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MODULAR INVARIANT PARTITION FUNCTIONS IN TWO DIMENSIONS

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We present a systematic study of modular invariance of partition functions, relevant both for two-dimensional minimal conformal invariant theories and for string propagation on a SU(2) group manifold. We conjecture that all solutions are labelled by simply laced Lie algebras.

1. Introduction

The minimal two-dimensional conformal invariant field theories [1] carry a set of representations of two Virasoro algebras of common central charge

$$c = 1 - \frac{6(p-p')^2}{pp'}, \tag{1.1}$$

with (p, p') a pair of coprime positive integers. Belavin, Polyakov and Zamolodchikov have shown that it is consistent to retain only a finite number of primary fields $\phi_{h,\bar{h}}$, of conformal dimensions h and $\bar{h}$ chosen among the Kac values [2]

$$h_{rs} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'} = h_{p'-r,\,p-s}, \tag{1.2a}$$

with

$$1 \leqslant r \leqslant p'-1, \qquad 1 \leqslant s \leqslant p-1. \tag{1.2b}$$

An important subset of these minimal theories consists of the unitary $c<1$ conformal theories, for which p and p' must be consecutive integers: $|p-p'|=1$ [3].

Cardy [4] has shown that putting such a conformal theory in a finite box with periodic boundary conditions, i.e. on a torus, gives stringent constraints on its

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operator content. These constraints arise from the requirement of modular invariance of the partition function which has the general form:

$$Z = \sum \mathcal{N}_{hh'} \chi_h(\tau) \chi^*_{h'}(\tau). \tag{1.3}$$

The conformal characters $\chi_h(\tau)$ (including a prefactor $e^{-2\pi i\tau c/24}$) are explicit functions of τ, the modular ratio of the torus; $\mathcal{N}_{hh'} = \mathcal{N}_{h'h}$ are non-negative integers, arising from the decomposition of the representation of the Virasoro algebras carried by the space of states into irreducible representations. Some modular invariant partition functions have been constructed so far [5–7]. We analyse this problem more systematically. We present some results and conjectures towards a complete classification of minimal $c<1$ conformal theories, summarized in table 3.

Amazingly, this problem is mathematically related to a different physical situation. Studying the propagation of a string on a group manifold, Gepner and Witten [8] have been led to study modular invariants sesquilinear in characters of an affine Lie algebra:

$$\sum \mathcal{N}_{ll'} \chi_l \chi^*_{l'}, \tag{1.4}$$

with again non-negative integers $\mathcal{N}_{ll'}$. In the case of $A_1^{(1)} = \widehat{su}(2)$, Gepner [7] has put forward the relation between the two problems by constructing conformal modular invariants from the $A_1^{(1)}$ ones. The connection between the two problems also underlies our analysis, based on arithmetical properties of the even integer $N = 2pp'$ in the conformal case, $N = 2(k+2)$ in the case of $A_1^{(1)}$ representations of level k.

Sect. 2 introduces the notations and clarifies the group theoretical setting. As the characters transform under a unitary projective representation of the *finite* group $PSL(2, \mathbb{Z}/2N\mathbb{Z})$, our problem is related to the decomposition of this representation into irreducible ones. Sect. 3 contains the discussion of the modular invariance of (1.3) and (1.4), when the positivity and integrality conditions on the $\mathcal{N}$'s are relaxed. A large class of solutions is obtained, associated with each factorization of $\frac{1}{2}N$ or of its divisors of the form $N/2\alpha^2$ into a product of two coprimes: see eq. (3.14) for a precise statement. We conjecture that this describes all modular invariants of either problem, and we shall present the elements of proof we have obtained so far. The applications are discussed in sect. 4. Our conjecture justifies and completes Gepner's procedure, in so far as it establishes that *all* conformal invariants are obtained from the $A_1^{(1)}$ ones. The integrality and positivity conditions on the coefficients $\mathcal{N}$ are then shown to reduce drastically the acceptable solutions. We have found only three classes of solutions for the $A_1^{(1)}$ case. The first which exists for any value of $\frac{1}{2}N = k+2$ corresponds to the trivial diagonal invariant $\sum |\chi_l|^2$ while the second appears only for even $k \geqslant 4$. In addition, there are three exceptional cases for $k+2 = 12$, 18 and 30, which are the Coxeter numbers of the exceptional Lie algebras E_6, E_7, and E_8. These two infinite series and three exceptional solutions are indeed in correspondence with the simply laced Lie algebras A_{k+1}, $D_{k/2+2}$ and E_6, E_7 and E_8.

Correspondingly, the "positive" conformal modular invariants are labelled by a pair of such algebras, relative to the values $p'-2$ and $p-2$ of level k. As p and p' must be coprimes, hence not simultaneously even, one of the two algebras at least must be an A algebra. For the unitary conformal theories ($p'=p-1=m$), we find two solutions for any value of m ($m \geqslant 5$ for the second): they are the principal and complementary series discussed in ref. [5] and completed in refs. [6,7]. In addition, there are six exceptional theories for m or $m+1=12$, 18 and 30. The principal series may be denoted (A_{m-1}, A_m), the complementary one ($m \geqslant 5$) $(A_{4\rho}, D_{2\rho+2})$ if $m=4\rho+1$, $(D_{2\rho+2}, A_{4\rho+2})$ if $m=4\rho+2$, $(A_{4\rho+2}, D_{2\rho+3})$ if $m=4\rho+3$, $(D_{2\rho+3}, A_{4\rho+4})$ if $m=4\rho+4$, and the exceptional ones (A_{10}, E_6) ($m=11$), (E_6, A_{12}) ($m=12$), (A_{16}, E_7) ($m=17$), (E_7, A_{18}) ($m=18$), (A_{28}, E_8) ($m=29$) and (E_8, A_{30}) ($m=30$). That this list exhausts all possible modular invariant $c<1$ unitary theories is our second conjecture. This conjecture is partially supported by a parallel work of Pasquier [9].

2. Notations and group-theoretic considerations

For the following discussion of conformal characters, it is convenient to trade the two indices (r,s) of eq. (1.2) for a single variable λ

$$\lambda = pr - p's \tag{2.1}$$

and to consider λ as defined modulo $N=2pp'$. All the ensuing expressions will indeed be periodic functions of λ of period N; the reason why the intervals (1.2b) (with $r=0$ and $s=0$ added) only represent half a period will soon become clear. Since p and p' are coprimes, two integers r_0 and s_0 exist such that

$$r_0 p - s_0 p' = 1\,. \tag{2.2}$$

We introduce the number $\omega_0 \in \mathbb{Z}/N\mathbb{Z}$

$$\omega_0 = r_0 p + s_0 p' \qquad \operatorname{mod} N\,. \tag{2.3}$$

It satisfies

$$\omega_0^2 = 1 \qquad \operatorname{mod} 2N\,, \tag{2.4a}$$

$$\omega_0 \lambda = pr + p's \qquad \operatorname{mod} N\,. \tag{2.4b}$$

(Beware of the mod condition on ω_0^2!) Hence, multiplication by ω_0 leaves unchanged the multiples of p, and changes sign of those of p'. Conversely, given λ

and $\omega_0\lambda \mod N$, one obtains a pair (r, s) modulo a two-dimensional lattice generated by (p, p') and $(p, -p')$. The appearance of ω_0 satisfying (2.4a) associated with the factorization into two coprimes of $\frac{1}{2}N = pp'$ is a typical feature of this problem and will be encountered again in the following. In general, we introduce the abelian group G_n for n an arbitrary *even* integer:

$$\mathrm{G}_n = \{\omega, \omega \in \mathbb{Z}/n\mathbb{Z}, \omega^2 = 1 \mod 2n\}.$$

We now turn to the conformal characters. For τ the modular ratio of the torus defined such that $\operatorname{Im}\tau > 0$, we need the Dedekind η function

$$\eta(\tau) = \exp\left(\frac{2\pi i}{24}\tau\right)\prod_{n=1}^{\infty}(1-\exp(2i\pi n\tau)) \tag{2.5}$$

and define the set of functions of τ:

$$K_\lambda(\tau) = \frac{\sum_{n=-\infty}^{\infty}\exp\{i\pi\tau(nN+\lambda)^2/N\}}{\eta(\tau)}. \tag{2.6}$$

Obviously K is even in λ and periodic of period N:

$$K_\lambda = K_{-\lambda} = K_{\lambda+N}. \tag{2.7}$$

Hence eq. (2.6) defines a set of $\frac{1}{2}N+1$ independent functions. The conformal characters are then [10]

$$\chi_\lambda^{\mathrm{conf}}(\tau) = K_\lambda(\tau) - K_{\omega_0\lambda}(\tau). \tag{2.8}$$

(Recall that the factor $\exp(-(2\pi i/24)c\tau)$ has been included into χ_λ, as in eq. (1.3).) $\chi_\lambda^{\mathrm{conf}}$ vanishes on all multiples of p and p', satisfies

$$\chi_\lambda^{\mathrm{conf}} = \chi_{-\lambda}^{\mathrm{conf}} = \chi_{\lambda+N}^{\mathrm{conf}} = -\chi_{\omega_0\lambda}^{\mathrm{conf}} \tag{2.9}$$

and assumes $\frac{1}{2}(p-1)(p'-1)$ distinct, linearly independent values, which may be labelled by the values of λ, or (r, s) in a fundamental domain:

$$B^{\mathrm{conf}}: 1 \leqslant r \leqslant p'-1, \qquad 1 \leqslant s \leqslant p-1, \qquad sp' \leqslant rp. \tag{2.10}$$

We also consider the Kac-Peterson characters of the affine Lie algebra $\mathrm{A}_1^{(1)}$ [11] for integrable representations (integer or half-integer spin $l \leqslant \frac{1}{2}k$) in their basic specialization multiplied by the factor $\exp(-(2\pi i/24)c_{\mathrm{aff}}\tau) = \exp(-(2\pi i/8)(k/k+2)\tau)$ (in short, affine characters). For level k and spin l, we use intentionally the same notation χ_λ by setting $N = 2(k+2)$ and $\lambda = 2l+1$

$$\chi_\lambda^{\mathrm{aff}}(\tau) = \frac{1}{\eta^3(\tau)}\sum_{n=-\infty}^{\infty}(nN+\lambda)\exp\left[i\pi\tau(nN+\lambda)^2/N\right]. \tag{2.11}$$

It is an odd, periodic function of λ

$$\chi_\lambda^{\text{aff}} = \chi_{\lambda+N}^{\text{aff}} = -\chi_{-\lambda}^{\text{aff}} \tag{2.12}$$

and therefore it assumes $\frac{1}{2}N-1$ distinct and linearly independent values labelled by λ in a fundamental domain

$$B^{\text{aff}}\colon 1 \leqslant \lambda \leqslant \tfrac{1}{2}N-1\,. \tag{2.13}$$

Note already the similarity of the properties (2.9)–(2.10), (2.12)–(2.13).

The modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ is the set of 2×2 matrices A with integral entries and unit determinant, with $\pm A$ identified. It acts on τ in the upper half-plane through

$$\tau \to \tau' = \frac{a\tau+b}{c\tau+d}\,, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.14}$$

This action is then carried to functions of τ. In particular

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_A (c\tau+d)^{1/2}\eta(\tau)\,, \tag{2.15}$$

where ε_A is a 24th root of unity. Using the freedom of sign of A, one can always choose $c \geqslant 0$ and $d=1$ if $c=0$, and $0 \leqslant \arg(c\tau+d)^{1/2} < \frac{1}{2}\pi$. With this convention, ε_A is a well-defined τ-independent phase. The modular group is generated by S and T

$$S\colon \tau \to -\frac{1}{\tau}\,, \qquad T\colon \tau \to \tau+1\,. \tag{2.16}$$

Using (2.15) as well as Poisson formula, one finds for the conformal characters:

$$T\colon\ K_\lambda(\tau+1) = \exp\left[2i\pi\left(\frac{\lambda^2}{2N} - \tfrac{1}{24}\right)\right] K_\lambda(\tau)\,, \tag{2.17a}$$

$$S\colon\ K_\lambda(-\tau^{-1}) = \frac{1}{\sqrt{N}} \sum_{\lambda'=0}^{N-1} \exp\left[2i\pi \frac{\lambda\lambda'}{N}\right] K_{\lambda'}(\tau)\,. \tag{2.17b}$$

Since

$$\begin{aligned} \lambda^2 &= (\omega_0\lambda)^2 && \bmod 2N\,, \\ \lambda\lambda' &= (\omega_0\lambda)(\omega_0\lambda') && \bmod N\,, \end{aligned} \tag{2.18}$$

eq. (2.17) still holds with $\chi_\lambda^{\text{conf}}$ substituted for K_λ.

For the affine characters, one finds similar formulae:

$$T:\quad \chi_\lambda^{\text{aff}}(\tau+1)=\exp\left[2i\pi\left(\frac{\lambda^2}{2N}-\tfrac{1}{8}\right)\right]\chi_\lambda(\tau), \tag{2.19a}$$

$$S:\quad \chi_\lambda^{\text{aff}}(-\tau^{-1})=\frac{-i}{\sqrt{N}}\sum_{\lambda'=0}^{N-1}\exp\left[2i\pi\frac{\lambda\lambda'}{N}\right]\chi_{\lambda'}(\tau). \tag{2.19b}$$

All these transformations are of course compatible with the mod N periodicity in λ and the parity properties under $\lambda\to-\lambda$. The symmetries of χ_λ enable us to rewrite (2.17b) and (2.19b) with λ,λ' in the fundamental domain B (cf. eqs. (2.10), (2.13))

$$\chi_{[r,s]}^{\text{conf}}(-\tau^{-1})=2\sqrt{\frac{2}{pp'}}\sum_{\substack{1\leqslant r'\leqslant p'-1\\ 1\leqslant s'\leqslant p-1\\ s'p'\leqslant r'p}}(-1)^{rs'+r's+1}\sin\frac{\pi prr'}{p'}\sin\left(\frac{\pi p'ss'}{p}\right)\chi_{[r',s']}^{\text{conf}}(\tau), \tag{2.17c}$$

$$\chi_\lambda^{\text{aff}}(-\tau^{-1})=\sqrt{\frac{2}{k+2}}\sum_{1\leqslant\lambda'\leqslant k+1}\sin\left(\frac{\pi\lambda\lambda'}{k+2}\right)\chi_{\lambda'}^{\text{aff}}(\tau). \tag{2.19c}$$

Notice the presence of the factor $-i$ in (2.19b), in contrast with (2.17b), which guarantees that the square of S acting on characters is the identity. Indeed, the Fourier operator

$$(\mathscr{F})_{\lambda\lambda'}=\frac{1}{\sqrt{N}}\exp\left[2i\pi\frac{\lambda\lambda'}{N}\right] \tag{2.20}$$

has its fourth power equal to the identity, but its square

$$(\mathscr{F}^2)_{\lambda\lambda'}=\delta_{\lambda',-\lambda}. \tag{2.21}$$

Therefore $\mathscr{F}^2$ acting on even functions of λ, or $(-i\mathscr{F})^2$ acting on odd ones are both equivalent to the identity. The presence of the phases $\exp(-\frac{1}{12}i\pi)$, $\exp(-\frac{1}{4}i\pi)$ in the T transformations ((2.17a) and (2.19a)) are also crucial to ensure the modular consistency (see Appendix A).

For n an arbitrary integer, the kernel of the surjective map [12] $\mathrm{PSL}(2,\mathbb{Z})\to\mathrm{PSL}(2,\mathbb{Z}/n\mathbb{Z})$ is called the modular group of level n and denoted Γ_n:

$$1\to\Gamma_n\to\mathrm{PSL}(2,\mathbb{Z})\to\mathrm{PSL}(2,\mathbb{Z}/n\mathbb{Z})\to 1.$$

Γ_n is therefore the invariant subgroup Γ of matrices

$$A=\pm\begin{pmatrix}1+\alpha n & \beta n\\ \gamma n & 1+\delta n\end{pmatrix}. \tag{2.22}$$

One may prove that under any transformation of Γ_{2N}, the functions K_λ and $\chi_\lambda^{\rm conf}$ are only multiplied by a λ- and τ-independent 24th root of unity

$$\tau \xrightarrow{A} \tau', \qquad A \in \Gamma_{2N}, \qquad K_\lambda(\tau') = \hat{\varepsilon}_A K_\lambda(\tau), \qquad \hat{\varepsilon}_A^{24} = 1. \tag{2.23}$$

A similar result holds for $\chi_\lambda^{\rm aff}$ with an 8th root of unity. To establish this result, it is *not* in general sufficient to verify that T^{2N} acts through a pure phase: indeed for $n \geqslant 6$, Γ_n is not generated by elements conjugate to T^n only [13]. The calculation is presented in appendix B. We conclude that the set of characters χ_λ with λ in a fundamental domain B carries a unitary projective (i.e. up to a phase) representation of the finite group

$$\mathrm{M}_{2N} \equiv \mathrm{PSL}(2, \mathbb{Z}/2N\mathbb{Z}) = \Gamma/\Gamma_{2N}:$$

$$A \in \mathrm{M}_{2N}, \qquad \tau \xrightarrow{A} \tau', \qquad \chi_\lambda(\tau') = \sum_{\lambda'=0}^{N-1} U_{\lambda\lambda'}(A)\chi_{\lambda'}(\tau), \tag{2.24}$$

$$\left.\begin{aligned} U(A)U(A') &= e^{i\varphi_{A,A'}}U(AA') \\ U(A)U^\dagger(A) &= \mathbb{1} \end{aligned}\right\}. \tag{2.25}$$

To check that an hermitian form

$$Z_{\mathcal{N}} = \sum (\chi_\lambda(\tau))^* \mathcal{N}_{\lambda\lambda'}\chi_{\lambda'}(\tau) \tag{2.26}$$

is invariant under modular transformations, it is therefore necessary and sufficient to check invariance under M_{2N}:

$$\mathcal{N}U(A) = U(A)\mathcal{N}, \qquad A \in \mathrm{M}_{2N}. \tag{2.27}$$

This could be done with the help of Schur's lemma, decomposing the representation (2.24)–(2.25) into irreducible parts and taking $\mathcal{N} \propto c \cdot \mathbb{1}$ in each of them. As no general result on these decompositions is known to us, our forthcoming analysis may be regarded as a study of the commutant of this representation of M_{2N}.

Alternatively, for any arbitrary hermitian $\mathcal{M}$,

$$\mathcal{N} = \frac{1}{|\mathrm{M}_{2N}|} \sum_{A \in \mathrm{M}_{2N}} U^\dagger(A)\mathcal{M}U(A) \tag{2.28}$$

satisfies (2.27) and conversely any solution of (2.27) is of this form. Unfortunately, the order $|\mathrm{M}_{2N}|$ of M_{2N} grows fast [12]

$$|\mathrm{M}_n| = \tfrac{1}{2}n^3 \prod_{\substack{p \text{ prime} \\ \text{divisor of } n}} \left(1 - \frac{1}{p^2}\right), \tag{2.29}$$

and the use of this remark requires some ingenuity [7].

3. The commutant

Let us first examine which linear transformations C of the χ's may commute with T. From eqs. (2.17a)–(2.19a) we learn that $C_{\lambda\lambda'} \neq 0$ only if

$$\lambda^2 = \lambda'^2 \qquad \mathrm{mod}\, 2N. \tag{3.1}$$

We shall prove that this implies the following proposition:

(1) λ and λ' have a common divisor $\alpha \geqslant 1$, such that α^2 divides $\frac{1}{2}N$, with $N' = N/\alpha^2$ even.

(2) There exists μ defined modulo N' such that $\mu^2 = 1 \bmod 2N'$ and

$$\frac{\lambda'}{\alpha} = \mu \frac{\lambda}{\alpha} \qquad \mathrm{mod}\, N', \tag{3.2a}$$

or equivalently

$$\lambda' = \mu\lambda + \xi \frac{N}{\alpha} \ \mathrm{mod}\, N, \qquad \xi \in \mathbb{Z}/\alpha\mathbb{Z}. \tag{3.2b}$$

Conversely any such λ' satisfies (3.1).

Assume $\lambda \neq \pm\lambda' \ \mathrm{mod}\, N$ since otherwise the proposition is obvious ($\alpha = 1$, $\mu = \pm 1$). Condition (3.1) amounts to

$$(\lambda' - \lambda)(\lambda' + \lambda) = 4pp'l.$$

$\lambda' + \lambda$ and $\lambda' - \lambda$ are both even, and all the divisors of $\frac{1}{2}N = pp'$ must appear either in $\frac{1}{2}(\lambda' + \lambda)$ or in $\frac{1}{2}(\lambda' - \lambda)$. Let $\frac{1}{2}N = \pi\pi'$, $\frac{1}{2}(\lambda + \lambda') = \pi\rho$, $\frac{1}{2}(\lambda' - \lambda) = \pi'\sigma$, $(\rho\sigma = l)$. Of course π or π' could be one. It then follows that

$$\lambda = \pi\rho - \pi'\sigma,$$

$$\lambda' = \pi\rho + \pi'\sigma, \tag{3.3}$$

very much like the original relation between λ and $\tilde{\lambda} = \omega_0\lambda$, except for the fact that π and π' *need not* be coprimes. Let α be their greatest common divisor:

$$\alpha = (\pi, \pi'). \tag{3.4}$$

Then α divides both λ and λ' and α^2 divides $\frac{1}{2}N = pp'$, proving the first point. We set

$$\pi = \alpha P, \qquad \pi' = \alpha P', \qquad (P, P') = 1,$$

$$N = \alpha^2 N', \qquad N' = 2PP'. \tag{3.5}$$

Repeating the procedure of sect. 2, we introduce R_0 and S_0 such that $R_0 P - S_0 P'$

$=1$, and define $\mu = R_0 P + S_0 P'$ (mod N'). Given $\lambda/\alpha = \rho P - \sigma P'$, it is easy to check that $\lambda'/\alpha = \mu\lambda/\alpha$ mod N', and that $\mu^2 = 1$ mod $2N'$, i.e. that $\mu \in \mathrm{G}_{N'}$, the group defined in sect. 2.

In the conformal case, the proposition implies that if p and p' are distinct primes, there are only four possibilities for $\lambda^2 = \lambda'^2$ mod $2N$:

$$\begin{aligned}
&P = pp', \qquad && P' = 1, \qquad && \lambda' = -\lambda \qquad && \operatorname{mod} N, \\
&P = p, && P' = p', && \lambda' = \omega_0 \lambda && \operatorname{mod} N, \\
&P = p', && P' = p, && \lambda' = -\omega_0 \lambda && \operatorname{mod} N, \\
&P = 1, && P' = pp', && \lambda' = \lambda && \operatorname{mod} N.
\end{aligned} \tag{3.6}$$

Likewise, in the affine case, if $k+2$ is a prime, $\lambda = \pm\lambda'$ mod N is the unique solution. In any of these cases $\chi_{\lambda'} = \pm\chi_\lambda$. This means that the projective representation U of M_{2N} is irreducible on the χ_λ, and the invariants must be diagonal: $\mathcal{N}_{\lambda\lambda'} = c_\lambda \delta_{\lambda\lambda'}$. Invariance under S, the second generator of the modular group then implies that the coefficients c_λ are independent of λ and equal, for λ in B. Then $\mathcal{N} = \mathbb{1}$ is the unique solution up a factor.

In general, the proposition suggests to consider the action of the group G_N on the χ's through

$$\omega \in \mathrm{G}_N \to (\Omega_\omega)_{\lambda\lambda'} = \delta_{\lambda', \omega\lambda \bmod N}, \tag{3.7a}$$

$$\Omega_\omega \Omega_{\omega'} = \Omega_{\omega\omega'}. \tag{3.7b}$$

These matrices are symmetric ($\omega = \omega^{-1}$ mod $2N$) and commute with T and S

$$(\Omega S)_{\lambda\lambda'} = \sum_{\lambda''} \delta_{\omega\lambda, \lambda''} \frac{1}{\sqrt{N}} e^{2\pi i \lambda''\lambda'/N} = \frac{1}{\sqrt{N}} e^{2i\pi\omega\lambda\lambda'/N},$$

$$(S\Omega)_{\lambda\lambda'} = \sum_{\lambda''} \frac{1}{\sqrt{N}} e^{2\pi i \lambda\lambda''/N} \delta_{\omega\lambda'', \lambda'} = \frac{1}{\sqrt{N}} e^{2i\pi\omega\lambda\lambda'/N}. \tag{3.8}$$

There is a subgroup H of such Ω's which acts trivially on characters χ_λ, $\lambda \in \mathrm{B}$. H is generated by $\Omega_{\pm 1}$ in the affine case, by $\Omega_{\pm 1}$ and $\Omega_{\pm\omega_0}$ in the conformal one. Therefore the commutant of the representation U contains at least the set of symmetric matrices:

$$C = \sum_{\omega \in \mathrm{G}_N/\mathrm{H}} c_\omega \Omega_\omega, \tag{3.9}$$

with c_ω arbitrary real numbers. There exists a correspondence between the group

G_N and the center of the group M_{2N}, i.e. the set of matrices $\pm\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, $a^2 = 1 \bmod 2N$. This correspondence is set up as follows: $\pm\omega = \pm a \bmod N$. (Recall that if $a^2 = 1 \bmod 2N$, then $(a+N)^2 = 1 \bmod 2N$.) In any irreducible representation of M_{2N}, the elements of the center must be represented by multiples of the identity. Indeed one finds that

$$U_{\lambda\lambda'}\left(\pm\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \bmod 2N,\ a^2 = 1 \bmod 2N\right) = \varepsilon\delta_{\lambda', \omega\lambda},$$

where ε is a λ- and λ'-independent phase.

Whenever $\frac{1}{2}N$ is square-free, i.e. when the only α such that α^2 divides $\frac{1}{2}N$ is $\alpha = 1$, we have proven that the only invariants with integral coefficients ($\mathcal{N}_{\lambda\lambda'} \in \mathbb{Z}$) are of the form (3.9) (with $c_\omega \in \mathbb{Z}$). At the present stage our proof involves a study of the equation $(\mathcal{N}S)_{\lambda\lambda'} - (S\mathcal{N})_{\lambda\lambda'} = 0$ considered as a polynomial in the variable $z = e^{2i\pi/N}$. We refrain from presenting the proof here, however, as it seems difficult to extend to the general case. We also believe that the assumption that the $\mathcal{N}$'s are integers is not crucial.

When $\frac{1}{2}N$ has non-trivial square divisors α^2, the proposition above suggests a larger commutant. For $\mu \in G_{N/\alpha^2}$, introduce the matrix

$$\begin{aligned} \left(\Omega_\mu^{(\alpha)}\right)_{\lambda\lambda'} &= \sum_{\xi \in \mathbb{Z}/\alpha\mathbb{Z}} \delta_{\lambda', \mu\lambda + \xi N/\alpha}, \qquad &&\text{if } \alpha|\lambda \text{ and } \alpha|\lambda' \\ &= 0 &&\text{otherwise}. \end{aligned} \tag{3.10}$$

Such a $\Omega_\mu^{(\alpha)}$ commutes with T, as stated in the proposition. It also commutes with S, as one readily checks. The interpretation of these new matrices hence of new invariants is provided by the following connection between the characters relative to the values N and $N/\alpha^2 = N'$. Return to the definition (2.6) of $K_\lambda(\tau; N)$, making explicit the dependence on N. Let us pick a $\bar{\lambda}$ in $\mathbb{Z}/(N/\alpha)\mathbb{Z}$ and compute the sum:

$$\begin{aligned} \sum_{\xi=0}^{\alpha-1} K_{\alpha\bar{\lambda}+\xi N/\alpha}(\tau; N) &= \frac{1}{\eta(\tau)} \sum_{\xi=0}^{\alpha-1} \sum_{n=-\infty}^{\infty} \exp\left(\frac{i\pi\tau}{N'\alpha^2}\left(nN'\alpha^2 + \alpha\bar{\lambda} + \xi\alpha N'\right)^2\right) \\ &= \frac{1}{\eta(\tau)} \sum_{\xi=0}^{\alpha-1} \sum_{n=-\infty}^{\infty} \exp\left(\frac{i\pi\tau}{N'}\left[(n\alpha+\xi)N' + \bar{\lambda}\right]^2\right) \\ &= K_{\bar{\lambda}}(\tau; N'). \end{aligned} \tag{3.11}$$

The result depends only on $\bar{\lambda} \bmod N'$. This relation carries over to the conformal characters pertaining to N and N'

$$\sum_{\xi=0}^{\alpha-1} \chi^{\text{conf}}_{\alpha\bar{\lambda}+\xi N/\alpha}(\tau; N) = \chi^{\text{conf}}_{\bar{\lambda}}(\tau; N'), \tag{3.12}$$

provided that $\omega_0 \in \mathbb{Z}/N\mathbb{Z}$ projects out on a non-trivial ω'_0 in $\mathbb{Z}/N'\mathbb{Z}$, i.e. $\omega'_0 \neq \pm 1$. The triviality of ω'_0 occurs whenever p (or p') is a square and p (or p') divides α^2. In such a case, the sum (3.12) vanishes by eq. (2.9). If $\omega'_0 \neq \pm 1$, any invariant at level N' gives rise to an invariant at level N by (3.12). In the affine case, we find similarly

$$\sum_{\xi=0}^{\alpha-1} \chi^{\mathrm{aff}}_{\alpha\lambda+\xi N/\alpha}(\tau;\, N) = \alpha\chi^{\mathrm{aff}}_{\lambda}(\tau;\, N'). \tag{3.13}$$

To summarize, it is conjectured that the most general form of modular invariants is

$$\mathcal{N}_{\lambda\lambda'} = \sum_{\substack{\alpha:\,\alpha^2|\frac{1}{2}N\\ \alpha|\lambda,\,\alpha|\lambda'}} \sum_{\mu\in G_{N/\alpha^2}} C^{(\alpha)}_{\mu} \sum_{\xi\in\mathbb{Z}\alpha/\mathbb{Z}} \delta_{\lambda',\mu\lambda+\xi N/\alpha}. \tag{3.14}$$

We know for sure that this is true for $\frac{1}{2}N$ square free and we intend to present details in a future publication.

4. Partition functions

The results of the previous sections will be now used to classify and study the various invariants

$$Z_{\mathcal{N}} = \sum_{\lambda,\lambda'\in B} \chi_\lambda(\tau)^* \mathcal{N}_{\lambda\lambda'} \chi_{\lambda'}(\tau), \tag{4.1}$$

with $\mathcal{N}_{\lambda\lambda'}$ given in (3.14).

As for the affine modular invariants, eq. (3.14) gives immediately the general form of the invariant. There are as many independent terms in eq. (3.14) as there are choices of α and μ with the exception already encountered at the end of sect. 3. If $\frac{1}{2}N$ is a square, taking $\alpha = \sqrt{\frac{1}{2}N}$ produces a vanishing contribution when contracted with a character. If $\frac{1}{2}N = \prod_{i=1}^{n} p_i^{r_i}$ is the decomposition of $\frac{1}{2}N$ into prime factors, the number of relevant factorizations of $\frac{1}{2}N$ and of its divisors $N/2\alpha^2$ into two coprimes is:

$$\psi\left(\tfrac{1}{2}N\right) = \tfrac{1}{2}\left\{ \prod_{i=1}^{n} (1+r_i) - \delta \right\}, \tag{4.2}$$

with $\delta = 1$ iff $\frac{1}{2}N$ is a square. We conclude that the number of independent affine modular invariants is $\psi(k+2)$. The function $\psi(m)$ is displayed in table 1 and a generating function is given by

$$\sum_{m=1}^{\infty} \frac{\psi(m)}{m^s} = \tfrac{1}{2}\left[\zeta^2(s) - \zeta(2s)\right],$$

from which it is seen that on the average $\psi(m)$ grows as $\frac{1}{2}\ln m$.

TABLE 1

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\psi(m)$	0	1	1	1	1	2	1	2	1	2	1	3	1	2	2	2
$\psi(m)\psi(m+1)$	0	1	1	1	2	2	2	2	2	2	3	3	2	4	4	2
m	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\psi(m)$	1	3	1	3	2	2	1	4	1	2	2	3	1	4	1	3
$\psi(m)\psi(m+1)$	3	3	3	6	2	2	4	4	2	4	6	3	4	4	3	6

To see better the structure of the invariants in the conformal case, we first undo what has been done in sect. 2 and reexpress the content of (3.14) in terms of the original pair of indices r, s. Let us show that $\mathcal{N}_{\lambda\lambda'} = \mathcal{N}_{rs,r's'}$ factorizes as a tensor product of similar expressions pertaining to the indices r, r' and s, s':

$$\begin{aligned}\mathcal{N}_{rs,r's'} &= \sum_{\substack{\alpha_1 \\ \alpha_1 | r, r' \\ \alpha_1^2 | p'}} \sum_{\mu_1 \in G_{2p'/\alpha_1^2}} \sum_{\xi_1 \in \mathbf{Z}/\alpha_1\mathbf{Z}} \sum_{\substack{\alpha_2 \\ \alpha_2 | s, s' \\ \alpha_2^2 | p}} \sum_{\mu_2 \in G_{2p/\alpha_2^2}} \sum_{\xi_2 \in \mathbf{Z}/\alpha_2\mathbf{Z}} \\ &\quad \times C^{(\alpha_1,\alpha_2)}_{\mu_1,\mu_2} \delta_{r',\mu r + 2p'\xi_1/\alpha_1} \delta_{s',\mu_2 s + 2p\xi_2/\alpha_2} \\ &= \sum_{\alpha_1\mu_1} \sum_{\alpha_2\mu_2} C^{(\alpha_1,\alpha_2)}_{\mu_1,\mu_2} \mathcal{N}^{(R)}_{rr'} \mathcal{N}^{(S)}_{ss'} .\end{aligned} \tag{4.3}$$

Consider a pair (λ, λ') such that α divides both λ and λ', and

$$\frac{\lambda'}{\alpha} = \mu \frac{\lambda}{\alpha} \mod \frac{2pp'}{\alpha^2} , \tag{4.4}$$

with $\mu \in G_{2pp'/\alpha^2}$. Let $\lambda = pr - p's$ and $\lambda' = pr' - p's'$. As p and p' are coprimes, $\alpha = \alpha_1\alpha_2$ with α_1^2 a divisor of p', α_2^2 a divisor of p. Then α_1 divides also r and r', α_2 divides s and s', and if $\mu_1 = \mu \mod(2p'/\alpha_1^2)$ and $\mu_2 = \mu \mod(2p/\alpha_2^2)$ condition (4.4) reads:

$$\frac{p}{\alpha_2} \frac{r' - \mu_1 r}{\alpha_1} - \frac{p'}{\alpha_1} \frac{s' - \mu_2 s}{\alpha_2} = 0 \mod \frac{2pp'}{\alpha_1^2\alpha_2^2} . \tag{4.5}$$

Taking this condition mod 2, $\mathrm{mod}(p'/\alpha_1^2)$ and $\mathrm{mod}(p/\alpha_2^2)$ leads to

$$\begin{aligned}\frac{r' - \mu_1 r}{\alpha_1} &= t \frac{p'}{\alpha_1^2} \mod \frac{2p'}{\alpha_1^2} , \\ \frac{s' - \mu_2 s}{\alpha_2} &= u \frac{p}{\alpha_2^2} \mod \frac{2p}{\alpha_2^2} ,\end{aligned} \tag{4.6}$$

where t and u may take the values 0 or 1 but satisfy

$$t\alpha_2 - u\alpha_1 = 0 \mod 2 . \tag{4.7}$$

Finally, eq. (4.6) may be recast as

$$r' = \mu_1 r + t\frac{p'}{\alpha_1} + \xi_1\frac{2p'}{\alpha_1},$$

$$s' = \mu_2 s + u\frac{p}{\alpha_2} + \xi_2\frac{2p}{\alpha_2}, \tag{4.8}$$

where $t, u = 0, 1$ satisfy (4.7), $\xi_1 \in \mathbb{Z}/\alpha_1\mathbb{Z}$, $\xi_2 \in \mathbb{Z}/\alpha_2\mathbb{Z}$.

Conversely, given (r, s, r', s') satisfying eq. (4.7)–(4.8) we may reconstruct $\lambda = pr - p's$, $\lambda' = pr' - p's'$ and $\mu = r_0 p \mu_1 - s_0 p' \mu_2$ where r_0 and s_0 are the Bezout multipliers of p and p' (cf. eq. (2.2)). One verifies that μ belongs to G_{2pp'/α^2} and λ and λ' satisfy (4.4). Finally, t and u may be set equal to zero; the other choices do not give new invariants, thanks to the periodicity and symmetry properties of the characters.

The form (4.3) is convenient to count and tabulate the independent invariants, as they are just obtained by tensor products of the p' and p contributions. Moreover this justifies the procedure followed by Gepner to generate conformal invariants from $A_1^{(1)}$ invariants. Namely, this proves (assuming the correctness of the conjecture of sect. 3) that *all* conformal invariants for the (p', p) theory may be obtained by tensor products of affine invariants pertaining to the levels $k = p - 2$ and $k' = p' - 2$. It implies that the number of independent invariants is $\psi(p)\psi(p')$. The values of $\psi(m)\psi(m+1)$ relevant for the unitary series are displayed in table 1.

Knowing that all conformal modular invariants are obtained from the $A_1^{(1)}$ ones, let us concentrate on the latter. We write their explicit form in the physically relevant cases, namely $\mathcal{N}_{\lambda\lambda'}$ non-negative integers, and $\mathcal{N}_{11} = 1$. (unicity of the vacuum state). To the best of our knowledge, the following list is exhaustive.

For any value of $\frac{1}{2}N = k + 2$

$$\mathcal{N}_{\lambda\lambda'} = \delta_{\lambda\lambda'} \tag{4.9}$$

is always a solution, corresponding to the trivial factorization $k + 2 = (k+2)\cdot 1$. Suppose now that k is of the form $k = 4\rho$. The number μ associated with the factorization $k + 2 = 2\cdot(2\rho + 1)$ is $\mu = 4\rho + 1 = k + 1$ so that the relation $\lambda' = \mu\lambda$ mod N splits into two sectors

$$\lambda \text{ (and } \lambda'\text{) odd: } \lambda' = k + 2 - \lambda,$$

$$\lambda \text{ (and } \lambda'\text{) even: } \lambda' = -\lambda.$$

As the affine characters are odd functions of λ, this means:

$$\mathcal{N}_{\lambda\lambda'} = \begin{cases} -\delta_{\lambda',\lambda} & \text{if } \lambda \text{ even} \\ +\delta_{\lambda',k+2-\lambda} & \text{if } \lambda \text{ odd} \end{cases}.$$

Adding the trivial solution $\mathcal{N}=\mathbb{1}$ gives

$$k=0 \bmod 4:\quad \mathcal{N}_{\lambda\lambda'}\begin{cases} =(\delta_{\lambda',\lambda}+\delta_{\lambda',k+2-\lambda}) & \text{if } \lambda \text{ and } \lambda' \text{ are odd} \\ =0 & \text{otherwise}\end{cases}. \tag{4.10}$$

If k is of the form $k=4\rho-2$ one may take $\alpha=2$ and the number μ associated with the trivial factorization $(k+2)/\alpha^2=\rho\cdot 1$ is $\mu=2\rho-1=\frac{1}{2}k$. One discards the possibility $\rho=1$, $k=2$ which corresponds to the forbidden case $\alpha=\sqrt{k+2}$. Then for $\rho>1$:

$$\begin{aligned} \mathcal{N}_{\lambda\lambda'} &= \delta_{\lambda',k+2-\lambda}-\delta_{\lambda',\lambda} && \text{iff } \lambda, \lambda' \text{ are even} \\ &=0 && \text{otherwise}. \end{aligned}$$

Adding again the trivial solution yields:

$$k=2 \bmod 4:\quad \mathcal{N}_{\lambda\lambda'}=\begin{cases} \delta_{\lambda',\lambda} & \text{if } \lambda,\lambda' \text{ odd} \\ \delta_{\lambda',k+2-\lambda} & \text{if } \lambda,\lambda' \text{ even}. \\ 0 & \text{otherwise}\end{cases} \tag{4.11}$$

The corresponding modular invariants are the SO(3)-invariants of ref. [7]. They also lead to the complementary series of conformal modular invariants of refs. [5,6] (see below).

To these two infinite series of invariants, we add three more exceptional ones, constructed by using eq. (3.14)

$k+2=12$

$$\begin{aligned} \mathcal{N}_{\lambda\lambda'} &= (\delta_{\lambda 1}+\delta_{\lambda 7})(\delta_{\lambda' 1}+\delta_{\lambda' 7}) \\ &\quad +(\delta_{\lambda 4}+\delta_{\lambda 8})(\delta_{\lambda' 4}+\delta_{\lambda' 8})+(\delta_{\lambda 5}+\delta_{\lambda 11})(\delta_{\lambda' 5}+\delta_{\lambda' 11}), \end{aligned} \tag{4.12}$$

$k+2=18$

$$\begin{aligned} \mathcal{N}_{\lambda\lambda'} &= (\delta_{\lambda 1}+\delta_{\lambda 17})(\delta_{\lambda' 1}+\delta_{\lambda' 17})+(\delta_{\lambda 5}+\delta_{\lambda 13})(\delta_{\lambda' 5}+\delta_{\lambda' 13}) \\ &\quad +(\delta_{\lambda 7}+\delta_{\lambda 11})(\delta_{\lambda' 7}+\delta_{\lambda' 11})+\delta_{\lambda 9}(\delta_{\lambda' 9}+\delta_{\lambda' 3}+\delta_{\lambda' 15})+(\delta_{\lambda 3}+\delta_{\lambda 15})\delta_{\lambda' 9}, \end{aligned} \tag{4.13}$$

$k+2=30$

$$\begin{aligned} \mathcal{N}_{\lambda\lambda'} &= (\delta_{\lambda 1}+\delta_{\lambda 11}+\delta_{\lambda 19}+\delta_{\lambda 29})(\delta_{\lambda' 1}+\delta_{\lambda' 11}+\delta_{\lambda' 19}+\delta_{\lambda' 29}) \\ &\quad +(\delta_{\lambda 7}+\delta_{\lambda 13}+\delta_{\lambda 17}+\delta_{\lambda 23})(\delta_{\lambda' 7}+\delta_{\lambda' 13}+\delta_{\lambda' 17}+\delta_{\lambda' 23}). \end{aligned} \tag{4.14}$$

The first two had already been discovered [15,6,7].

After inspection of all invariants generated by eq. (3.14) up to $k+2=100$, we conjecture that the previous list ((4.9)–(4.14) and table 2) exhausts the set of positive integral invariants, satisfying $\mathcal{N}_{11}=1$. This conjecture is supported and embellished by the observation that there is a connection between these invariants and the simply-laced simple Lie algebras A_n, D_n, E_6, E_7 and E_8. Indeed the values of the labels $\lambda=\lambda'$ taken by the *diagonal* terms in eqs. (4.9)–(4.14) coincide with the

TABLE 2
List of known partition functions in terms of $A_1^{(1)}$ characters

$k \geqslant 1$	$\sum_{\lambda=1}^{K+1} \lvert\chi_\lambda\rvert^2$	A_{k+1}
$k=4\rho,\ \rho \geqslant 1$	$\sum_{\substack{\lambda\ \mathrm{odd}=1 \\ \lambda \neq 2\rho+1}}^{4\rho+1} \lvert\chi_\lambda\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2 + \sum_{\lambda\ \mathrm{odd}=1}^{2\rho-1} (\chi_\lambda \chi^*_{4\rho+2-\lambda} + \mathrm{c.c.})$	$D_{2\rho+2}$
	$= \sum_{\lambda\ \mathrm{odd}=1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	
$k=4\rho-2,\ \rho \geqslant 2$	$\sum_{\lambda\ \mathrm{odd}=1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda\ \mathrm{even}=2}^{2\rho-2} (\chi_\lambda \chi^*_{4\rho-\lambda} + \mathrm{c.c.})$	$D_{2\rho+1}$
$k+2=12$	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	E_6
$k+2=18$	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + [(\chi_3+\chi_{15})\chi_9^* + \mathrm{c.c.}]$	E_7
$k+2=30$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	E_8

exponents (or Betti numbers) of these algebras, including their multiplicities. Recall that these exponents give the degrees (minus 1) of a system of independent generators of the ring of invariant polynomials in these algebras [14]. In eq. (4.9), λ takes all values $1 \leqslant \lambda \leqslant k+1$, corresponding to the exponents of A_{k+1}. In (4.10), for $k=4\rho$ the diagonal terms have $\lambda=\lambda'=1,3,\ldots,4\rho+1$, exponents of $D_{2\rho+2}$, with the middle value $\lambda=\lambda'=2\rho+1$ appearing twice as it should, whereas for $k+2=4\rho$ the values of $\lambda=\lambda'$ run over the exponents $1,3,\ldots,4\rho-1$ and 2ρ of $D_{2\rho+1}$. Finally the values $\lambda=\lambda'=1,4,5,7,8,11$, $\lambda=\lambda'=1,5,7,9,11,13,17$ and $\lambda=\lambda'=1,7,11,13,17,19,23,29$ appearing in eqs. (4.12)–(4.14) are respectively the exponents of E_6, E_7 and E_8★. This relation between the modular invariant sesquilinear forms in the characters of the $A_1^{(1)}$ Kac-Moody algebra and the simply laced Lie algebras seems at present rather mysterious. In particular, one would like to understand the group theoretical meaning of the off-diagonal terms.

In order to produce the positive invariants in the conformal case, we simply need to combine the previous results. For a minimal theory with central charge $c=c(p,p')$ (cf. eq. (1.1)) we need a pair of affine invariants of levels $k=p-2$ and $k'=p'-2$. Since p and p' must be coprimes, they cannot be both even, and this forces at least one of $\mathcal{N}^R$ or $\mathcal{N}^S$ in eq. (4.3) to be $\mathcal{N}=\mathbb{1}$. (algebra A). This leads to a classification in two infinite series and three pairs of exceptional invariants, which we call respectively the principal (or A–A), the complementary (A–D) and the exceptional (A–E) series (see table 3). In the unitary series, we have $p'=m-1$, $p=m$ or $p'=m$, $p=m-1$, $m \geqslant 3$, and the exceptional values are $m=$

★ This observation has been first made in the $k+2=12$ theory by Kac [15].

TABLE 3
List of known partition functions in terms of conformal characters

	$\frac{1}{2}\sum_{r=1}^{p'-1}\sum_{s=1}^{p-1}\lvert\chi_{rs}\rvert^2$	$(A_{p'-1}, A_{p-1})$
$p'=4\rho+2$, $p\geqslant 1$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{\substack{r\,\mathrm{odd}=1\\ r\neq 2\rho+1}}^{4\rho+1}\lvert\chi_{rs}\rvert^2+2\lvert\chi_{2\rho+1,s}\rvert^2+\sum_{r\,\mathrm{odd}=1}^{2\rho-1}(\chi_{rs}\chi^*_{r,p-s}+c.c.)\right\}$	$(D_{2\rho+2}, A_{p-1})$
$p'=4\rho$, $p\geqslant 2$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\,\mathrm{odd}=1}^{4\rho-1}\lvert\chi_{rs}\rvert^2+\lvert\chi_{2\rho,s}\rvert^2+\sum_{r\,\mathrm{even}=1}^{2\rho-2}(\chi_{rs}\chi^*_{p'-r,s}+c.c.)\right\}$	$(D_{2\rho+1}, A_{p-1})$
$p'=12$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{7s}\rvert^2+\lvert\chi_{4s}+\chi_{8s}\rvert^2+\lvert\chi_{5s}+\chi_{11s}\rvert^2\}$	(E_6, A_{p-1})
$p'=18$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{17s}\rvert^2+\lvert\chi_{5s}+\chi_{13s}\rvert^2+\lvert\chi_{7s}+\chi_{11s}\rvert^2+\lvert\chi_{9s}\rvert^2+[(\chi_{3s}+\chi_{15s})\chi^*_{9s}+c.c.]\}$	(E_7, A_{p-1})
$p'=30$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{11s}+\chi_{19s}+\chi_{29s}\rvert^2+\lvert\chi_{7s}+\chi_{13s}+\chi_{17s}+\chi_{23s}\rvert^2\}$	(E_8, A_{p-1})

The unitary series corresponds to $p'=m+1,\ p=m$ or $p=m+1,\ p'=m,\ m=3,4,\dots$.

11, 12, 17, 18, 29, 30. In the expressions of table 3, the summation over s may be rewritten as

$$\frac{1}{2}\sum_{s=1}^{p-1}=\sum_{p's<pr}^{s},$$

using $\chi_{rs}=\chi_{p'-r,p-s}$ and the symmetry under $r\to p'-r$ of all the summands.

In the non-unitary theories, i.e. for $|p-p'|>1$, the table of conformal weights (1.2) always contains at least one negative value, namely h_{r_0,s_0} with r_0 and s_0 satisfying (1.2b) and (2.2) [5, 16]. It turns out that all the partition functions of table 3 contain the scalar operator of dimensions $h=\bar{h}=h_{r_0s_0}$. This is obvious in the (A, A) case, where all scalar operators contribute to Z. In the other cases, assuming for example p' to be even, r_0 is odd and coprime to p'. In the (D, A) cases, which involves all operators with $r=r'$ odd, as well as in the (E, A) cases, where all odd values of r coprime to p' appear in the diagonal terms, $|\chi_{r_0s_0}|^2$ shows up. We conclude that the presence of negative dimension operators is an unavoidable feature of *minimal* non-unitary theories.

One is left with the question of finding concrete realizations and interpretations for all these invariants. In the affine case, this has already been discussed in refs. [8, 7]. In the conformal case, the unitary models corresponding to the (A, A) series are the Ising model and its RSOS generalizations [17], whereas (A_4, D_4) and

(D_4, A_6) describe the 3-state critical and tricritical Potts models. (Is there a role of the A_2 and A_3 Lie algebras in the Ising case?) In the unitary case, the other theories have not yet been totally identified, although there is some current work of Pasquier [9] who constructs integrable models based upon simply laced Lie algebras. According to Pasquier, one expects the discrete symmetry of these theories to be at most the symmetry group of their Dynkin diagram. We have verified that the conformal theories $m = 11$ and $m = 12$ associated with the E_6 algebra do have a $\mathbb{Z}_2$ symmetry. According to [18, 19, 6], this amounts to showing the existence of partition functions with $\mathbb{Z}_2$ twisted boundary conditions. These frustrated partition functions read, for example for $m = 11$, with the notations of refs. [18, 6]:

$$Z_{1,0} = \sum_{r\ \mathrm{odd}=1}^{9} \left\{ (\chi_{r1} + \chi_{r7})(\chi_{r5} + \chi_{r11})^* + \text{c.c.} + |\chi_{r4} + \chi_{r8}|^2 \right\},$$

$$Z_{0,1} = \sum |\chi_{r1} + \chi_{r7}|^2 + |\chi_{r5} + \chi_{r11}|^2 - |\chi_{r4} + \chi_{r8}|^2 .$$

On the latter expression, one reads [19] that the operators with $s, \bar{s} = 1, 7, 5, 11$ are even under the $\mathbb{Z}_2$ transformation while those with $s = \bar{s} = 4, 8$ are odd.

Our final comment concerns the compatibility of these new $A_1^{(1)}$ and conformal theories with the "fusion rules" of conformal algebra [1, 8]. For the two infinite series and the first two exceptional theories, this has been checked by Gepner. The same holds for the $k = 28$ or $m = 29$ or 30 theories.

It may be appropriate as a conclusion to quote Arnold [20]. "The relations between all the A, D, E classifications are used for the simultaneous study of all simple objects regardless of the fact that many of them [...] remain an unexplained manifestation of the mysterious unity of the Universe".

It is a pleasure to thank Vincent Pasquier for explaining us his work and for making us realize the importance of Dynkin diagrams in that construction. It is in the course of these discussions and in collaboration with him that our second conjecture was born. We are also indebted to V. Kac for communicating to us his results on the $m = 11$ modular invariants prior to publication. One of us (A.C.) acknowledges the Angelo Della Riccia foundation for partial support.

Appendix A

The functions χ^{conf} and χ^{aff} have modular transformations under S and T of the form:

$$^{T}\chi_\lambda(\tau) = \chi_\lambda(\tau + 1) = e^{i\varphi_T} \exp\left(2i\pi \frac{\lambda^2}{2N}\right) \chi_\lambda(\tau),$$

$$^{S}\chi_\lambda(\tau) = \chi_\lambda(-\tau^{-1}) = e^{i\varphi_S} \frac{1}{\sqrt{N}} \sum_{\lambda' \in \mathbb{Z}/N\mathbb{Z}} \exp\left(2i\pi \frac{\lambda\lambda'}{N}\right) \chi_{\lambda'}(\tau), \qquad \text{(A.1)}$$

with φ_s and φ_T well defined phases in each case (compare eqs. (2.17), (2.19)). Let us examine in general what are the consistency conditions on these phases so that (A.1) defines an action of the modular group on these functions. The transformations S and T acting on τ satisfy

$$S^2 = (ST)^3 = 1, \tag{A.2}$$

hence the same condition must be satisfied by their action on the χ's. It is easy to calculate:

$$\begin{aligned}
{}^{S^2}\chi_\lambda(\tau) &= e^{2i\varphi_S}\frac{1}{N}\sum_{\lambda_1,\lambda'}\exp\left(2i\pi\frac{\lambda_1(\lambda+\lambda')}{N}\right)\chi_{\lambda'}(\tau) \\
&= e^{2i\varphi_S}\chi_{-\lambda}(\tau),
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
{}^{(ST)^3}\chi_\lambda(\tau) &= e^{3i(\varphi_S+\varphi_T)}\frac{1}{N^{3/2}}\sum_{\lambda_1,\lambda_2,\lambda'}\exp\bigg(\frac{2i\pi}{2N}(2\lambda\lambda_1+\lambda_1^2+2\lambda_1\lambda_2+\lambda_2^2 \\
&\qquad\qquad +2\lambda_2\lambda'+\lambda'^2)\bigg)\chi_{\lambda'}(\tau) \\
&= e^{3i(\varphi_S+\varphi_T)}\frac{1}{N}\sum_{\lambda_1,\lambda'}\exp\left(\frac{2i\pi}{N}\lambda_1(\lambda-\lambda')\right) \\
&\quad\times\frac{1}{\sqrt{N}}\sum_{\lambda_2}\exp\left(\frac{2i\pi}{2N}(\lambda_2+\lambda_1+\lambda')^2\right)\chi_{\lambda'}(\tau) \\
&= G\,e^{3i(\varphi_S+\varphi_T)}\chi_\lambda(\tau).
\end{aligned} \tag{A.4}$$

G stands for the Gauss sum

$$\begin{aligned}
G &= \frac{1}{\sqrt{N}}\sum_{\lambda\in\mathbb{Z}/N\mathbb{Z}}\exp\left(\frac{2i\pi}{2N}\lambda^2\right) = \frac{1}{\sqrt{2}\sqrt{2N}}\sum_{\lambda\in\mathbb{Z}/2N\mathbb{Z}}\exp\left(2i\pi\frac{\lambda^2}{2N}\right) \\
&= \sqrt{\tfrac{1}{2}}\,(1+i) = \exp\left(\tfrac{1}{4}i\pi\right),
\end{aligned}$$

which holds for even N. Eqs. (A.2), (A.3) and (A.4) imply

$$\begin{aligned}
&\varphi_S = 0 \text{ or } \pi && \text{on even functions}, \\
&\varphi_S = \pm\tfrac{1}{2}\pi && \text{on odd functions}, \\
&\varphi_T = \tfrac{1}{12}(8j-1)\pi - \varphi_S, && j = 0, 1 \text{ or } 2.
\end{aligned}$$

The phases relative to $\chi^{\text{conf}}, \chi^{\text{aff}}$ are thus two realizations out of these twelve possibilities:

$$\chi^{\text{conf}}: \varphi_S = 0, \qquad \varphi_T = -\tfrac{1}{12}\pi \qquad (j=0),$$

$$\chi^{\text{aff}}: \varphi_S = -\tfrac{1}{2}i\pi, \qquad \varphi_T = -\tfrac{1}{4}\pi \qquad (j=2).$$

Appendix B

We want to study the action on the K-function of modular transformations belonging to the subgroup Γ_{2N} of level $2N$:

$$A \in \Gamma_{2N}, \qquad A = \pm \begin{pmatrix} 1+2\alpha N & 2\beta N \\ 2\gamma N & 1+2\delta N \end{pmatrix}. \tag{B.1}$$

The theta function:

$$\Theta(\zeta; t) = \sum_{n=-\infty}^{\infty} \exp\left[2i\pi\left\{\tfrac{1}{2}t\left(n+\tfrac{1}{2}\right)^2 + \left(n+\tfrac{1}{2}\right)\left(\zeta - \tfrac{1}{2}\right)\right\}\right] \tag{B.2}$$

enjoys the following two properties:

$$\forall\, \mu, \nu \in \mathbb{Z}: \qquad \Theta(\zeta + \mu t + \nu; t) = (-1)^{\mu+\nu} \exp\left(-2i\pi\left(\tfrac{1}{2}\mu^2 t + \mu\zeta\right)\right)\Theta(\zeta; t), \tag{B.3a}$$

$$\forall\, A \in \Gamma: \; \Theta\left(\frac{\zeta}{ct+d}; \frac{at+b}{ct+d}\right) = \varepsilon_A^3 (ct+d)^{1/2} \exp\left(i\pi c \frac{\zeta^2}{ct+d}\right)\Theta(\zeta; t), \tag{B.3b}$$

where ε_A is the same phase that occurred in (2.15) [21]. Defining

$$t = N\tau,$$
$$2\zeta = 1 + \tau(2\lambda - N), \tag{B.4}$$

we have the relation for K_λ introduced in (2.6)

$$\eta(\tau) K_\lambda(\tau) = \exp\left[\frac{i\pi\tau}{4N}(2\lambda - N)^2\right]\Theta(\zeta; t). \tag{B.5}$$

We apply formula (B.3b) for a particular element $\tilde{A} \in \Gamma$ of the form

$$\tilde{A} = \begin{pmatrix} 1+2\alpha N & 2\beta N^2 \\ 2\gamma & 1+2\delta N \end{pmatrix}, \tag{B.6}$$

acting on t as

$$t \xrightarrow{\tilde{A}} t' = \frac{(1+2\alpha N)t + 2\beta N^2}{2\gamma t + (1+2\delta N)}. \tag{B.7}$$

Using the relation $t = N\tau$, this means that on τ acts A of eq. (B.1), according to

$$\tau \xrightarrow{A} \tau' = \frac{(1+2\alpha N)\tau + 2\beta N}{2\gamma N\tau + (1+2\delta N)}. \tag{B.8}$$

From (B.4), the corresponding ζ' reads:

$$\zeta' = \tfrac{1}{2} + \tau'\left(\lambda - \tfrac{1}{2}N\right). \tag{B.9}$$

The unimodularity of the matrix A ensures that two integers μ and ν may be found such that

$$\zeta' - \frac{\zeta}{2\gamma t + (1+2\delta N)} = N(\mu\tau' + \nu). \tag{B.10}$$

One may then compute $K_\lambda(\tau')$ using (2.15) and (B.3) and one obtains after some calculation

$$K_\lambda\left(\frac{(1+2\alpha N)\tau + 2\beta N}{2\gamma N\tau + (1+2\delta N)}\right) = \frac{\varepsilon_{\tilde{A}}^3}{\varepsilon_A} \exp(i\pi\gamma/2) K_\lambda(\tau). \tag{B.11}$$

Modular transformations of Γ_{2N} change K_λ and $\chi_\lambda^{\mathrm{conf}}$ by a λ- and τ-independent phase which is a 24th root of unity. A similar calculation for the affine characters yields an analogous result with a 8th root of unity:

$$\chi_\lambda^{\mathrm{aff}}\left(\frac{(1+2\alpha N)\tau + 2\beta N}{2\gamma N\tau + (1+2\delta N)}\right) = \left(\frac{\varepsilon_{\tilde{A}}}{\varepsilon_A}\right)^3 \exp(i\pi\gamma/2) \chi_\lambda^{\mathrm{aff}}(\tau). \tag{B.12}$$

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
[2] V.G. Kac, Lecture Notes in Phys., vol. 94 (Springer, 1979) p. 441;
B.L. Feigin and D.B. Fuchs, Functs. Anal. Pril. 16 (1982) 47 (Funct. Anal. and Appl. 16 (1982) 114)
[3] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; *in* Vertex operators in mathematics and physics, ed. J. Lepowsky, S. Mandelstam and I. Singer (Springer, NY, 1985)
[4] J.L. Cardy, Nucl. Phys. B270 [FS16] (1986) 186
[5] C. Itzykson and J.-B. Zuber, Nucl. Phys. B275 [FS17] (1986) 580
[6] J.-B. Zuber, Proc. Symp. on Topological and geometrical methods in field theory, Espoo, Finland (June 1986) (World Scientific) to be published
[7] D. Gepner, Princeton preprint
[8] D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493
[9] V. Pasquier, Saclay preprint PhT 86/124
[10] A. Rocha-Caridi, *in* Vertex operators in mathematics and physics (Springer, 1985)
[11] V.G. Kac and D. Peterson, Adv. Math. 53 (1984) 125;
V.G. Kac, Infinite dimensional Lie algebras (Cambridge University Press, London, 1985)
[12] R.C. Gunnings, Lectures on modular forms (Princeton University Press, 1962);
M.I. Knopp, Modular functions in analytic number theory (Markham, Chicago, 1970)

[13] H.S.M. Coxeter and W.O.J. Moser, Generators and relations for discrete groups (Springer, Berlin, 1965)
[14] J.E. Humphreys, Introduction to Lie algebras and representation theory (Springer, Berlin, 1972)
[15] V.G. Kac, private communication
[16] C. Itzykson, H. Saleur and J.-B. Zuber, Europhys. Lett. 2 (1986) 91
[17] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193;
D.A. Huse, Phys. Rev. B30 (1984) 3908
[18] J.-B. Zuber, Phys. Lett. 176B (1986) 127
[19] J.L. Cardy, Nucl. Phys. B275 [FS17] (1986) 200
[20] V.I. Arnold, Catastrophe theory (Springer, Berlin, 1984)
[21] A. Weil, Elliptic functions according to Eisenstein and Kronecker (Springer, 1976)