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Asymptotic estimates in quantum electrodynamics. II

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The order of growth of the fermion determinant in quantum electrodynamics, $\det(\not{p} - e\not{A})$, as an entire function of e is reconsidered. This order is of utmost importance in the high-order estimates of the perturbative expansion in QED. Explicit examples and some general arguments lead us to the conclusion that this determinant is generally of order 4 in four dimensions. The behavior $\alpha^n(n/2)!(-a)^n$ of the nth order of perturbation theory follows, disregarding ultraviolet problems.

I. INTRODUCTION

In a previous paper by three of us, under the same title,[1] we presented a general investigation of the large orders of the perturbative expansion in quantum electrodynamics.

Two points were found crucial, namely, the role of gauge invariance on the one hand and of Fermi statistics of the charge-carrying field on the other. It must be stressed that we are interested in the behavior of Green's functions at fixed, Euclidean momenta and we study implicitly a regularized theory. This was not stated perhaps with sufficient clarity in previous papers dealing with this subject, as the main estimate was found independent of ultraviolet divergences. The latter only occurred in the subsequent terms of order k^0, $k^{-1}, \ldots$, where k ($\gg 1$) denotes the order of perturbation and could be accounted for by introducing counterterms in a standard fashion. In other words, the method implies an interchange of the large-cutoff ($\Lambda \to \infty$) and large-order ($k \to \infty$) limits. Within these limitations it was shown in I that if a gauge-invariant amplitude, such as the vacuum polarization for instance, is expanded in powers of e^2, where e is the elementary charge, its kth order behaves as $\Gamma(k(1-2/\rho))$. The real number ρ is the order of the entire function (in e)

$$\Delta(e) = \det(\not{\partial} - e\not{A} + m) = \exp[L(e)], \qquad (1.1)$$

describing the (Euclidean) vacuum-persistence amplitude in the presence of the external c-number potential A.

An attempt was made in I to evaluate ρ by analogy with the simpler case of a Yukawa interaction. Under such circumstances the order of the corresponding determinant could be found by using a quasiconstant (or locally constant) approximation which was shown to be a relativistic generalization of the Thomas-Fermi method of atomic physics. Its natural interpretation is an expansion of $\Delta(e)$ with respect to Planck's constant $\hbar$; the leading term is a classical contribution in $\hbar^0$. The extension of these ideas to QED, however, was not straightforward since a locally constant potential may simply be eliminated by a gauge transformation. This is analogous to the absence of classical diamagnetism (the Bohr-Van Leeuven theorem) in statistical mechanics.[2] Therefore, the next possibility was to assume a quasiconstant field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, leading to a function $L(e)$ behaving as $e^{D/2}$ in space-time dimension D, or $\rho = D/2$.

However, this result was at variance with an explicit computation of Adler,[3] who found in his case an order $\rho = 4$ for $D = 4$. It is very likely that this is the correct result even though we have not succeeded in finding a mathematically clean proof of this property. Nevertheless, we shall present a number of arguments and explicit examples which all support this view, namely, that in dimension D the determinant is of order $\rho = D$ as in the Yukawa case. What seems to happen is that for real e (and A) $\Delta(e)$ does not assume its maximal growth but behaves as exp (const $\times e^{D/2}$) in agreement with classical expectations. This circumstance would reconcile our previous analysis with the general behavior for arbitrary complex coupling.

The organization of the paper suffers unfortunately from our lack of a rigorous proof. It is roughly divided in two parts. Sections II and III use extremely simplified examples as a basis for a general conjecture. The latter is then tested in two specific instances where calculations can be performed in detail (Secs. IV and V). In Sec. VI we summarize our conclusions. The role of spin in this context is not altogether clear. Indeed some of the intuitive arguments developed in I with some precaution have to be revised in the light of the present work.

Beyond their relevance to perturbation theory we believe that the questions raised here may have some bearing on various physical problems connected with strong fields.

II. ELEMENTARY EXAMPLES

We are interested in the study of the analytic properties of the functions

$$\Delta(e) = \det[\not p - e\not A + m]$$

or

$$\Delta(e) = \det[(p - eA)^2 + m^2], \tag{2.1}$$

i.e., the functional determinants of the Dirac or Klein-Gordon equations in the presence of a smooth external electromagnetic potential A of fast decrease. We consider the Euclidean situation and use the same notations as in I. When expanded in powers of e^2 the logarithm

$$L(e) = \ln\Delta(e) \tag{2.2}$$

is given by the set of Feynman diagrams with one charged-particle loop and an arbitrary (even) number of external photon lines. By definition we set $L(0) = 0$ so that $\Delta(0) = 1$. If the Euclidean space dimension D is greater than or equal to two, some of these diagrams are divergent and the determinant stands for a renormalized one. However, when $D < 4$, insisting on gauge invariance eliminates the divergent part without any recourse to a normalization convention. If $D = 4$, gauge invariance alone is not sufficient to remove the genuine logarithmic divergence in the e^2 term and some conventional subtraction has to be performed, which amounts to a charge renormalization.

As explained in I, $\Delta(e)$ is an entire function in e. We observe for instance that the above arbitrariness in the subtraction procedure only alters $\Delta(e)$ by a factor $\exp(\text{const} \times e^2 \int d^4 x F^2)$. We want to compute its order, i.e., its behavior for large e, its distribution of zeros, and finally the asymptotic behavior of its derivatives at the origin. Of course these properties are not independent, and the theory of entire functions may be used to investigate their mutual relations.

To get some feeling on the problem, it is instructive to consider some elementary cases. For the moment, let us restrict our attention to the Klein-Gordon case. Consider the trivial situation of a circular ring of length L where $|A_\mu|$ is constant and A_μ is tangent to the ring. The problem is effectively one dimensional if we ignore the transverse direction, with

$$\Delta(e) = \det[(P_x - eA)^2 + m^2]/\det[P_x^2 + m^2]. \tag{2.3}$$

The differential operator $P_x = -i\partial_x$ is defined on periodic functions of period L and as a result $\Delta(e)$ is a periodic function in e with period $2\pi/AL$. Indeed the equation

$$[(\partial_x + ieA)^2 - m^2]\psi(x) = 0 \tag{2.4}$$

admits periodic solutions only for $eA = 2\pi l/L \pm im$, where l is an integer. The entire function can now be reconstructed from the knowledge of its zeros and from the requirement of being even in e as follows:

$$\Delta(e) = \prod_{l=-\infty}^{+\infty} \left[1 - \frac{e^2A^2}{(2\pi l/L + im)^2}\right] = 1 + \frac{\sin^2(eAL/2)}{\sinh^2(mL/2)}. \tag{2.5}$$

We find that $\Delta(e)$ in an entire function of order one ($D = 1$), and moreover, for e real, we observe that $\Delta(e)$ remains bounded (indeed it is periodic) while for complex e, $L(e) = \ln\Delta(e)$ behaves as

$$L(e) \sim LA\,|\mathrm{Im}\,e| \tag{2.6}$$

for large e in agreement with the statements made in the Introduction.

In passing we note that the two infrared zeros $eA = \pm im$ together with the normalization condition $\Delta(0) = 1$ prevent Δ from having a finite limit as $m \to 0$. However, the mass does not affect the leading asymptotic behavior in e. This one-dimensional problem can readily be generalized to a nonconstant but periodic potential. If $A(x + L) = A(x)$ and ϕ stands for the corresponding "flux"

$$\phi = \int_0^L dx A(x), \tag{2.7}$$

then $A(x) - (1/L)\phi$ can be written as $(d/dx)\Lambda(x)$ with Λ periodic. A gauge transformation implying $\Lambda(x)$ will effectively replace $A(x)$ with the constant ϕ/L. Therefore, the general result, valid for any field is

$$L(e) \underset{|e| \to \infty}{\simeq} \phi\,|\mathrm{Im}\,e|. \tag{2.8}$$

We notice an interesting phenomenon which will also be met in the sequel. Quantum effects produce oscillations along the real line which can be neglected in a first approximation. They are, however, the signal for the existence of zeros near the real axis. For imaginary values of e these oscillatory terms are transformed into exponentially increasing ones which dominate the behavior of $\Delta(e)$.

Let us try now to generalize this to a two-dimensional example with a constant potential along the x direction. The corresponding operator

$$H=(P_x-eA)^2+P_y^2+m^2 \tag{2.9}$$

is defined in a rectangular box of size L in the x direction and T in the y direction. The physically interesting case occurs when we impose periodic boundary conditions in the x direction. In the y direction we can impose any type of boundary conditions. To avoid infrared problems in the zero-mass limit, we require the wave function to vanish for $y=0, T$, and for simplicity set $m=0$. The "eigencouplings" e, corresponding to the vanishing of H, are classified by two quantum numbers $l=p_x L/2\pi$ and $n=p_y T/\pi$ $(n\neq 0)$ corresponding to wave functions

$$\psi_{l,n}(n,y)=\exp\left(il\,\frac{2\pi x}{L}\right)\sin\left(n\frac{\pi y}{T}\right),$$
$$n\geq 1,\quad l=0,\pm 1,\ldots, . \tag{2.10}$$

The zeros of $\Delta(e)$ are then given by

$$e_{l,n}=\frac{2\pi l}{AL}+\frac{i\pi n}{AT},\quad n\neq 0\,. \tag{2.11}$$

The smoothed density of zeros in the complex e plane is therefore given by

$$d\nu(e_1,e_2)=\frac{LTA^2}{2\pi^2}\,de_1de_2=de_1de_2\int\frac{dx\,dy\,A^2}{2\pi^2}\,, \tag{2.12}$$

where we have set $e=e_1+ie_2$. The order ρ of the entire function $\Delta(e)$ is, of course, 2 $(D=2)$. It may be computed from the distribution of zeros. The zeros occur in pairs with opposite values; therefore we can reconstruct the function from a once-subtracted Hadamard representation as follows:

$$\Delta(e)=\prod_{n>1,\,l}\left[\left(1-\frac{e^2}{e_{l,n}^2}\right)\exp\left(\frac{e^2}{e_{l,n}^2}\right)\right]\exp(Ce^2)\,. \tag{2.13}$$

Note that $\sum_{l,n} 1/e_{l,n}^2$ is not absolutely convergent and that

$$\sum_l\sum_{n\geq 1}\frac{1}{e_{l,n}^2}\neq\sum_{n\geq 1}\sum_l\frac{1}{e_{l,n}^2}\,.$$

The use of a subtracted Hadamard representation is therefore compulsory. The constant C should in principle be fixed by requiring that the second derivative of $L(e)$ at $e=0$ coincides with the result of a Feynman-diagram gauge-invariant computation. We may also expect $\Delta(e)$ to be a periodic function of e with period $2\pi/AL$, a condition which fixes C unambiguously. To clarify the situation we note that $L(e)$ can be written as

$$L(e)=-\operatorname{tr}\int_0^\infty\frac{dt}{t}\left(\exp\{-t[(P_x-eA)^2+P_y^2]\}-\exp[-t(P_x^2+P_y^2)]\right). \tag{2.14}$$

Expanding $L(e)$ as a power series in e we reproduce the contribution of perturbation theory. The ambiguity in the definition of the second-order term is solved in a gauge-invariant way by interchanging the integration over t with the trace. In the language of Feynman diagrams this amounts to using the representation $(p^2+m^2)^{-1}=\int_0^\infty dt\times\exp[-t(p^2+m^2)]$ and then exchanging the orders of integrations over p and t.

We therefore find that

$$L(e)=-\int_0^\infty\frac{dt}{t}\sum_{l=-\infty}^{+\infty}\sum_{n=1}\left\{\exp\left(-t\left[\left(\frac{2\pi l}{L}+eA\right)^2+\left(\frac{\pi n}{T}\right)^2\right]\right)-\exp\left(-t\left[\left(\frac{2\pi l}{L}\right)^2+\left(\frac{\pi n}{T}\right)^2\right]\right)\right\}. \tag{2.15}$$

It is trivial to see that when $eAL/2\pi$ is an integer $L(e)$ vanishes, and that for real e it admits the period $2\pi/AL$. We can also verify in the representation (2.15) that $\Delta(e)$ indeed satisfies the condition of minimal growth along the real axis:

$$\Delta(e)=\prod_{n=1}^\infty\left[1+\frac{\sin^2(eAL/2)}{\sinh^2(n\pi L/2T)}\right]$$
$$=\frac{\theta_1(eAL/2\pi;e^{-\pi L/2T})}{2\sin(eAL/2)\,q_0^3}\exp(\pi L/8T)\,, \tag{2.16}$$

where $\theta_1(v;q)$ is the Jacobi function of modulus q (Ref. 4):

$$\theta_1(v;q)=2\sum_{j=1}^\infty(-1)^{j-1}q^{(j-1/2)^2}\sin[(2j-1)\pi v]$$
$$=2q_0q^{1/4}\sin\pi v\prod_1^\infty(1-2q^{2n}\cos 2\pi v+q^{4n})$$

and

$$q_0\equiv\prod_{n=1}^\infty(1-q^{2n})=\prod_1^\infty[1-\exp(-n\pi L/T)]\,. \tag{2.17}$$

One can then verify the following properties of $\Delta(e)$:

$$\sup_{|e|\to\infty}\ln|\Delta(e)|\simeq\frac{1}{2\pi}(\operatorname{Im}e)^2A^2LT\,, \tag{2.18}$$

and

$$\Delta(e)=\sum_0^\infty \Delta_k \frac{e^k}{k!}\ ,$$

$$\Delta_k \underset{k\to\infty}{\sim} \Gamma\left(\frac{k+1}{2}\right)\cos\frac{\pi k}{2}\left(\frac{2A^2LT}{\pi}\right)^{k/2}.$$

This analysis can readily be extended to a D-dimensional tube with longitudinal variable x and $d=D-1$ transverse dimensions. A constant potential points along the x direction with magnitude A and periodic conditions are imposed at the two ends of the tube. We now study the function

$$\Delta(e)=\det[(P_x-eA)^2+P_\perp^2+m^2]\times\det{}^{-1}[P_x^2+P_\perp^2+m^2]\ . \qquad (2.19)$$

On the surface of the tube we may impose any type of conditions provided they are invariant under translation along the x direction. Let y stand for the transverse variables. We look for solutions of the differential equation

$$[-(\partial_x-ieA)^2-\Delta_\perp+m^2]\psi(x,y)=0\ , \qquad (2.20)$$

having the form

$$\psi(x,y)=\exp\left(i\frac{2\pi l}{L}x\right)\psi(y)\ . \qquad (2.21)$$

If λ_n^2 are the positive eigenvalues of the operator $-\Delta_\perp+m^2$ corresponding to the required boundary conditions, we find the eigencoupling condition

$$e_{l,n}=\frac{1}{A}\left(\frac{2\pi l}{L}\pm i\lambda_n\right)\ .$$

Let S be the area of the transverse section of the tube. The asymptotic density of eigenvalues is, according to a famous theorem of Weyl,[5] independent of the boundary conditions and given by

$$\frac{d\rho}{d\lambda}=\frac{\sigma_d S\lambda^{d-1}}{(2\pi)^d}\ , \qquad (2.22)$$

with σ_d the area of the unit sphere in d-dimensional space: $\sigma_d=2\pi^{d/2}/\Gamma(d/2)$. As a consequence, the density of zeros of $\Delta(e)$ is given asymptotically by the expression

$$d\nu(e_1,e_2)=\frac{LSA^D}{(2\pi)^D}\sigma_{D-1}(e_2)^{D-2}de_1de_2\ . \qquad (2.23)$$

For $D=2$ this agrees of course with (2.12). The function $\Delta(e)$ is then of order $\rho=D$ and if we again impose periodicity, it can be written

$$\Delta(e)=\prod_n\left[1+\frac{\sin^2(eLA/2)}{\sinh^2(\lambda_n L/2)}\right]\ . \qquad (2.24)$$

With some work one can show that

$$\sup_{|e|\to\infty}\ln|\Delta(e)|\simeq(\mathrm{Im}e)^D A^D LS\times\frac{1}{(4\pi)^{(D-1)/2}D\Gamma((D+1)/2)} \qquad (2.25)$$

and

$$\Delta(e)=\sum^\infty\Delta_k\frac{e^k}{k!}\ ,$$

$$\Delta_k\simeq\cos\left(\frac{\pi k}{2}\right)\left[\frac{A^DLS}{\Gamma((1+D)/2)}\right]^{k/D}\times\left[\frac{(4\pi)^{1/2}(D-1)}{D}\right]^{(1/D-1)k}\Gamma((k+1)(1-1/D)), \qquad (2.26)$$

up to a factor behaving at most like a power of k. We note that in all the previous estimates, the quantity of interest appears to be $\int d^Dx A^D$.

III. GENERAL CASE

We would like to extend the results of the previous section to the general case where A_μ is not constant. Rigorous results have been recently summarized by Martin[6] for the similar problem of the asymptotic distribution of eigenvalues in the case of a scalar external potential. We will try to use the same technique as far as possible. At a crucial point, strong modifications are needed and our results will only apply to a limited class of potentials. Our arguments are far from rigorous, however, we will be able to test the correctness of the results in some particular cases in the next two sections.

Let us first present a simple form of Martin's arguments neglecting any point of rigor. Consider the functional determinant

$$\Delta(g)=\det H\ ,\qquad H=-\Delta+m^2-gV,\quad V(x)\geq 0. \qquad (3.1)$$

One wants to find an estimate of $N(R)$ the number of zeros of $\Delta(g)$ for $|g|<R$. The self-adjointness of $(-\Delta+m^2)^{-1/2}V(-\Delta+m^2)^{-1/2}$ implies that the zeros are concentrated on the positive real axis. Therefore the problem becomes equivalent to finding the number of bound states in the attractive potential $-gV$ with energy less than m^2. Assume that one is able to find two Hamiltonians $H_\pm$, such that $H_-\leq H\leq H_+$, for which the number of bound states can be exactly computed. If it turns out that one gets the same estimate for $N_+(R)$ and $N_-(R)$ when R gets large, the problem will be solved.

The Hamiltonians $H_\pm$ can be constructed as follows: Space is divided into cubic cells of size δ. The operators H_+ and H_- coincide with H, except that extra boundary conditions are imposed in such a way as to decouple the Schrödinger equations in two different cells. Thus for H_+ we impose that the wave function vanishes at the surface of the cubes, while H_- is constructed by requiring the normal derivative to vanish, disregarding the continuity of the wave function from one cube to the next.

While it is immediately seen that $H_+ > H$ (one has added an infinite repulsive potential along the cell boundaries), it can also be proved that $H > H_-$.

If we let the (linear) size δ of the cells go to zero when g goes to infinity, the potential may be assumed to be almost constant throughout each cube. Moreover, if $\delta^2 g$ also becomes infinite, so does the number of bound states in each cell (independent of the boundary conditions), and one finds the following in dimension D:

$$N_\pm(R) \sim \frac{v_D}{(2\pi)^D} R^{D/2} \int d^D x\, V^{D/2}(x)\,[1 + O(g^{-\epsilon})], \tag{3.2}$$

where ϵ is a computable positive number and v_D is the volume of the unit sphere, $v_D = 2\pi^{D/2}/D\Gamma(D/2)$. The same result also holds for $N(R)$. The reader will note that the above method yields a justification of the so-called quasiconstant field method to obtain large-g estimates.

Let us try to use similar methods in our case. We call $H = (P - eA)^2 + m^2$, again divide the space into cubes of size δ, and define two operators H_+ and H_- as in the previous case. Although we are now unable to bound the properties of H in terms of those of $H_\pm$, we feel that the two cases considered are rather extremal. If it turns out that $\Delta_\pm(e) = \det H_\pm$ have equal asymptotic behavior when $|e| \to \infty$, the same result should be valid for $\Delta(e)$. Now $\Delta_\pm(e)$ factorizes into a product of determinants $\Delta_\pm^i(e)$ pertaining to each cell and

$$\Delta_\pm(e) = \prod_i \Delta_\pm^i(e)\ . \tag{3.3}$$

To zeroth order in $\hbar$ we may approximate the potential A_μ by a constant in each box, and given the boundary conditions we therefore find $\Delta_\pm^i(e) = 1$. A nontrivial result follows when one takes into account the second order in $\hbar$ which amounts to assuming a constant field within each box. To be precise we ought to discuss each (entire) dimension separately, as the number of scalar invariants of the field increases with D. Qualitatively one finds, however, that for large *real* e we obtain

$$\Delta_\pm(e) \sim \exp\left(\text{const} \times e^{D/2} \int d^D x\, F^{D/2} \right), \tag{3.4}$$

while for $D = 4$ an extra logarithmic term arises from ultraviolet divergences and

$$\Delta_\pm(e) \sim \exp\left(\text{const} \times e^2 \int d^4 x\, F^2 \ln e^2 F^2 \right) \quad (D = 4). \tag{3.5}$$

These are the approximations discussed in I. The careful reader has noticed that here we have stressed the fact that Eqs. (3.4) and (3.5) are only expected to be valid for real e (assuming of course that A is real). For complex values of e, the asymptotic behavior of the determinant depends crucially on the boundary conditions along the lines of flow. By lines of flow we mean, in the present context, the integral lines of the A_μ potential (i.e., the curves $x_\mu(s)$ such that $\dot{x}_\mu(s) = A_\mu[x(s)]$). The discussion of the preceding section shows indeed that $\Delta_+(e)$ is presumably an entire function of order $D/2$ since with zero boundary conditions a constant potential may be gauged away. It is not likely that this is the case for $\Delta_-(e)$; the condition of a vanishing normal derivative is not seriously different from periodic boundary conditions, provided that the cell substructure is adapted to the lines of flow. In the latter case (periodic boundary conditions), we already know from the preceding section that

$$\Delta_{\text{p.b.c.}}(e) \sim \exp\left(\text{const} \times |\text{Im}\, e|^D \int d^D x\, |A(x)|^D \right). \tag{3.6}$$

Furthermore, this cannot still be the correct answer in that it has not yet been given a gauge-invariant expression. From this disaster (very different behavior in complex directions according to innocent-looking extra conditions) we learn that we are not allowed to impose arbitrary conditions along the lines of flow.

For our purposes it is strongly suggested that one study the case where the vector potential satisfies the Lorentz condition

$$\partial \cdot A(x) = 0\ . \tag{3.7}$$

We note that if in Euclidean space we insist on a regular potential with fast decrease at infinity, the Lorentz condition fixes the gauge unambiguously. Using this choice the above lines of flow are well defined.

We shall assume that Eq. (3.7) entails that the lines of flow are closed without trying to extend the discussion to a more generic situation. It is then natural to divide space into tubes of small (and in general varying) cross-sectional area S, the surface of which is made of lines of flow. Using our previous assumptions, these tubes will have the topology of tori.

We now define operators $H_\pm$ by imposing, respectively, the vanishing of its normal derivative along the surface of the tube, in such a way that the determinant factorizes into contributions pertaining to each tube separately. Even when the cross section is vanishingly small, the magnitude of the potential A_μ will vary along the tube. Therefore, to obtain an estimate it would be necessary to extend the reasoning of the previous section. We shall not do that here, and will further restrict ourselves to the case where $|A|$ is constant along the lines of flow. If V_i denotes the volume of the ith tube, in the limit of large e, we find the following, independent of boundary conditions:

$$\Delta_i(e) \sim \exp\left[a_D(\mathrm{Im}e)^D \int_V d^Dx\,|A^T(x)|^D\right], \qquad (3.8)$$

for the contribution of the ith tube. Therefore we have the following estimates for the complete determinant:

$$\Delta(e) \sim \exp\left[a_D(\mathrm{Im}e)^D \int d^Dx\,|A^T(x)|^D\right]. \qquad (3.9)$$

In Eqs. (3.8) and (3.9) a_D stands for the quantity

$$a_D^{-1} = (4\pi)^{(D-1)/2} D\,\Gamma\left(\frac{D+1}{2}\right), \qquad (3.10)$$

and the symbol T is to remind us that we deal with the Lorentz (or transverse) gauge. Finally we expect the Taylor series of $\Delta(e)$ to be given as in Eq. (2.26) with $A^D LS$ replaced by $\int d^Dx\,|A^T(x)|^D$. As far as the density of zeros is concerned, it is presumed to be of the form (2.23) with the same substitution as before.

Let us stress the fact that the above results are derived in the Lorentz gauge when the lines of flow are closed curves, and when the A field has a constant length along these lines, although it is possible that their domain of validity is larger.

In the next two sections we show with specific examples that the above estimates are indeed the correct ones.

IV. TWO-DIMENSIONAL MODEL

The general statements made previously will be tested in two steps. In this section we study a two-dimensional nontrivial example referring to the Klein-Gordon equation (see, however, some remarks on the Dirac case at the end of this section). In the next section a similar situation is studied in a four-dimensional context with respect to the Dirac spin-$\frac{1}{2}$ case.

Consider first a two-dimensional Hamiltonian of the form

$$H = (P - eA)^2 + m^2, \qquad (4.1)$$

with $A_\mu(x)$ a prescribed external potential such that the corresponding field $F = \partial_1 A_2 - \partial_2 A_1$ is integrable:

$$\int d^2x\,|F(x)| < \infty. \qquad (4.2)$$

This condition excludes the possibility of a constant field throughout space. As recalled above, the Feynman perturbation expansion of $L(e)$ is convergent due to gauge invariance but infrared divergent if we set $m = 0$. We therefore have to keep $m \neq 0$, and we keep only the leading nonvanishing contributions as $m \to 0$. Also, since we are interested in alternative means of computing $\Delta(e)$, we shall use for $\Delta(e)$ and $L(e) = \ln\Delta(e)$ an index F when referring to the results of computations done within the context of gauge-invariant Feynman perturbation theory. In particular the second-order term in $L(e)$, call it $e^2 L_F^{(2)}$, is finite and given by

$$L_F{}^2 = \pi \int d^2q\,\tilde{A}_\mu(q)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \times \tilde{A}_\nu(-q)\left(\ln\frac{q^2}{m^2} - 2\right), \qquad (4.3)$$

with

$$A_\mu(x) = \int d^2q\,\tilde{A}_\mu(q)\exp(iq\cdot x). \qquad (4.4)$$

Dimension $D = 2$ allows rotationally invariant field configurations which we expect to simplify the calculations seriously, hopefully without destroying the asymptotic large-e behavior. Define the antisymmetric symbol $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ such that $\epsilon_{12} = 1$ and set

$$A_\mu(x) = \epsilon_{\mu\nu}\frac{x_\nu}{x^2}\phi(|x|). \qquad (4.5)$$

Note that $2\pi\phi(|x|)$ is the magnetic flux through the circle of radius $|x|$. Using the notation

$$|x| = e^t \qquad (4.6)$$

we shall abusively use the same symbol for the function expressed in terms of $|x|$ or t variables. Note that $A_\mu(x)$ defined through (4.5) fulfills the Lorentz condition $\partial\cdot A(x) = 0$, and is also transverse in x space, $x\cdot A(x) = 0$. Its Fourier transform reads

$$\tilde{A}_\mu(q) = \frac{i}{2\pi}\epsilon_{\mu\nu}\frac{q_\nu}{q^2}\int_{-\infty}^{+\infty} dt\,J_0(qe^t)\dot{\phi}(t), \qquad (4.7)$$

and may be substituted in (4.3) to yield

$$L_F^{(2)} = \int_0^\infty \frac{dq}{q}\ln q\left[\int_{-\infty}^{+\infty} dt\,\dot{\phi}(t)J_0(qe^t)\right]^2 - (\ln m + 1)\int_{-\infty}^{+\infty} dt\,\phi^2(t). \qquad (4.8)$$

In writing (4.8) we have assumed, of course, that ϕ vanishes sufficiently rapidly both when $t\to\infty$ ($|x|\to\infty$) or $t\to-\infty$ ($|x|\to 0$). We turn now to a determination of the eigencouplings to reconstruct $\Delta(e)$ from a knowledge of its zeros. To this end we use the rotation-invariant form (4.5) to introduce an angular decomposition:

$$(P-eA)^2=P_r^{\,2}+\frac{(L-e\phi)^2}{r^2}=\frac{1}{r^2}\left[-\frac{\partial^2}{\partial t^2}+(L-e\phi)^2\right], \tag{4.9}$$

where the operator L is the orbital angular momentum with integer eigenvalues. For each partial wave, we can compute a partial determinant $\delta_l(e)=\delta_{-l}(-e)$ in such a way that

$$\Delta_f(e)=\delta_0(e)\prod_{l=1}^{\infty}\delta_l(e)\delta_l(-e), \tag{4.10}$$

using the fact that Δ is even in e. This time we have used an index f to allow for a possible difference in normalization between this procedure and the perturbative one (see below). In any case $\Delta_f(e)$ and $\Delta_f(e)$ have the same zeros (of equal multiplicity).

Let us consider the cases $l\neq 0$, and $l=0$ separately. In the former one we may assume $l>1$ without loss of generality and set $m=0$ without encountering any singularity. The equation

$$\left[-\frac{d^2}{dt^2}+(l-e\phi)^2\right]u(t)=0 \tag{4.11}$$

admits solution regular at $t=-\infty$, i.e., behaving as e^{lt} for $t\to-\infty$; this fixes their normalization. Such solutions contain in general a term increasing as $\delta_l(e)e^{lt}$ for $t\to\infty$. Clearly the coefficient $\delta_l(e)$ has the following properties:

(i) It is equal to one if $e=0$,

(ii) it vanishes for an eigenvalue $e_{n,l}$ that is a coupling constant such that Eq. (4.11) admits solutions regular at both $t=\pm\infty$,

(iii) it is an entire function according to Poincaré's theorem. Indeed we recognize a method of computing the partial determinant akin to the method of Jost functions.

In the case $l=0$ we cannot set $m=0$ blindly for otherwise $e=0$ would appear as a zero of Δ, thus conflicting with the normalization $\delta_0(0)=1$. Otherwise the same method as above does apply.

To be specific consider the "square well" case

$$\phi(t)=\begin{cases}\phi, & |t|<T\\ 0, & |t|>T.\end{cases} \tag{4.12}$$

An easy calculation yields indeed the entire functions

$$\delta_l(e)=\exp(-2e\phi T)\left(1+\frac{e^2\phi^2}{4l(l-e\phi)}\times\{1-\exp[-4T(l-e\phi)]\}\right), \tag{4.13}$$

$$\delta_0(e)=\cosh(2e\phi T)-e\phi\sinh(2e\phi T)\left(\ln\frac{m}{2}+T+\gamma\right).$$

Here γ is Euler's constant and δ_0 has been computed for vanishing m. The pattern of zeros is shown in Fig. 1. It exhibits asymptotically an almost double periodicity with a constant density characteristic of an entire function of order two:

$$e_{ln}\approx\frac{l}{\phi}+i\,\frac{n\pi}{2\phi T}\quad(n\text{ integer}),$$

$$d\nu\simeq\frac{2\phi^2T}{\pi}\,de_1de_2,$$

in agreement with (2.12) and (2.23). The function can be reconstructed by inserting expressions (4.13) into Eq. (4.10), the result of which is an absolutely convergent product when the contributions $\delta_l(e)$ and $\delta_l(-e)$ are grouped together. Let $L_f(e)$ stand for the logarithm of $\Delta_f(e)$ and denote $e^2L_f^{(2)}$ its second-order coefficient in a Taylor expansion around $e=0$. From the theory of entire functions it follows that

$$\begin{aligned}\Delta_F(e)&=\Delta_f(e)\exp[e^2(L_F^{(2)}-L_f^{(2)})],\\ L_F(e)&=L_f(e)+e^2(L_F^{(2)}-L_f^{(2)}).\end{aligned} \tag{4.14}$$

The reshuffling implied by (4.14) may modify the asymptotic behavior along certain directions, but of course does not modify the overall order of growth of the function.

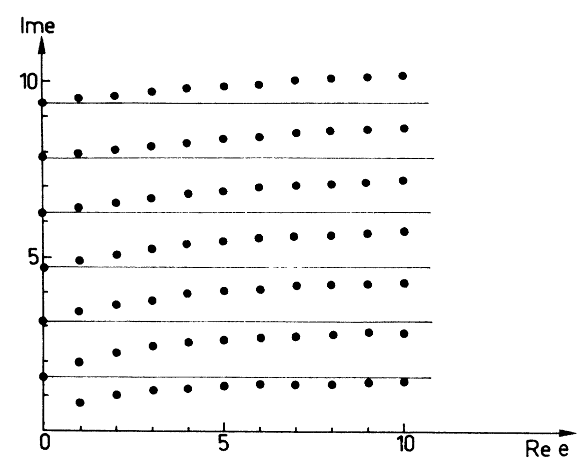

FIG. 1. The pattern of zeros of a two-dimensional determinant arising from expressions (4.13), for the values $2T=1$, $\phi=1$

It is suggested that for an arbitrary smooth ϕ, the same type of behavior applies. We expect $\Delta_F(e)$ to be of order two and therefore Eq. (4.14) applies. Arguments to this effect can be based on the analogy with the operator P^2+e^2V, where the corresponding determinant is of order two in dimension two.[1] The gauge-invariant minimal coupling certainly does not increase the order. The above explicit example shows, on the other hand, that it does not decrease it.

Let us, however, study in more detail the behavior along the real axis. This can be done explicitly for the above example, but can even be generalized to a smooth arbitrary ϕ.

For the lth partial wave the solution of the equation

$$\left(-\frac{\partial^2}{\partial t^2}+[l-e\phi(t)]^2\right)\psi_l(t)=0, \tag{4.15}$$

behaving as e^{lt} when $t\to-\infty$, can be estimated for large e and $l>0$ as follows:

$$\psi_l(t)\simeq\exp\left\{lt+\int_{-\infty}^{t}dt'[|l-e\phi(t')|-l]\right\}. \tag{4.16}$$

This gives

$$\delta_l(e)\simeq\exp\int_{-\infty}^{+\infty}dt\,[|l-e\phi(t)|-l]. \tag{4.17}$$

The leading behavior of $L_f(e)$ for real e is obtained by inserting (4.17) into (4.10). This yields

$$L_f(e)\underset{e^2\to\infty}{\simeq}e^2\int_{-\infty}^{+\infty}dt\,\phi^2(t). \tag{4.18}$$

The corresponding result for $L_F(e)$ has to take Eq. (4.14) into account so that one finds

$$\frac{L_F(e)}{e^2}\underset{e^2\to\infty}{\to}a=\int_{-\infty}^{+\infty}dt\,\phi^2(t)+L_F^{(2)}-L_f^{(2)}. \tag{4.19}$$

In the expression for a, $L_F^{(2)}$ is given by the perturbative result (4.8), while $L_f^{(2)}$ is easily obtained as follows:

$$L_f^{(2)}=-\int_{-\infty}^{+\infty}dt\,\phi^2(t)\left(\ln\frac{m}{2}+\gamma+t\right)+\sum_1^{\infty}\frac{1}{2l^2}\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{+\infty}dt_2\theta(t_2-t_1)\dot{\phi}(t_1)\dot{\phi}(t_2)\exp[-2l(t_2-t_1)]. \tag{4.20}$$

It is then a matter of straightforward (but tedious) calculation to verify that when these integrals are substituted into relation (4.19) giving a, they cancel so that

$$a=0. \tag{4.21}$$

It is noteworthy that this was the expectation from the general considerations of Secs. II and IV. This result may be interpreted as meaning that for a smooth ϕ (i.e., a smooth A_μ leading to a smooth $F_{\mu\nu}$) the leading behavior along the real axis may be estimated by using classical methods which amount to assuming that the field intensity $F(t)\simeq\phi(t)e^{-2t}$ is a slowly varying function. In two dimensions this leads to the following estimate:

$$L_F(e)\underset{e^2\to+\infty}{\simeq}|e|\frac{\ln 2}{4\pi}\int d^2x|F(x)|$$

$$=|e|\frac{\ln 2}{2}\int_{-\infty}^{+\infty}dt\,|\dot{\phi}(t)|. \tag{4.22}$$

Unfortunately, the explicit "square well" case does not fulfill the above assumptions which we were therefore unable to test directly. Indeed for this singular case we found that while (4.22) would lead to $|e|\phi\ln 2$, direct calculations yields $2|e|\phi$. By a more accurate evaluation of the partial determinants it would be interesting to verify whether (4.22) is obeyed for a general smooth ϕ.

Until now the effects of Fermi statistics have been discussed at length, but no mention has been made of the influence of spin. The reader may wonder how the previous considerations extend to that case. In particular the analogous "determinants" for two-dimensional QED (where the Dirac operator replaces the Klein-Gordon one) may seem at first deceptive. Indeed the limit $m=0$ does not lead to any singularity any more and for this value $\ln[\Delta(e)]$ is simply given by second-order perturbation theory, a well-known result due to Schwinger.[7] In other words one has then

$$\Delta_F(e)=\exp[L(e)],\qquad L(e)=-\frac{e^2}{2\pi}\int d^2x\,A^T(x)^2. \tag{4.23}$$

Recall that this is the origin of the fact that the Schwinger solution generates a "mass" term $|e|/\sqrt{\pi}$ for the photons. Of course Eq. (4.23) also defines an entire function of order two in this case. Therefore this general feature is preserved. However, expression (23) is of course very special being only due to an anomaly of perturbation theory. The genuine nontrivial part of the determinant is reduced to the identity in this massless fermionic case. One may wonder whether for any nonvanishing m the behavior for large real e is utterly modified.

V. FOUR-DIMENSIONAL EXAMPLE

We now turn to the case of real interest, namely, QED in four dimensions, and we study the Dirac determinant. We choose to do so for an external field configuration of a type already introduced in our previous work.[1] This is again because of its great symmetry; it allows a complete angular analysis. The vector potential is taken of the form

$$A_\mu(x) = M_{\mu\nu} x^\nu a(x^2), \tag{5.1}$$

where M is a 4×4 antisymmetric constant matrix with square equal to -1. In I we have exhibited solutions of the Dirac equation for specific instances of $a(x^2)$. We note that the fermion mass may be set equal to zero without encountering any divergence. To compute $\Delta(e) = \det(i\not\partial - e\not A)$, we perform an angular analysis of the Euclidean Dirac equation using its SU(2)×SU(2) invariance from which we shall construct a complete set of pairs (ψ_n, e_n) of regular wave functions and associated eigencouplings. As in I we introduce the matrices

$$\overline{\sigma} = (I, i\vec{\sigma}), \quad \underline{\sigma} = (I, -i\vec{\sigma}) \tag{5.2}$$

[for typographical reasons, the notations of I have been changed, $\overline{\sigma}$ $(\underline{\sigma})$ stands now for what used to be noted σ $(\not\sigma)$] and the notation

$$\overline{n} = n_\mu \overline{\sigma}^\mu, \quad \underline{n} = n_\mu \underline{\sigma}^\mu \tag{5.3}$$

for any four-vector n. The Euclidean massless Dirac equation decouples into two two-dimensional spinor equations:

$$(i\overline{\partial} - e\overline{A})u = 0, \quad (i\underline{\partial} - e\underline{A})v = 0. \tag{5.4}$$

To an arbitrary rotation $R \in O(4)$

$$x_\mu \to x'_\mu = R_{\mu\nu} x_\nu \tag{5.5}$$

is associated a couple (U, V) of SU(2) matrices such that

$$\overline{x} \to \overline{Rx} = U\overline{x}V^\dagger. \tag{5.6}$$

Of course (U, V) and $(-U, -V)$ yield the same R. From the orthogonality of R it follows that

$$U\overline{\sigma}_\mu V^\dagger = R^{-1}{}_{\mu\nu}\overline{\sigma}_\nu, \tag{5.7}$$

from which we derive the transformation law for the spinors u and v:

$$u_R(x) = Vu(R^{-1}x), \quad v_R(x) = Uv(R^{-1}x), \tag{5.8}$$

to which one may add

$$A_{\mu,R}(x) = R_{\mu\nu} A_\nu(R^{-1}x) \tag{5.9}$$

so that the Eqs. (4) are covariant. If u and v satisfy (5.4) it follows that

$$[i\overline{\partial} - e\overline{A}_R(x)]u_R(x) = 0, \quad [i\underline{\partial} - e\underline{A}_R(x)]v_R(x) = 0. \tag{5.10}$$

For an $A_\mu(x)$ of the form (5.1), Eqs. (5.4) are left invariant by the set of matrices (U, V) such that $\overline{A}(x) = \overline{A}_R(x)$ or $\underline{A}(x) = \underline{A}_R(x)$, respectively. If we use some explicit form of $M_{\mu\nu}$, for instance

$$M_{\mu\nu} = \begin{bmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}, \tag{5.11}$$

such that $\overline{A}(x) = i\sigma_3 \overline{x} a(x^2)$, the above condition reduces to $U\sigma_3 U^\dagger = \sigma_3$ or $V\sigma_3 V^\dagger = \sigma_3$, respectively. Thus the groups in question are U(1)×SU(2) or SU(2)×U(1). This will enable us to diagonalize the equations with respect to the conserved quantum numbers.

From the relation

$$\det(i\underline{\partial} - e\underline{A}) = [\det(i\overline{\partial} - e^*\overline{A})]^* \tag{5.12}$$

it follows that

$$\Delta(e) = \det(\not p - e\not A) = D(e)D^*(e^*), \tag{5.13}$$

where we focus our attention on

$$D(e) = \det(i\overline{\partial} - e\overline{A}) \tag{5.14}$$

and correspondingly on the first of Eqs. (5.4). Similar considerations apply of course to the second one.

Call m and j the conserved quantum numbers characterizing the representation of the invariance group U(1)×SU(2). Both can be integers or half-integers. Here m corresponds to the U(1) phase, while j labels the representation of SU(2). Now each (m, j) representation must occur in the decomposition of a $(l, l\pm\frac{1}{2})$ representation of the SU(2) × SU(2) group (l integer or half-integer) since these are all the relevant ones occurring for the two-spinors defined on four-space. From this we learn that $-l \le m \le l$ and $|j-m|$ must be half-integers. Moreover, each pair (j, m) occurs twice corresponding to the values $l = j \pm \frac{1}{2}$ except when $m = \pm(j + \frac{1}{2})$.

Spinor harmonics belonging to the representation $(l, l\pm\frac{1}{2})$ may be written as follows:

$$u^{(l, l\pm 1/2)}_{m, M}(x) = \sum_\lambda \mathfrak{D}^l_{m,\lambda}(\overline{x}) \begin{bmatrix} \langle l, \tfrac{1}{2}; \lambda, \tfrac{1}{2} \,|\, l\pm\tfrac{1}{2}, M\rangle \\ \langle l, \tfrac{1}{2}; \lambda, -\tfrac{1}{2} \,|\, l\pm\tfrac{1}{2}, M\rangle \end{bmatrix}, \tag{5.15}$$

in terms of Clebsch-Gordan coefficients and homogeneous rotation matrices. Corresponding to the choice (5.11) of the matrix M, let us denote by ξ

and η the combinations

$$\xi = x_0 + ix_3, \quad \eta = x_2 + ix_1. \tag{5.16}$$

Then we obtain

$$\bar{x} = \begin{pmatrix} \xi & \eta \\ -\bar{\eta} & \bar{\xi} \end{pmatrix}, \quad \overline{Mx} = i\sigma_3 \bar{x} = i\begin{pmatrix} \xi & \eta \\ \bar{\eta} & -\bar{\xi} \end{pmatrix},$$
$$\bar{\partial} = 2\begin{pmatrix} \partial_{\bar{\xi}} & \partial_{\bar{\eta}} \\ -\partial_{\eta} & \partial_{\xi} \end{pmatrix}, \tag{5.17}$$

where a bar over a complex number means complex conjugation. Furthermore, we have explicitly

$$\mathfrak{D}^l_{\mu,\mu'} = [(l+\mu)!(l-\mu)!(l+\mu')!(l-\mu')!]^{1/2} \times \sum_{\substack{n_1+n_2=l+\mu \\ n_1+n_3=l+\mu' \\ \Sigma n_i=2}} \frac{\xi^{n_1}\eta^{n_2}(-\bar{\eta})^{n_3}\bar{\xi}^{n_4}}{n_1!n_2!n_3!n_4!}. \tag{5.18}$$

When $|m| \leq j-1/2$, u spinors with quantum numbers j, M, and m (of course $-j \leq M \leq j$ characterizes the degeneracy) may be decomposed as follows:

$$u(x) = f(x^2) u^{(j-1/2,j)}_{m,M}(x) + g(x^2) u^{(j+1/2,j)}_{m,M}. \tag{5.19}$$

Given (j,m) the Dirac equation reexpressed in terms of the rotationally invariant $f(x^2)$ and $g(x^2)$ is most easily derived by selecting a particular value $M=j$. Reabsorbing some constant coefficients in f and g, $u(x)$ takes the simple form

$$u(x) = f(x^2)\begin{pmatrix} \xi^{j-1/2+m}\bar{\eta}^{j-1/2-m} \\ 0 \end{pmatrix} + g(x^2)\begin{pmatrix} (|\xi|^2 - |\eta|^2)\xi^{j-1/2+m}\bar{\eta}^{j-1/2-m} \\ 2\xi^{j+1/2+m}\bar{\eta}^{j+1/2-m} \end{pmatrix}. \tag{5.20}$$

When this is inserted into the first Eq. (4) together with $\bar{A}(x) = i\sigma_3\bar{x}a(x^2)$, one finds that

$$\frac{df}{dx^2} - \left(2m + x^2\frac{ea}{2}\right)g = 0,$$
$$x^2\frac{dg}{dx^2} + 2(j+1)g - \frac{ea}{2}f = 0. \tag{5.21}$$

After a convenient change of variable

$$t = \ln x^2, \tag{5.22}$$

and of functions

$$f(x^2) = \exp[-(j+\tfrac{1}{2})t]\varphi(t),$$
$$g(x^2) = \exp[-(j+\tfrac{3}{2})t]\gamma(t), \tag{5.23}$$
$$a(x^2) = \exp[-t]\alpha(t)$$

[where $\alpha(t)$ is analogous to what was denoted $\phi(t)$ in the previous section: it has the dimension of a flux] the system reads as follows:

$$\frac{d\varphi}{dt} = (j+\tfrac{1}{2})\varphi + 2m\gamma + \frac{e\alpha}{2}\gamma,$$
$$\frac{d\gamma}{dt} = -(j+\tfrac{1}{2})\gamma + \frac{e\alpha}{2}\varphi. \tag{5.24}$$

We recognize a generalization of the system studied in I which, however, did correspond to the lowest quantum numbers $j=\frac{1}{2}$, $m=0$.

By forming suitable combinations of Eqs. (5.24) it is easy to show that

$$\frac{1}{2}\frac{d}{dt}(\varphi^2 - \gamma^2) = (j+\tfrac{1}{2})(\varphi^2+\gamma^2) + 2m\varphi\gamma, \tag{5.25}$$

where the potential has been eliminated.

Since $|m| < j - \frac{1}{2}$ the right-hand side is positive definite for real φ and γ. Hence there exists no real solution (i.e., no solution for e real) such that the functions φ and γ vanish at infinity.

Moreover, similar combinations yield

$$\mathrm{Re}[e] = -4m \frac{\int_{-\infty}^{+\infty} dt\, |\gamma|^2}{\int_{-\infty}^{+\infty} dt\, \alpha(t)(|\gamma|^2 + |\varphi|^2)}. \tag{5.26}$$

Thus for $m=0$, zeros lie along the imaginary axis in the e plane, but move into the complex plane for $m \neq 0$. If α is positive, $\mathrm{Re}[e]$ has a definite sign. With the above conventions $m\,\mathrm{Re}[e] < 0$. We remark that if (φ,γ,e) is a solution to (5.24) corresponding to (j,m), then $(\varphi,-\gamma,-e)$ is also a solution pertaining to the quantum numbers $(j,-m)$.

For completeness let us mention what happens when $|m| = j+\frac{1}{2}$, corresponding to a unique set of spinor harmonics transforming according to the representation $(l=j+\frac{1}{2}, l-\frac{1}{2}=j)$. Again when $M=j$,

$$u = f(x^2)\begin{pmatrix} \xi^{2j}\eta \\ -\xi^{2j+1} \end{pmatrix}, \tag{5.27}$$

and the Dirac equation leads to

$$x^2\frac{d}{dx^2}f + 2(j+1)f + x^2\frac{ea}{2}f = 0. \tag{5.28}$$

Since this equation is readily solved for arbitrary $a(x^2)$, it is easy to see that it admits no solutions regular both at $x^2=\infty$ and $x^2=0$. We may therefore discard this case and henceforth restrict m to the range

$$-j+\tfrac{1}{2} \leq m \leq j - \tfrac{1}{2}. \tag{5.29}$$

As in the two-dimensional case, we may compute the contribution $d_{j,m}(e)$ of each partial wave

(j,m,M) to the determinant and reconstruct $D(e)$ formally as

$$D(e)=\prod_{2j=1}^{\infty}\prod_{m=-j+1/2}^{j-1/2}[d_{j,m}(e)]^{2j+1}, \qquad (5.30)$$

where the $(2j+1)$-fold degeneracy in M has been taken into account.

We may now illustrate various properties on the "square well" case

$$\alpha(t)=\begin{cases}\alpha, & |t|<T\\ 0, & |t|>T\,.\end{cases} \qquad (5.31)$$

Then $d_{j,m}$ is seen to be equal to

$$d_{j,m}(e)=\exp[-2T(j+\tfrac{1}{2})]\times\left\{\cosh 2KT+\frac{(j+\frac{1}{2})^2+me\alpha/2}{K(j+\frac{1}{2})}\sinh 2KT\right\}, \qquad (5.32)$$

where

$$K^2=(m+\tfrac{1}{2}e\alpha)^2-m^2+(j+\tfrac{1}{2})^2\,. \qquad (5.33)$$

This explicit expression enables us to find the zeros of $d_{j,m}$. The nearest ones in the quadrant $\mathrm{Re}(e)<0$, $\mathrm{Im}(e)>0$ are plotted in Fig. 2. The asymptotic distribution satisfies in this case for $j\gg 1$, and $|n|$ integer $\gg 1$,

$$\frac{\alpha}{2}e_{j,m,n}\simeq -m\pm i\left[\frac{n^2\pi^2}{4T^2}+(j+\tfrac{1}{2})^2-m^2\right]^{1/2}, \qquad (5.34)$$

from which we may verify that $\sum_{j,m,n}e_{j,m,n}^{-4}$ is only semiconvergent, but $\sum|e_{j,m}|^{-4-\epsilon}$ is finite for arbitrary positive ϵ. Thus $\Delta_f(e)$ constructed from (5.13) and (5.30) using these expressions for $d_{j,m}$ is an entire function of order four as was expected. Since in the four-dimensional case $L_F(e)$ differs from $L_f(e)$ by a polynomial of degree four at most, we can conclude that $\Delta_F(e)$ is also of order four.

The asymptotic distribution of zeros (5.34) is in agreement with the results of Sec. II and III, which means that spin effects may be neglected for large e:

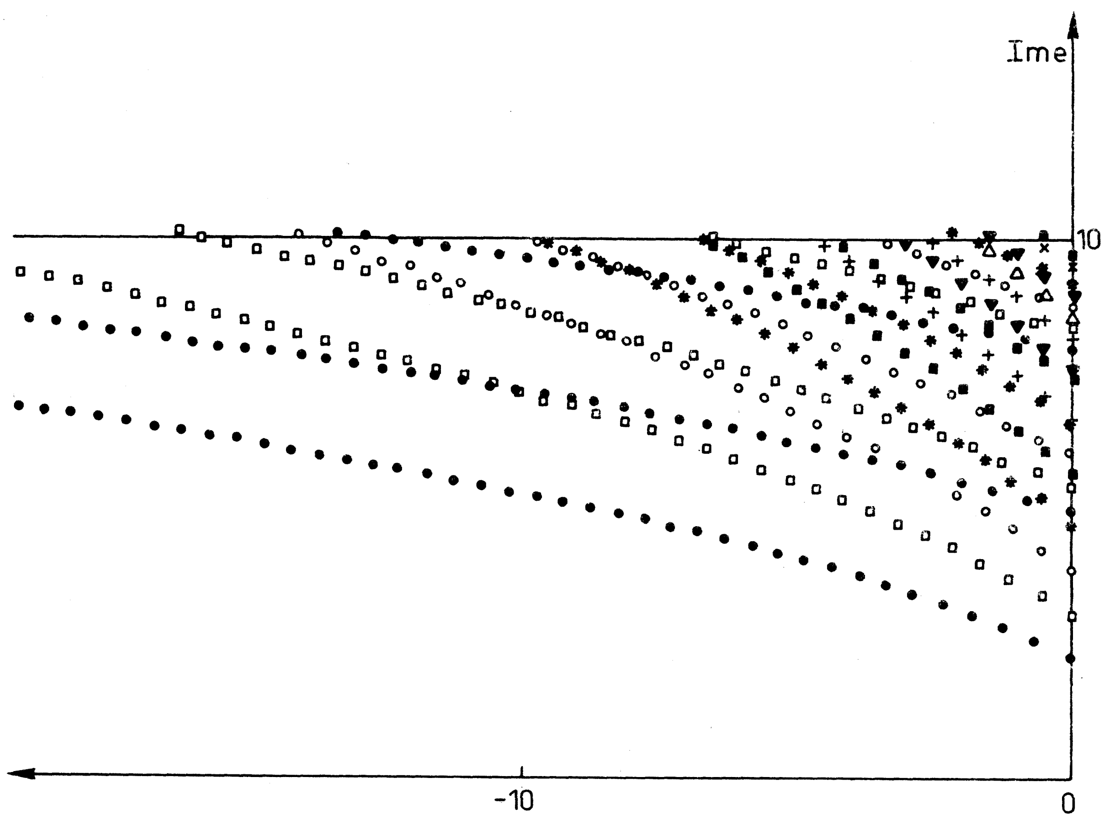

FIG. 2. The pattern of zeros of a four-dimensional determinant arising from expressions (5.32), for $2T=1$, $\alpha=1$. The various symbols correspond to different channels (j,m) as follows, where the number indicates the value of $(j-m)$:
●1/2 □3/2 ○5/2 *7/2 ■9/2 +11/2 ▼13/2 △15/2 ×17/2

$$\det(\bar{p} - e\bar{A}) \times \det(\underline{p} - e\underline{A}) \underset{|e|\to\infty}{\approx} [\det(p - eA)^2]^2 \,. \tag{5.35}$$

We also attempted to see whether the cancellation mechanism along the real axis worked for $L_F(e)$ as it did in the two-dimensional situation. Unfortunately the calculation is long and tedious and we cannot report any definite result here.

VI. CONCLUSION

The main conclusion to be drawn from the present work is that the order in the coupling constant e of the fermionic determinant in four-dimensional QED is four, very much as in Yukawa theory. Therefore, within the limitations indicated in the introduction, perturbation theory in QED diverges at least as badly as $\alpha^k\Gamma(k/2)$. More precisely, if the fermionic determinant behaves as

$$\Delta(e) \sim \exp[(\mathrm{Im}e)^4 L(A)]$$

for large values of e, where $L(A)$ is a homogeneous functional of degree four, then according to I the kth order of perturbation theory, i.e., the coefficient of α^k, is asymptotic, up to powers of k and constant factor, to

$$Z_k \sim (-1)^k \Gamma\left(\frac{k}{2}\right) a^k \,,$$

where

$$a = \frac{1}{8\pi} \max\left\{[L(A)]^{1/2} \Big/ \int \tfrac{1}{4} F^2 d^4x\right\}.$$

Taking

$$L(A) = a_4 \int A_T^4 d^4x \,,$$

the Sobolev inequality[1] tells us that a is less than or equal to $1/\pi$. To continue this study, two directions at least require attention. First, one would like to improve the above estimate by computing the coefficients beyond $\Gamma(k/2)a^k$. Second, it remains to investigate with greater care the effect of renormalization.[8]

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[1] C. Itzykson, G. Parisi, and J. B. Zuber, Phys. Rev. D 16, 996 (1977). This paper will be referred to as I; it includes a list of references which will not be repeated here. The general problem was also formulated and discussed by J. Schwinger, Phys. Rev. 94, 1362 (1954).

[2] J. H. Van Vleck, *The Theory of Electric and Magnetic Susceptibilities* (Oxford Univ. Press, London, 1932).

[3] S. Adler, Phys. Rev. D 10, 2399 (1974).

[4] *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi *et al.* (Mc Graw-Hill, New York, 1953), Vol. 2, Chap. 13.

[5] H. Weyl, Nachr. Akad. Wiss. Göttingen 110 (1911).

[6] A. Martin, CERN Report No. TH. 2294, 1977 (unpublished).

[7] J. Schwinger, Phys. Rev. 128, 2425 (1962).

[8] B. Lautrup, Phys. Lett. 69B, 109 (1977).