Singular vectors of the Virasoro algebra

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Received 26 February 1991

We present an explicit construction of the singular (or null) vectors in highest weight Verma modules.

In this short communication, we give an explicit construction of singular vectors in Verma modules over the Virasoro algebra. We proceed in two steps, the first being a reformulation of a series of such vectors obtained by Benoit and Saint-Aubin [1], in a formalism inspired by the study of W-algebras. The second step uses the fusion method of Belavin, Polyakov and Zamolodchikov [2], a rephrasing of Wilson's short-distance expansion applied to primary fields. This yields the general singular vectors. Detailed proofs and examples will be presented in a more comprehensive exposition [3].

Verma modules $V(c, h)$ of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}cm(m^2-1)\delta_{m+n,0} \tag{1}$$

are characterized by the existence of a unique highest weight vector f (up to a factor) such that

$$L_0 f = hf, \quad L_m f = 0, \quad m > 0 . \tag{2}$$

In the Verma module freely generated by elements

$$f_{p_1,p_2,\dots,p_k} = L_{-p_1} L_{-p_2} \dots L_{-p_k} f, \quad 1 \leqslant p_1 \leqslant p_2 \leqslant \dots \leqslant p_k , \tag{3}$$

a "singular" vector arises at level n when a combination F of the form

$$F = \sum_{p_1+p_2+\dots=n} C_{p_1,p_2,\dots} f_{p_1,p_2,\dots} = \mathscr{Q}(L_{-1}, L_{-2}, \dots) f \tag{4}$$

satisfies the properties of a highest weight:

$$L_0 F = (h+n)F, \quad L_m F = 0, \quad m > 0 . \tag{5}$$

Parametrize c, h as follows, with j and j' integers or half-integers:

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{[r(m+1)-sm]^2-1}{4m(m+1)}, \quad r = 2j'+1 , \quad s = 2j+1 \tag{6}$$

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and m is any complex number, the values 0 and ∞ being included in a limiting sense (for $j'=0$ and $j=0$ respectively). From a theorem of Kac [4] and Feigin and Fuchs [5], one knows that $V(c, h)$ admits a non-zero singular vector at level $n=rs$, and conversely, all singular vectors are obtained in this way. The general form of the singular (or "null") vector was, however, now known in general. These expressions have a practical interest in writing differential equations satisfied by correlation functions (after quotienting by the null vectors) [2].

In the particular case where either j or j' vanishes, an explicit formula was given by Benoit and Saint-Aubin [1]. We shall rewrite their expression in a more compact way.

Let us concentrate on the case $r=1$, $n=s=2j+1$ for definiteness. Then (6) reduces to

$$c=13+6(t+t^{-1}), \quad h=-j-tj(j+1), \quad t=-\frac{m}{m+1}, \quad n=2j+1\,, \tag{7}$$

where $j=\frac{1}{2}, 1, \frac{3}{2}, \ldots$. In $V(c, h)$ we introduce a sequence of elements denoted

$$f=f_{-j}, f_{-j+1}, \ldots, f_j, f_{j+1}=F\,, \tag{8}$$

where f_{-j} is the highest weight state f, f_{j+1} is the singular vector F and f_M satisfies

$$L_0 f_M=(j+M)f_M\,. \tag{9}$$

We define the n-dimensional vectors

$$\boldsymbol{f}=(f_j, f_{j-1}, \ldots, f_{-j})^{\mathrm{T}}, \quad \boldsymbol{F}=(F, 0, \ldots, 0)^{\mathrm{T}}\,. \tag{10}$$

This n-dimensional space carries a representation of spin j of sl(2) and we choose generators of the form

$$J_-=\begin{pmatrix} 0 & 0 & \ldots & & 0\\ 1 & 0 & & & 0\\ 0 & 1 & & & 0\\ \vdots & \ddots & \ddots & & \vdots\\ 0 & & \ldots & 1 & 0\end{pmatrix}, \quad J_+=\begin{pmatrix} 0 & 1\cdot(n-1) & 0 & \ldots & & 0\\ 0 & 0 & 2\cdot(n-2) & & & 0\\ 0 & 0 & 0 & 3\cdot(n-3) & & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ 0 & \ldots & & 0 & 0 & (n-1)\cdot 1\\ 0 & & \ldots & & 0 & 0\end{pmatrix},$$

$$J_0=\mathrm{diag}(j, j-1, \ldots, -j+1, -j)\,. \tag{11}$$

We claim that the set of equations embodied in the linear system:

$$\boldsymbol{F}=\left(-J_-+\sum_{k=0}^{n-1} L_{-k-1}(tJ_+)^k\right)\boldsymbol{f} \tag{12}$$

defines $F=f_{j+1}=\mathscr{Q}f$ as a non-vanishing singular state at level n in $V(c, h)$. Moreover the successive components of $\boldsymbol{f}$ satisfy the relation

$$p>0, \quad L_p\boldsymbol{f}=\{[J_0-\tfrac{1}{2}(3p+1)]-\tfrac{1}{4}t^{-1}(3p+1)\}(tJ_+)^p\boldsymbol{f} \tag{13}$$

obtained by induction on the index M of the components of $\boldsymbol{f}$ starting from the last one, for which both sides of (13) vanish (highest weight). Eq. (13) for $p=1, 2$ implies it for higher p by commutation. It also extends to $M=j+1$ for which it means that F is annihilated by the L_p, $p>0$. The details of the proof will be presented elsewhere in greater detail [3]. Eliminating all components f_M for $-j+1\leqslant M\leqslant j$ leads to an explicit expression of $\mathscr{Q}$ which reproduces the result of ref. [1].

An unexpected similarity with *classical* W-algebras, i.e. algebras of deformations of differential operators, appears in the form (12). Following Drinfeld and Sokolov [6] it is appropriate to represent an nth order linear differential operator in terms of a first-order matrix differential operator. The substitution

$$L_{-1} \to \mathrm{d}/\mathrm{d}x, \quad t^k L_{-k-1} \to W_{k+1} \tag{14}$$

on the RHS of (12) produces the covariant $n\times n$ differential operator attached to the $\mathrm{W}(\mathrm{A}_{n-1})$-algebra [7] with a change of the W-basis [8,3].

The case where both j and j' are different from zero may be obtained by fusion of $(0, j)$ with $(j', 0)$. It is known [2] that the fusion of the two corresponding primary fields gives rise to a single conformal family. In general the leading term in the operator product expansion of two primary fields f_1 and f_2 is also a primary field, denoted $f \equiv f^{(0)}$:

$$f_1(x_1) f_2(x_2) = \frac{1}{z^{h_1+h_2-h}} \sum_{l=0}^{\infty} z^l f^{(l)}(u) , \tag{15}$$

where we expand (for example) around the mid-point $u = \frac{1}{2}(x_1+x_2)$ and $z = x_1 - x_2$. The covariance of eq. (15) determines unambiguously all $f^{(l)}$, modulo the ideal generated by possible singular vectors in the $V(h, c)$ module. This is the heart of our method to obtain the singular vector. The Virasoro generators acting on the field at x_1 may be reexpressed in terms of those acting at u according to

$$L^{(1)}_{-p} = (-z)^{-p}[h_2(p-1) + \tfrac{1}{2}zL_{-1} - z\partial_z] + \sum_{k\geqslant 0} (\tfrac{1}{2}z)^k \binom{p+k-2}{k} L_{-p-k} , \tag{16}$$

and likewise for the $L^{(2)}_{-p}$, with $h_1 \leftrightarrow h_2$, $z \leftrightarrow -z$. Given the operator $\mathcal{Q}^{(1)}(L^{(1)})$ of (12) that constructs the singular vector F_1 attached to f_1 at level $n_1 = 2j+1$, one may thus reexpress it in terms of L's and compute the operator product expansion of F_1 and f_2:

$$F_1(x_1) f_2(x_2) = \mathcal{Q}^{(1)}(L) \frac{1}{z^{h_1+h_2-h}} \sum_{l=0}^{\infty} z^l f^{(l)}(u) . \tag{17}$$

The leading term in this expansion must be a primary field. An explicit calculation shows that the coefficient of the singular term $1/z^{h_1+h_2-h+n_1}$ vanishes [3]. This implies that the actual leading term is a primary field among the descendents of $f = f^{(0)}$, hence (in the generic case) the unique singular field attached to f (corresponding to the singular vector in the module). The vanishing of all coefficients of the intermediate singular terms $1/z^{h_1+h_2-h+n_1-r}$, $r = 0, ..., n-1$, determines the $n = (2j+1)(2j'+1)$ first terms $f^{(0)}, ..., f^{(n-1)}$ in the expansion (15) and agrees with the general method discussed in ref. [2]. In practice, to determine the expression of the singular field of f, one eliminates the contribution of $f^{(n)}$ by forming the following combination of $\mathcal{Q}^{(i)}$, $i = 1, 2$:

$$F(u) = \lim_{z\to 0} z^{h_1+h_2-h-n} (\alpha_2 z^{n_1} \mathcal{Q}^{(1)} - \alpha_1 z^{n_2} \mathcal{Q}^{(2)}) \frac{1}{z^{h_1+h_2-h}} \sum_{l=0}^{n-1} z^l f^{(l)}(u) ,$$

$$\alpha_1 = \prod_{-j\leqslant M\leqslant j} \{(j-M)[2j'+1+t(j+M+1)] - n\} , \tag{18}$$

and α_2 is given by a similar formula with $j \leftrightarrow j'$, $t \leftrightarrow t^{-1}$.

To summarize, we have obtained a complete description of all singular vectors in the Verma modules of the Virasoro algebra. Although we do not have in general formulas as explicit as in the case j or $j' = 0$, we have a well-defined algorithm to construct the singular vector, which involves the application of an explicit operator on explicit expressions. This is to be contrasted with the preexisting situation in which the determination of the singular vector of level n involved solving a linear system of $P(n)$ equations ($P(n)$ is the number of partitions of n). Conversely the method can perhaps be turned around to give not only the singular vectors but the ingredients which yield the Kac determinantal formula as well as the possible embeddings of Verma modules.

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