}{2}$	$\frac{898}{5}$	$\frac{3349}{3}$	$\frac{52853}{7}$	$\frac{217693}{4}$	$\frac{1236806}{3}$	$\frac{16251146}{5}$

Note: Numbers of spherical OO immersions: line $\#\,\mathcal{C}'_\rho$-orbits in Z''.
Total numbers of general OO immersions (all genera): line $\#\,\mathcal{C}'_\rho$-orbits in Z'.

It is readily seen that $V(r)$ is a left coset of $\mathcal{C}_\rho$, since $\sigma, \sigma' \in V(r) \Leftrightarrow \sigma\rho\sigma^{-1} = \sigma'\rho\sigma'^{-1} \Leftrightarrow \sigma'^{-1}\sigma\rho = \rho\sigma'^{-1}\sigma$, hence $\sigma'^{-1}\sigma \in \mathcal{C}_\rho$ and $\sigma \in \sigma'\mathcal{C}_\rho$. This property of being a left coset will be used shortly. This implies that $|V(r)| = |\mathcal{C}_\rho|$ and from the fact that Y' may be partitioned into $V(r)$, $Y' = \bigsqcup_{r\in R} V(r)$, it follows, using the values of $|Y'|$ and $|\mathcal{C}_\rho|$ calculated above, that $|R| = |Y'|/|\mathcal{C}_\rho| = 2^{n-1}(n-1)!$.

3.2.2. *Further gauge fixing*

One may now restrict further the set of admissible σ by imposing the additional condition (on top of ρ fixed as above)

$$\tilde{\rho}\rho = \sigma\rho\sigma^{-1}\rho = \alpha \text{ fixed in } R\rho$$

or equivalently $\sigma \in V(\alpha\rho)$. *For example*, one may demand that $\sigma\rho\sigma^{-1}\rho$ be the product of the two cycles

$$\sigma\rho\sigma^{-1}\rho = \alpha_0 := (1, 3, 5, \ldots, 2n-1)(2, 2n, 2n-2, \ldots, 4). \tag{3.3}$$

This latter choice α_0 corresponds to a sequential labeling of edges by $(1, 2, 3, \ldots, 2n)$ as the curve is traveled one way or the other. We call U the set of σ, such that

$$U = \{\sigma \,|\, \sigma\rho\sigma^{-1}\rho = \alpha_0\} = V(\alpha_0\rho) \tag{3.4}$$

and we recall that

$$|U| = |C_\rho| = 2^n n!.$$

Proposition 1. *The general solution of* (3.4) *is* $\sigma = \beta\xi$, *with* β *the cyclic permutation* $\beta = (1, 2, 3, \ldots 2n)$ *and* ξ *arbitrary in* $\mathcal{C}_\rho$. *In other words,* $U = \beta\mathcal{C}_\rho$, *a particular* $\mathcal{C}_\rho$*-left coset, in agreement with the previous argument.*

Proof. It is easy to check that $\beta\rho\beta^{-1}\rho = \alpha_0$, hence upon the change of variable $\sigma = \beta\xi$, Eq. (3.4) reads $\beta\xi\rho\xi^{-1}\beta^{-1}\rho = \beta\rho\beta^{-1}\rho$, hence $\xi\rho\xi^{-1} = \rho$, $\xi \in \mathcal{C}_\rho$. $\square$

The parametrization (1) of σ as an element of the left coset U, therefore automatically implies condition (I′). This is very useful in practice (see a comment at the end of Appendix A).

3.2.3. *The remaining reparametrization groups*

In this new "gauge", the remaining labeling freedom on a given σ is the choice of the origin (edge number 1), and the direction of travel, if one considers unoriented curves. Accordingly the group of reparametrization, $\mathcal{C}_\rho \cap \mathcal{C}_\alpha$, where $\mathcal{C}_\alpha$ is the centralizer of α in S_{2n}, is the dihedral group D_n (of order $2n$), if one considers unoriented curves, and the cyclic group $\mathbb{Z}_n$, if the curves are oriented. Unlabeled

curves are thus in one-to-one correspondance with orbits of the set U under the adjoint action of D_n (unoriented curves) or of $\mathbb{Z}_n$ (oriented ones).
Remark. This occurrence of the dihedral or cyclic group makes clear that the length of orbits, which must be divisors of the orders of these groups, are divisors of $2n$ or n, a "well known" fact.

Warning: Y' is stable, under the adjoint action of $\mathcal{C}_\rho$ and can be decomposed into the corresponding orbits, but its subset U is not stable under this action, although, it intersects all the orbits of Y' (not only once, in general); U is however stable under the action of D_n. See Appendix A for more details.

3.2.4. *Back to* $|Y'|$ *and* $|R|$

The number of left cosets contained in a double coset $K\backslash g/K$, for g an element of a group G and K, a subgroup of G, is equal to the index, in G, of the subgroup $K \cap K^g$, where $K^g = gKg^{-1}$. In the present situation, with $G = S_{2n}$, $K = \mathcal{C}_\rho$ and $g = \beta$ (the above cyclic permutation), we have $\mathcal{C}_\rho \cap \mathcal{C}_\rho^\beta = D$, where D is the dihedral subgroup of $\mathcal{C}_\rho$ (see also [19, p. 402]). The previous index is therefore $|\mathcal{C}_\rho|/2n$. Since all $\mathcal{C}_\rho$ left cosets have the same number of elements, the number of permutations contained in the double coset $Y' = \mathcal{C}_\rho\backslash\beta/\mathcal{C}_\rho$ is equal to $|\mathcal{C}_\rho| \times |\mathcal{C}_\rho|/2n$: we recover the number of elements of Y'. As a double coset, Y' is a disjoint union of left cosets $V(r)$ parametrized by the homogenous space $R = \mathcal{C}_\rho/\mathcal{C}_\rho \cap \mathcal{C}_\rho^\beta = \mathcal{C}_\rho/D$.

3.2.5. *The set* U_g *of long curves*

The set U just defined may be partitioned into U_g according to genus, as was done before for Y', and each U_g may be interpreted as the set of *rooted maps* on an oriented surface Σ of genus g, or in other words, of (equivalent classes of) *open* (and oriented) curves drawn in Σ, sometimes dubbed *long curves*. In genus $g = 0$, their number have been computed in [20] up to $n = 10$, and in [21] up to $n = 19$ crossings using transfer matrix techniques. (Their asymptotic behavior has also been studied using a method of random sampling [22].)

Proof. Consider a *rooted* 4-valent genus g map with n crossings and one component: the marked half-edge, that we label by 0 may be regarded as cut open, which transforms the map into a "long curve". We then label by $1, 2, \ldots, 2n$ the successive edges encountered along the curve. The curve may then be bicolored by assigning to the left of the marked edge, the color say white, and then alternating colors as we go from a face to an adjacent one. To completely describe the pattern of crossings of the curve, it remains to give the permutation σ satisfying the rules of the previous formalism, namely conditions (3.3) and genus g. There is a bijection between these rooted maps and elements of the set U_g. There is no reparametrization freedom left, hence no orbit to take, once the root has been fixed. Then any such open curve may be closed by identifying edges of labels 0 and $2n$. Topologically distinct closed curves, i.e. images of immersions, correspond to orbits of the set U_g by the reparametrization group, namely the cyclic group $\mathbb{Z}_n$ or the dihedral group D_{2n} depending on whether the curve is oriented or not (OOc respectively, UOc). $\square$

Thus one finds a decomposition of the $2^n n!$ curves of U, $(n = 0, 1, \ldots, 9)$, according to genus as

$$\begin{aligned}
\#\text{ open curves } &= \begin{pmatrix} 1 \\ 2 \\ 8 \\ 48 \\ 384 \\ 3840 \\ 46080 \\ 645120 \\ 10321920 \\ 185794560 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & & & \\ 2 & & & & \\ 8 & & & & \\ 42 & 6 & & & \\ 260 & 116 & 8 & & \\ 1796 & 1700 & 344 & & \\ 13396 & 22528 & 9700 & 456 & \\ 105706 & 284284 & 220570 & 34560 & \\ 870772 & 3488904 & 4392820 & 1506576 & 62848 \\ 7420836 & 42074568 & 79951716 & 49572528 & 6774912 \end{pmatrix}
\end{aligned} \tag{3.5}$$

with the first column (genus $g = 0$) in agreement with [20, 21]. Notice that the sum over all genera is of course equal to $|\mathcal{C}_\rho| = 2^n n!$.

3.2.6. *Orbits of* U *and* UOc *and* OOc *immersions*

The same sort of counting of orbits, that was done in the sets Y' and Y'_g may be carried out in the sets U and U_g. From the previous discussion, it follows that UOc immersions are orbits of U under the action of D_n, while its $\mathbb{Z}_n$-orbits are what may be called OOc *immersions*. The numbers of UOc immersions have been computed before, see Table 2, using the $\mathcal{C}_\rho$ action on Y', but can be recovered in a more economic way, using the D_n action on the set U. Here are the numbers of

OOc immersions and their distribution according to genus for $n = 1, \ldots, 9$.

$$\#\text{ curves of type OOc } = \begin{pmatrix} 2 \\ 6 \\ 20 \\ 108 \\ 776 \\ 7772 \\ 92172 \\ 1291048 \\ 20644140 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & & & & \\ 6 & & & & \\ 18 & 2 & & & \\ 74 & 32 & 2 & & \\ 364 & 340 & 72 & & \\ 2286 & 3780 & 1630 & 76 & \\ 15106 & 40612 & 31510 & 4944 & \\ 109118 & 436368 & 549334 & 188356 & 7872 \\ 824612 & 4675012 & 8883620 & 5508120 & 752776 \end{pmatrix} \tag{3.6}$$

See also Table 9. For even n, the numbers of such orbits are just the double of those of type UOc, while for $g = 0$, these numbers are the double of OO immersions, see the proof below in Sec. 4.5, Theorem 4.

4. From Orbits to Various Types of Immersions

4.1. *Preamble*

In this section, we examine the effect of three involutive transformations on orbits of bicolored immersions: the color swapping or *swap* in short, denoted by s; the *mirror* transformation, m; and the *orientation reversal* r. These three involutions commute. Their explicit form depends on the class of orbits on which they act, as we shall see below. Given an orbit o belonging to a set O and an involution I, if o_I denotes the transform of o under I, there are two cases: either $o = o_I$, or $o \neq o_I$, a truism!, and we define

$$r_I = \#\{o \in O \,|\, o = o_I\}, \quad s_I = \#\{\{o, o_I\} \,|\, o \neq o_I\} \tag{4.1}$$

(i.e. $s_I = \#$ *unordered* pairs of distinct o, o_I). In the case of two commuting involutions I and J, there are five cases:

$$(1) \quad o = o_I = o_J = o_{IJ}, \quad (2) \quad o = o_I \neq o_J = o_{IJ},$$

$$(3) \quad o = o_J \neq o_I = o_{IJ}, \quad (4) \quad o = o_{IJ} \neq o_I = o_J,$$

$$(5) \quad o, o_I, o_J, o_{IJ} \text{ all distinct}$$

and we call

$$\begin{aligned} x_{IJ} = x_{JI} &= \#\{o \in O \,|\, o = o_I = o_J = o_{IJ}\} \\ y_{IJ} &= \#\{\{o, o_J\} \,|\, o = o_I \neq o_J = o_{IJ}\} \\ z_{IJ} = y_{JI} &= \#\{\{o, o_I\} \,|\, o = o_J \neq o_I = o_{IJ}\} \\ v_{IJ} = v_{JI} &= \#\{\{o, o_I\} \,|\, o = o_{IJ} \neq o_I = o_J\} \\ w_{IJ} = w_{JI} &= \#\{\{o, o_I, o_J, o_{IJ}\} \,|\, o, o_I, o_J, o_{IJ} \text{ all distinct}\}. \end{aligned} \tag{4.2}$$

For I and J standing for the mirror and the orientation reversal, those are the five cases discussed by Arnold [2]. We note that the relation between (4.1) and (4.2) is

$$r_I = x_{IJ} + 2y_{IJ}, \quad s_I = z_{IJ} + v_{IJ} + 2w_{IJ}.$$

For three involutions, there would be 15 cases (in general, the number of cases is given by a Bell number), but we shall refrain from listing them here.

4.2. *The swap image of a map*

We first examine the effect of (color) swapping (or equivalently, of interchanging all overcrossings and undercrossings in a knot diagram). Consider a bicolored curve described by some $\sigma \in Y'$ and its (color) swap described by σ_s. What is the relation between σ and σ_s? Let ρ be fixed equal to ρ_0 as above, $\tau = \sigma\rho_0$ and $\tilde\rho = \sigma\rho_0\sigma^{-1}$. Then swapping colors implies to exchange ρ and $\tilde\rho$, and σ and τ^{-1}, but also to change the labeling of edges, in such a way that $\tilde\rho$ takes the form ρ_0. A permutation γ that carries over that change of labeling must satisfy

$$\tilde\rho = \sigma\rho_0\sigma^{-1} = \gamma^{-1}\rho_0\gamma$$

the general solution of which is $\gamma = \gamma'\sigma^{-1}$ with $\gamma' \in \mathcal{C}_\rho$. Up to $\mathcal{C}_\rho$-equivalence, we may just choose $\gamma = \sigma^{-1}$. Then after conjugation by γ,

$$\begin{aligned} \sigma_s = \gamma\tau^{-1}\gamma^{-1} = \gamma\rho_0\sigma^{-1}\gamma^{-1} &= \gamma\sigma^{-1}\gamma^{-1}\,\gamma\sigma\rho_0\sigma^{-1}\gamma^{-1} \\ &= \gamma\sigma^{-1}\gamma^{-1}\,\gamma\tilde\rho\gamma^{-1} = \gamma\sigma^{-1}\gamma^{-1}\rho_0 \end{aligned} \tag{4.3}$$

which for the above choice $\gamma = \sigma^{-1}$ reduces to

$$\sigma_s = \sigma^{-1}\rho_0 \Leftrightarrow \sigma\sigma_s = \rho_0. \tag{4.4}$$

Hence the colored curve (or the alternating knot diagram) and its swapped version are described by σ and $\sigma_s = \sigma^{-1}\rho_0$. We refer to the $\mathcal{C}_\rho$-orbits of σ and σ_s as *swapped orbits* o and o_s.

If n is odd, the signature of ρ_0, a product of an odd number of transpositions, is -1, and σ and $\sigma_s = \sigma^{-1}\rho_0$ cannot be conjugate in S_{2n}, and *a fortiori* cannot belong to the same orbit, under the action of $\mathcal{C}_\rho$: $\sigma \nsim \sigma_s$, where $\sim$ and its negate $\nsim$

refer to conjugacy with respect to the group $\mathcal{C}_\rho$. Another argument is that $\sigma \sim \sigma_s$ would imply that the numbers of white and shaded faces are equal, hence # faces is even, in contradiction with Euler formula for n odd.

In general, using the terminology of (4.1), for given n and genus g, let r_s be the number of *self-swapped orbits*, i.e. such that $o = o_s$, and s_s be the number of *pairs* of non self-swapped orbits $\{o, o_s\}$, i.e. such that $o \neq o_s$. Thus $r_s = 0$ for n odd and all genera, while for example, in genus 0, we find

$$\begin{array}{lll} n=2 & r_s=1 & s_s=1 \\ n=4 & r_s=5 & s_s=16 \\ n=6 & r_s=33 & s_s=555 \\ n=8 & r_s=249 & s_s=27155 \\ n=10 & r_s=2036 & s_s=1624602. \end{array} \tag{4.5}$$

For any genus g, the number of Y'_g orbits, i.e. of *bicolored* UO *curves* of genus g is thus given by $r_s + 2s_s$, while those in which, we identify the two colors, namely the *bicolorable* UO curves, have a cardinality equal to $r_s + s_s$. As we discussed already, for $g = 0$, bicolorability is not a constraint, and we recover the number of UO curves found in Sec. 2, while for $g > 0$, the UOc bicolorable curves are a subset of the UO curves, see below Sec. 4.4 for a general discussion.

4.3. *Mirror image of a map*

On maps/orbits of Y'_g, we may also define a mirror transformation. The latter swaps σ and τ, hence, if $\rho = \rho_0$ is fixed, changes σ into $\sigma\rho_0$. Maps are either "achiral", if σ and $\sigma_m := \sigma\rho_0$ belong to the same orbit, and we write $o = o_m$, or appear in chiral pairs $\{\sigma, \sigma_m\}$, when $\sigma_m \nsim \sigma$, or $o \neq o_m$. Again, for n odd, as ρ_0 has an odd signature, σ and σ_m cannot belong to the same orbit.

In general, for given n and genus g, let r_m be the number of achiral orbits, i.e. such that $o = o_m$, and s_m be the number of chiral *pairs* of orbits $\{o, o_m\}$, $o_m \neq o$. Thus $r_m = 0$ for n odd and all genera, while for example, in genus 0, we find

$$\begin{array}{lll} n=2 & r_m=1 & s_m=1 \\ n=4 & r_m=5 & s_m=16 \\ n=6 & r_m=15 & s_m=564 \\ n=8 & r_m=97 & s_m=27231 \\ n=10 & r_m=592 & s_m=1625324. \end{array} \tag{4.6}$$

The number of orbits in Y'_g, i.e. of bicolored UOc curves of genus g is thus given by $r_m + 2s_m$, while those in which, we identify the two mirror images, i.e. the two orientations of the target surface, dubbed UUc, have a cardinality equal to $r_m + s_m$, see below Sec. 4.4 for a general discussion.

4.4. *Discrete operations on* UOc *immersions*: *from* UOc *to* UOb, UUc *and* UUb

Following the discussion of Sec. 4.1, we may analyze the interplay between swap and mirror transformations on $\mathcal{C}_\rho$-orbits of Y'_g (UOc immersions) by introducing

$$\begin{aligned}
x_{sm} &= \#\{\text{orbits } o \,|\, o = o_s = o_m = o_{sm}\}, \text{ i.e.} \\
&= \#\{\text{orbits that are both achiral and self-swapped}\}, \\
y_{sm} &= \#\{\text{unordered pairs } \{o, o_m\} \,|\, o = o_s \neq o_m = o_{sm}\}, \text{ i.e.} \\
&= \#\{\text{chiral pairs of self-swapped orbits}\}, \\
z_{sm} &= \#\{\text{unordered pairs } \{o, o_s\} \,|\, o = o_m \neq o_s = o_{sm}\}, \text{ i.e.} \\
&= \#\{\text{swap pairs of achiral orbits}\}, \\
v_{sm} &= \#\{\text{unordered pairs } \{o, o_s\} \,|\, o \neq o_s,\ o = o_{sm} \text{ and } o_s = o_m\}, \\
w_{sm} &= \#\{\text{4-plets of orbits } \{o, o_s, o_m, o_{sm}\} \,|\, o, o_s, o_m, o_{sm} \text{ all distinct}\}
\end{aligned}$$

hence

$$\begin{aligned}
r_s &= x_{sm} + 2y_{sm} \\
s_s &= z_{sm} + v_{sm} + 2w_{sm} \\
r_m &= x_{sm} + 2z_{sm} \\
s_m &= y_{sm} + v_{sm} + 2w_{sm}.
\end{aligned}$$

In particular for n odd, the vanishing of r_s and r_m implies $x_{sm} = y_{sm} = z_{sm} = 0$.

The five independent quantities x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm} must be determined in each Y'_g, their values are gathered in Appendix B.1. Then counting how many times each class of orbits contributes to each type of immersions, one obtains, for every genus:

$$\begin{aligned}
|\text{UOc}| &= r_s + 2s_s = r_m + 2s_m = x_{sm} + 2y_{sm} + 2z_{sm} + 2v_{sm} + 4w_{sm} \\
|\text{UOb}| &= r_s + s_s = x_{sm} + 2y_{sm} + z_{sm} + v_{sm} + 2w_{sm} \\
|\text{UUc}| &= r_m + s_m = x_{sm} + y_{sm} + 2z_{sm} + v_{sm} + 2w_{sm} \\
|\text{UUb}| &= x_{sm} + y_{sm} + z_{sm} + v_{sm} + w_{sm}.
\end{aligned} \tag{4.7}$$

For example in genus 0, and for $n = 1, \ldots, 10$, we obtain:

Unoriented S^1 in unoriented S^2:

$$\#\text{UU immersions } = 1, 2, 6, 19, 76, 376, 2194, 14614, 106421, 823832 \tag{4.8}$$

thus extending the OEIS sequence A008989 [4] and Valette's recent results [11].

From the values given in Appendix B.1, we see that $x_{sm} = y_{sm} = z_{sm} = 0$ for odd g, an empirical observation for which, we have no explanation yet.

4.5. *Discrete operations on* OOc *immersions*: *from* OOc *to* OOb, UOc, UOb, OUc *and* OUb

The previous discussion, that was applied to the set Y' and its $\mathcal{C}_\rho$ orbits (or, equivalently, to the set U and its D_n-orbits) of type UOc may be applied to the set U of Sec. 3.2 and its $\mathbb{Z}_n$-orbits of type OOc.

Proposition 2. *For σ belonging to some $\mathbb{Z}_n$-orbit of U*

(i) *$\sigma \mapsto \sigma_r := r\sigma r$ belongs to the reversed orbit, with $r = [2n, 2n-1, \ldots, 2, 1]$, (remember that $r^2 = 1$);*
(ii) *$\sigma \mapsto \sigma_m := \tau = \sigma\rho$ belongs to the mirror image of the orbit.*
(iii) *$\sigma \mapsto \sigma_s := \beta^{-1}\tau^{-1}\beta = \beta^{-1}\rho\sigma^{-1}\beta$ belongs to the swap orbit, with β the cyclic permutation $(1, 2, \ldots, 2n)$ as above.*
(iv) *$\sigma \mapsto \sigma_{rm} := r\tau r = r\sigma r\rho$ belongs to the reversed mirror orbit, and likewise for the other compositions of the commuting involutions s, r, m.*

Proof. In each case, it is clear that the transform of σ carries out the required transformation. The important point is that if σ belongs to U, i.e. satisfies (3.3), so do σ_r, σ_m and σ_s. This is obvious for σ_m; it follows from the identities $r\rho r = \rho$ for σ_r and $r\alpha_0 r = \alpha_0$ for σ_r; and from $\beta^{-1}\rho\beta\rho = \alpha_0$ for σ_s, if one remembers that $\sigma = \beta\xi$, $\xi \in \mathcal{C}_\rho$:

$$\sigma_s\rho\sigma_s^{-1}\rho = \beta^{-1}\rho\sigma^{-1}\beta\rho\beta^{-1}\sigma\rho\beta\rho = \beta^{-1}\rho\xi^{-1}\rho\xi\rho\beta\rho = \beta^{-1}\rho\beta\rho = \alpha_0. \qquad \square$$

Now define once again along the lines of (4.2)

$$\begin{aligned}
x_{sr} &= \#\{\mathbb{Z}_n\text{−orbits } o \,|\, o = o_s = o_r = o_{sr}\} \\
y_{sr} &= \#\{\text{pairs } \{o, o_r\} \,|\, o = o_s \neq o_r = o_{sr}\} \\
z_{sr} &= \#\{\text{pairs } \{o, o_s\} \,|\, o = o_r \neq o_s = o_{sr}\} \\
v_{sr} &= \#\{\text{pairs } \{o, o_s\} \,|\, o = o_{sr} \neq o_s = o_r\} \\
w_{sr} &= \#\{\text{quadruplets } (o, o_{sr}, o_r, o_s),\ \text{all non equal}\}.
\end{aligned} \tag{4.9}$$

Their values are gathered in Appendix B.2.

Then

$$\begin{aligned}
|\text{OOc}| &= x_{sr} + 2y_{sr} + 2z_{sr} + 2v_{sr} + 4w_{sr} \\
|\text{OOb}| &= x_{sr} + 2y_{sr} + z_{sr} + v_{sr} + 2w_{sr} \\
|\text{UOc}| &= x_{sr} + y_{sr} + 2z_{sr} + v_{sr} + 2w_{sr} \\
|\text{UOb}| &= x_{sr} + y_{sr} + z_{sr} + v_{sr} + w_{sr}.
\end{aligned} \tag{4.10}$$

A similar discussion can be carried out on the action of the involutions s and m on the orbits of OOc, expressing $|\text{OOc}|, |\text{OOb}|, |\text{OUc}|$ and $|\text{OUb}|$ in terms of new numbers x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm}.[f] The values of these five parameters are gathered in Appendix B.3. Then

$$\begin{aligned}
|\mathrm{OOc}| &= x_{sm} + 2y_{sm} + 2z_{sm} + 2v_{sm} + 4w_{sm} \\
|\mathrm{OOb}| &= x_{sm} + 2y_{sm} + z_{sm} + v_{sm} + 2w_{sm} \\
|\mathrm{OUc}| &= x_{sm} + y_{sm} + 2z_{sm} + v_{sm} + 2w_{sm} \\
|\mathrm{OUb}| &= x_{sm} + y_{sm} + z_{sm} + v_{sm} + w_{sm}.
\end{aligned} \tag{4.11}$$

From the values given in Appendices B.2 and B.3, one observes that x_{sr}, x_{sm}, y_{sr} and y_{sm} vanish for all (n,g), meaning that $\sigma \sim \sigma_s$ never occurs. Moreover, for n even, $z_{sr} = v_{sm} = 0$, and for n odd, $z_{sm} = v_{sr} = 0$. Those are general features:

Proposition 3. *If $\sim$ means the equivalence with respect to the adjoint action of $\mathbb{Z}_n$,*

(i) *for any n and g, $\#\{\sigma \in U \,|\, \sigma \sim \sigma_s\} = 0$, hence $x_{sr} = y_{sr} = x_{sm} = y_{sm} = 0$;*
(ii) *for any **even** n and any genus g, $\#\{\sigma \in U \,|\, \sigma \sim \sigma_r\} = 0$, hence $z_{sr} = 0$; and $\#\{\sigma \in U \,|\, \sigma \sim \sigma_{sm}\} = 0$, hence $v_{sm} = 0$.*
(iii) *for any **odd** n and any genus, $\#\{\sigma \in U \,|\, \sigma \sim \sigma_m\} = 0$, hence $z_{sm} = 0$; and $\#\{\sigma \in U \,|\, \sigma \sim \sigma_{sr}\} = 0$, hence $v_{sr} = 0$.*

Proof. First, one notices that $\sigma \nsim \sigma_s$ is certainly true for n odd, see Sec. 4.2. We thus turn to n even. We write $\sigma = \beta\xi$ as above in (1) and note that β^2 is a generator of the $\mathbb{Z}_n$ group and that ρ, β^2 and ξ are in the centralizer $\mathcal{C}_\rho$. By the homomorphism ϕ introduced in Sec. 3.1, ρ is mapped to the identity permutation of S_n and β^2 to the cyclic permutation $(1,2,\ldots,n)$, which is odd for n even. Then

(i) $\sigma \sim \sigma_s = \beta^{-1}\tau^{-1}\beta = \beta^{-1}\rho\,\xi^{-1}$ means $\exists\, p \in \{0,\ldots,n-1\}$ s.t. $\beta^{2p}(\beta\xi)\beta^{-2p} = \beta^{-1}\rho\,\xi^{-1}$, or $\beta^{2p+2}\xi\beta^{-2p} = \rho\,\xi^{-1}$. If we take the image of both sides by ϕ, the signature of the left-hand side is minus the signature of $\phi(\xi)$, while the right-hand side has the signature of $\phi(\xi)$. There is a contradiction, q.e.d.

(ii) Suppose likewise that $\sigma \sim \sigma_r = r\sigma r = \beta^{2p}(\beta\xi)\beta^{-2p}$. Conjugation of a permutation by r shifts the labels by -1 and reverses its cycles; in particular $r\beta r = \beta^{-1}$. Thus the images of $r\xi r$ and ξ by ϕ have the same signature. Notice that $r\sigma r = r\beta\xi r = (r\beta r)(r\xi r)$, using the fact that $r^2 = 1$, so that $\sigma_r = \beta^{-1} r\xi r$. Supposing that σ and σ_r are $\mathbb{Z}_n$-conjugates therefore amounts to supposing that $r\xi r$ is conjugate with $\beta^{2p+2}\xi\beta^{-2p} = \beta^2\beta^{2p}\xi\beta^{-2p}$. However the image of β^2 by ϕ is odd for n even. This contradiction completes the proof of the first part of (ii). For the second part, $\sigma_{sm} \overset{?}{\sim} \sigma$, i.e. $\sigma_{sm} = \beta^{-1}\xi^{-1} \overset{?}{=} \beta^{2p+1}\xi\beta^{-2p}$, it leads to $\xi^{-2} \overset{?}{=} \beta^2$ again in contradiction with signatures for n even, q.e.d.

[f] By "new", we mean that they are relative to the OOc orbits and that their values differ from those defined and listed below in Appendix B.1.

(iii) is again a trivial consequence of the parity of permutations: for n odd, $\sigma \sim \sigma_m = \sigma\rho$, or $\sigma \sim \sigma_{sr} = r\beta^{-1}\rho\sigma^{-1}\beta r^{-1}$ are impossible, since ρ is odd. □

Theorem 4. *For any genus g,*

$$|\mathrm{OOc}| = 2|\mathrm{OOb}| \text{ for any } n \tag{4.12}$$

$$|\mathrm{UOc}| = \begin{cases} |\mathrm{OOb}| & \text{if } n \text{ even} \\ 2|\mathrm{UOb}| & \text{if } n \text{ odd,} \end{cases} \tag{4.13}$$

$$|\mathrm{OUc}| = \begin{cases} 2|\mathrm{OUb}| & \text{if } n \text{ even} \\ |\mathrm{OOb}| & \text{if } n \text{ odd,} \end{cases} \tag{4.14}$$

$$|\mathrm{UUc}| = \begin{cases} |\mathrm{OUb}| & \text{if } n \text{ even} \\ |\mathrm{UOb}| & \text{if } n \text{ odd} \end{cases} \tag{4.15}$$

Proof. Those are consequences of relations (4.10) and of Proposition 3: (4.12) follows from $x_{sr} = y_{sr} = 0$; (4.13) follows from $z_{sr} = 0$ for n even and from $v_{sr} = 0$ for n odd. For (4.14), we perform a similar analysis relating the sets OOc, OOb, OUc, OUb: one finds that

$$2|\mathrm{OUb}| - |\mathrm{OUc}| = v_{sm} = \#\{\text{pairs } \{o, o_s\} \,|\, o = o_{sm} \neq o_s\}$$

which vanishes for n even, according to Proposition 3(ii), and that

$$|\mathrm{OUc}| - |\mathrm{OOb}| = z_{sm} = \#\{\text{pairs } \{o, o_s\} \,|\, o = o_m \neq o_s\}$$

which vanishes for n odd, according to Proposition 3(iii).

Finally (4.15), may be derived from the same analysis for the sets OUc, OUb, UUc and UUb: one finds that

$$|\mathrm{OUb}| - |\mathrm{UUc}| = \#\{o \in \mathrm{OUc} \,|\, o = o_r\}$$

which vanishes for n even; for the second relation (4.15) one may appeal to (4.7), together with $x_{sm} = y_{sm} = z_{sm} = 0$ for n odd. □

Remark. Recall that for genus 0, $|\mathrm{OOb}| = |\mathrm{OO}|$ and thus, from Theorem 4, we have $|\mathrm{UOc}| = |\mathrm{OO}|$ if n is even, and $|\mathrm{UOc}| = 2|\mathrm{UO}|$ if n is odd.

Note that as a by-product of this discussion, we have obtained now the number of spherical ($g = 0$) immersions of types OO and OU,

Oriented S^1 in oriented S^2:

$$\#\text{OO immersions } = 1, 3, 9, 37, 182, 1143, 7553, 54559, 412306, 3251240, \ldots \tag{4.16}$$

Oriented S^1 in unoriented S^2:

$$\#\text{OU immersions } = 1, 2, 6, 21, 97, 579, 3812, 27328, 206410, 1625916, \ldots \tag{4.17}$$

thus extending the OEIS sequences A008986, A008988 [4] and Valette's recent results [11].

5. Immersions of Types OO, UO, UO and UU from Cyclic Permutations of S_{2n}

5.1. *The subset $Z' = [2n]$ of $Z = S_{2n}$ and its orbits for the adjoint action of a particular S_n subgroup ("Z method")*

In the present section, we shall count the number of orbits in a particular conjugacy class of $Z = S_{2n}$, namely the set Z' of its cyclic permutations, under the action of a particular subgroup C'_ρ isomorphic with S_n. Our goal is to determine the numbers of immersions of a circle in a Riemann surface of given genus, **irrespective of the bicolorability condition**, that we introduced in the previous section. To achieve this, we first consider an oriented circle and make use of another labeling of maps by permutations of S_{2n}.

The "Z method"

Consider a map, the edges of which are oriented in a consistent way for our purpose, namely with incoming edges at each vertex next to one another, see Fig. 11. Let us label the edges of such a map by an index i running from 1 to $2n$. At each vertex, there is an involution $\rho \in [2^n] \subset S_{2n}$, which exchanges the labels of the two incoming edges, and a permutation π that yields the labels of the outgoing edges, see Fig. 11. The condition that the map has a single component amounts to saying that π has a single cycle, $\pi \in [2n]$. As before, we can fix ρ, for example to be equal to $\rho_0 = (1, 2)(3, 4) \cdots (2n-1, 2n)$, a product of n disjoint transpositions. With that convention, the integers $(2a - 1, 2a)$, $a = 1, \ldots, n$, label the ath pair of incoming edges, ordered, say, in a clockwise way. Then the number of topologically inequivalent oriented maps equals the number of orbits of $Z' = [2n]$ under the conjugate action of a subgroup of S_{2n}, made of permutations that map odd (respectively even) labels onto odd (even) labels and commute with ρ_0. As it consists of permutations of the n pairs $(2a - 1, 2a)$, $a = 1, 2, \ldots, n$, it is isomorphic with S_n and has order $n!$. We shall usually denote it by $\mathcal{C}'_\rho$.

In order to study the genus of the corresponding map, we now associate with the permutation $\pi \in S_{2n}$ another permutation $\psi_\pi \in S_{4n}$. The idea is to duplicate the edge labels, so as to label separately the left and the right-hand sides of each edge (or in the fat graph picture [28], to label independently each of the double lines): we choose to label the right-hand side of the oriented edge originally labeled by $i \in \{1, 2n\}$ by i, and its left side by $i + 2n$, see Fig. 11 bottom for an illustration.

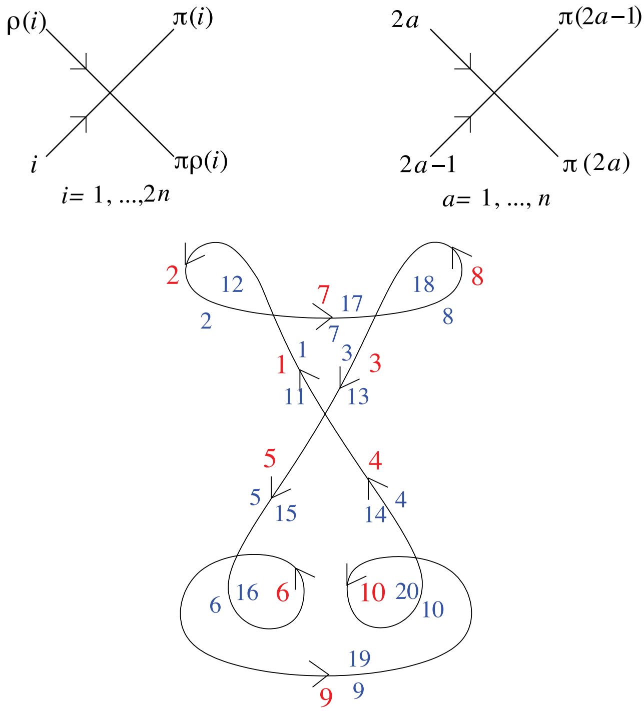


Fig. 11. Above, left: labeling oriented edges; right: special choice of ρ. Below, defining the new labeling of edges: in red the original labeling from 1 to $2n$, in blue the new one from 1 to $4n$.

The permutation ψ_π then describes the succession of these labels as each face is travelled clockwise. The transformation $\pi \mapsto \psi_\pi$ (not a group homomorphism) is easily implemented, for $1 \le i \le 2n$,

$$\psi_\pi(i) = \begin{cases} \pi(i+1) & \text{if } i \text{ is odd,} \\ i - 1 + 2n & \text{if } i \text{ is even} \end{cases}$$

$$\psi_\pi(i+2n) = \begin{cases} \pi^{-1}(i) + 1 + 2n & \text{if } \pi^{-1}(i) \text{ is odd} \\ \pi(\pi^{-1}(i) - 1) & \text{if } \pi^{-1}(i) \text{ is even.} \end{cases} \tag{5.1}$$

Example of encoding

As an example, the bottom diagram of Fig. 11 is encoded by permutations

$$\begin{aligned} \pi &= [2,7,5,1,6,9,8,3,10,4] = (1,2,7,8,3,5,6,9,10,4) \\ \psi_\pi &= [7,11,1,13,9,15,3,17,4,19,5,12,8,10,14,16,2,18,6,20] \\ &= (1,7,3)(2,11,5,9,4,13,8,17)(6,15,14,10,19)(12)(16)(18)(20). \end{aligned} \tag{5.2}$$

The genus of the map is then given by the Euler characteristics,

$$c(\psi_\pi) = n + 2 - 2g. \tag{5.3}$$

Filtering the set $Z' = [2n]$, respectively, its orbits under the action of $\mathcal{C}'_\rho$, with that criterion yields the sets Z'_g, respectively, their orbits; the number of orbits of $Z'' := Z'_0$ is the number of immersions of an oriented circle in the oriented sphere.

Theorem 5. *Call* $\rho = (1,2)(3,4)\ldots(2n-3,2n-2)(2n-1,2n) \in [2^n] \subset S_{2n}$, *using cycle notation, and* $\mathcal{C}'_\rho$, *the subgroup of* $Z = S_{2n}$, *isomorphic with* S_n, *that commutes with* ρ *and permutes odd respectively even integers among themselves. Circle immersions of the oriented circle in an oriented surface of genus* g, *or OO immersions for short, are in bijection with the orbits of* $\mathcal{C}'_\rho$ *acting by conjugacy on the set of permutations* π *that belong to* $Z' = [2n]$, *the subset of cyclic permutations of* Z, *and such that the associated permutation* $\psi_\pi \in S_{4n}$, *defined previously, satisfies the condition*

$$c(\psi_\pi) = n + 2 - 2g \tag{5.4}$$

$c(x)$ *being the function that gives the number of cycles* (*including singletons*) *of the permutation* x.

Remarks. The group $\mathcal{C}'_\rho$, isomorphic with S_n, is contained in the centralizer $\mathcal{C}_\rho$ of ρ in S_{2n}. It is generated by the pairs of transpositions $(1,3)(2,4), (1,5)(2,6), (1,7)(2,8), \ldots, (1,2n-1)(2,2n)$. Notice that $C(S_{2n}, \mathcal{C}'_\rho) = \{1, \rho\}$. One can see that $\mathcal{C}'_\rho$ is precisely the subgroup S_n of S_{2n} that allows one to build the subgroup $\mathcal{C}_\rho$ used in the previous two sections as a wreath product (see the comments in Sec. 3.1).

As already mentioned in the introduction, the 4-valent maps that we consider also define cellular embeddings of particular graphs called "simple assembly graphs without endpoints" in [9]. In this reference, the authors introduce the notion of genus range of a given graph (the set of all possible genera of surfaces in which the graph can be embedded cellularly), a notion that is also studied and generalized in [10].[g] Their work uses the same ribbon (or fat) graph construction as ours, a construction that was described in a quantum field theory context [28], and in [5] as a tool for classification of immersed curves; their encoding of maps use chord diagrams and Gauss codes. In contrast, the methods presented in this section do not use chord diagrams but introduce a way to encode the relevant graphs (and their fat partners) in terms of permutations and relate systems of representatives for different types of immersions to double cosets of appropriate finite groups (see below).

From cyclic permutations on $2n$ elements to simple closed curves with n crossings in Riemann surfaces

We described how to associate a cyclic permutation to the image of an immersion in an oriented Riemann surface of genus g, more precisely to a closed curve,

[g]We thank an anonymous referee for pointing out these two references.

drawn in the plane, with n regular crossings, and some number of virtual crossings. Conversely, associating a closed simple curve with a given cyclic permutation π belonging to S_{2n} is straightforward. One draws n four-valent vertices: four half-edges at each vertex, two ingoing, two outgoing, obeying the usual transversality (crossing) condition. If j is odd, $\pi(j)$ labels an in-going half-edge, $\pi(j)+1$ labels the in-going half-edge at the same vertex and located immediately to the right of the previous half-edge (using a clockwise orientation), $\pi(j+1)$ labels the outgoing half-edge corresponding to $\pi(j)$, and $\pi(\pi(j)+1)$ the outgoing half-edge corresponding to $\pi(j)+1$. One starts with $j=1$ and obtains in this way the four half-edges associated with some vertex. One then considers, in turns, $j=3, j=5$ etc. and the construction terminates since there is a finite number n of vertices. The last operation is to connect the half-edges carrying the same labels. Of course, the obtained closed curve, drawn on a plane, will have usually more that n crossings, but only n of them — those defined by the permutation π — should be considered as regular crossings (the others being virtual). The genus is determined by considering the associated fat graph, i.e. the permutation $\psi(\pi)$, and using the Euler formula. The fact that the obtained curve indeed corresponds to the image of an immersion is taken care of by the necessity of choosing the possibly non-zero genus determined by $\psi(\pi)$.

5.2. *A partition of Z'*

In order to get representatives, for each genus, of the orbits of Z', one may first determine the orbits, and then filter them according to the genus, this is what we shall actually do. However one can also start by partitioning the set Z' according to the genus (sets Z'_g) and determine the orbits, for each g, in a second step. This latter method is, in practice, slower than the first. It produces as a by-product, and for each positive integer n, a family of numbers $|Z'_g|$ that add up to $(2n-1)!$ since this is the cardinality of $Z' = [2n]$. The same numbers could also be obtained with the first method, proceeding backwards, by using the orbit-stabilizer theorem for each orbit of Z'_g. These integers are gathered, for the first values of n, in Fig. 12. Notice that each member of the above partition can itself be decomposed into strata corresponding to different sizes of the orbits: for instance, taking $n=5$, one gets 21552 orbits corresponding to the union of 21480 and 72 orbits with respective centralizers of order 1 and 5. We shall not display that information.

5.3. *Counting orbits and their lengths*

In order to obtain the genus decomposition for the various kinds of immersions we are interested in (OO, UO, OU and UU types), one has to use explicit cyclic permutations for these different kinds of immersions, together with the method (filtering by genus) previously described in Sec. 5.1 for OO immersions. We shall see later, in 5.5, how the introduction of discrete transformations (orientation reversal

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & 4 \quad 2 & & & \\ & & 42 & 66 & 12 & & \\ & & 780 & 2652 & 1608 & & \\ & 21552 & 132240 & & 183168 & 25920 & \\ 803760 & 7984320 & & 20815440 & & 10313280 & \\ & & & \ldots & & & \end{array}$$

Fig. 12. Numbers of elements in the sets Z'_g. Each line adds up to an odd factorial.

and mirror symmetry) on OO orbits allows us to refine the method and obtain the numbers of immersions for the types OU, UO and UU. The appearance of nontrivial stabilizers complicates the counting of orbits: in practice, one possibility is to select a random permutation σ from the given set (Z'), determine its conjugates, see whether or not one of them has already been selected, take the decision about keeping σ, or not, throw away its conjugates, and start again (a similar method amounts to implementing an appropriate variant of the Jeu de Taquin). Initially, this is what the authors did. However, the problem of selecting one representative for each orbit is simply related to a similar problem for double cosets, and one can take advantage of the fact that several computer algebra programs (in particular Magma, see our comments in Appendix A), provide fast algorithms to determine such representatives. This other approach allowed the authors to recover and extend their previous results. The relation between the two problems is summarized in the following proposition:

Proposition 4. *For each type OO, UO, OU, or UU, of immersions of the circle, a system of representatives for the orbits of the subgroup $\mathcal{C}'_\rho \simeq S_n$ acting by conjugation on the set of cyclic permutations $[2n]$, is given by the elements of the set*[h] *$\{\beta^{1/x}\}$, with $x \in H\backslash G/K$, where $\beta = (1, 2, 3, \ldots, 2n)$, where the subgroups H and K of $G = S_{2n}$ are as indicated below, and where it is understood that, we choose one representative element x (a permutation) in each double coset.*

- *For* OO, *one takes $H = Z_\beta$, the centralizer of the cyclic permutation β in G, and $K = \mathcal{C}'_\rho$.*
- *For* UO, *one takes $H = \langle Z_\beta, \sigma_r\rangle$, the subgroup of S_{2n} generated by Z_β and the permutation $(2, 2n)(3, 2n-1)(4, 2n-2)\ldots(n, n+2)$ that conjugates β and β^{-1} in S_{2n} and implements orientation reversal of the source (circle), and $K = \mathcal{C}'_\rho$.*

[h] There are two possible conventions for the product of two permutations. In this paper, we use the right-to-left product, which is *not* the convention used in [29] or [30]. Also, $\beta^y = y\beta y^{-1}$. With the other convention, one should replace $1/x = x^{-1}$ by x in the next formula, or replace the pair (H, K) by (K, H).

- *For* OU, *one takes* $H = Z_\beta$ *and* $K = \langle C'_\rho, \rho\rangle$, *the subgroup of* S_{2n} *generated by* C'_ρ *and the permutation* ρ *which describes mirroring in the target*, *see Sec.* 5.5.
- *For* UU, *one takes* $H = \langle Z_\beta, \sigma_r\rangle$ *and* $K = \langle C'_\rho, \rho\rangle$.

The proof of this proposition, in the OO case, relies on Theorem 2. We already know that the orbits for the adjoint action of K on the conjugacy class of β (the cyclic permutations) are in one-to-one correspondence with the double cosets of $H\backslash G/K$. The above proposition makes this correspondence explicit in the present situation: taking x and x' in the same double coset, we write $x' = hxk$, with $h \in H = Z_\beta$ and $k \in K = \mathcal{C}'_\rho$ and see immediately that $\beta^{1/x'} = k^{-1}\,\beta^{1/x}\,k$, so the two permutations $\beta^{1/x'}$ and $\beta^{1/x}$, which are cyclic since both conjugated of β — which is cyclic itself — by an element of G, are also conjugated by K and therefore characterize the same OO orbit. The justification for the choice of the other subgroups, appropriate to handle the immersions of types OU, UO and UU, ultimately relies on a discussion that will be carried out in the next section (discrete transformations).

Notice that, if one is interested only in counting the *total* number of immersions, i.e. summing over all genera for each of the types OO, OU, UO and UU, one does not need to determine double coset representatives since only the total number of double cosets matters. The latter can be computed up to large values of n by using Frobenius' formula (2.2) — we remind the reader that, it uses only the knowledge of the cyclic structure and size for the usual conjugacy classes: this is both simpler and faster. The results for OO, i.e. the number of orbits in Z' are displayed in Table 4, up to $n = 20$. The corresponding results for types OU, UO, UU, can also be determined[i] by using Frobenius' formula in virtually no time up to $n = 20$, those up to $n = 10$, are given in Table 8.

The drawback of this last method is that it is not constructive, so that one has to rely on the previous approaches (brute force determination of the orbits or use of double coset representatives) to go to the next step: filtering according to the

Table 4. Number of orbits in Z' (OO case).

1	1	11	1279935820810
2	4	12	53970628896500
3	22	13	2490952020480012
4	218	14	124903451391713412
5	3028	15	6761440164391403896
6	55540	16	393008709559373134184
7	1235526	17	24412776311194951680016
8	32434108	18	1613955767240361647220648
9	980179566	19	113146793787569865523200018
10	33522177088	20	8384177419658944198600637096

[i]They have been added to the OEIS: OO A260296, UU A260912, UO A260847, OU A260887.

genus. Actually, this last part is the bottleneck of the process as the function ψ_π defined in the previous section takes its values in S_{4n}.

For n a prime integer, we found an explicit formula for this number of orbits.

Proposition 5. *For n a prime integer,*

$$\#\text{orbits in } Z' = n - 1 + \frac{(2n-1)!}{n!}. \tag{5.5}$$

Proof. For any n, orbits of Z' have length $|\mathcal{C}'_\rho|/d$ with d a divisor of n, and we claim there are exactly n orbits of length $|\mathcal{C}'_\rho|/n$. The diagrams of these orbits have cyclic n-fold symmetry. At the possible price of introducing "virtual crossings", (see Introduction), these diagrams may always be drawn in the plane, in such a way that the outmost edges form a convex regular n-gon, traveled in the clockwise or counter-clockwise way, with vertices numbered from 1 to n, and with one pair of oriented edges connecting vertices i and $i+1$, and the other pair i and $i+k \mod n$, for $k = 0, 1, \ldots n-2$ ($k = n-1$ yields a n-component diagram). See Figs. 13 and 14 for illustration. □

5.4. *From orbits of Z'' to spherical immersions of type* OO

According to Theorem 5, the numbers of spherical immersions of type OO may be determined from the numbers of orbits of $Z'' = Z'_0$ and agree with those computed above in (4.16). They also appear in Table 3. The numbers in red are still to be double-checked.

5.5. *Discrete transformations of* OO *immersions. Immersions of* OU, UO *and* UU *types*

We now consider the effect of the discrete transformations r and m on immersions of OO type.

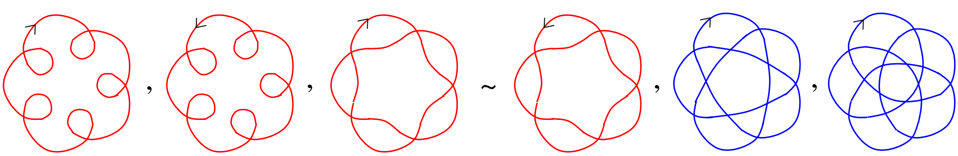


Fig. 13. The $n = 5$ orbits of Z' with 5-fold symmetry. Only the first three (in red) are spherical, the two others have higher genus (2 here); the last three are equivalent to (i.e. in the same orbit as) their reversed; in the last two, only the outmost vertices are double points, the others are "virtual crossings" as explained in the Introduction.

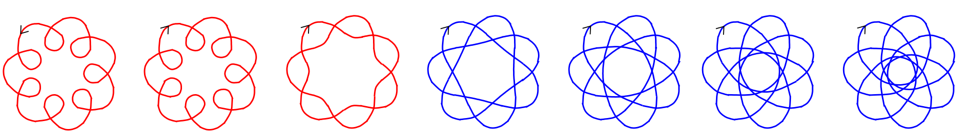


Fig. 14. The $n = 7$ orbits of Z' with symmetry of order 7. Only the first three (in red) are spherical, the others have higher genus (3 here).

Orientation reversal

Consider the effect of changing the orientation of the circle: it simply corresponds to $\pi \mapsto \pi^{-1}$. Orbits of Z' (for a given genus, in particular those of Z'') split into two classes: those for which π and π^{-1} belong to the same orbit may be called *reversible* immersions; the others form pairs of irreversible immersions.

Mirror symmetry

If some immersion is described (for ρ fixed as above) by (the orbit of) some π, its mirror image is associated with the orbit of $\pi' = \rho\pi\rho$. We call again achiral the immersions such that π and $\pi' = \rho\pi\rho$ belong to the same orbit, while the other form chiral pairs.

The five types of symmetries

o being an orbit of Z' (of given genus), we call

- o_r the orientation reversal image of o,
- o_m the chiral image of o,
- o_{rm} the chiral image of the orientation reversal image of o (or the other way around).

By combining the two previous transformations, we thus find five types of immersions that match Arnold's classification of symmetries [2]. Following our notations of Sec. 4.1, we call $x_{rm}, y_{rm}, z_{rm}, v_{rm}, w_{rm}$ their numbers of elements:

$$\begin{aligned}
x_{rm} &= \#\{\text{orbits} \,|\, o = o_r = o_m = o_{rm}\},\\
&= \text{number of orbits that are both achiral and reversible.}\\
y_{rm} &= \#\{\text{orbit pairs}\,\{o, o_m\} \,|\, o = o_r, o_m = o_{rm}\},\\
&= \text{number of chiral pairs of reversible orbits.}\\
z_{rm} &= \#\{\text{orbit pairs}\,\{o, o_r\} \,|\, o = o_m, o_r = o_{rm}\},\\
&= \text{number of irreversible pairs of achiral orbits.}\\
v_{rm} &= \#\{\text{orbit pairs }\{o, o_r\} \,|\, o \neq o_r \text{ but such that } o = o_{rm} \Leftrightarrow o_r = o_m\}.\\
w_{rm} &= \#\{\text{4-plets of orbits }\{o, o_r, o_m, o_{rm}\},\\
&\qquad \text{where all members of each 4-plet should be distinct.}
\end{aligned}$$

The values of those five parameters are gathered in Appendix B.4, and one obtains, for every genus:

$$\begin{aligned}
|\mathrm{OO}| &= x_{rm} + 2y_{rm} + 2z_{rm} + 2v_{rm} + 4w_{rm},\\
|\mathrm{UO}| &= x_{rm} + 2y_{rm} + z_{rm} + v_{rm} + 2w_{rm},\\
|\mathrm{OU}| &= x_{rm} + y_{rm} + 2z_{rm} + v_{rm} + 2w_{rm},\\
|\mathrm{UU}| &= x_{rm} + y_{rm} + z_{rm} + v_{rm} + w_{rm}.
\end{aligned} \tag{5.6}$$

In that way, we recover the number of immersions for $g = 0$ found in Sec. 4, which yields a nontrivial check on both methods.

From Theorem 4 (Eq. (4.15)), one shows that, for $g = 0$ and n odd, $v_{sm} = x_{rm} + z_{rm} + v_{rm}$ and $w_{sm} = y_{rm} + w_{rm}$, the v_{sm} and w_{sm} parameters being those defined in Sec. 4.4 — see also Appendix B1 (not B3 !); remember also that $x_{sm} = y_{sm} = z_{sm} = 0$ in that case. For $g = 0$ and n even, we notice a one-to-one equality between the same 5-plet $(x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm})$ and $(x_{rm}, y_{rm}, z_{rm}, v_{rm}, w_{rm})$, but this is only an observation for which we have no explanation, and it suggests a direct correspondence between the corresponding orbit types.

6. Miscellaneous Comments

6.1. *Asymptotics*

The number of points in the three sets X', Y', U or Z' are known explicitly and grow factorially

$$\begin{aligned} |X'_n| &= (4n-2)!! \\ |Y'_n| &= 2^{2n-1} n!(n-1)! \\ |U_n| &= 2^n n! \\ |Z'_n| &= (2n-1)!. \end{aligned} \tag{6.1}$$

Then, using the classical fact that "almost all maps are asymmetric" [15], the asymptotic numbers of orbits in X', Y' or Z' are given by

$$\begin{aligned} &\#\mathcal{C}_\sigma\text{—orbits in } X'_n \sim \frac{(4n-2)!!}{4^n n!} = \frac{1}{2}\frac{(2n-1)!}{n!} \sim n!\,\frac{2^{2n-1}}{2\pi^{1/2} n^{3/2}} \\ &\#\mathcal{C}_\rho\text{—orbits in } Y'_n \sim \#D_n\text{—orbits in } U_n \sim 2^{n-1}(n-1)! \\ &\#\mathbb{Z}_n\text{—orbits in } U_n \sim 2^n (n-1)! \\ &\#\mathcal{C}'_\rho\text{—orbits in } Z'_n \sim \frac{(2n-1)!}{n!} \sim 2\,\#\mathcal{C}_\sigma\text{—orbits in } X'_n. \end{aligned} \tag{6.2}$$

Unfortunately, we have no similar exact formulae for orbits in X'', Y'' or Z'', and we have to appeal to empirical estimates derived by physicists in similar contexts, see for example [25]. For each of the above quantities, one expects an exponential growth of the form

$$\#_n \sim \kappa n^{\gamma-3} a^n \tag{6.3}$$

with a, γ the "string susceptibility" and κ some constant, depending on the problem at hand.

Here, according to Schaeffer and Zinn-Justin [22], $\gamma = -\frac{1+\sqrt{13}}{6}$, corresponding to a central charge $c = -1$ in KPZ formula

$$\gamma_{\mathrm{KPZ}}(c) = \frac{c - 1 - \sqrt{(25-c)(1-c)}}{12} \tag{6.4}$$

(see for instance [25], Eq. (4.2)).

In genus g, one expects γ in asymptotic behavior (6.3) to be replaced by

$$\gamma \mapsto \gamma(g) = (1-g)\gamma \tag{6.5}$$

(which makes the "double scaling limit" possible). Unfortunately, the order $n = 10$ that we have reached is certainly much too low to enable one to observe the onset of this asymptotic behavior. See [22] for a discussion of the logarithmic corrections to that asymptotic behavior.

6.2. *Knot census*

Applying the previous counting of maps to the census of (alternating) knots requires eliminating various types of redundancies, kinks aka nugatory crossings, nonprime and flype equivalent diagrams, following Sundberg and Thistlethwaite's procedure [26]. See [27] for a beautiful implementation including virtual knots and links.

7. Results and Conclusion

Our results on the numbers of curves of different types and different genera are gathered in Tables 9 (bicolorable and/or bicolored immersions) and 8 (general immersions). Subtables of Table 9 can all be obtained from the U method, and also from the Y method for cases UOc, UOb, UUc and UUb. Subtables of genus 0 are also obtained from the X method. Subtables of Table 8 are obtained from the Z method. Subtable UO is also obtained from the X method. The reader will verify the various identities stated in Theorem 4 between numbers of different types of immersions.

As the case of immersions of a circle in the sphere is of particular interest, we summarize their numbers in Tables 5–7. Recall that in that case (genus 0), there is no distinction between bicolorable and general immersions. As the vast majority of immersion diagrams contain a "simple loop" (aka "kink", see for example all the diagrams in Fig. 1), it is suggested to discard them and to count only diagrams without such simple loops. In a next step, one may concentrate on diagrams that are *irreducible* and *indecomposable*, i.e. not made disconnected by removal of a vertex, respectively by cutting two distinct lines.[j]

In the formalism of Sec. 3, no simple loop means that neither σ nor $\tau = \sigma\rho$ has a cycle of length 1. Imposing indecomposability and irreducibility requires a more detailed analysis, incorporated in a Mathematica code.[k]

[j] In the knot theory terminology, diagrams with a simple loop or reducible are referred as having a "nugatory" crossing, and indecomposable ones as "prime".

[k] The numbers for OU and UU irreducible and indecomposable immersions (Table 7) have already appeared in the literature, actually up to $n = 11$ crossings, see OEIS sequences A089752 and A007756 and [23]. After completion of the first version of the present work, we learnt from Bétréma that he had been able to compute the numbers of OO, OU and UU irreducible and indecomposable immersions up to $n = 14$ [36].

Table 5. Counting of spherical immersions.

n	1	2	3	4	5	6	7	8	9	10
OO	1	3	9	37	182	1143	7553	54559	412306	3251240
UO	1	2	6	21	99	588	3829	27404	206543	1626638
OU	1	2	6	21	97	579	3812	27328	206410	1625916
UU	1	2	6	19	76	376	2194	14614	106421	823832
UOc	2	3	12	37	198	1143	7658	54559	413086	3251240

Table 6. Counting of spherical immersions with no simple loop.

n	1	2	3	4	5	6	7	8	9	10
OO immersions	0	0	1	1	2	9	29	133	594	2864
UO immersions	0	0	1	1	2	6	19	74	320	1469
OU immersions	0	0	1	1	2	5	18	70	313	1440
UU immersions	0	0	1	1	2	5	16	52	205	863
bicolored UO immersions	0	0	2	1	4	9	38	133	640	2864

Table 7. Counting of irreducible indecomposable spherical immersions.

n	1	2	3	4	5	6	7	8	9	10
OO immersions	0	0	1	1	2	6	17	73	290	1274
UO immersions	0	0	1	1	2	4	12	41	161	658
OU immersions	0	0	1	1	2	3	11	38	156	638
UU immersions	0	0	1	1	2	3	10	27	101	364
bicolored UO immersions	0	0	2	1	4	6	24	73	322	1274

Since our method is constructive and not only enumerative, we not only have the number of orbits or of immersions, but also their list, encoded in the various ways explained in the previous sections (the full listing up to $n = 10$ is available on request). This also enables us to *draw* images of these immersions. See the UU immersions for $n = 8$ and $n = 9$ in Figs. 15–17 respectively. These figures have been prepared using `DrawPD`, a routine to draw planar diagrams, within the Mathematica package "KnotTheory" written by Redelmeier [24] (the distinction between under- and over-crossings is irrelevant in the current discussion).

To summarize, what we have achieved in this paper,

- we have emphasized the role of bicolorability and made explicit 12 different types of immersions that may be considered;
- we have extended existing series of numbers of spherical immersions to $n = 10$ crossings;
- we have given tables of immersions (given here for $n = 8$ and 9 for irreducible indecomposable immersions, see Figs. 15–17, but they are available on request for the other known cases);
- we have extended to nonzero genus the counting of immersions and provided their cardinals up to $n = 9$ or 10 crossings;
- we have discovered and proved novel relations between numbers of immersions of different types, see Theorem 4.

Table 8. Counting of **general** immersions of a circle in a surface of arbitrary genus g, up to stable geotopy. U = Unoriented, O = Oriented. Figures in red should be confirmed.

n	1	2	3	4	5	6	7	8	9	10
OO, total	1	4	22	218	3028	55540	1235526	32434108	980179566	33 522 177088
OO, $g=0$	1	3	9	37	182	1143	7553	54559	412306	3251240
OO, $g=1$	0	1	11	113	1102	11114	112846	1160532	12038974	
OO, $g=2$	0	0	2	68	1528	28947	491767	7798139	117668914	
OO, $g=3$	0	0	0	0	216	14336	554096	16354210	407921820	
OO, $g=4$	0	0	0	0	0	0	69264	7066668	397094352	
OO, $g=5$	0	0	0	0	0	0	0	0	45043200	
UO, total	1	3	13	121	1538	28010	618243	16223774	490103223	16761330464
UO, $g=0$	1	2	6	21	99	588	3829	27404	206543	1626638
UO, $g=1$	0	1	6	64	559	5656	56528	581511	6020787	
UO, $g=2$	0	0	1	36	772	14544	246092	3900698	58838383	
UO, $g=3$	0	0	0	0	108	7222	277114	8180123	203964446	
UO, $g=4$	0	0	0	0	0	0	34680	3534038	198551464	
UO, $g=5$	0	0	0	0	0	0	0	0	22521600	
OU, total	1	3	14	120	1556	27974	618824	16223180	490127050	16761331644
OU, $g=0$	1	2	6	21	97	579	3812	27328	206410	1625916
OU, $g=1$	0	1	6	62	559	5614	56526	580860	6020736	
OU, $g=2$	0	0	2	37	788	14558	246331	3900740	58842028	
OU, $g=3$	0	0	0	0	112	7223	277407	8179658	203974134	
OU, $g=4$	0	0	0	0	0	0	34748	3534594	198559566	
OU, $g=5$	0	0	0	0	0	0	0	0	22524176	
UU, total	1	3	12	86	894	14715	313364	8139398	245237925	8382002270
UU, $g=0$	1	2	6	19	76	376	2194	14614	106421	823832
UU, $g=1$	0	1	5	45	335	3101	29415	295859	3031458	
UU, $g=2$	0	0	1	22	427	7557	124919	1961246	29479410	
UU, $g=3$	0	0	0	0	56	3681	139438	4098975	102054037	
UU, $g=4$	0	0	0	0	0	0	17398	1768704	99304511	
UU, $g=5$	0	0	0	0	0	0	0	0	11262088	

Table 9. Counting of **bicolorable** immersions of a circle of arbitrary genus g, up to stable geotopy. U = Unoriented, O = Oriented, Oc = Oriented bicolored, Ob = Oriented bicolorable, etc. Figures in red should be confirmed.

n	1	2	3	4	5	6	7	8	9	10
OOc, total	2	6	20	108	776	7772	92172	1291048	20644140	
OOc, $g=0$	2	6	18	74	364	2286	15106	109118	824612	6502480
OOc, $g=1$	0	0	2	32	340	3780	40612	436368	4675012	
OOc, $g=2$	0	0	0	2	72	1630	31510	549334	8883620	
OOc, $g=3$	0	0	0	0	0	76	4944	188356	5508120	
OOc, $g=4$	0	0	0	0	0	0	0	7872	752776	
OOb, total	1	3	10	54	388	3886	46086	645524	10322070	
OOb, $g=0$	1	3	9	37	182	1143	7553	54559	412306	3251240
OOb, $g=1$	0	0	1	16	170	1890	20306	218184	2337506	
OOb, $g=2$	0	0	0	1	36	815	15755	274667	4441810	
OOb, $g=3$	0	0	0	0	0	38	2472	94178	2754060	
OOb, $g=4$	0	0	0	0	0	0	0	3936	376388	
UOc, total	2	3	14	54	420	3886	46470	645524	10328214	
UOc, $g=0$	2	3	12	37	198	1143	7658	54559	413086	3251240
UOc, $g=1$	0	0	2	16	186	1890	20516	218184	2340106	
UOc, $g=2$	0	0	0	1	36	815	15812	274667	4443518	
UOc, $g=3$	0	0	0	0	0	38	2484	94178	2754988	
UOc, $g=4$	0	0	0	0	0	0	0	3936	376516	
UOb, total	1	2	7	30	210	1973	23235	323182	5164107	
UOb, $g=0$	1	2	6	21	99	588	3829	27404	206543	1626638
UOb, $g=1$	0	0	1	8	93	945	10258	109092	1170053	
UOb, $g=2$	0	0	0	1	18	421	7906	137585	2221759	
UOb, $g=3$	0	0	0	0	0	19	1242	47089	1377494	
UOb, $g=4$	0	0	0	0	0	0	0	2012	188258	
OUc, total	1	4	10	60	388	3920	46086	645928	10322070	
OUc, $g=0$	1	4	9	42	182	1158	7553	54656	412306	3251832
OUc, $g=1$	0	0	1	16	170	1890	20306	218184	2337506	
OUc, $g=2$	0	0	0	2	36	834	15755	274922	4441810	
OUc, $g=3$	0	0	0	0	0	38	2472	94178	2754060	
OUc, $g=4$	0	0	0	0	0	0	0	3988	376388	
OUb, total	1	2	7	30	210	1960	23276	322964	5165732	
OUb, $g=0$	1	2	6	21	97	579	3812	27328	206410	1625916
OUb, $g=1$	0	0	1	8	93	945	10256	109092	1170002	
OUb, $g=2$	0	0	0	1	20	417	7948	137461	2222562	
OUb, $g=3$	0	0	0	0	0	19	1260	47089	1378256	
OUb, $g=4$	0	0	0	0	0	0	0	1994	188502	
UUc, total	1	2	7	30	210	1960	23235	322964	5164107	
UUc, $g=0$	1	2	6	21	99	579	3829	27328	206543	1625916
UUc, $g=1$	0	0	1	8	93	945	10258	109092	1170053	
UUc, $g=2$	0	0	0	1	18	417	7906	137461	2221759	
UUc, $g=3$	0	0	0	0	0	19	1242	47089	1377494	
UUc, $g=4$	0	0	0	0	0	0	0	1994	188258	
UUb, total	1	2	7	26	152	1168	12548	165742	2605526	
UUb, $g=0$	1	2	6	19	76	376	2194	14614	106421	823832
UUb, $g=1$	0	0	1	6	63	539	5508	56067	592457	
UUb, $g=2$	0	0	0	1	13	242	4183	70118	1119180	
UUb, $g=3$	0	0	0	0	0	11	663	23907	692749	
UUb, $g=4$	0	0	0	0	0	0	0	1036	94719	

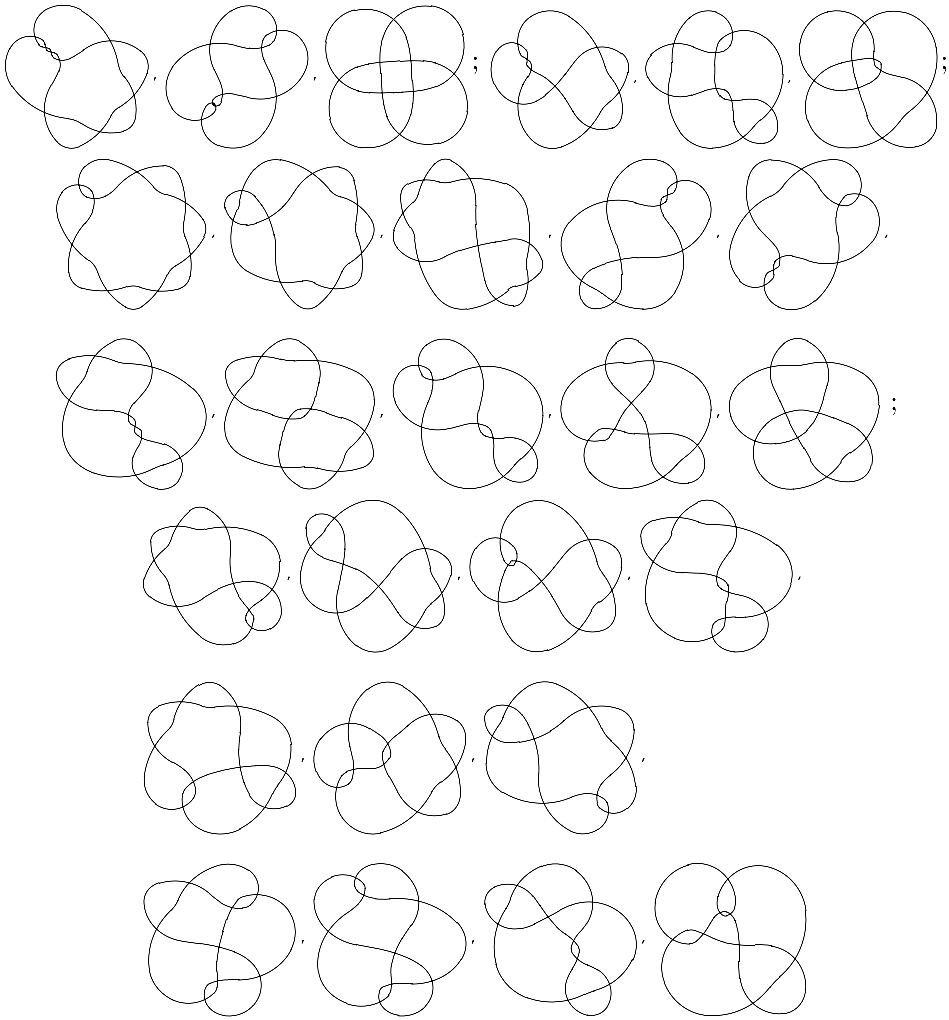


Fig. 15. The 27 indecomposable irreducible immersions of an unoriented circle in an unoriented sphere with $n = 8$ double points. They may also serve as colored and/or oriented immersions: the first three are invariant both under swapping (color swap or undercrossing $\leftrightarrow$ overcrossing) and mirror symmetry; the next three are swapping invariant but have a mirror partner; the next 10 have identical swap, mirror and orientation-reversal images; and the last 11 give rise to four images under swapping and mirror symmetry. In the notations of Sec. 4.4, the values of $x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm}$ restricted to this set of indecomposable irreducible immersions read $(3, 3, 0, 10, 11)$. (For all, the effect of orientation-reversal is the same as swapping.) We thus have $3+3+10+11 = 27$ immersions of type UU; $3+6+10+22 = 41$ of type UO; $3+3+10+22 = 38$ of type OU; and $3+6+20+44 = 73$ OO or bicolored UO immersions.

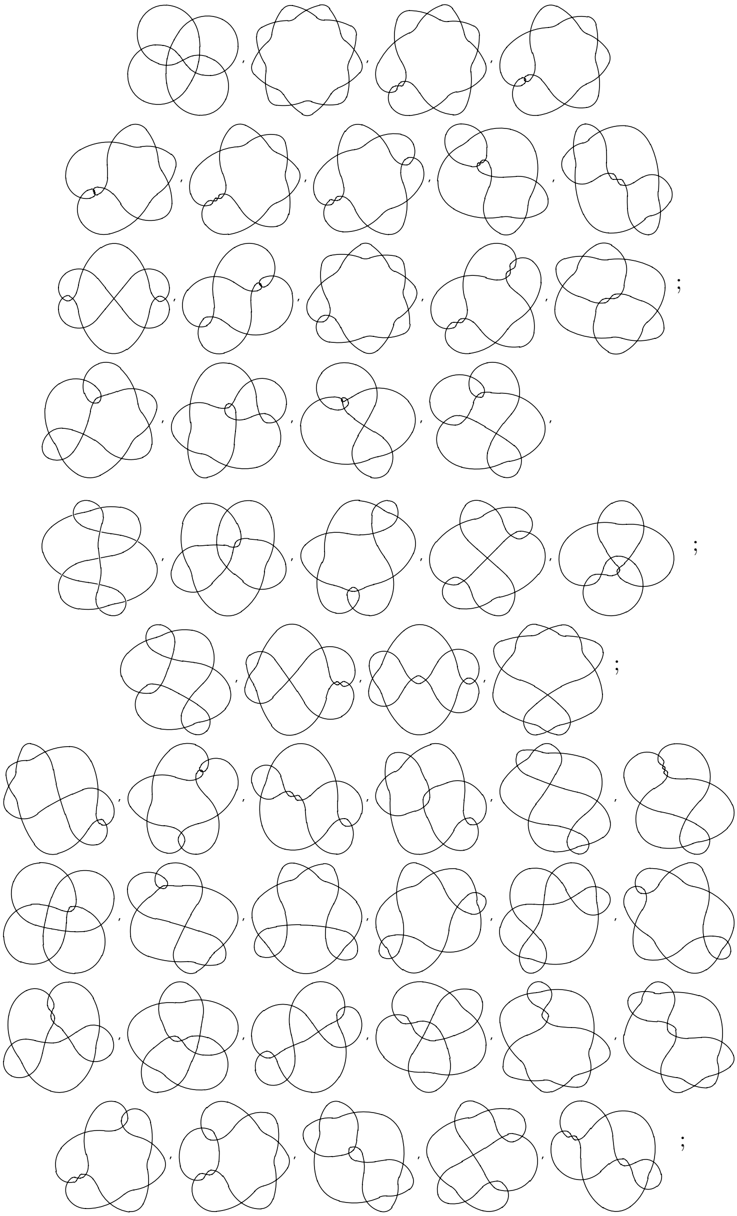


Fig. 16. The 101 indecomposable irreducible immersions of an unoriented circle in an unoriented sphere with $n = 9$ double points: on that figure, the $x_{rm} = 14$ immersions such that $\sigma \sim \sigma_m \sim \sigma_r$; the $y_{rm} = 9$ ones such that $\sigma \sim \sigma_r \nsim \sigma_m$; the $z_{rm} = 4$ ones, such that $\sigma \sim \sigma_m \nsim \sigma_r$; the $v_{rm} = 23$ ones such that $\sigma \sim \sigma_{rm} \nsim \sigma_r$; next figure, the $w_{rm} = 51$ with no symmetry.

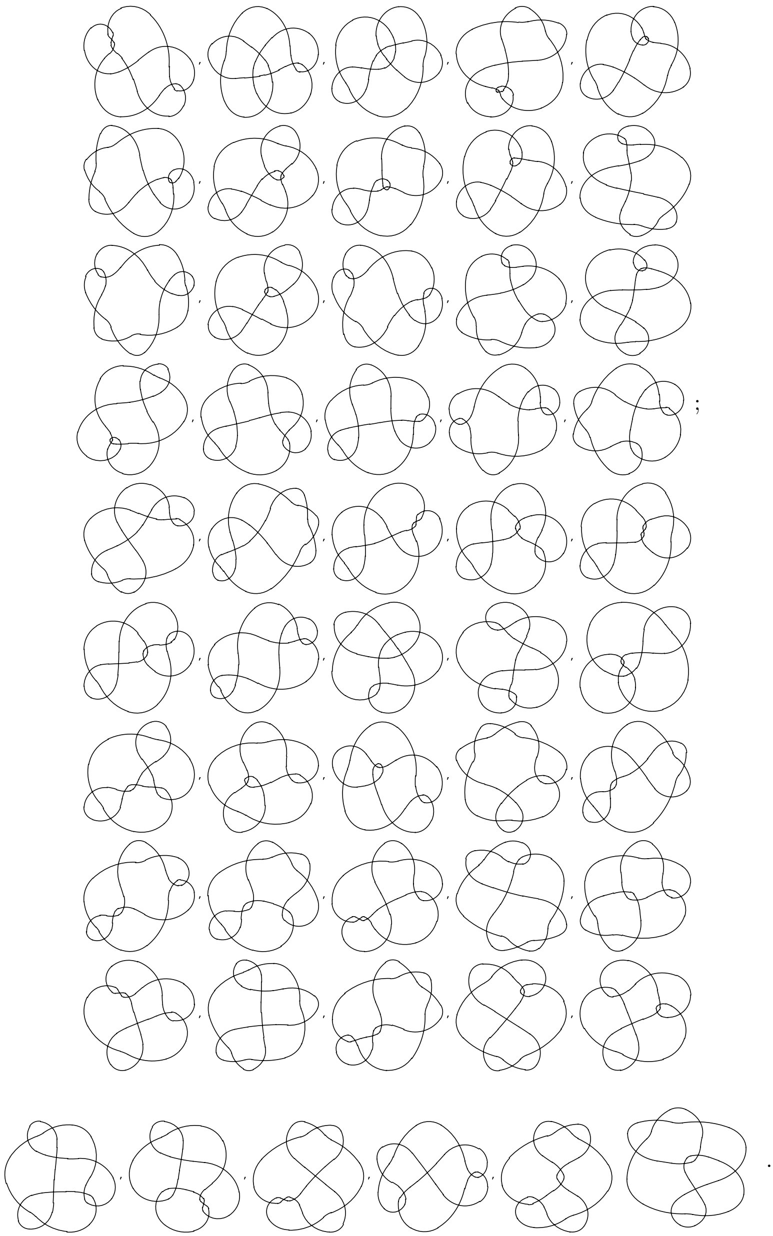


Fig. 17. The 101 indecomposable irreducible immersions of an unoriented circle in an unoriented sphere with $n = 9$ double points (continued): the 51 immersions with no symmetry.

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Appendix A. Details About the Algorithms

All the tables found in the present paper, using the methods and algorithms discussed in the different sections, have been generated using computer programs written both in Mathematica [29] and Magma [30]. Magma implements fast algorithms to determine explicitly the conjugates of a chosen group element with respect to some subgroup of the permutation group, and to test, whether two elements are conjugated, this allows one to determine orbit representatives. Magma can also determine very quickly the centralizer of a group element in a given subgroup of a permutation group; this feature is used in many places in our calculations, for instance, when we determine the orbit sizes. We implemented in Magma the Frobenius formula (2.2), that only uses the cardinality of the absolute or relative conjugacy classes (i.e. relative to the whole permutation group, or relative to specific subgroups); as the determination of the size of such conjugacy classes, together with representatives elements for each class, is very fast in Magma, our algorithm turns out to be much faster than the available commands giving the size of double cosets.

In Sec. 2.1, we work in S_{4n} to study immersions with n crossings, and the number of permutations to be handled becomes unfortunately very high, even for modern processors; it becomes time and memory consuming to go beyond $n = 6$ by this technique.

In Sec. 3.1, for low values of n (up to 6) a direct enumeration of all elements of Y' and an explicit construction of their orbits, together with the different kinds of immersions, was possible both in Mathematica and Magma. Initially, our first method, for larger values of n, up to 9, was to perform, using Mathematica, a random sampling of Y' followed by the determination of a typical representative of each orbit of Y', therefore giving a list of orbits. The sampling was continued until the results stabilize and the procedure was finally certified by checking the sum rule $\sum_{\text{orbits}\, o} \ell_o = |Y'| = 2^{2n-1}(n-1)!n!$, where the length ℓ_o of each orbit o was determined independently by use of Magma (determination of the order of stabilizers of orbits points).

Replacing the sets Y (actually Y') by the $\mathcal{C}_\rho$ left coset U, and the adjoint action of $\mathcal{C}_\rho$ by the action of its dihedral subgroup or of the appropriate cyclic subgroup of the latter, allowed us, at a later stage (see Sec. 3.2), to recover all these results, including the determination of representatives for all orbits of all kinds of immersions, up to $n = 9$, by a direct enumeration of all elements of U, using

Magma. A comparison between the lengths of orbits obtained for these different group actions will be done below, together with a particular example. For $n = 10$, we could not determine representatives for the orbits of Y' or U. We used again the same random sampling method for genus 0 until the results stabilize, but, unfortunately as we have no *a priori* knowledge of $|Y''|$, we had no way to check the correctness of the result by using a sum rule.

Finally, the orbits for the adjoint action of (a particular subgroup) S_n on the cyclic permutations of S_{2n}, leading to the number of immersions of type OO, OU and UO, with no constraint of bicolorability (the "Z method" of Sec. 5), were obtained both using Mathematica (random sampling) and Magma (full enumeration of orbits), up to $n = 9$ for all genera, and $n = 10$ in genus 0. Remember that representative elements of orbits are needed in order to consider the effects of the five types of symmetries that match Arnold's classification. The number of orbits in $Z' = [2n]$ itself (Table 4), aka the total number of immersions of OO type (summing over all genera), and its variants of types UO, OU, UU, were calculated using both Magma and a Frobenius formula on double cosets, see also our comments in Sec. 5.3. The number of immersions of type OO, OU, UO and UU were then quickly recovered by using double cosets and Proposition 4; this latter method gives however slighly less information than the former (full enumeration of orbits), since it does not determine the five parameters $x_{rm}, y_{rm}, z_{rm}, v_{rm}, w_{rm}$ describing symmetries of orbit types.

More on the action of $\mathcal{C}_\rho$ and D_n on U

The restriction of the action of $\mathcal{C}_\rho$ to its subgroup D_n, defines an action of the latter for which the set U is stable: the points of intersection between U and a given orbit of $\mathcal{C}_\rho$ define an orbit of D_n, of length $|D_n|/k = 2n/k$, where $x \in U$ and $k = |C(D_n, x)|$. On the other hand, the orbit of $\mathcal{C}_\rho$ going through x has length $|\mathcal{C}_\rho|/k'$, where $k' = |C(\mathcal{C}_\rho, x)|$.

We shall prove below that $k' = k$, but let us take this property for granted at the moment. As the discussion can be carried out independently for the different genera, let us call $\mu(k)$, the number of elements of the left coset U, of fixed genus (call U_g this subset), whose centralizer in D_n has order k. These $\mu(k)$ elements can be gathered into $\mu(k)/(2n/k)$ orbits of D_n, but each orbit of D_n determines one orbit of $\mathcal{C}_\rho$, so these $\mu(k)$ elements of U determine $\lambda(k) = \frac{k}{2n}\mu(k)$ orbits of $\mathcal{C}_\rho$, of length $|\mathcal{C}_\rho|/k$. This discussion is summarized in the following proposition:

Proposition A.1. *For any genus g, and for all x in U_g, the following two centralizer subgroups are equal: $C(\mathcal{C}_\rho, x) = C(D_n, x)$. Denoting by k their common order, we call $\mu(k) = \#\{x \in U_g : |C(D_n, x)| = k\}$. The number of orbits of length $2n/k$, for the adjoint action of D_n on U_g, is equal to $\lambda(k) = \frac{k}{2n}\,\mu(k)$. With the notations of the text, $\lambda(k)$ is also the number of orbits of length $|\mathcal{C}_\rho|/k$, for the adjoint action of $\mathcal{C}_\rho$ on the set Y'_g.*

Proof. It remains to prove that $C(\mathcal{C}_\rho, x) = C(D_n, x)$ for all $x \in U$, hence $k = k'$ as stated previously. One inclusion ($C(D_n, x) \subset C(\mathcal{C}_\rho, x)$) is obvious, since $D_n \subset \mathcal{C}_\rho$. Now take y in $C(\mathcal{C}_\rho, x)$, so $y \in \mathcal{C}_\rho$ and $yx = xy$; since $U = \beta\mathcal{C}_\rho$, one can write $x = \beta z$ for some $z \in \mathcal{C}_\rho$, and the commutation property reads $y\beta z = \beta z y$, equivalently $y = \beta(zyz^{-1})\beta^{-1}$. But $zyz^{-1} \in \mathcal{C}_\rho$ so $y \in \mathcal{C}_\rho^\beta$. The conclusion is that $y \in \mathcal{C}_\rho \cap \mathcal{C}_\rho^\beta$, but the latter subgroup coincides with D_n (this way of defining D_n was used in Secs. 3.1 and 3.2). So, we have also $C(\mathcal{C}_\rho, x) \subset C(D_n, x)$, hence the equality. □

Proposition A.1 has a practical value: identifying distincts orbits of Y', under the adjoint action of $\mathcal{C}_\rho$ is a time-consuming task that is replaced by the calculation of the order of a (small) finite group associated with the elements of a left coset U of that group: this is much faster. The result is illustrated on the following example: With k, the order of the centralizer $C(D_n, x)$ of x in U_0 (the subset of the permutations of genus 0 belonging to the left coset U), and using the notation $k^{\mu(k)=\#\text{orbits of length}|D_n|/k}$, one obtains, for $n = 5$, the following sizes and numbers of D_n orbits: $1^{1640}\ 2^{150}\ 5^4\ 10^2$, with $|U_0| = 1796$, the number of long (open) spherical curves. The number and sizes of orbits of Y', under the adjoint action of the group $\mathcal{C}_\rho$ is given by a similar formula with the "exponent" multiplied by the correcting factor $k/2n$, so that we get, instead, $1^{164}\ 2^{30}\ 5^2\ 10^2$, for a total of 198 orbits (UOc bicolored spherical immersions). A similar analysis can be done, if we replace the dihedral subgroup D_n by its cyclic subgroup Z_n, the correcting factor being this time equal to k/n: we have $1^{1790}5^6$ Z_n-orbits in U_0 and $1^{358}5^6$ Z_n-orbits in Y', for a total of 364 orbits (OOc bicolored spherical immersions).

Typical CPU time (*T*) and memory (*M*) for calculations done on a MacBookPro 2.8 GHz Intel Core i7, leading to the results given in Table 9 (bicolorable and/or bicolored immersions) are as follows: $n \le 4 : T < 0.4\,\text{s}$, $M < 32\,\text{MB}$; $n = 5 : T = 0.63\,\text{s}$, $M < 32\,\text{MB}$; $n = 6 : T = 4.37\,\text{s}$, $M < 32\,\text{MB}$; $n = 7 : T = 70.64\,\text{s}$, $M = 116.88\,\text{MB}$; $n = 8 : T = 3285\,\text{s}$, $M = 1316.81\,\text{MB}$. For $n = 9$, calculations were done genus by genus on a faster machine, with a large amount of available random access memory, but the results for each genus nevertheless required several hours of computer time. For $n = 10$, the enumerative algorithm was traded for a sampling method (see above), implemented in Mathematica, and required several weeks of CPU. With the exception of the total number of immersions (summing over genera) of all types, obtained (up to $n = 20$) by a fast algorithm using double cosets, calculations leading to Table 8 (general immersions) are significantly slower and use more memory than the previous ones because, we use the whole class of cyclic permutation (growing like $(2n-1)!$ for n crossings). They could nevertheless be performed with enumerative methods up to $n = 9$. Typical values are as follows: $n = 6 : T = 12\,\text{s}$, $M < 32\,\text{MB}$; $n = 7 : T = 340\,\text{s}$, $M = 258\,\text{MB}$; $n = 8 : T = 6293\,\text{s}$, $M = 4934\,\text{MB}$; $n = 9 : T = 106893\,\text{s}$, $M = 124.5\,\text{GB}$.

Appendix B

B.1. *The five parameters x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm} for the $\mathcal{C}_\rho$-orbits of Y' (or for the D_n-orbits of U)*

$$x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad n = 1$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad n = 2$$

$$\begin{pmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad n = 3$$

$$\begin{pmatrix} 5 & 0 & 0 & 12 & 2 \\ 0 & 0 & 0 & 4 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad n = 4$$

$$\begin{pmatrix} 0 & 0 & 0 & 53 & 23 \\ 0 & 0 & 0 & 33 & 30 \\ 0 & 0 & 0 & 8 & 5 \end{pmatrix} \qquad n = 5$$

$$\begin{pmatrix} 9 & 12 & 3 & 152 & 200 \\ 0 & 0 & 0 & 133 & 406 \\ 7 & 10 & 6 & 50 & 169 \\ 0 & 0 & 0 & 3 & 8 \end{pmatrix} \qquad n = 6$$

$$\begin{pmatrix} 0 & 0 & 0 & 559 & 1635 \\ 0 & 0 & 0 & 758 & 4750 \\ 0 & 0 & 0 & 460 & 3723 \\ 0 & 0 & 0 & 84 & 579 \end{pmatrix} \qquad n = 7$$

$$\begin{pmatrix} 39 & 105 & 29 & 1756 & 12685 \\ 0 & 0 & 0 & 3042 & 53025 \\ 47 & 228 & 104 & 2500 & 67239 \\ 0 & 0 & 0 & 725 & 23182 \\ 10 & 39 & 21 & 29 & 937 \end{pmatrix} \qquad n = 8$$

$$\begin{pmatrix} 0 & 0 & 0 & 6299 & 100122 \\ 0 & 0 & 0 & 14861 & 577596 \\ 0 & 0 & 0 & 16601 & 1102579 \\ 0 & 0 & 0 & 8004 & 684745 \\ 0 & 0 & 0 & 1180 & 93539 \end{pmatrix} \qquad n = 9,$$

where, for each n, successive rows correspond to increasing genus

$$g = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor \quad \text{(for } n > 2\text{)}.$$

For $n = 10$, we have only the genus 0 data:

$$(x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm})_{g=0} = \begin{pmatrix} 98, 969, 247, 20681, 801837 \end{pmatrix} \quad n = 10.$$

B.2. *The five parameters x_{sr}, y_{sr}, z_{sr}, v_{sr}, w_{sr} for the Z_n-orbits of U*

$$x_{sr},\ y_{sr},\ z_{sr},\ v_{sr},\ w_{sr}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad n = 1$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad n = 2$$

$$\begin{pmatrix} 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad n = 3$$

$$\begin{pmatrix} 0 & 0 & 0 & 5 & 16 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad n = 4$$

$$\begin{pmatrix} 0 & 0 & 16 & 0 & 83 \\ 0 & 0 & 16 & 0 & 77 \\ 0 & 0 & 0 & 0 & 18 \end{pmatrix} \qquad n = 5$$

$$\begin{pmatrix} 0 & 0 & 0 & 33 & 555 \\ 0 & 0 & 0 & 0 & 945 \\ 0 & 0 & 0 & 27 & 394 \\ 0 & 0 & 0 & 0 & 19 \end{pmatrix} \qquad n = 6$$

$$\begin{pmatrix} 0 & 0 & 105 & 0 & 3724 \\ 0 & 0 & 210 & 0 & 10048 \\ 0 & 0 & 57 & 0 & 7849 \\ 0 & 0 & 12 & 0 & 1230 \end{pmatrix} \qquad n = 7$$

$$\begin{pmatrix} 0 & 0 & 0 & 249 & 27155 \\ 0 & 0 & 0 & 0 & 109092 \\ 0 & 0 & 0 & 503 & 137082 \\ 0 & 0 & 0 & 0 & 47089 \\ 0 & 0 & 0 & 88 & 1924 \end{pmatrix} \qquad n = 8$$

$$\begin{pmatrix} 0 & 0 & 780 & 0 & 205763 \\ 0 & 0 & 2600 & 0 & 1167453 \\ 0 & 0 & 1708 & 0 & 2220051 \\ 0 & 0 & 928 & 0 & 1376566 \\ 0 & 0 & 128 & 0 & 188130 \end{pmatrix} \quad n = 9.$$

B.3. *The five parameters*[1] x_{sm}, y_{sm}, z_{sm}, v_{sm}, w_{sm} *for the* Z_n*-orbits of* U

$$x_{sm},\ y_{sm},\ z_{sm},\ v_{sm},\ w_{sm}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad n = 1$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad n = 2$$

$$\begin{pmatrix} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad n = 3$$

$$\begin{pmatrix} 0 & 0 & 5 & 0 & 16 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad n = 4$$

$$\begin{pmatrix} 0 & 0 & 0 & 12 & 85 \\ 0 & 0 & 0 & 16 & 77 \\ 0 & 0 & 0 & 4 & 16 \end{pmatrix} \qquad n = 5$$

$$\begin{pmatrix} 0 & 0 & 15 & 0 & 564 \\ 0 & 0 & 0 & 0 & 945 \\ 0 & 0 & 19 & 0 & 398 \\ 0 & 0 & 0 & 0 & 19 \end{pmatrix} \qquad n = 6$$

$$\begin{pmatrix} 0 & 0 & 0 & 71 & 3741 \\ 0 & 0 & 0 & 206 & 10050 \\ 0 & 0 & 0 & 141 & 7807 \\ 0 & 0 & 0 & 48 & 1212 \end{pmatrix} \qquad n = 7$$

$$\begin{pmatrix} 0 & 0 & 97 & 0 & 27231 \\ 0 & 0 & 0 & 0 & 109092 \\ 0 & 0 & 255 & 0 & 137206 \\ 0 & 0 & 0 & 0 & 47089 \\ 0 & 0 & 52 & 0 & 1942 \end{pmatrix} \qquad n = 8$$

[1]They should not be confused with those of Appendix B.1.

$$\begin{pmatrix} 0 & 0 & 0 & 514 & 205896 \\ 0 & 0 & 0 & 2498 & 1167504 \\ 0 & 0 & 0 & 3314 & 2219248 \\ 0 & 0 & 0 & 2452 & 1375804 \\ 0 & 0 & 0 & 616 & 187886 \end{pmatrix} \quad n = 9,$$

where, for each $n > 2$, successive rows correspond to increasing genus $g = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

B.4. *The five parameters x_{rm}, y_{rm}, z_{rm}, v_{rm}, w_{rm} for the S_n-orbits of Z'*

$$x_{rm},\ y_{rm},\ z_{rm},\ v_{rm},\ w_{rm}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad n = 1$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad n = 2$$

$$\begin{pmatrix} 3 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad n = 3$$

$$\begin{pmatrix} 5 & 0 & 0 & 12 & 2 \\ 7 & 4 & 2 & 17 & 15 \\ 2 & 1 & 2 & 4 & 13 \end{pmatrix} \qquad n = 4$$

$$\begin{pmatrix} 10 & 3 & 1 & 42 & 20 \\ 10 & 3 & 3 & 98 & 221 \\ 4 & 6 & 22 & 56 & 339 \\ 0 & 0 & 4 & 0 & 52 \end{pmatrix} \qquad n = 5$$

$$\begin{pmatrix} 9 & 12 & 3 & 152 & 200 \\ 34 & 82 & 40 & 472 & 2473 \\ 25 & 58 & 72 & 473 & 6929 \\ 12 & 48 & 49 & 79 & 3493 \end{pmatrix} \qquad n = 6$$

$$\begin{pmatrix} 35 & 35 & 18 & 506 & 1600 \\ 60 & 75 & 73 & 2169 & 27038 \\ 53 & 182 & 421 & 3272 & 120991 \\ 12 & 60 & 353 & 1397 & 137616 \\ 0 & 48 & 116 & 0 & 17234 \end{pmatrix} \qquad n = 7$$

$$\begin{pmatrix} 39 & 105 & 29 & 1756 & 12685 \\ 160 & 1165 & 514 & 9533 & 284487 \\ 199 & 1529 & 1571 & 20024 & 1937923 \\ 194 & 2921 & 2456 & 15177 & 4078227 \\ 36 & 686 & 1242 & 2092 & 1764648 \end{pmatrix} \quad n = 8$$

and in genus 0,

$$n = 9 \quad (x_{rm}, y_{rm}, z_{rm}, v_{rm}, w_{rm}) = \big(124\ 328\ 195\ 5980\ 99794\big),$$

$$n = 10 \quad (x_{rm}, y_{rm}, z_{rm}, v_{rm}, w_{rm}) = \big(98\ 969\ 247\ 20681\ 801837\big).$$

Appendix C

The following appendix recalls how certain integrals over real, complex or matrix variables enable one, through their Feynman diagram interpretation, to construct generating functions of maps and in some cases, to compute the cardinals of some classes of maps.

C.1. *The diagrammatic expansion of matrix integrals*

Let us consider the integral over a set of f $N \times N$ Hermitian matrices $M_a, a = 1, \dots, f$

$$Z_X = \int \left(\prod_{a=1}^{f} DM_a \right) \exp -N \left[\frac{1}{2} \sum_{a=1}^{f} \mathrm{tr}(M_a)^2 - \frac{\gamma}{4} \sum_{a,b=1}^{f} \mathrm{tr}(M_a M_b)^2 \right] \qquad \text{(C.1)}$$

(initially defined for $\Re\gamma \leq 0$ and implicitly normalized by dividing by the Gaussian integral at $\gamma = 0$). The measure DM is the natural integration measure over Hermitian matrices, $DM = \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}\Re M_{ij}\, \mathrm{d}\Im M_{ij}$. The integrand and the measure are clearly invariant, under orthogonal transformations of the M's, $M_a \mapsto M'_a = \sum_{a'=1}^{f} O_{aa'} M_{a'}$, $O \in \mathrm{O}(f)$.

We are mostly interested in the series expansion in powers of γ of $F = \log Z_X$, the "free energy" in physicists' parlance,

$$F = \sum_{n=1}^{\infty} \gamma^n F_n. \qquad \text{(C.2)}$$

This expansion may be obtained by diagrammatic rules, in terms of 4-valent connected maps. As is well known, since 't Hooft [28], it is fruitful to represent Feynman diagrams arising from the expansion of F with double lines, associated with the matrix indices of the M's; for reviews, see [13, 31, 33]. The resulting diagrams, sometimes called "fat graphs", are in fact maps in the combinatorial sense; moreover, here, each line across a vertex is decorated with an index a (or b) running over f values, referred to as "flavor". This flavor will enable us to identify the number of components when we regard the map as a multi-component curve or an alternating link or knot diagram.

The diagrammatic rules are the following (see Fig. C1): for a given map, to each vertex, assign a weight $\frac{\gamma}{4}N$; to each "component", assign a weight f (arising from the summation over the running index a); to each "index loop", i.e. each face of the map, assign a weight N (arising from the summation over matrix indices $i, j = 1, \dots, N$); and to each edge, a factor N^{-1}. Each map then carries a power of N equal to the Euler characteristics of the closed compact orientable Riemann surface spanned by its faces, namely N^{2-2g}.

If F in (C.2) is written as

$$F = \sum_{g\ge 0}\sum_{c\ge 1} N^{2-2g} f^c \sum_{n\ge 1}\gamma^n F_n^{(g,c)} \tag{C.3}$$

then $F_n^{(g,c)}$, is by the previous rules the product of $\frac{1}{n!}\left(\frac{1}{4}\right)^n$ times the number of *labeled maps* with genus g, n vertices and c components. In other words $F^{(g,c)}(\gamma) := \sum_n \gamma^n F_n^{(g,c)}$ is the exponential generating function of labeled maps of given genus g and number c of components, and with n vertices. In the present paper, we are focusing on one-component diagrams, whose generating function is

$$F^{[1\mathrm{c}]} = f\sum_{n\ge 1}\gamma^n \sum_{g\ge 0} N^{2-2g}\, F_n^{(g,1)}. \tag{C.4}$$

In the formalism of Sec. 2,

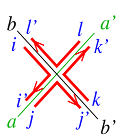
$$F^{(g,1)}(\gamma) = \sum_{n\ge 1}\frac{1}{n!}\left(\frac{\gamma}{4}\right)^n \#\{\tau \text{ satisfying (I) and (II)}_{\mathrm{g}}\} \tag{C.5}$$

hence

$$F_n^{(g,1)} = \frac{1}{4^n n!}|X'_{gn}| = \sum_{\substack{\mathcal{C}_\sigma\text{-orbits } o\\ \text{of } X'_g}} \frac{\ell_o}{4^n n!} = \sum_{\substack{\text{orbits } o\\ \text{of } X'_g}} \frac{1}{d_o} \tag{C.6}$$

with now a sum over $\mathcal{C}_\sigma$-*orbits* o, i.e. *unlabeled* maps, of length ℓ_o. Thus $d_o = \frac{4^n n!}{\ell_0} = \frac{|\mathcal{C}_\sigma|}{\ell_o}$, the "symmetry factor" in Feynman rules, is the order of the stabilizer group of the orbit o. As an independent argument shows, (see for example Sec. 3.2.c), d_o turns out to be a divisor of $2n$.

$= \frac{1}{N}\delta_{ab}\delta_{il}\delta_{jk}$



$= \gamma N/4\, \delta_{aa'}\delta_{bb'}\delta_{ii'}\delta_{jj'}\delta_{kk'}\delta_{ll'}.$

Fig. C.1. Feynman rules.

For genus $g = 0$ (planar maps), the first terms of the series expansion (C.4) read

$$\frac{1}{fN^2}F^{[\mathrm{pl,1c}]} := \sum_n \gamma^n F_n^{(0,1)} = \frac{1}{4}2\gamma + \frac{1}{4^2 2!}32\gamma^2 + \frac{1}{4^3 3!}1344\gamma^3 + \frac{1}{4^4 4!}99840\gamma^4 + \frac{1}{4^5 5!}11034624\gamma^5 + \cdots \tag{C.7}$$

and more terms appear in Tables 1 and 2.

Unfortunately, there exists no closed formula for this series, in contrast with the cases $f = 1$ for which we have Tutte's result ([32], see also [34] or [35, Eq. (3.9)]).

$$\lim_{N\to\infty}\frac{1}{N^2}F^{[\mathrm{pl}]} = \sum_{n\geq 1}\gamma^n \sum_{c\geq 1} F_n^{(0,c)} = \sum_{n=1}^{\infty}(3\gamma)^n \frac{(2n-1)!}{n!(n+2)!} \tag{C.8}$$

or with the sum over all genera of one-component maps (i.e. the case $N = 1$ of (C.4))

$$F^{[1\mathrm{c}]}|_{N=1} = f\sum_{n=1}^{\infty}\left(\frac{\gamma}{4}\right)^n \frac{(4n-2)!!}{n!} \quad \text{see below Eq. (7.19).} \tag{C.9}$$

Two other matrix integrals

The reader will convince him/her-self, that the case of bicolorable curves or of alternating knots and links is related in the same way to another matrix integral,

$$Z_Y = \int\left(\prod_{a=1}^{f} D(M_a, M_a^\dagger)\right)\exp -N\left[\sum_{a=1}^{f}\mathrm{tr}\,(M_a M_a^\dagger) - \frac{\gamma}{4}\sum_{a,b=1}^{f}\mathrm{tr}\,(M_a M_b^\dagger)^2\right] \tag{C.10}$$

with now an integration over complex $N \times N$ matrices. From the fact that, any *planar* map may be bicolored in two different ways, it follows that the free energy $\log Z_Y$ coincides up to a factor 2 with $F = \log Z_X$ considered above. Thus, using the formalism of Sec. 3,

$$F_n^{(0,1)} = \frac{1}{2(2n)!}\#\{\sigma,\tau\in S_{2n}\,|\,\rho\in[2^n]\cap(\mathrm{I}')\cap(\mathrm{II}')_0\} \tag{C.11}$$

$$= (2n-1)!!\frac{1}{2(2n)!}\#\{\sigma\in S_{2n}\,|\,(\mathrm{I}')\cap(\mathrm{II}')_0 \text{ with } \rho=\rho_0,\ \tau=\sigma\rho\}, \tag{C.12}$$

where in the first line the factor $2(2n)!$ comes from the two possible bicolorations along with a general relabeling of the $2n$ edges, and in the second, the factor $(2n-1)!!$ comes from the possible choices of ρ, (pairings at vertices). The bottom line of Table 2 is that $F_n^{(0,1)} = \frac{1}{2^{n+1}n!}\#\{\sigma\cdots\}$. (Fortunately the results coincide with those of Table 1!)

Finally, the counting of general oriented curves in Sec. 5 is related to the following integral

$$Z_Z = \int \left(\prod_{a=1}^{f} D(M_a, M_a^\dagger)\right) \times \exp -N\left[\sum_{a=1}^{f} \operatorname{tr}(M_a M_a^\dagger) - \frac{\gamma}{4}\sum_{a,b=1}^{f} \operatorname{tr}(M_a M_b M_a^\dagger M_b^\dagger)\right]. \quad \text{(C.13)}$$

For *one-component* maps, there are two ways of orienting the corresponding curve, hence the free energy $F_Z^{[1c]} = \log Z_Z|_{\text{term } f^1}$ coincides up to a factor 2 with F considered above, for any genus.

C.2. *The cardinal of X′ through a simple integral*

In this section, we compute the cardinal of the set X' of Sec. 3 through a simple integral. Recall that X' gathers maps of all genera. Since, we are not concerned by the genus of the graph/map, we may use an integration over real vectors ϕ of $\mathbb{R}^f$ rather than matrices, i.e. the case $N = 1$ of the integral (C.1). Let

$$Z = (2\pi)^{-f/2}\int d^f\phi\, \exp\left[-\frac{1}{2}\phi^2 + \frac{\gamma}{4}(\phi^2)^2\right] \quad \text{(C.14)}$$

in which the terms linear in f yield the contribution of one-component graphs. As above, we assume that $\Re\gamma < 0$ and we have explicitly normalized Z to be 1 for $\gamma = 0$. Following a standard trick, we rewrite Z as

$$Z = \int_{\mathbb{R}} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2} \int_{\mathbb{R}^f} \frac{d^f\phi}{(2\pi)^{f/2}}\, \exp\left[-\frac{1}{2}\phi^2(1 + 2i\sqrt{-\gamma}\alpha)\right]. \quad \text{(C.15)}$$

Integrating over the f-dimensional ϕ gives

$$Z = \int_{\mathbb{R}} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2}(1 + 2i\sqrt{-\gamma}\alpha)^{-f/2}. \quad \text{(C.16)}$$

In the series expansion of the term $(1 + 2i\sqrt{-\gamma}\alpha)^{-f/2}$, we keep only the term of order f^1, hence

$$Z|_{f\ \text{term}} = \frac{f}{2}\sum_{n\geq 1} \frac{(2i\sqrt{-\gamma})^n \langle\alpha^n\rangle}{n}, \quad \text{(C.17)}$$

where $\langle\alpha^m\rangle$ denote the moments of the Gaussian measure $\frac{d\alpha}{\sqrt{\pi}}e^{-\alpha^2}$. Only even moments are nonvanishing and we find

$$Z|_{f\ \text{term}} = f\sum_{n=1} \frac{(2\gamma)^n}{4n}(2n-1)!! \quad \text{(C.18)}$$

which may be recast as

$$Z|_{f\ \text{term}} = F^{[1c]}|_{N=1} = f\sum_{n=1} \frac{1}{n!}\left(\frac{\gamma}{4}\right)^n (4n-2)!!. \quad \text{(C.19)}$$

By comparing this calculation with formula (7.6), one sees that the coefficient $(4n-2)!!$ is nothing else than the number of points in the set X'. See also next appendix for a direct combinatorial argument.

C.3. *The set $X' \subset [2^{2n}]$*

In this section, we reproduce the previous result on $|X'|$ by a purely combinatorial argument. The set X' of permutations $\tau \in [2^{2n}]$ that satisfy $\sigma^2\,\tau \in [(2n)^2]$ may be constructed explicitly. We choose $\sigma = (1324)\cdots(4n-3, 4n-1, 4n-2, 4n)$, so that $\sigma^2 = (12)(34)\cdots(4n-1, 4n)$. We note that $i_1 := \tau(1)$ is different from 1 (since $\tau \in [2^{2n}]$), and from $2 = \sigma^2(1)$, otherwise $\sigma^2\,\tau$ would have a 1-cycle. We thus have $4n-2$ possible choices for i_1.

By recursion, suppose that after $r < 2n$ iterations, we choose $i_r := \tau(\sigma^2(i_{r-1}))$ different from $i_0 := 2, i_1, \ldots, i_{r-1} := \tau(\sigma^2(i_{r-2}))$ and from their images by σ^2, with these $2r$ numbers assumed to be all different: we thus have $4n-2r$ choices for i_r.

Then for any $0 \le s \le r-1, \sigma^2(i_r) \neq \sigma^2(i_s)$ since $i_r \neq i_s$; and $\sigma^2(i_r) \neq i_s \Leftrightarrow i_r \neq \sigma^2(i_s)$ by assumption.

Moreover $i_{r+1} := \tau\sigma^2(i_r) = (\tau\sigma^2)^{r+1}(2)$ must be different from the $2(r+1)$ numbers $i_0 := 2, i_1, \ldots, i_r$ and their images by σ^2:

(1) for $0 \le s \le r-1$, $i_{r+1} = \tau\sigma^2(i_r) \neq \sigma^2(i_s) \Leftrightarrow \sigma^2(i_r) \neq \tau\sigma^2(i_s) = i_{s+1} \Leftrightarrow i_r \neq \sigma^2(i_{s+1})$ by the assumption on i_r for $s < r-1$, and the fact that σ^2 has no fixed point for $s = r-1$;
(2) $i_{r+1} = \tau\sigma^2(i_r) \neq \sigma^2(i_r)$ since τ has no fixed point;
(3) for $1 \le s \le r$, $i_{r+1} = \tau\sigma^2(i_r) \neq i_s = \tau(\sigma^2(i_{s-1}))$ since $i_r \neq i_{s-1}$;
(4) finally, $i_{r+1} = \tau\sigma^2(i_r) = (\tau\sigma^2)^{r+1}(2)$ may be equal to $i_0 = 2$, iff 2 is a fixed point of $(\tau\sigma^2)^{r+1}$, which occurs iff $r+1 = 2n$ (remember that $\tau\sigma^2 \in [(2n)^2]$).

Hence, for $r < 2n-1$, the recursion assumption is verified, and there are $4n-2(r+1)$ choices for i_{r+1}. At the end of this iterative procedure, we have constructed a $\tau = ((1, i_1), (\sigma^2(i_1), i_2), \ldots, (\sigma^2(i_{2n-1}), i_{2n})$, and all $\tau \in X'$ are obtained that way. This completes the construction of the set X' and the proof that $|X'| = \prod_{r=0}^{2n-2}(4n-2(r+1)) = (4n-2)!!$.

C.4. *The set $Y' \subset S_{2n}$*

By lack of a direct combinatorial construction of the set Y' (as we had for X', see previous appendix), we resort again to a simple integral to compute the cardinality of Y'. In the same spirit as in Appendix C.2, let us consider the integral over vectors of $\mathbb{C}^f$

$$Z = \frac{1}{\pi^f}\int d^f(z,\bar z)\exp\left[-z\cdot\bar z + \frac{\gamma}{4}z^2\bar z^2\right], \tag{C.20}$$

where $z\cdot\bar z = \sum_{a=1}^{f} z_a\bar z_a$, $z^2 := \sum_{a=1}^{f}(z_a)^2$ and likewise for $\bar z^2$. Note that this may be regarded as the $N = 1$ version of integral (C.10). We take $\gamma < 0$ to ensure convergence. Using again the same trick, we rewrite Z, up to a factor, as

$$Z = \int \frac{d(\alpha, \bar{\alpha})}{\pi} e^{-\alpha\bar{\alpha}} \int \frac{d^f(z, \bar{z})}{\pi^f} \exp\left[-z \cdot \bar{z} - -i\sqrt{\frac{\gamma}{4}} z^2 \bar{\alpha} - i\sqrt{\frac{\gamma}{4}} \bar{z}^2 \alpha\right] \quad \text{(C.21)}$$

which upon integration over $z, \bar{z}$ gives

$$Z = \int \frac{d(\alpha, \bar{\alpha})}{\pi} e^{-\alpha\bar{\alpha}} (1 - \gamma\alpha\bar{\alpha})^{-f/2}. \quad \text{(C.22)}$$

Keeping again the term of order f^1 in the expansion of $(1 - \gamma\alpha\bar{\alpha})^{-f/2}$ gives

$$Z|_{f \text{ term}} = \frac{f}{2} \sum_{n \geq 1} \frac{\langle (\alpha\bar{\alpha})^n \rangle}{n}, \quad \text{(C.23)}$$

where $\langle (\alpha\bar{\alpha})^n \rangle = n!$ are the moments of the measure $\frac{d(\alpha,\bar{\alpha})}{\pi} e^{-\alpha\bar{\alpha}}$, hence

$$Z|_{f \text{ term}} = f \sum_{n=1} \frac{1}{n!} \left(\frac{\gamma}{4}\right)^n 2^{2n-1} n!(n-1)! \quad \text{(C.24)}$$

which (in view of the diagrammatic interpretation *à la* Feynman of this computation) shows that the number of points in the set Y' is indeed $2^{2n-1} n!(n-1)! = (2n)!!(2n-2)!!$.

As a little exercise left to the reader, one may check that the same reasoning applied to integral (C.13), i.e. consideration of the integral

$$Z = \int d^f(z, \bar{z}) \exp\left[-z \cdot \bar{z} + \frac{\gamma}{4}(z \cdot \bar{z})^2\right] \quad \text{(C.25)}$$

and computation of its f^1 term will reproduce the counting of points in the set Z' of Sec. 5, namely $(2n-1)!$.

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