CLASSIFICATION OF MODULAR INVARIANTS MADE OF CHARACTERS OF THE VIRASORO OR KAC-MOODY ALGEBRAS

J.-B. Zuber
Service de Physique Théorique
CEN Saclay
91191 Gif-sur-Yvette cedex France

Abstract
A short account is given of the classification of modular invariants of the form $\sum N_{h\bar{h}}\, \chi_h \chi_{\bar{h}}^{*}$, where χ is a character of the Virasoro or of the $A_1^{(1)}$ algebras. Their physical implications on the classification of universality classes of 2D critical phenomena are also discussed.

Recent years have seen a new irruption of infinite dimensional Lie algebras in physics. The Virasoro algebra, which had been introduced by physicists [1] some fifteen years ago in the context of string theory, and Kac-Moody algebras, alias two-dimensional current algebras, familiar to field theorists, are in the foreground of a countless amount of theoretical papers. The physics concerned by these new developments encompasses string theory of course, but also two-dimensional critical phenomena, integrable models, both classical and quantum (for reviews, see for example [2,3,4]). The purpose of this note is to report on some recent results on modular invariants made of characters of such infinite dimensional algebras and on the classification of field theories having the corresponding invariances.

In a critical system, one expects the continuous field theory describing long range physics to be not only scale, but also conformal invariant [6]. In two dimensions, where the conformal group is infinite dimensional, one uses complex coordinates z and $\bar{z}$ in Euclidean space. The conformal symmetry is realized in the quantum theory by the central extension of the algebras of z and $\bar{z}$ diffeomorphisms, i.e.

by the product of two commuting Virasoro algebras $Vir_L \times Vir_R$ with the same central charge c

$$[L_n, L_m] = (n-m)L_{n+m} + c/_{12}\, n(n^2-1)\, \delta_{n+m,0} \qquad (1)$$

with a similar relation for the $\bar{L}_n$'s, and commutation of the L's and $\bar{L}$'s [7]. So in this case, conformal invariance is a dynamical symmetry, anomalously realized because of the central charge. In string theory, on the other hand, conformal invariance comes in as a remnant of reparametrization invariance, hence as a constraint. A consistent string theory must have its total central charge, including the contribution of ghosts, equal to zero, but the physical sector of the theory may have $c \neq 0$.

At any rate, the Hilbert space of such a conformal theory must split into irreducible representations of the Virasoro algebra(s) and in physics, sensible representations must be highest weight (h.w.) representations. A h.w. state satisfies

$$L_0\,|h\rangle = h\,|h\rangle \quad , \quad L_n\,|h\rangle = 0 \quad \text{for all } n>0 \qquad (2)$$

and the h.w. (Verma) module is the linear span of the states

$$M = \{\, L_{-1}^{\alpha_1} \ldots L_{-n}^{\alpha_n}\,|h\rangle \,\}$$

and is graded by the level $\sum k\alpha_k$. This module may, however, be reducible, in which case the state $|h\rangle$ is said to be degenerate. This occurs whenever at a certain level in M there is a state satisfying (2), hence bearing its own h.w. submodule. The characterization of the degenerate h.w. states of the Virasoro algebra has been an important step in the modern developments [8]. If the central charge is parametrized in the form

$$c = 1 - 6/x(x+1) \qquad (3)$$

where x may be real or complex, then the values of h corresponding to degenerate representations are given by

$$h = ((r(x+1)-sx)^2-1)\,/\,4x(x+1), \quad r,\ s \text{ positive integers}, \qquad (4)$$

the case c=1 being obtained in the limit $x\to\infty$. In such a degenerate case, one builds up the irreducible representation by factoring out the submodules. This reflects on the form of the irreducible character which counts, up to an overall factor, the number d_n of states at a given level:

$$\chi_h(q) = \mathrm{tr}\,(\, q^{L_0 - c/24}\,) = q^{h-c/24} \sum d_n\, q^n \qquad (5)$$

(q is here a dummy variable). If h is not of the form (1.4), $d_n = p(n)$, the number of partitions of n, otherwise $d_n \leq p(n)$. From the structure of embeddings of submodules [9], one may derive explicit formulae for the characters [10].

A conformal field theory is characterized by the value of its central charge c and the set of highest weights $\{h,\bar{h}\}$ of its irreducible representations. As $L_0+\bar{L}_0$ and $L_0-\bar{L}_0$ generate dilatations and rotations in the plane, their eigenvalues $h+\bar{h}$ and $h-\bar{h}$ represent the scaling dimension and spin of fields present in the theory. Finally, one must also specify the operator product algebra that these scaling fields satisfy [7].

All this discussion may also be carried out for a Kac-Moody algebra G. The generators are denoted $J^a{}_n$

$$[J^a{}_n, J^b{}_m] = f^{abc}\, J^c{}_{n+m} + {}^k/_2\, n\, \delta_{n+m,0} \tag{6}$$

where k is the Kac-Moody central charge. This is just the current algebra with a Schwinger term, expressed in terms of the moments of the current $J^a(z) = \sum_n z^{-n-1} J^a{}_n$. In a unitary h.w. representation, the K.M. central charge k takes a fixed positive integral value, called the level of this representation. It is possible to construct a Virasoro algebra from the Kac-Moody generators using the Sugawara formula:

$$L_n = \text{const.} \sum_m \sum_a J^a{}_{n-m}\, J^a{}_m \quad . \tag{7}$$

This enables one to consider theories endowed with a $G_L x G_R$ current algebra as a particular class of conformal theories, which plays actually a central role in the analysis of general conformal theories. Their Virasoro central charge is computed to be [11]:

$$c = k \dim G\, /\, (k+g) \tag{8}$$

where g stands for the dual Coxeter number of the algebra G; the values of the possible conformal weights h and $\bar{h}$ may be determined and formulae for the characters (1.5) computed in a K.M. highest weight representation have been given in [12]. Physically, such a conformal theory describes the critical point of the Wess-Zumino-Witten Lagrangian. In string theory, it accounts for the propagation of the string on the group manifold [15].

To examine what are the consistent choices of $c, \{h_i, \bar{h}_i\}$, Cardy [13] has proposed to construct the theory in a finite box with periodic boundary conditions, i.e. on a torus. If ω_1, ω_2 denote the two periods of the torus, $\tau = \omega_1/\omega_2$ its modular ratio, $q = \exp 2i\pi\tau$, one may show that the partition function of the system takes the form

$$Z = \text{tr}\; q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24} \tag{9}$$

where the trace is taken in the Hilbert space of states. The latter may be decomposed on irreducible representations and Z may be recast as

$$Z = \sum_{h,\bar{h}} N_{h,\bar{h}}\, \chi_h(q)\, \chi_{\bar{h}}(\bar{q}) \tag{10}$$

in terms of Virasoro characters, with $N_{h,\bar{h}}$ denoting the multiplicity of the $(h;\bar{h})$ representation of $Vir_L x Vir_R$. If the theory possesses a larger symmetry such as the K.M. algebra, it must be possible to decompose the Hilbert space in irreps of this algebra, and thus to reexpress Z in the form (10) in terms of the corresponding characters. In any event, Z contains all the information about c, which is probed by the small q limit [14], and about the h.w. content (set $\{h_i, \bar{h}_i\}$, multiplicities) of the theory. Moreover Z satisfies the important constraint of modular invariance which expresses the fact that it is intrinsically attached to the torus and thus insensitive to a change of basis of periods ω_1, ω_2

$$\omega'_2 = a\,\omega_1 + b\,\omega_2$$

$$\omega'_1 = c\,\omega_2 + d\,\omega_1 \qquad\qquad a\,d - b\,c = 1$$

or

$$\tau' = (a\tau+b)/(c\tau+d) \tag{11}$$

This constraint, together with the character expansion (10), turns out to be powerful enough to allow a general classification of all possible partition functions, at least whenever the sum (10) is finite. This includes the case of the "minimal" conformal theories which occur when the parameter x in (3) is a positive rational:

$$x = p'/(p-p') \qquad p,p' \text{ coprime integers} \quad . \tag{12}$$

The finite set of h.w. representations given by formula (4) with the additional constraints $1 \leq r \leq p'-1$, $1 \leq s \leq p-1$, is then known to provide us with a consistent field theory [7]. A finite sum in Equ. (10) is also provided by the conformal current algebras introduced above: there the finite set of representations which is summed over are the integrable representations of the K.M. algebra. For the simplest case of the $\widehat{su}(2) = A_1^{(1)}$ algebra, the only case we shall discuss in detail here, these representations are labelled by the value k of the central charge and by the spin l for the finite algebra, satisfying $0 \leq l \leq k/2$ [15].

What make the problem feasible are the remarkable modular properties of the characters. The finite set of characters of minimal Virasoro or of K.M. are known [12,13] to form a finite dimensional representation of the modular group. To the best of my knowledge, this feature has no good a priori raison d'être. Maybe its best intuitive interpretation is precisely provided by these consistency properties of conformal theories. The classification program has now been completed for the two classes of theories discussed above: minimal conformal theories of central charge as in Equ. (3,12)

$$c = 1 - 6(p-p')^2/pp' \qquad (13)$$

and $A_1^{(1)}$ theories of central (Virasoro) charge

$$c = 3k / (k+2) \qquad (14)$$

and also for various derived classes. Another remarkable and rather unexpected feature is the tight connection between these two classes of theories [16]. In a certain sense, the minimal conformal theories may be regarded as made of tensor products of $A_1^{(1)}$ theories. More precisely let $N^{aff}_{\lambda\bar{\lambda}}(k)$ be the matrix describing in (10) a K.M. $A_1^{(1)}$ modular invariant of level k; instead of labelling the representations by the "isospin" l for the finite dimensional su(2) algebra, we denote them by the dimension λ=2l+1. Then a possible solution to the conformal invariant is:

$$N^{conf}_{h_{rs},h_{\bar{r}\bar{s}}}(p,p') = N^{aff}_{r,\bar{r}}(p'-2)\ N^{aff}_{s,\bar{s}}(p-2) \qquad (15)$$

and the most general invariant of minimal theories is obtained in this way (this property was conjectured in [16] and proved in [17]). This property has to do with the Goddard-Kent-Olive [18] "coset construction" of unitary minimal representations from the K.M. ones, but seems to be more general, as it is not restricted to unitary representations and also applies to unphysical modular invariants with indefinite coefficients $N_{h\bar{h}}$. At any rate this property enables us to first study the $A_1^{(1)}$ affine problem before turning to the minimal conformal one.

The study of these $A_1^{(1)}$ affine modular invariants is done in two steps. One first characterizes the general modular invariants, relaxing the condition of positivity of the $N_{\lambda\bar{\lambda}}$ matrix elements. One finds essentially one independent modular invariant associated with each factorization of the integer k+2 = $\delta \times \bar{\delta}$. The condition of positivity is then reintroduced and shown to reduce drastically the possible invariants. An amazing result is that these positive invariants are in one-to-one correspondence with simply laced algebras A-D-E. I want to only give some clues to these results without entering the details of the proofs which may be found elsewhere [17,19-21].

Let us first examine the affine $A_1^{(1)}$ characters of level k and spin l $(0 \le l \le k/2)$. Let λ=2l+1 and K=2(k+2). Then on the explicit form of χ :

$$\chi(q) = 1/\eta^3(q) \sum_n (nK+\lambda)\ q^{(nK+\lambda)^2/2K} \qquad (16)$$

where $\eta(q) = q^{1/24} \prod(1-q^n)$ is Dedekind's function, one sees clearly that χ extends to an odd, K-periodic function of λ:

$$\chi_\lambda(q) = -\chi_{-\lambda}(q) = \chi_{\lambda+K}(q) \ . \qquad (17)$$

As mentionned above, these characters form a finite dimensional representation of the modular group Γ. This representation turns out to be unitary, and we denote it by $g \in \Gamma \to U(g)$. Under the action of two generators of Γ:

$$\begin{aligned} &T:\ \tau \to \tau+1 && \chi_\lambda \to \varepsilon_T\ e^{2\pi i \lambda^2/2K}\ \chi_\lambda \\ &S:\ \tau \to -\tau^{-1} && \chi_\lambda \to \varepsilon_S \sum 1/\sqrt{K}\ e^{2\pi i \lambda\lambda'/K}\ \chi_{\lambda'} \end{aligned} \qquad (18)$$

Here ε_T and ε_S are phases which are not terribly relevant for our analysis as they disappear in the sesquilinear forms (10). In the latter sum, the summation runs over the integers λ'modulo K. Therefore the action of the S-generator is nothing but the finite Fourier transform. Actually the infinite invariant subgroup Γ_{2K} of the modular group Γ:

$$\Gamma_{2K} = \{ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = \pm 1 \bmod 2K \ ;\ ad - bc = 1 \} \qquad (19)$$

(subgroup of level 2K) acts trivially on the χ's [12,15-17]:

$$\text{if } g \in \Gamma_{2K} :\ U(g)\ \chi_\lambda(q) = e^{i\alpha(g)}\ \chi_\lambda(q) \qquad (20)$$

where the phase $\alpha(g)$ is independent of λ and q. Therefore the characters χ_λ form a projective representation of the coset group:

$$\Gamma / \Gamma_{2K} = PSL(2, Z/Z_{2K}) \qquad (21)$$

which is a finite group. In other words, the problem under study, i.e. the construction of the most general modular invariant sesquilinear form in the χ's:

$$\chi^\dagger N \chi = \chi^\dagger U^\dagger N U \chi \iff N = U^\dagger N U \iff N U = U N \qquad (22)$$

amounts to finding the commutant of the representation of this finite group afforded by the characters.

As mentioned above, there is an independent element of this commutant associated with each factorization of K/2=k+2. A clue to this result is provided by the analysis of the commutativity of the matrix N with the generator T of $\tau \to \tau+1$. From Equ. (18), we learn that the only non vanishing matrix elements $N_{\lambda\bar{\lambda}}$ are such that $\lambda^2 = \bar{\lambda}^2 \bmod 2K$, i.e. (recalling that $K=2(k+2)$ is even):

$$\frac{\lambda+\bar{\lambda}}{2}\;\frac{\lambda-\bar{\lambda}}{2} = 0 \mod K/2 \quad . \tag{23}$$

All the divisors of $K/2$ must therefore appear either in $(\lambda+\bar{\lambda})/2$ or in $(\lambda-\bar{\lambda})/2$. This is the way the divisors of K come in. Now take any decomposition $K/2 = \delta . \bar{\delta}$, with $\alpha = \text{g.c.d.}(\delta, \bar{\delta})$; since δ/α and $\bar{\delta}/\alpha$ are coprimes, we may find two integers u and v such that $u\delta/\alpha - v\bar{\delta}/\alpha = 1$ and define $\mu = (u\delta + v\bar{\delta})/\alpha$. With this factorization of $k+2$ we associate the matrix

$$\Omega^{(\delta)}_{\lambda\bar{\lambda}} \begin{cases} = \sum_{\xi} \delta_{\bar{\lambda},\, \mu\lambda + \xi K/\alpha} & \text{if } \alpha \text{ divides both } \lambda \text{ and } \bar{\lambda} \\ = 0 & \text{otherwise} \end{cases} \tag{24}$$

where the sum runs over integers ξ mod α. It is easy to check that such a matrix is an element of the commutant and yields a non vanishing invariant provided $\delta \neq \sqrt{k+2}$. It was conjectured in [17] and proved in [19] that the commutant is spanned by these matrices.

We now reinstate the condition that N should have only non negative matrix elements, when the form (10) is expressed in terms of independent characters $1 \le \lambda \le K/2-1$. A tedious analysis leads to the conclusion that the only solutions are as follows:

$$\begin{array}{rll} k \ge 0 & N = \Omega^{(k+2)} & (A_{k+1}) \\ k \text{ even} \ge 4 & N = \Omega^{(k+2)} & (D_{k/2+2}) \\ k = 10 & N = \Omega^{(12)} + \Omega^{(2)} + \Omega^{(3)} & (E_6) \\ k = 16 & N = \Omega^{(18)} + \Omega^{(2)} + \Omega^{(3)} & (E_7) \\ k = 28 & N = \Omega^{(30)} + \Omega^{(2)} + \Omega^{(3)} + \Omega^{(5)} & (E_8) \,. \end{array} \tag{25}$$

This is the complete classification of $A_1^{(1)}$ modular invariants, i.e. of two-dimensional conformal theories with a su(2) current algebra. This was conjectured in [17] and proved in [20-21]. As explained above, by tensoring these expressions one derives the classification of other families of conformal theories [22] : the minimal discussed previously, but also "superminimal" [23], i.e. endowed with a finite number of irreps of the Neveu-Schwarz-Ramond algebra, or of its N=2 extension [24], or parafermionic theories of various types [25,19,26], etc... To summarize what has been achieved from a physical point of view, I restate that we now have a complete classification of several families of universality classes of two-dimensional critical phenomena. This is so far restricted to simple theories:

typically the minimal theories imply a finite number of Virasoro irreps, hence have only a finite number of independent critical exponents (not differing by integers). It seems clear, however, that these restrictions should be soon disposed of, and that a general classification is within reach.

There is a hidden connection between the modular invariants of (25) and simply laced Lie algebras. One first notices that the values of k+2 coincide with the Coxeter numbers of the algebras A_{k+1}, $D_{k/2+2}$, E_6, E_7, E_8. Moreover, as the explicit form of the invariants displayed in Table 1 reveals, the value of λ for the non vanishing diagonal elements of the matrix N are nothing but the Coxeter exponents of these algebras with their multiplicities!!

This intriguing connection has not yet been well understood. Attempts at interpreting it in terms of another ADE classification, namely of finite subgroups of SU(2) [17,27], has not produced any illuminating explanation. Could the whole matrix $N_{\lambda\bar{\lambda}}$, and not only its diagonal elements, receive a Lie-algebraic interpretation? So far the most successful connection with another A-D-E classification is the work of Pasquier [28]: using a totally different approach, he constructs completely integrable lattice models. In his study of the algebra of microscopic transfer matrices (Temperley-Lieb-Jones algebra) he uses the results of [29]. His models are attached to Dynkin diagrams of simply laced algebras and their critical behaviour is described by the minimal models classified above. This leaves open, however, the question of a direct interpretation of the A-D-E classification of modular invariants.

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TABLE I

List of affine partition functions in terms of $A_1^{(1)}$ characters

(here, n stands for k+2)

$n \geq 2$	$\sum_{\lambda=1}^{n-1} \lvert\chi_\lambda\rvert^2$	A_{n-1}	Ω_n
$n = 4\rho+2 \quad \rho \geq 1$	$\sum_{\substack{\lambda\ \mathrm{odd}\ =1 \\ \lambda \neq 2\rho+1}}^{4\rho+1} \lvert\chi_\lambda\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2 + \sum_{\lambda\ \mathrm{odd}\ =1}^{2\rho-1} (\chi_\lambda\chi^*_{4\rho+2-\lambda}+c.c.)$ $= \sum_{\lambda\ \mathrm{odd}\ =1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$D_{2\rho+2}$	$\Omega_n+\Omega_2$
$n = 4\rho \quad \rho \geq 2$	$\sum_{\lambda\ \mathrm{odd}\ =1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda\ \mathrm{even}\ =2}^{2\rho-2} (\chi_\lambda\chi^*_{4\rho-\lambda}+c.c.)$	$D_{2\rho+1}$	$\Omega_n+\Omega_2$
$n = 12$	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	E_6	$\Omega_{12}+\Omega_3+\Omega_2$
$n = 18$	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + [(\chi_3+\chi_{15})\chi_9^*+c.c.]$	E_7	$\Omega_{18}+\Omega_3+\Omega_2$
$n = 30$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	E_8	$\Omega_{30}+\Omega_5+\Omega_3+\Omega_2$