MINIMAL MODEL CORRELATION FUNCTIONS ON THE TORUS

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We generalize the Feigin–Fuchs construction to the torus, and propose an ansatz for certain correlation functions of the minimal conformal models. Our ansatz is periodic, modular-covariant and has the correct short-distance behavior. As an example, we compute the one-point function of the Ising-model energy operator on the torus.

The computation of correlation functions on arbitrary-genus Riemann surfaces is an important problem in conformal field theory. On the sphere, the minimal model correlation functions can be calculated using the Feigin–Fuchs construction [1]. For higher-genus surfaces, there is no known method for systematically computing the correlation functions. In this letter we generalize the Feigin–Fuchs construction and propose an expression for certain minimal model two-point functions on the torus.

In the Feigin–Fuchs construction, the minimal models with central charge [2]

$$c=1-\frac{6(p-p')^2}{pp'} \tag{1}$$

are represented in terms of a free scalar field $\phi(z,\bar z)$ interacting with a background charge $-2\alpha_0=-2(p-p')/p'$ at infinity. The scalar field is normalized so that its planar two-point function is given by

$$\langle\phi(z,\bar z)\phi(w,\bar w)\rangle=-\frac{p'}{p}\log|z-w|. \tag{2}$$

The spin-zero primary fields in these models have conformal dimensions

$$h_{rs}=\bar h_{rs}=\frac{(pr-p's)^2-(p-p')^2}{4pp'}. \tag{3}$$

They are represented in terms of vertex operators

$$A_{rs}(z,\bar z)=\exp[\mathrm{i}\alpha_{rs}\phi(z,\bar z)], \tag{4}$$

where $\alpha_{rs}=\frac{1}{2}(1-r)\alpha_++\frac{1}{2}(1-s)\alpha_-$, $\alpha_+=2p/p'$ and $\alpha_-=-2$.

The planar four-point functions of the spin-zero fields can be represented in terms of the vertex operators $A_{rs}(z,\bar z)$ [1]. In particular, we have

$$\begin{aligned}&\langle \mathrm{B}|A_{rs}(1)A_{rs}(z,\bar z)|\mathrm{B}\rangle\\&=\langle \mathrm{B}(\infty)A_{rs}(1)A_{rs}(z,\bar z)\mathrm{B}(0)\rangle\\&=\mathcal{N}_4\langle\exp[-\mathrm{i}\beta\phi(\infty)]\exp[\mathrm{i}\alpha_{rs}\phi(1)]\\&\times\exp[\mathrm{i}\alpha_{rs}\phi(z,\bar z)]\exp[\mathrm{i}\beta\phi(0)]Q_+^{r-1}Q_-^{s-1}\rangle,\end{aligned} \tag{5}$$

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where $\mathcal{N}_A$ is a normalization constant. The $Q_\pm = \int d^2w \exp[i\alpha_\pm \phi(w, \bar{w})]$ are screening operators. They have conformal dimension zero and are introduced to balance the charge in the correlation function (5). The conformal blocks can be exhibited by converting the two-dimensional integrals into contour integrals on the plane [3].

The explicit formulas of ref. [1] show that $\mathcal{N}_A$ does not depend on the state $|B\rangle$. Thus we can use (5) to write the two-point functions in terms of screening charges

$$\langle A_{rs}(1) A_{rs}(z, \bar{z}) \rangle$$
$$= \mathcal{N}_A \langle \exp[i\alpha_{rs}\phi(1)] \exp[i\alpha_{rs}\phi(z, \bar{z})]$$
$$\times Q_+^{r-1} Q_-^{s-1} \rangle. \tag{6}$$

Power-counting arguments show that (6) scales as $1/|1-z|^{4h_{rs}}$. In the two-point functions, $\mathcal{N}_A$ is chosen so that the singularity has residue one.

In what follows we generalize these two-point functions to the torus. Before we do this, however, let us first recall the possible forms for the zero-point functions on the torus. These are just the possible partition functions, which can be naturally expressed in terms of the Virasoro characters $\chi_\lambda(q)$.

The possible modular-invariant partition functions are classified by pairs of simply-laced Lie algebras $(A_{p-1}, G_{p'-1})$ with Coxeter numbers p and p' [4,5]. The allowed partition functions then take the form [6]

$$Z_{(A_{p-1}, G_{p'-1})}$$
$$= \tfrac{1}{2} \sum_n \sum_{m,m' \in \mathbb{Z}} \cos \frac{2\pi n}{p'} \langle m, m' \rangle Z_{m,m'}(p/p'), \tag{7}$$

where $Z_{m,m'}(g)$ is the partition function for a free scalar field, compactified on a circle, with action $(g/\pi) \int d^2z\, \partial_z \phi \partial_{\bar{z}} \phi$. The different sectors of the partition function are labeled by integers m and m', corresponding to the boundary conditions $\phi(z+1, \bar{z}+1) = \phi(z, \bar{z}) + 2\pi m$ and $\phi(z+\tau, \bar{z}+\bar{\tau}) = \phi(z, \bar{z}) + 2\pi m'$, where τ is the modular parameter on the torus. The sum over n runs over the exponents of the $G_{p'-1}$ algebra, and $\langle m, m' \rangle$ denotes the greatest common divisor of m and m'.

On the torus, the generalization of the background charge at infinity is given by the finite set of values n/p'. For unitary theories, $|p-p'|=1$ and the background charge can be identified with $1/p'$. The coupling of the charges to the winding numbers m and m' through $\cos 2\pi n \langle m, m' \rangle / p'$ arises naturally in the discussion of the corresponding lattice models [6–9]. This coupling is also consistent with modular invariance.

To obtain our ansatz for the two-point functions of the spin-zero fields on the torus, we combine (6) and (7). We restrict ourselves to primary fields with [#1] $r=1$. (The case $r \neq 1$ and $s=1$ can be obtained by interchanging p and p' in the formulas below.) Our expression for the two-point functions on the torus is

$$Z \langle A_{r1}(z, \bar{z}) A_{r1}(0) \rangle_{\text{torus}}$$
$$= \tfrac{1}{2} \mathcal{N}_A \sum_n \sum_{m,m' \in \mathbb{Z}} \cos \frac{2\pi n}{p'} \langle m, m' \rangle Z_{m,m'}(p/p')$$
$$\times \langle \exp[i\alpha_{1s}\phi(z, \bar{z})] \exp[i\alpha_{1s}\phi(0)]\, Q_-^{s-1} \rangle_{m,m'}. \tag{8}$$

The screening charges are integrated over the fundamental region of the torus T. Similar expressions can be written for other n-point functions.

In eq. (8) the expectation value is computed in the winding number sector labeled by m and m'. There are two contributions to the expectation value; one classical and one quantum mechanical. The classical contribution is found using the solution

$$\phi_{\text{cl}}(z, \bar{z}) = 2\pi\, \text{Im}[(m' - m\bar{\tau})z] / \text{Im}\, \tau. \tag{9}$$

The quantum contribution is obtained using Wick's theorem and the doubly-periodic propagator

$$\langle \phi_{\text{qu}}(z, \bar{z}) \phi_{\text{qu}}(0) \rangle_{\text{torus}}$$
$$= -\frac{p'}{p} \log \left| \frac{\vartheta_1(z|\tau)}{\vartheta_1'(0)|\tau)} \right| + \pi \frac{p'}{p} \frac{(\text{Im}\, z)^2}{\text{Im}\, \tau}. \tag{10}$$

In what follows we shall only consider the diagonal (A, A) minimal theories, where all fields have spin zero. (The other cases can be treated in a similar fashion.) For (A, A) theories, the sum over n in (8) runs from 1 to p'^{-1}, and we have

[#1] When we study the periodicity properties of the correlation functions, we shall see why we restrict ourselves to these values.

$$\sum_{n=1}^{p'-1} \sum_{m,m'\in\mathbb{Z}} \cos\frac{2\pi n}{p'} \langle m, m'\rangle Z_{m,m'}(p/p')$$
$$\times \langle \exp[i\alpha_{1s}\phi(z,\bar{z})] \exp[i\alpha_{1s}\phi(0)]\, Q_-^{s-1}\rangle_{m,m'}$$
$$= \sum_{m,m'\in p'\mathbb{Z}} p' Z_{m,m'}(p/p')$$
$$\times \langle \exp[i\alpha_{1s}\phi(z,\bar{z})] \exp[i\alpha_{1s}\phi(0)]\, Q_-^{s-1}\rangle_{m,m'}$$
$$- \sum_{m,m'\in\mathbb{Z}} Z_{m,m'}(p/p')$$
$$\times \langle \exp[i\alpha_{1s}\phi(z,\bar{z})] \exp[i\alpha_{1s}\phi(0)]\, Q_-^{s-1}\rangle_{m,m'}. \quad (11)$$

After a Poisson transformation on the summation index m', the integrand factorizes into holomorphic and antiholomorphic contributions

$$Z\langle A_{1s}(z,\bar{z})A_{1s}(0)\rangle_{\text{torus}}$$
$$= \tfrac{1}{2}\mathcal{N}_1 \left(\sum_{\substack{m\in p'\mathbb{Z}\\ e\in\mathbb{Z}/p'}} - \sum_{m,e\in\mathbb{Z}}\right) \frac{q^{h_{em}}}{\eta(q)} \frac{\bar{q}^{\bar{h}_{em}}}{\eta(\bar{q})} |E(z)|^{\alpha^2/g}$$
$$\times \int \prod_{i=1}^{s-1} d^2u_i \prod_{i<j} |E(u_i-u_j)|^{\alpha_-^2/g}$$
$$\times \prod_i |E(z-u_i)E(u_i)|^{\alpha\alpha_-/g}$$
$$\times \exp\frac{2\pi i}{\sqrt{g}}\left[\Delta_{em}\left(\alpha_- \sum u_i + \alpha z\right)\right.$$
$$\left. -\bar{\Delta}_{em}\left(\alpha_- \sum \bar{u}_i + \alpha\bar{z}\right)\right], \quad (12)$$

where $\alpha=\alpha_{1s}$ and $q=\exp(2\pi i\tau)$. In (12), $\eta(q)$ is the Dedekind η-function, $E(z)=\vartheta_1(z|\tau)/\vartheta_1'(0|\tau)$ is the prime form and

$$\Delta_{em} = \frac{1}{2}\left(\frac{e}{\sqrt{g}} + m\sqrt{g}\right), \quad \bar{\Delta}_{em} = \frac{1}{2}\left(\frac{e}{\sqrt{g}} - m\sqrt{g}\right). \quad (13)$$

The exponents $h_{em}=\Delta_{em}^2$ and $\bar{h}_{em}=\bar{\Delta}_{em}^2$ are the conformal weights of the gaussian field.

The conformal weights Δ_{em} and $\bar{\Delta}_{em}$ can be expressed in terms of p, p', integers n, $\bar{n}$ and a residue λ [4]. When $e\in\mathbb{Z}/p'$ and $m\in p'\mathbb{Z}$, we have

$$\Delta_{em} = \frac{2pp'n+\lambda}{2\sqrt{pp'}}, \quad \bar{\Delta}_{em} = \frac{2pp'\bar{n}+\lambda}{2\sqrt{pp'}}, \quad (14)$$

while for $e\in\mathbb{Z}$ and $m\in\mathbb{Z}$, we find

$$\Delta_{em} = \frac{2pp'n+\lambda}{2\sqrt{pp'}}, \quad \bar{\Delta}_{em} = \frac{2pp'\bar{n}-\omega_0\lambda}{2\sqrt{pp'}}. \quad (15)$$

Here $\omega_0=ap+bp'$, where the integers a and b satisfy $ap-bp'=1$. Following ref. [4], we can trade the sums over e and m in (12) for sums over n, $\bar{n}$ and λ. This gives the final formula for the spin-zero two-point functions on the torus,

$$Z\langle A_{1s}(z,\bar{z})A_{1s}(0)\rangle_{\text{torus}}$$
$$= \tfrac{1}{4}\mathcal{N}_1 |E(z)|^{\alpha^2/g} \int \prod_{i=1}^{s-1} d^2u_i \prod_{i<j} |E(u_i-u_j)|^{\alpha_-^2/g}$$
$$\times \prod_i |E(z-u_i)E(u_i)|^{\alpha\alpha_-/g} \sum_{\lambda \bmod 2pp'} |\chi_\lambda(q,w)|^2, \quad (16)$$

where

$$\chi_\lambda(q,w) = \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} q^{(2pp'n+\lambda)^2/4pp'}$$
$$\times \exp[2\pi i(2pp'n+\lambda)w]$$
$$- \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} q^{(2pp'n-\omega_0\lambda)^2/4pp'}$$
$$\times \exp[2\pi i(2pp'n-\omega_0\lambda)w] \quad (17)$$

and

$$w = -\frac{1}{2p}\sum_{i=1}^{s-1}(2u_i - z). \quad (18)$$

The function $\chi_\lambda(q,w)$ is a generalization of the Virasoro character, which is obtained by setting $w=0$. It immediately follows from the definition (17) that the generalized Virasoro character satisfies

$$\chi_\lambda(q,w) = \chi_{-\lambda}(q,-w). \quad (19)$$

It is also easy to verify that χ_λ obeys the following property:

$$\chi_\lambda(q,w) = \chi_{\lambda+2pp'}(q,w) = -\chi_{-\omega_0\lambda}(q,w). \quad (20)$$

Because $\omega_0 p' = -p' \pmod{2pp'}$, the generalized character χ_λ vanishes when λ is a multiple of p'. Unlike the Virasoro character, χ_λ does not vanish when λ is a multiple of p. Therefore the expressions for the correlation functions contain more terms than the corresponding formulas for the partition functions.

Let us now return to the two-point functions and study their properties. It is not hard to check that the integrand in (16) is doubly-periodic in each of the variables u_i, with periods 1 and τ. This property is what allowed us to restrict our u-integrals to fundamental domain of the torus T. Note, however, that the integrand is not manifestly periodic if we relax the condition $r=1$. This is the reason why we restricted the parameters s and r in the above discussion. It is also easy to check that the two-point functions (16) are doubly-periodic in the variable z. Furthermore, they are even functions of z, as follows from (19).

The modular properties of the correlation functions are obtained from the corresponding properties of the generalized Virasoro character. For example, one can show that the two-point functions transform covariantly

$$\langle A(z/\tau, \bar{z}/\bar{\tau})A(0)\rangle_{-1/\tau}=|\tau|^{4h}\langle A(z,\bar{z})A(0)\rangle_{\tau} \tag{21}$$

under the modular transformation $\tau\to-1/\tau$ and $z\to z/\tau$ by using the following property of the generalized character $\chi_\lambda(q, w)$:

$$\chi_\lambda(\exp(-2\pi i/\tau), w/\tau)=\exp(2\pi i p p' w^2/\tau\sqrt{2pp'})\times\sum_{\lambda' \bmod 2pp'}\exp(-i\pi\lambda\lambda'/2pp')\,\chi_{\lambda'}(q,w). \tag{22}$$

Note that the above correlation functions also have the correct behavior at short distances. This can be seen by taking $z\to0$, where we recover the universal behavior $\langle A(z,\bar{z})A(0)\rangle\to1/|z|^{4h}$. This follows from the fact that the u-integrals are dominated by the region $|u|\sim|z|$. The generalized characters decouple in the integral and reproduce the partition functions. In this limit we recover the result (6) on the plane. The normalization constant $\mathcal{N}_A$ is determined by demanding that the residue be equal to one. The normalization $\mathcal{N}_A$ is independent of q and coincides with its value on the plane.

A second interesting limit is Im $\tau\to\infty$, where the torus degenerates to a long cylinder. Expanding the correlation function in a small q-expansion, one can read off the two- and four-point functions $\langle AA\rangle$ and $\langle BAAB\rangle$ on the plane. For example, in the Ising model we have checked that the two-point function of the energy operator $\epsilon(z,\bar{z})$ has the following small q-expansion:

$$\begin{aligned}Z\langle\epsilon(z,\bar{z})\epsilon(0)\rangle_{\text{torus}}&=(2\pi)^2\exp[i\pi(z-\bar{z})](q\bar{q})^{-1/48}\\&\times[\langle\epsilon(w,\bar{w})\epsilon(1)\rangle_{\text{plane}}\\&+(q\bar{q})^{1/16}\langle\sigma|\epsilon(w,\bar{w})\epsilon(1)|\sigma\rangle_{\text{plane}}\\&+(q\bar{q})^{1/2}\langle\epsilon|\epsilon(w,\bar{w})\epsilon(1)|\epsilon\rangle_{\text{plane}}+\ldots],\end{aligned} \tag{23}$$

where σ is the spin operator, $w=\exp(2\pi iz)$, and the normalization $\mathcal{N}_A$ is chosen as in (6). The correlation functions on the right-hand side of (23) are all evaluated on the plane.

The two-dimensional integrals over u can be transformed into contour integrals. As mentioned earlier this form is useful for displaying the holomorphic and antiholomorphic conformal blocks. To show how this works, we shall consider the one-point function of the Ising-model energy operator. (The two-point functions can be treated using similar techniques.) The energy operator can be represented by the vertex operator A_{31}, and its one-point function follows trivially from (8), after exchanging p with p'. On the torus, this operator has a nonvanishing expectation value.

To compute the expectation value of the energy operator, we must insert one screening charge into the correlation function. We then have

$$\begin{aligned}Z\langle\epsilon\rangle&\propto\int d^2u\,\langle\exp[i\alpha_+\phi(u,\bar{u})]\exp[-i\alpha_+\phi(0)]\rangle\\&\propto\int d^2u\,|E(u)|^{\alpha_+^2/g}\sum_\lambda|\chi_\lambda(q,u/4)|^2.\end{aligned} \tag{24}$$

To transform the integral over u to a contour integral, we use Stokes theorem. A simple calculation gives

$$Z\langle\epsilon\rangle\propto\frac{1}{2i}\sum_\lambda\oint_{\partial T}du\,E(u)^{-3/2}\chi_\lambda(q,u/4)\,f_\lambda(\bar{u}), \tag{25}$$

where

$$f_\lambda(\bar{u})=\int_0^{\bar{u}}dv\,\bar{E}(v)^{-3/2}\bar{\chi}_\lambda(\bar{q},v/4). \tag{26}$$

The integral in (25) is over the boundary of the fundamental region of the torus, defined by the parallelogram $(0, 1, 1+\tau, \tau)$. Neither $E^{-3/2}\chi_\lambda$ nor f_λ is doubly-periodic, but we have

$$E^{-3/2}\chi_\lambda|_{u+1}=\exp[i\pi(\lambda+3)/2]E^{-3/2}\chi_\lambda|_u \tag{27}$$

and

$$E^{-3/2}\chi_\lambda|_{u+\tau}=\exp(-3\pi i/2)\,E^{-3/2}\chi_{\lambda-6}|_u. \quad (28)$$

Using (25), (27) and (28), we can express $Z\langle\epsilon\rangle$ as a sum of contour integrals of $E^{-3/2}\chi_\lambda$ around the 1- and τ-cycles. We can eliminate the integral over the τ-cycle using Cauchy's theorem,

$$\oint_{\partial T} du\, E^{-3/2}(u)\chi_\lambda(q, u/4)=0. \quad (29)$$

After some algebra we find that only $\lambda=\pm 2$ contributes to the one-point function

$$Z\langle\epsilon\rangle \propto \left|\int_0^1 du\, E(u)^{-3/2}\chi_2(q, u/4)\right|^2. \quad (30)$$

This integral is defined by an analytic continuation of the exponent $-\alpha_+^2/2g$ to $-3/2$. As expected, only one conformal block contributes to $\langle\epsilon\rangle$. This block corresponds to the propagation of the spin field and its descendants around the torus.

Our results for the two-point functions give rise to a host of interesting identities for integrals over theta functions. These are obtained by comparing the integral representations presented here to known expressions [10].

In this letter we have proposed an expression for certain minimal model two-point functions on the torus. Our ansatz has the correct periodicity, modular properties and singularity structure. However, we have not proven that our expression is indeed correct. In particular, it would be desirable to prove that (8) or (12) satisfy the partial differential equations of ref. [11].

After this work was completed we received a paper by Felder [12] which uses an algebraic approach to the Feigin–Fuchs construction to derive an integral representation for the conformal blocks on the torus. Felder uses the fact that no null states propagate along the torus, so his integrals automatically satisfy the differential equations of ref. [11]. We have checked that our integrand coincides with that of Felder. It must therefore be possible to reduce our two-dimensional integrals to those of ref. [12].

Our expressions can be readily generalized to more complicated situations. For example, they can be applied to non-diagonal minimal models with operators of nonzero spin [13], to Feigin–Fuchs representations of coset theories [14–18], as well as to models for which representations similar to (7) are known [8,19]. One would also like to extend our results to higher-point functions and to the case of operators with r and $s>1$. (This restriction does not appear in ref. [12].) It would also be interesting to find a derivation of (8) starting from the corresponding lattice model, thus extending the Coulomb gas techniques of ref. [20] and the arguments of ref. [6].

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