CONFORMAL FIELD THEORY AND APPLICATIONS

by

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Abstract. These lectures present an introduction to conformal field theories, with an emphasis on applications to two-dimensional critical phenomena. After a review of the formalism of Belavin, Polyakov and Zamolodchikov, finite size effects, modular invariance and the classification of minimal theories are discussed.

1. INTRODUCTION

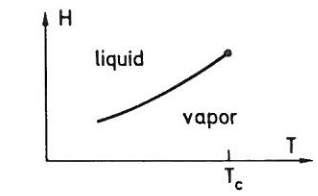
Conformal field theory applies to two different physical situations:

$*$ string theory: if the metric is written as $g_{ab}(x) = \rho(x)\, g^{(0)}_{ab}(x)$, we do not want the conformal factor $\rho(x)$ to play any role in the dynamics; the theory must depend only on conformal classes of metrics.

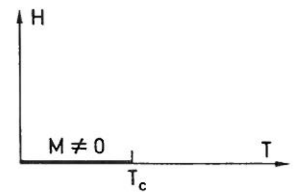
$*$ critical phenomena in two dimensions: Polyakov (1970) has shown that in arbitrary dimension, the dilatation invariance expected in critical phenomena implies conformal invariance. This remark is particularly fruitful in two dimensions.

1.1 A little reminder on critical phenomena.

We are interested in systems undergoing a second order i.e. continuous phase transition, for example :



liquid-vapor system
under T_c : two distinct phases;
the order parameter $=\rho_{\rm liq}-\rho_{\rm vap}$
vanishes continuously at T_c.



ferromagnet
under T_c (Curie temperature), as $H \to 0$,
a spontaneous magnetization M appears;
this order parameter M vanishes at T_c.

Such critical systems exhibit a singular behavior at T_c : for example, the specific heat c behaves as $c \sim |T-T_c|^{-\alpha}$, the order parameter as $\mathrm{M} \sim (T_c - T)^{\beta}$ etc...; α, β etc... are *critical exponents.*

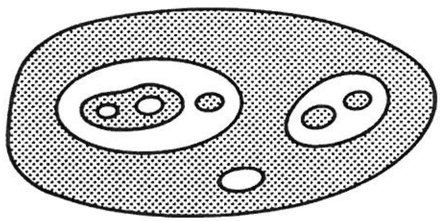
At some arbitrary temperature, correlation functions of physical quantities decrease exponentially :

$$\langle \varphi(r)\varphi(o)\rangle \sim \mathrm{e}^{-r/\xi} \qquad \xi = \text{correlation length}$$

What happens is that at T_c, the correlation length diverges. (There may be several ξ for the various physical quantities, and we assume that they all diverge in the same way) :

$$\xi \sim |T-T_c|^{-\nu}$$

This is the *fundamental* property of a second order phase transition. It implies that the system has lost a scale ξ (or is massless : mass $\sim \xi^{-1}$). This means that in the critical system, fluctuations occur at all scales : bubbles of one phase inside bubbles of the other inside bubbles ... at all scales ranging between the distance of physical observation and the microscopic scale (see Figure). One has *scale invariance.* Just at T_c, the correlation functions have a power law fall-off : $\langle \varphi(r)\varphi(o)\rangle \sim r^{-\eta}$ and one may regard $\frac{1}{2}\eta$ as the scaling dimension of φ : $\dim\varphi = \frac{1}{2}\eta$.



Following Wilson, one may use renormalization group (R.G.) ideas : if physics at scale a is described by a theory with coupling(s) g , at scale λa, it is described by the same theory with effective couplings $g(\lambda)$. As $\lambda \to \infty$, one assumes that $g(\lambda)$ approaches some fixed point g^* : $\beta(g^*) = 0$. In general, one may show that a critical system may be described by a (massless) scale-invariant Euclidean field theory.

The concept of fixed point of the R.G. also explains the properties of *universality* : modifications of microscopic interactions (that do not affect the symmetries of the problem) do not change the critical behaviour (critical exponents etc...), although they may change the value of the critical temperature. Classes of universality are sets of systems with different microscopic physics but the same critical behaviour. Conformal field theory will describe the universality classes only.

1.2 Example: Ising model.
The partition function of this spin ± 1 ferromagnet reads:

$$Z = \sum_{\{\sigma_i = \pm 1\}} \mathrm{e}^{\beta \sum_{\text{bonds}} \sigma_i \sigma_j}$$

In 2 dimensions, $\alpha = 0$ (logarithmic singularity), $\beta = \frac{1}{8}$, $\eta = \frac{1}{4}$, etc... The addition of a term $\gamma \sigma_i \sigma_j \sigma_k \sigma_\ell$ to the 2-spin interaction shouldn't modify the exponents. Actually the Ising model is also appropriate to describe the phase transition of a binary fluid system.

In d $\geq$ 2, Wilson and his followers have developed very powerful R.G. improved perturbative techniques to compute critical exponents. The results fit very well with experimental data. In d = 2 , however, conformal field theory will enable us to go farther and:

- explain why rational exponents or exponents varying continuously occur so frequently in statistical mechanical models,
- possibly achieve a classification of all universality classes,
- compute all the correlation functions at T_c , etc...

2. <u>GENERAL FEATURES OF CONFORMAL FIELD THEORY</u>

2.1 Conformal transformations.
Conformal transformations are point transformations of the Euclidean space which preserve the angles, i.e. they may be regarded as local dilatations. In any dimension d $\geq$ 3, the conformal group is made of

translations : infinitesimal form : $\delta x_\mu = a_\mu$
rotations : $\delta x_\mu = \omega_{\mu\nu} x^\nu$, $\omega_{\mu\nu}$ antisymmetric
dilatations : $\delta x_\mu = \gamma x_\mu$
"special conformal transformation" = inversion $*$ translation $*$ inversion:

$$\delta x_\mu = b_\mu x^2 - 2x_\mu (b.x)$$

In d = 2, there are more conformal transformations, as we shall see below.

In field theory, let us write a Ward identity to describe the effect of a change of coordinates. Consider some correlation function

$$\langle A_1(1) \ldots A_n(n)\rangle = \int \mathcal{D}\varphi e^{-S(\varphi)} A_1(\varphi(1)) \ldots A_n(\varphi(n))$$

assumed to be given by some functional integral; A is a local function of the basic field(s) φ . Perform a change of coordinates $x \to x'$: the form of the function $A(x)$ changes into $A'(x')$ and

$$\delta A(x) = A'(x') - A(x) .$$

Write that the physical quantity (the l.h.s.) is unaffected by this change of coordinates. In the r.h.s., the measure $\mathcal{D}\varphi e^{-S(\varphi)}$ contributes a term linear in $\partial_\mu \delta x_\nu(x)$ (it must vanish for $\delta x_\nu = a_\nu$ by translation invariance) :

$$0 = \sum_i \langle A_1 \ldots \delta A_i \ldots A_n\rangle + \int \mathrm{d}^d x \, \langle T_{\mu\nu}(x) A_1 \ldots A_n\rangle \, \partial^\mu \delta x^\nu(x)$$

which defines $T_{\mu\nu}$ (or rather $\langle T_{\mu\nu}A_1...A_n\rangle$). $T_{\mu\nu}$ is the energy-momentum tensor. Imposing rotation invariance and dilatation invariance namely :

$$\sum_i \langle A_1...\delta A_i...A_n\rangle = 0 \quad \text{for rotations and dilatations}$$

shows that

$$T_{\mu\nu}\delta\omega^{\mu\nu} = 0 \qquad T_{\mu\nu} \ \text{symmetric}$$
$$T_{\mu\nu}\delta^{\mu\nu} = 0 \qquad T_{\mu\nu} \ \text{traceless}$$

up to total derivatives. We assume that after possibly a suitable modification ("improvement"), $T_{\mu\nu}$ is symmetric and traceless. (One may show that this improvement is always possible in two dimensions, which is the only case which will concern us here.)

Moreover, the variation δA_i must be local in δx and its derivatives: taking a functional derivative of the Ward identity with respect to $\delta x(x)$ gives

$$\sum_i [...\delta(x-x_i) + ...\bar{\nabla}\delta(x-x_i) + ...] = \partial_\mu \langle T_{\mu\nu}(x) A_1(1)...A_n(n)\rangle$$

Up to coinciding-point singularities, $T_{\mu\nu}$ is *conserved.* Finally if in the Ward identity, δx^μ is chosen to be a special conformal transformation, it is easy to check that the symmetry and the tracelessness of T imply the vanishing of the second term :

$$\left.\begin{array}{c}\text{translation}\\ \text{rotation}\\ \text{dilatation}\end{array}\right\} \text{invariances imply conformal invariance.}$$

2.2 In 2 dimensions (Euclidean plane) :
Two new features emerge:

1) One can use complex coordinates $z = x + iy \quad \bar{z} = x - iy$ and write

$$ds^2 = dx^2 + dy^2 = dz d\bar{z}$$

as

$$ds^2 = g_{ab} d\xi^c d\xi^b \qquad \xi^a = \{z, \bar{z}\}$$
$$= (g_{z\bar{z}} + g_{\bar{z}z})\, dz d\bar{z}$$

hence $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$, $\quad g_{zz} = g_{\bar{z}\bar{z}} = 0$ and the inverse tensor is $g^{z\bar{z}} = 2$. The energy momentum tensor satisfies $T_{z\bar{z}} = T_{\bar{z}z} = 0$ as a consequence of its tracelessness.

2) Any analytic (holomorphic) change of coordinates is a conformal transformation

$$z \to z' = f(z) \qquad ds^2 \to |f'(z)|^2 dz d\bar{z}$$

This changes locally the lengths, but not their ratios, hence not the angles.

Important : The transformations discussed before: translations, rotations, dilatations, special conformal transformations form the group $SL(2,\mathrm{C})$ of Moebius transformations:

$z \to \frac{az+b}{cz+d} \quad a,b,c,d \in \mathrm{C},\ ad - bc = 1$. Those are the *only* conformal transformations which are one-to-one on the Riemann sphere (completed plane).

2.3 Transformation of fields.
Dealing with complex coordinates enables one to define tensors of non-integer rank: A *primary field* is by definition a $(h, \bar{h})$ tensor, i.e. under $z \to z'$:

$$A(z,\bar{z}) = A'(z',\bar{z}') \left(\frac{dz'}{dz}\right)^h \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\bar{h}}$$

where $h, \bar{h}$ are two arbitrary (real) numbers, called the conformal weights of A. We shall soon interpret $h+\bar{h}$ as the scaling dimension of A and $h-\bar{h}$ as its spin. For example, a free Majorana-Weyl field $\psi(z)$ has $h = \frac{1}{2}$, $\bar{h} = 0$ (or vice versa, depending on the chirality). The infinitesimal transformation of a primary field reads:

$$\delta z = \varepsilon(z) \qquad \delta\bar{z} = \bar{\varepsilon}(\bar{z})$$

$$\delta A(z,\bar{z}) = [\varepsilon(z)\partial_z + h\varepsilon'(z)]\, A(z,\bar{z}) + [\bar{\varepsilon}(\bar{z})\partial_{\bar{z}} + \bar{h}\bar{\varepsilon}'(z)]\, A(z,\bar{z}) \, .$$

This result is actually quite typical : the contribution of the z and $\bar{z}$ variables decouple, which enables one to treat them as independent variables. In many respects, the conformal field theory behaves as made of two one-dimensional components. Ultimately the "physical world" is recovered by imposing that $\bar{z}$ = complex conjugate of z.

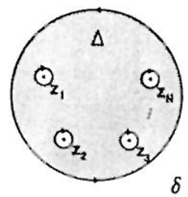
Important warning. Not all fields behave as primary fields. Compute for example the transformation of $\partial_z A(z,\bar{z})$. Another important non primary field is $T(z,\bar{z})$ as we shall see below.

2.4 Two-dimensional conformal field theory
Let us assume $T_{z\bar{z}} = 0$ (scale invariance) and return to the Ward identity. In 2 dimensions, it reads, after a little change in the normalization, suited for what follows :

$$\delta\langle A_1(1)...A_n(n)\rangle = \frac{1}{2\pi i}\int dz d\bar{z} \;\; \partial_{\bar{z}}\varepsilon(z,\bar{z}) \langle T_{z,z}(z,\bar{z}) A_1(1)...A_n(n)\rangle + c.c.$$

For the reason mentioned before (singularities for transformations which are not Moebius), one cannot take $\varepsilon(z,\bar{z})$ analytic right away. By the same argument as presented above, T is conserved everywhere but at the points $1, 2, ...n$:

$$\partial_{\bar{z}} T_{zz} = 0 \qquad \partial_z T_{\bar{z}\bar{z}} = 0 \quad \text{i.e.} \quad T_{zz} = T(z) \text{ is analytic and } \ T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}) \ \text{ is antianalytic}$$



except at the points $1,2...n$. Take ε analytic in the vicinity of $z_1, ..., z_n$ and vanishing outside a compact domain Δ'. $\langle TA_1...A_n\rangle$ is therefore analytic in a domain Δ excluding the points $z_1, ..., z_n$ and using Stokes theorem one may write :

$$\delta\langle A_1...A_n\rangle = -\frac{1}{2\pi i}\int_{\partial\Delta} dz d\bar{z} \;\varepsilon(z,\bar{z}) \langle T_{zz}(z,\bar{z}) A_1...A_n\rangle + c.c.$$

$$= \sum_i \frac{1}{2\pi i} \oint_{z_i} \mathrm{d}z\, \varepsilon(z) \langle T(z) \ldots A(n) \rangle + c.c.$$

because ε vanishes on the external border. One may then identify separately the analytic and antianalytic contributions in ε on both sides of the equation. For a primary field, for example:

$$\sum_i \left[\varepsilon(z_i)\, \partial_{z_i} + h_i \varepsilon'(z_i)\right] \langle A_1(z_1, \bar{z}_1) \ldots A_n(z_n, \bar{z}_n) \rangle$$

$$= \frac{1}{2\pi i} \oint \mathrm{d}z \varepsilon(z) \langle T(z) \ldots A(n) \rangle$$

where the contour encircles the points $z_1, \ldots, z_n$. Finally one uses Cauchy theorem to interpret this as :

$$\langle T(z)\, A_1(z_1, \bar{z}_1) \ldots A_n(z_n, \bar{z}_n) \rangle = \sum_i \left(\frac{h_i}{(z - z_i)^2} + \frac{1}{(z - z_i)} \partial_{z_i} \right) \langle A_1 \ldots A_n \rangle$$

(which is the unique solution vanishing as $z \to \infty$). Notice that this expression of $\langle T(z) \ldots A_n \rangle$ is indeed holomorphic everywhere except at $z_1, \ldots, z_n$.

One may also rephrase this result in terms of the short distance expansion :

$$T(z)\, A_1(z_1, \bar{z}_1) = \frac{h_1}{(z - z_1)^2} A_1(z_1, \bar{z}_1) + \frac{1}{(z - z_1)} \partial_{z_1} A_1(z_1, \bar{z}_1) + \ldots$$

where ... means less singular terms. This expansion is to be understood as valid when inserted into a correlation function, i.e. in the presence of spectator fields.

2.5 Transformation of $\mathbf{T}(z)$.

If one wants to consider variations of $\langle T(z)\, A_1 \ldots A_n \rangle$ itself under changes of coordinates, one needs the short distance expansion of T with itself. The expression:

$$T(z)\, T(w) = \frac{c}{2(z - w)^4} + \frac{2}{(z - w)^2} T(w) + \frac{1}{(z - w)} \partial_w T(w) + \ldots$$

is the most general form consistent with dimensional counting. The energy momentum tensor is a conserved current, hence has no anomalous dimension: its canonical dimension is 2, and the second and third terms in the expression above are just the normal terms expected for a primary field of dimension $h = 2$, $\bar{h} = 0$. The first term, however, is an anomalous term, allowed by dimensional analysis, hence generically present: c is the "central charge", a denomination to be justified soon.

This short distance expansion of $T.T$ corresponds to an infinitesimal transformation of T of the form

$$\delta T(z) = \left[\varepsilon(z)\, \partial_z + 2\varepsilon'(z)\right] T(z) + \frac{c}{12} \varepsilon'''(z)$$

It integrates for finite transformations to:

$$T(z) = T'(z') \left(\frac{\mathrm{d}z'}{\mathrm{d}z} \right)^2 + \frac{c}{12} \{z', z\}$$

where the "schwarzian derivative" is

$$\{z', z\} = \frac{\mathrm{d}^3 z'}{\mathrm{d}z^3} \Big/ \frac{\mathrm{d}z'}{\mathrm{d}z} - \frac{3}{2} \left[\frac{\mathrm{d}^2 z'}{\mathrm{d}z^2} \Big/ \frac{\mathrm{d}z'}{\mathrm{d}z} \right]^2$$

This integration to finite transformations is non trivial. One may show that $\{z', z\}$ is the unique term (up to a factor) satisfying :

- antisymmetry i.e. $\{z', z\} = -\{z, z'\} \left(\frac{\mathrm{d}z'}{\mathrm{d}z} \right)^2$
- $SL(2, \mathbf{C})$ invariance, i.e. if f is a Moebius transformation :

$$\{f(z'), z\} = \{z', z\}$$

- cochain condition : changes of coordinates $z \to z' \to z''$ and $z \to z''$ are consistent
- dimension = 2 in units of $[length]^{-1}$.

This transformation law will have important consequences when we shall use conformal transformations to map some domain on some other. For example, map the punctured plane on the cylinder by $z = \mathrm{e}^{2i\pi w}$ and find that the relation between $T_{\mathrm{plane}}(z)$ and $T_{\mathrm{cyl}}(w)$ reads

$$T_{\mathrm{cyl}}(w) = (2i\pi)^2 \left[z^2 T_{\mathrm{plane}}(z) - \frac{c}{24} \right]$$

2.6 Descendants of a primary field

Only the first two terms in the "operator product expansion" (o.p.e.) of $T(z)$ and of the primary field A have been identified in the expression above. More generally, we write :

$$\begin{aligned} T(z)\, A(z_1, \bar{z}_1) &= \sum_{k=0}^{\infty} (z - z_1)^{-2+k} A^{(-k)}(z_1, \bar{z}_1) \\ A^{(-k)}(z_1, \bar{z}_1) &\equiv (L_{-k} A)(z_1, \bar{z}_1) \\ &= \oint \frac{\mathrm{d}z}{2\pi i (z - z_1)^{k-1}} T(z)\, A(z_1, \bar{z}_1) \end{aligned}$$

and by further multiplications by T and/or by $\bar{T}$ and expansions, define also

$$A^{(-k_1, -k_2)}(z_1, \bar{z}_1) = L_{-k_2} A^{(-k_1)}(z_1, \bar{z}_1)$$

$$A^{(-k_1, -\bar{k}_1)}(z_1, \bar{z}_1) = \bar{L}_{-\bar{k}_1} A^{(-k_1)}(z_1, \bar{z}_1)$$

etc... The family of fields A, $A^{(-k_1)}, \ldots, A^{(-k_1, \ldots, -\bar{k}_1, \ldots)}, \ldots$ constitutes the set of descendants of A, graded by theirs *levels* : $|k| = \sum_\ell \ell k_\ell$, $|\bar{k}| = \sum_\ell \ell \bar{k}_\ell$.

2.7 Operator algebra

An axiom of conformal field theory is that a product of two primary fields may be expanded on the (finite or infinite) set of primary fields and on their descendants:

$$A(z, \bar{z})\, B(w, \bar{w}) = \sum_{\substack{C \\ \{k\}, \{\bar{k}\}}} C_{ABC}^{(\{k\}, \{\bar{k}\})} (z - w)^{h_C - h_A - h_B + |k|} \times$$

$$\times (\bar{z}-\bar{w})^{\bar{h}_C-\bar{h}_A-\bar{h}_B+|\bar{k}|}\, C^{(\{-k\},\{-\bar{k}\})}(w,\bar{w})$$

where the sum runs over primary fields C. The coefficient $C_{ABC}^{(\{k\},\{\bar{k}\})}$ may be determined by use of the Ward identity, once the "structure constants" C_{ABC} pertaining to the primary fields are known.

2.8 Two- and three-point functions
Using translation and dilatation invariance, show that the 2-point function of a primary field reads:

$$\langle A(z,\bar{z})\, A(w,\bar{w})\rangle = \frac{1}{(z-w)^{2h_A}(\bar{z}-\bar{w})^{2\bar{h}_A}}$$

(The residue 1 results from a choice of normalization). Using $SL(2,\mathbf{C})$ invariance to map the three points z_A, z_B and z_C to $0, 1, \infty$ or any permutation thereof, show that (for A, B, C primary):

$$\langle A(z_A,\bar{z}_A)\, B(z_B,\bar{z}_B)\, C(z_C,\bar{z}_C)\rangle = \frac{C_{ABC}}{(z_A-z_B)^{h_A+h_B-h_C} \times \ perm. \ \times c.c.}$$

and that C_{ABC} is completely symmetric.

The form of the 2-point function above justifies our earlier statement : in radial coordinates $z-w=\rho e^{i\theta}$

$$\langle A(z,\bar{z})\, A(w,\bar{w})\rangle = \frac{e^{-2(h_A-\bar{h}_A)i\theta}}{\rho^{2(h_A+\bar{h}_A)}}$$

$h_A+\bar{h}_A$ is the scaling dimension of A, $h_A-\bar{h}_A$ its spin.

Examples : 1) Free boson field.
Its action reads $S=\frac{1}{4\pi}\int d^2x\,(\partial_\mu\phi)^2$, whence the propagator $\langle\phi(z,\bar{z})\,\phi(0)\rangle = -\frac{1}{2}\log z\bar{z}$. *Caution !* ϕ is NOT a primary field (it has a logarithmic propagator!) but $O_e(z,\bar{z}) =\, :\exp ie\phi(z,\bar{z}):$ is. This is the vertex operator in the language of string theory, the electric or spin-wave operator in the context of statistical mechanics. Check that $h=\bar{h}=e^2/4$ using Wick theorem. Also using Wick theorem, check that $T(z) = -(\partial_z\phi)^2$ has the right o.p.e. with $O_e(z,\bar{z})$ and from the o.p.e. $T(z)\,T(w)$, determine the value of c.

2) Free Majorana-Weyl fermion.
The action $S=\frac{1}{2\pi}\int d^2x\ \ \psi\bar{\partial}\psi$ corresponds to the propagator $\langle\psi(z)\,\psi(w)\rangle = \frac{1}{z-w}$, hence $h=1/2$, $\bar{h}=0$ as announced above. Check that the o.p.e. with $T(z) = -\frac{1}{2}\psi(z)\,\partial_z\psi(z)$ is what it should be, and determine c.

2.9 Operator language. Virasoro algebra
We want now to reinterpret the previous correlation functions in an operator language, as resulting from the vacuum expectation value of suitably ordered products of field operators:

$$\langle A_1(1)\ldots A_n(n)\rangle = \left\langle 0\left|\mathcal{P}\hat{A}_1(z_1,\bar{z}_1)\ldots\hat{A}_n(z_n,\bar{z}_n)\right|0\right\rangle$$

It turns out that the radial quantization, in which fields are ordered from right to left according to growing values of $|z_i|$ is particularly well suited. It corresponds by the exponential mapping to the standard quantization on a cylinder, where the time evolution along the axis of the cylinder describes either the propagation of a closed string, or a statistical mechanical system subject to a periodic boundary condition along the "space" direction.

All the short distance expansions found above must be regarded in this operator formalism as applying to the $\mathcal{P}$-ordered products. Expand $\hat{T}(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$ hence $L_n = \oint_0 \frac{dz}{2i\pi} z^{n+1}\hat{T}(z)$ and compute the commutator:

$$\left[L_n, \hat{A}(w)\right] = \left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\frac{dz}{2i\pi} z^{n+1}\,\mathcal{P}\hat{T}(z)\,\hat{A}(w)$$

The two $w-$contours may be then deformed into a single one circling around z. For a primary field A or the energy momentum tensor, respectively, this leads to

$$\left[L_n, \hat{A}(z)\right] = z^{n+1}\partial_z\hat{A}(z) + h\,(n+1)\,z^n\hat{A}(z)$$

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{n+m,0}$$

The latter equation defines the *Virasoro algebra.* One may regard L_n as the quantum realization of the differential operators $l_n = -z^{n+1}\frac{\partial}{\partial z}$, which generate the diffeomorphisms of the circle, with an algebra :

$$[l_n, l_m] = (n-m)\,l_{n+m}$$

The Virasoro algebra is therefore the "central extension" of the algebra of the diffeomorphisms of the circle by the c-number (Schwinger-term) $\frac{c}{12}n\left(n^2-1\right)\delta_{n+m,0}$. Notice that in the plane, L_{-1}, L_0, L_1 generate the $SL(2,\mathbf{C})$ transformations $\delta z =$ respectively ε, εz, εz^2. Together with their $\bar{z}$ counterparts, generating an isomorphic $\bar{L}$ algebra, they correspond to respectively translations, rotations and dilatations, and special conformal transformations. Notice also that this subalgebra does not "see" the central term: this reflects the fact that the Moebius transformations have a vanishing schwarzian derivative and are the only true symmetries of the theory.

To summarize, in a conformal field theory, there is a natural action of two Virasoro algebras, relative to the z and $\bar{z}$ coordinates (left-moving and right-moving components). It is thus natural to wonder what are the...

2.10 Representations of the Virasoro algebra
In physics, the interesting representations are the so-called highest weight (h.w.) representations: for reasons which will become clear soon, they correspond to systems whose energy is bounded from below. Remember from the discussion of ordinary Lie algebras, $SU(2)$ say, the construction of h.w. representations. The generators of the algebra are J_+, J_- and J_z. One chooses an eigenvector of J_z of eigenvalue j annihilated by J_+ :

$$J_z\,|j\rangle = j\,|j\rangle \qquad J_+\,|j\rangle = 0$$

Repeated action of J_- builds a sequence of eigenvectors of J_z

$$J_z (J^-)^m |j\rangle = (j-m)(J^-)^m |j\rangle$$

until $(J^-)^m |j\rangle = 0$ which occurs for $m = 2j+1$. Likewise, here, we start from a highest weight state, eigenstate of L_0 and annihilated by all L_n, $n > 0$:

$$L_0 |h\rangle = h |h\rangle \qquad L_{n>0} |h\rangle = 0$$

The action of the $L_{n<0}$ then generates a representation space ("Verma module") of the Virasoro algebra. Among these h.w. states, one plays a particular role : $h = 0$:

$$L_0 |0\rangle = 0 \qquad L_{n>0} |0\rangle = 0$$

and hence also (by the Virasoro algebra) $L_{-1} |0\rangle = 0$, which expresses that $|0\rangle$, the ground state, is invariant under $SL(2, \mathbf{C})$ transformations.

The h.w. representation built on $|h\rangle$ is infinite-dimensional, (contrary to the case of $SU(2)$ recalled above). It is also *graded* : this means that the eigenvalues of L_0 are integer spaced; as the Virasoro commutation relations show immediately :

$$L_0 L_{-1}^{\alpha_1} ... L_{-k}^{\alpha_k} |h\rangle = \left(h + \sum j\alpha_j\right) L_{-1}^{\alpha_1} ... L_{-k}^{\alpha_k} |h\rangle$$

and $\sum j\alpha_j$ is the level. The actual states of the theory are made of ***tensor products*** of representations of the left and right Virasoro algebras:

$$\mathcal{H} = \oplus_h V_h \otimes V_{\bar{h}}$$

There is a correspondence between the language of fields and the one of states :
primary fields $A_{h\bar{h}} \leftrightarrow$ h.w. state $|h, \bar{h}\rangle$
descendant fields $\leftrightarrow$ descendant states
through :

$$\lim_{z,\bar{z}\to 0} A_{h\bar{h}}(z, \bar{z}) |0\rangle = |h, \bar{h}\rangle .$$

Check using the commutator $[L_n, A]$ that this state is indeed a h.w. state; one has similar expressions for the descendants.

2.11 Irreducibility and Unitary of Virasoro representations

Two deep and remarkable theorems control these questions.

.Theorem 1. (Kac-Feigin-Fuchs)
Write the central charge as $c = 1 - \frac{6}{x(x+1)}$ where $x \in \mathbf{C}^*$. Then the only reducible representations occur for the discrete set of weights :

$$h_{rs} = \frac{(r(x+1) - sx)^2 - 1}{4x(x+1)} \qquad r, s \in \mathbf{N}$$

A Verma module gives a reducible representation whenever it contains a state of non-zero level ℓ which behaves itself as a highest weight :

$$L_0 |\psi\rangle = (h + \ell) |\psi\rangle \qquad L_{n>0} |\psi\rangle = 0.$$

This state $|\psi\rangle$ generates its own Verma module which forms a reducible subspace of the original representation. What the previous theorem tells us is a necessary and sufficient condition on h for this to happen. Moreover the theorem tells us that if h is as above, the lowest $|\psi\rangle$ occurs at level $\ell = \inf(r.s)$, where the infinimum is to be taken on all pairs (r, s) such that $h = h_{rs}$.

Whenever h gives a reducible representation, one constructs an irreducible representation by subtracting the reducible submodule(s). It turns out that conformal theories made of irreducible representations of that type, with the parameter x rational, display the greatest simplicity (see below) and play an important role in statistical mechanics.

Unitary must certainly be of fundamental importance in systems actually realized in Nature. On the one hand, positivity of probability amplitudes is desirable. On the other hand, there must be a certain consistency between the conformal symmetry and the hermitian conjugation with respect to the norm which is positive:

$$L_n^\dagger = L_{-n}$$

.Theorem 2. (Friedan-Qiu-Shenker)
A necessary and sufficient condition of unitarity is

i) either $\quad c \geq 1 \qquad h \geq 0 \qquad$ arbitrary
ii) or $\quad 0 \leq c < 1 \qquad$ and c, h, as in theorem 1
with $x = m$ an integer ≥ 2 and $1 \leq r \leq m-1 \qquad 1 \leq s \leq m$.

Using the Virasoro commutation relation, and imposing $L_n^\dagger = L_{-n}$, show that if $h < 0$ or if $c < 0$ there are certainly states of negative squared norm. Examine the case $m = 2$: which values of c, h are possible?

In words, this theorem shows that unitarity restricts representations with $c < 1$ to a discrete set. As a last comment, let us add that Friedan, Qiu and Shenker have established the necessary part of the statement. That unitary representations with these values of c and h do exist was shown by the so-called "coset construction" of Goddard, Kent and Olive.

Although unitarity is certainly a physically important constraint, some intermediate stages in string theory (ghosts) or some theoretical problems in statistical mechanics (for example, the behaviour of a ferromagnet in an *imaginary* field : Lee-Yang singularity) may involve non-unitary representations. Incidentally, let us recall that consistency of string theory requires the vanishing of the *total* central charge : $c_{\text{total}} = 0$ whereas in statistical mechanics we may a priori encounter any c !

2.12 Characters of a representation

A rough description of a representation of the Virasoro consists in enumerating the states at a given level, i.e. in evaluating the generating function :

$$f(h) = \sum_{\ell=0}^{\infty} (\text{dimension of space of level } \ell)\, q^\ell$$

Here q is a dummy variable. Multiplying for later convenience by $q^{h-c/24}$, we define the character of the *irreducible* representation built on the h.w. $|h\rangle$ as :

$$\chi_h(q) = \mathrm{tr}\ q^{L_0-c/24} = q^{h-c/24} f(q)$$

Whenever h is *not* of the form given by Theorem 1, i.e. whenever the Verma module is irreducible, the dimension of the space of level ℓ is simply the number of partitions of the integer $\ell = \sum j\alpha_j$, whose generating function is easily seen to be

$$f(q) = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{\ell=0}^{\infty} (\text{number of partitions of } \ell)\ q^{\ell}$$

$$\text{and} \qquad \chi_h(q) = \frac{q^{h-c/24}}{\Pi(1-q^n)}.$$

When h, however, is of the form in Theorem 1, we have to substract some reducible submodule(s) and this results in a more complicated formula

$$\chi_h(q) = \frac{q^{h-c/24} P(q)}{\Pi(1-q^n)}$$

with $P(q) = 1 - q^{\min(r.s)} \pm ...$ is a finite polynomial or an infinite series, depending on the case. Notice, however, that $\chi_h(q)$ must by definition have a q-expansion with integer spaced powers and positive coefficients.

2.13 An explicit example: the $m = 3$ unitary conformal field theory alias the Ising model.

Consider the first non-trivial case in the unitary series of Theorem 2: $m = 3$, $c = 1/2$. The possible values of h are 0, 1/16 and 1/2. It turns out that the conformal theory built from tensor products of left and right copies of these representations is nothing but the Ising model.

$$
\begin{aligned}
(h,\bar h) &= (0,0) && \text{is the identity operator} \\
&= \left(\frac{1}{16},\frac{1}{16}\right) && \text{is the spin operator } \sigma \quad (\text{compare } \eta = 2 \dim \sigma = \frac{1}{4}) \\
&= \left(\frac{1}{2},\frac{1}{2}\right) && \text{is the ''energy operator'' } \varepsilon
\end{aligned}
$$

We can also construct half-integer spin operators $(h,\bar h) = (\frac{1}{2},0)$ or $(0,\frac{1}{2})$: they correspond to the Majorana-Weyl free fermion ψ and $\bar\psi$ which underlie the two-dimensional Ising model and make it exactly solvable.

What is the meaning of the spin and energy operators σ and ε, in the context of critical phenomena? The field σ is the continuum version of the original spin $\sigma_i = \pm 1$. But it also describes the response of the critical system to an external magnetic field which would drive the system out of criticality :

$$\mathcal{H} = \mathcal{H}_{\text{crit}} + \int d^2x\, \sigma(x)\, h(x)$$

Likewise, the energy operator $\varepsilon(x)$ is coupled to the departure from the critical temperature $\Delta T = T - T_c$

$$\mathcal{H} = \mathcal{H}_{\text{crit}} + \Delta T \int d^2x\, \varepsilon(x)$$

σ and ε are *relevant* operators. This means that they have a scaling dimension $d = h + \bar h$ between 0 and 2. Accordingly, the scaling dimension of their conjugate variable h or ΔT is positive, hence the magnitude of the effective h or ΔT grows at large distances. To ensure criticality, both h and ΔT have to be fixed equal to zero.

The fact that Theorem 2 gives only the values of h corresponding to relevant operators is a peculiarity of the Ising model. In other models, other values may appear, corresponding to *irrelevant* operators $(h + \bar h > 2)$, unimportant for the criticality, but which may contribute to finite size effects (see below).

Using scaling arguments, one may derive the values of all critical exponents α, β... from the dimensions of the relevant operators σ and ε. For example, from the fact that

$$\dim \Delta T = 2 - \dim \varepsilon$$

and that the free energy per unit volume F has dimension $\dim F = 2$, show that the exponent α of the specific heat $c \sim \partial^2 F/\partial T^2$ is given by $2 - \alpha = 2/(2 - \dim \varepsilon)$ and that of the magnetization (spin) by $\beta = \frac{(2-\alpha)}{4}\eta$.

3. FINITE SIZE EFFECTS. MODULAR INVARIANCE

3.1 The partition function on a torus.

Given a conformal field theory, we want to study the effects of restricting it to a finite domain. A simple situation is provided by a torus, i.e. a box with periodic boundary conditions. Alternatively, the torus $\mathbf{T}$ may be regarded as the complex plane in which pairs of points differing by integral combinations of the periods 1 and τ are identified $z \sim z + n1 + p\tau$ or $\mathbf{T} = \mathbf{C}/\Lambda$ where Λ is the lattice $\Lambda = \mathbf{Z} \oplus \tau\mathbf{Z}$. (Thanks to scaling invariance, one of the periods may always be chosen equal to 1, and moreover $Im\tau > 0$: τ is the modular parameter of the torus). Let us also define $q = \exp 2i\pi\tau$, $\bar q$ its complex conjugate. A natural definition of the partition function of the system consists in writing:

$$Z = tr\ e^{\mathcal{T}\tau + c.c.}$$

where $\mathcal{T}$ is the translation operator in the w-variable on the cylinder. Notice that the segment of cylinder is closed into a torus by the trace operation.

Since $\mathcal{T} = L_{-1}^{\text{cyl}} = 2i\pi\left(L_0^{\text{plane}} - \frac{c}{24}\right)$ as follows from the transformation of T_{plane} to T_{cyl} computed at the end of section 2.5, we see that $L_0 + \bar L_0$ plays the role of Hamiltonian: this justifies our endeavour at restricting our attention to highest weight representations. Returning to the partition function Z we can now write it as

$$Z = tr\ q^{L_0 - c/24} \bar q^{\bar L_0 - c/24}.$$

The trace is to be taken on $\mathcal{H}$, the Hilbert space of states of the theory. The only information we have on $\mathcal{H}$ is that it decomposes into a (finite or infinite) sum of tensor products of irreducible representations of the left and right Virasoro algebras, characterized by their highest weights h and $\bar{h}$. Hence

$$Z = \sum N_{h\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}) \qquad (*)$$

where $N_{h\bar{h}}$ is the multiplicity of $(h,\bar{h})$, hence a non-negative integer. In particular, $N_{0,0}$, the multiplicity of the vacuum, must be equal to one.

It is important to realize that Z contains all the information about the operator content of the theory: all the pairs of h.w. states, or equivalently all the pairs of conformal weights of primary fields. More precisely, since we have by construction imposed periodic boundary conditions along the "space" direction (period 1), only the states of the periodic (untwisted) sector of the theory are encoded in Z ; stated differently only the integer spin fields of the theory appear in Z . Other sectors of the theory, and the corresponding fields, may be exposed by considering twisted (antiperiodic, etc...) boundary conditions. To summarize, in the context of string theory, the spectrum of energies of the closed string is encoded in Z; in the context of statistical mechanics, it is the set of conformal weights of the theory, i.e. essentially its critical exponents.

The next important observation about Z is that it must be intrinsically attached to the torus, and does not depend on the choice of periods which define Λ . In other words, Z must be invariant under modular transformations:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \qquad \begin{array}{l} a,b,cd \in \mathbf{Z} \\ ad - bc = 1 \end{array}$$

(The condition $ad - bc = 1$ ensures the invertibility of the change of basis). The modular group is actually generated by the two transformations

$$\tau \rightarrow \tau + 1 \qquad \text{and} \qquad \tau \rightarrow -1/\tau$$

So it is necessary and sufficient to ensure that:

$$Z(\tau) = Z(\tau + 1) = Z(-1/\tau).$$

This property of modular invariance, together with the general form $(*)$ $Z = \sum N_{h\bar{h}} \chi_h \chi_{\bar{h}}$ turns out to be very restrictive and opens the route to the classification of families of conformal field theories.

Before discussing this, let us examine the implication of the expression $(*)$ on finite size effects of the system. For this, choose for simplicity $\tau = i\delta$ purely imaginary, hence $q = \exp - 2\pi\delta$ real, and let $\delta \rightarrow \infty$, $q \rightarrow 0$. From the definition of characters, as $q \rightarrow 0$

$$\chi_h(q) \sim q^{h - c/24} [1 + \mathcal{O}(q)]$$

and we can perform an expansion in powers of $q = \bar{q}$. In unitary theories, the leading contribution comes from the identity operator $h = \bar{h} = 0$:

$$Z = (q\bar{q})^{-c/24} \left[1 + \mathcal{O}(q,\bar{q}) + \sum_{h,\bar{h} \neq 0} N_{h\bar{h}} q^h \bar{q}^{\bar{h}} (1 + \mathcal{O}(q,\bar{q})) \right] \sim \exp + \frac{\pi\delta c}{6} [1 + ...]$$

This has to be compared with the partition function expected for a system in a box $L \times T$:

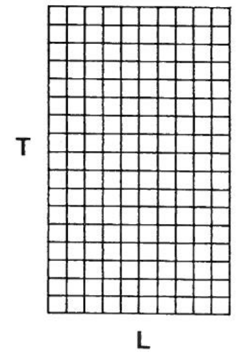
$$Z = \exp LTF(L)$$

In the thermodynamic limit $L, T \rightarrow \infty$, F approaches a finite limit F_0, the free energy per unit volume. Implicitly, this F_0 has been set equal to zero (at $T = T_c$) in our construction, and the expression above, with $\delta = T/L$, yields the finite size contribution to F :

$$F \sim F_0 + \frac{\pi c}{6L^2}$$

We conclude that the finite size contribution to the free energy is proportional to c, the central charge. In other words, c controls the Casimir effect on the cylinder. Other types of boundary conditions may modify the proportionality factor.

The subleading terms in the above expansion also contain an interesting information. Compare with the expression of Z obtained for a lattice model using the transfer matrix $\mathcal{U}$, i.e. the discrete time evolution operator:



$$Z = tr\,\mathcal{U}^T = \lambda_0^T \left(1 + \left(\frac{\lambda_1}{\lambda_0}\right)^T + \left(\frac{\lambda_2}{\lambda_0}\right)^T + ... \right)$$

if $|\lambda_0| > |\lambda_1| > |\lambda_2| ...$ are the ordered eigenvalues of $\mathcal{U}$. In the thermodynamic limit $\lambda_0^T = e^{LTF}$ is the dominant contribution, but we may identify (for the lattice model close to criticality) the successive ratios $\left(\frac{\lambda_i}{\lambda_0}\right)^T$ with the terms of the q-expansion:

$$\left(\frac{\lambda_i}{\lambda_0}\right)^T = \exp - 2\pi \frac{T}{L} \left(h + \bar{h} + |n| + |\bar{n}|\right)$$

for some integers n and $\bar{n}$. This is a method of practical importance. By numerical calculations of the eigenvalues of the transfer matrix, or by studies of the finite size corrections to the Bethe Ansatz results whenever it is possible, one may "read" the values of $(h,\bar{h})$ and hence identify the conformal theory which describes a given statistical mechanical model.

3.2 Modular invariance of the Ising model

In the case of the Ising model, as discussed above, there are only three primary fields (of integer spin) : $\mathbf{1}, \varepsilon, \sigma$, and the partition function of the system reads :

$$Z = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2$$

One may reshuffle this expression into :

$$Z = \left|\frac{\chi_0 + \chi_{1/2}}{\sqrt{2}}\right|^2 + \left|\frac{\chi_0 - \chi_{1/2}}{\sqrt{2}}\right|^2 + \left|\chi_{1/16}\right|^2$$

These combinations of characters have simple expressions in terms of elliptic θ -functions:

$$\begin{aligned}
\left|\chi_0 + \chi_{1/2}\right|^2 &= \left|\frac{\theta_3(0,\tau)}{\eta(\tau)}\right| = \left|\frac{\eta^4(\tau)}{\eta^2(2\tau)\,\eta^2(\tau/2)}\right| \\
\left|\chi_0 - \chi_{1/2}\right|^2 &= \left|\frac{\theta_4(0,\tau)}{\eta(\tau)}\right| = \left|\frac{\eta^2(\tau/2)}{\eta^2(\tau)}\right| \\
2\left|\chi_{1/16}\right|^2 &= \left|\frac{\theta_2(0,\tau)}{\eta(\tau)}\right| = 2\left|\frac{\eta^2(2\tau)}{\eta^2(\tau)}\right|
\end{aligned}$$

where $\eta(\tau) = q^{1/24}\prod^{\infty}(1-q^n)$ is the Dedekind function whose modular transformation under $S : \tau \to -\frac{1}{\tau}$ is as follows :

$$\eta\left(-\frac{1}{\tau}\right) = \left(\mathrm{e}^{-i\pi/2}\tau\right)\eta(\tau)$$

Check that under S and $T : \tau \to \tau + 1$, these combinations are permuted according to :

$$S \circlearrowleft \left|\frac{\chi_0 + \chi_{1/2}}{\sqrt{2}}\right| \overset{T}{\leftrightarrow} \left|\frac{\chi_0 - \chi_{1/2}}{\sqrt{2}}\right| \overset{S}{\leftrightarrow} \left|\chi_{1/16}\right| \circlearrowright T$$

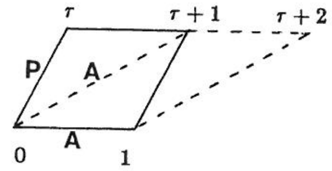
There is a good reason for these simple transformations. The two-dimensional Ising model may be interpreted in terms of a free Majorana-Weyl fermion $\psi(z)$ and its chiral partner $\bar{\psi}(\bar{z})$. The partition function of this fermionic system may be evaluated in terms of the functional integral

$$\begin{aligned}
Z &= \int \mathcal{D}(\psi,\bar{\psi}) \exp - \frac{1}{2\pi}\int \mathrm{d}^2x\,(\psi\bar{\partial}\psi + \bar{\psi}\partial\psi) \\
&= \det{}^{1/2}(-\partial\bar{\partial})
\end{aligned}$$

On a torus, however, this integral must be supplemented by the boundary conditions assigned to ψ and $\bar{\psi}$: there are four such boundary conditions (spin structures), corresponding to periodic (P) or antiperiodic (A) boundary conditions in either direction (for both ψ and $\bar{\psi}$). The P.P. sector contains a zero-mode and $\det - \Delta = 0$, while the three other identify with the three previous combinations of characters :

$$\begin{aligned}
(A,P) &= \frac{1}{2}(\det - \Delta)^{1/2}_{AP} = \left|\frac{\chi_0 - \chi_{1/2}}{\sqrt{2}}\right|^2 \\
(P,A) &= \frac{1}{2}(\det - \Delta)^{1/2}_{PA} = \left|\chi_{1/16}\right|^2 \\
(A,A) &= \frac{1}{2}(\det - \Delta)^{1/2}_{AA} = \left|\frac{\chi_0 + \chi_{1/2}}{\sqrt{2}}\right|^2
\end{aligned}$$

and $Z = (P,A)+(A,P)+(A,A)$. With this interpretation in terms of fermionic boundary conditions, the modular transformations become evident. Changing τ into $\tau+1$ in (A,P) for example, changes the boundary conditions to (A,A) :



whereas $\tau \to -1/\tau$ exchanges the two periods hence leads to (P,A) etc...

We conclude that the untwisted (spinless) sector of the free Majorana theory contains the operators $\mathbf{1}$, $\varepsilon = \psi\bar{\psi}$ and the spin operator. The later is responsible for the change of boundary conditions of the fermion field. As such, it plays a fundamental role in any superconformal (and superstring) theory. (Notice that $\frac{1}{16}$ is the difference of the ground state energies in the Ramond and Neveu-Schwarz sectors).

3.3 Free field compactified on a circle

Let us return to the free Gaussian field of action

$$\mathcal{A} = \frac{1}{4\pi}\int \mathrm{d}^2x\,(\partial\phi)^2$$

We have already constructed a few primary operators : $\exp ie\phi(x)$. If now we assume that ϕ lives on a circle of radius R, we want to identify

$$\phi \sim \phi + 2\pi\mathrm{R}$$

We may define new fields O_{om}, solitons in the language of strings, vortex operators in statistical mechanics, which create a discontinuity of ϕ of $2\pi\mathrm{R}m$. Their 2-point correlation function is computed as

$$\langle O_{om}(x)\,O_{o-m}(0)\rangle = \frac{\int \mathcal{D}\phi\,\mathrm{e}^{-\mathcal{A}}}{\int \mathcal{D}\phi\,\mathrm{e}^{-\mathcal{A}}}$$

where the functional integral in the numerator is over fields which are multivalued in the plane, with a discontinuity $2\pi\mathrm{R}m$ between 0 and x, with $m \in \mathbf{Z}$. One looks for a classical field satisfying these boundary conditions; show that $\phi_c(y) = m\mathrm{R}\mathrm{Arg}\left|\frac{y-x}{y}\right|$ is a solution. Use the Cauchy-Riemann equations on $\hat{\phi}(\varsigma) = m\mathrm{R}\,\mathrm{Log}\frac{\varsigma - x}{\varsigma}$ to compute its action. The fluctuations around ϕ_c cancel with the functional integral in the denominator with the result :

$$\langle O_{om}(x)\,O_{o-m}(0)\rangle = |x|^{-m^2\mathrm{R}^2}$$

hence $h = \bar{h} = m^2\mathrm{R}^2/4$. More generally, operators with both "electric" and "magnetic" charges have conformal weights

$$\begin{aligned}
h_{em} &= \frac{1}{4}(e + m\mathrm{R})^2 \\
\bar{h}_{em} &= \frac{1}{4}(e - m\mathrm{R})^2
\end{aligned}$$

Check this result by inserting in the numerator of the previous functional integral the electric operators

$$e^{ie\phi(x)}e^{-ie\phi(o)}$$

Let us now compute the partition function on a torus of such a compactified field : we must allow changes of the determination of ϕ by integral multiples of $2\pi\mathrm{R}$, as z winds around the torus:

$$Z_{mm'} = \int_{\substack{\phi(z+1)=\phi(z)+2\pi m\mathrm{R} \\ \phi(z+\tau)=\phi(z+1)+2\pi m'\mathrm{R}}} D\phi \, \mathrm{e}^{-A(\phi)}$$

We use the same method: find a field $\phi_c(z,\bar{z}) = \alpha z + \bar{\alpha}\bar{z} + \beta$ which satisfies the boundary conditions and the equation of motion; expand around it. This gives :

$$Z_{mm'} = \mathrm{e}^{-A(\phi_c)} \int_{\substack{\phi \text{ doubly} \\ \text{periodic}}} D\phi \, \mathrm{e}^{-A(\phi)}$$

The functional integral is equal to :

$$\int_{\substack{\phi \text{ doubly} \\ \text{periodic}}} D\phi \, \mathrm{e}^{-A(\phi)} = \frac{1}{\eta\bar{\eta}\,(Im\tau)^{1/2}}$$

and therefore

$$Z_{mm'} = \frac{1}{\eta\bar{\eta}\,(Im\tau)^{1/2}} \;\exp - 2\pi\mathrm{R}^2\frac{|m'-m\tau|^2}{Im\tau}.$$

Any of these partition functions $Z_{mm'}$, however, is not modular invariant. It is easy to see that under the S and T modular transformations, $Z_{mm'}$ is changed into $Z_{-m'm}$ and $Z_{m,m+m'}$, respectively. A natural way to construct a modular invariant is to sum the contributions of all the winding numbers (m,m'). One may then use the Poisson transformation to reexpress the sum over m' in terms of the Fourier conjugate variable. For any function f , one has indeed:

$$\sum_{m'\in Z} f(m') = \int \mathrm{d}x \sum_{e\in Z} e^{2i\pi e x} f(x)$$

After some algebra, one finds

$$Z = \sum_{m,m'\in Z} Z_{mm'}(\mathrm{R}) = \frac{1}{\eta(q)\,\eta(\bar{q})} \sum_{e,m\in Z} q^{h_{em}} \bar{q}^{\bar{h}_{em}}$$

Therefore this physical modular invariant partition function displays all the spectrum of conformal weights computed above for the electric-magnetic operators. Its small q limit exhibits the expected behaviour for a central charge equal to 1.

3.4 Classification of minimal theories.

As shown by Cardy, a simple consequence of the modular invariance of the formula (*) is that the conformal theory may contain a finite set of primary fields only if the central charge is less than one. It turns out that a class of theories with this property may be constructed using the representations of Theorem 1 (section 2.11) when the parameter x is rational. Let us write

$$x = \frac{p'}{(p-p')} \qquad p \text{ and } p' \text{ coprime integers}$$

that is

$$c = 1 - \frac{6\,(p-p')^2}{pp'}$$

$$h,\bar{h} \in \{h_{rs} = \frac{(rp-sp')^2 - (p-p')^2}{4pp'}$$

with $1 \le r \le p'-1$ and $1 \le s \le p-1$. There are clearly only a finite number of fields; actually the pairs (r,s) and $(p'-r, p-s)$ describe the same field. Belavin, Polyakov and Zamolodchikov have shown that the operator algebra of these fields is closed. Moreover, as we shall now discuss, one may construct modular invariant partition functions with the corresponding characters. These theories are called minimal. They are believed to be the only conformal theories with a finite number of primary fields, although no proof of it exists to the best of my knowledge. Notice that a subclass of these minimal theories is made by the $c<1$ unitary theories, for which $|p-p'| = 1$.

The characters of the representations occurring in minimal theories are explicitly known. They form a finite dimensional representation of the modular group. It is therefore a well posed problem to look for all possible modular invariant sesquilinear combinations of these characters, of the form (*), with the constraints that $N_{h\bar{h}}$ be a non negative integer and that $N_{00} = 1$. As the representation of the modular group afforded by the characters is unitary, an obvious solution is the diagonal combination

$$Z = \sum_h |\chi_h(q)|^2$$

This partition function describes a theory with only spinless ($h = \bar{h}$) fields. There are, however, other solutions. It turns out that they may be described in a very simple (and totally unexpected!) way. In fact each modular invariant is in correspondence with a pair of simply laced simple Lie algebras of Coxeter numbers p and p'. I recall that simply laced algebras are those Lie algebras, all the roots of which have the same length. They have been classified by Cartan in two infinite series A_n and D_n corresponding respectively to $su(n+1)$ and $so(2n)$, and three exceptional cases, denoted E_6, E_7, E_8. The Coxeter numbers take the values $n+1$, $2(n-1)$, 12, 18 and 30 respectively. The fact that the Coxeter number is even in the D and E series and that the numbers p and p' must be coprime integers imply that one of the two algebras describing the minimal theory must always be of the A type. The diagonal invariant written above actually corresponds to the (A,A). This classification by means of pairs of Lie algebras must not be misleading: it does not imply that these algebras play any role as a symmetry algebra of these theories. Actually it is the Dynkin diagrams, rather than the algebras itself, which classify the theories. The deep reason behind this classification remains still mysterious, although there exist explicit realizations of the corresponding theories as lattice models in which the Dynkin diagram plays a central role. At any rate, one has a complete classification of all minimal theories, i.e. (presumably) of all universality classes with only a finite number of primary fields. Strictly speaking, what has been classified at this stage is only the operator content (in the untwisted sector of the Hilbert space). A complete classification must also include a discussion of the couplings, the "structure constants", and possibly of the twisted sectors of the theory. The structure constants of the minimal theories have recently been completely determined and shown to be unique, given the partition function. This classification program has been also applied to other families of conformal theories with $c > 1$ and and is still actively pursued.

As a last remark, I mention that the partition functions of the minimal theories may be also expressed in terms of the Gaussian partition functions introduced in section 3.3. For the theory labelled by the pair of algebras (A, G), of Coxeter numbers p and p', it reads:

$$Z = \frac{1}{2} \sum_{\substack{n \\ \text{exponent}}} \sum_{mm'} \cos \frac{2\pi n \langle m, m' \rangle}{p'} Z_{mm'} \left(\mathrm{R} = \sqrt{\frac{p}{2p'}} \right)$$

where $\langle m, m' \rangle$ denotes the greatest common divisor of m and m' : this occurrence of the greatest common divisor may look surprising but actually guarantees the modular invariance of the previous expression. The summation on n runs over the exponents of the simply laced Lie algebra of Coxeter number p'; I recall that adding 1 to the exponents yields the degrees of the independent invariant polynomials of the algebra. This relation is not a coincidence but reflects the possible construction of the underlying lattice models by suitable modifications of the free Gaussian theory ("Coulomb gas"). We refer the reader to references quoted hereafter for a further discussion.

A FEW REFERENCES.

Several hundreds of papers have appeared over the last three years on various aspects of conformal theories. Here is a very small (and extremely biased) list of references which may be useful in connection with the topics discussed above. A review on critical phenomena and the renormalization group may be found in: K. Wilson and J. Kogut, Phys. Rep. **C12** (1974) 75. The original observation by A.M. Polyakov of the conformal symmetry of critical systems appeared in JETP Lett. **12** (1970) 381. Section 2 reviews the seminal paper by A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov in Nucl. Phys. **B241** (1984) 333. I have to a large extent followed the presentation given by J.-M. Drouffe and C. Itzykson in their book *"Statistical Field Theory"*, Cambridge University Press, to appear. The reader should also consult for a thorough discussion the lecture notes by J.L. Cardy and by P. Ginsparg at the 1988 Les Houches Summer School. The conditions for conformal invariance are discussed by J. Polshinski, Nucl. Phys. **B303** (1988) 226. The two theorems on the irreducibility and unitarity of Virasoro representations are found in V.G. Kac, in Lect. Notes in Physics **94** (1979) 441 and in B.L. Feigin and D.B. Fuchs, Funct. Anal. and Appl. **16** (1982) 114 for the first one, and in D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575 and Comm. Math. Phys. **107** (1986) 535, and in P. Goddard, A. Kent and D. Olive, Phys. Lett. **152B** (1985) 88 and Comm. Math. Phys. **103** (1986) 105 for the second. The structure of the Virasoro representations and the ensuing formulae for the characters may be read in B.L. Feigin and D.B. Fuchs, Funct. Anal. and Appl. 17 (1983) 241 and Lect. Notes in Math. 1060 (1984) 230, and in A. Rocha-Caridi, in *"Vertex Operators in Mathematics and Physics"*, J. Lepowski, S. Mandelstam and I. Singer eds, Springer (1985).

The interpretation of the central charge as the coefficient of the finite size contribution to the free energy appeared in two letters by H.W.J. Blöte, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. **56** (1986) 742, and by I. Affleck, id. **746**. The expression of the partition function in terms of characters and its modular invariance are due to J.L. Cardy, Nucl. Phys. **B270** [**FS16**] (1986) 186. The classification of the minimal theories may be found in A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. **B280** [**FS18**] (1987) 445 and Comm. Math. Phys. **113** (1987) 1 and in A. Kato, Mod. Phys. Lett. **A2** (1987) 585 and their structure constants have been determined by P. Di Francesco, Phys. Lett. to appear. The reinterpretation of these minimal models in terms of free fields has been done by P. Di Francesco, H. Saleur and J.-B. Zuber, J. Stat. Phys. **49** (1987) 57, and more details and references on this last topic may be found in my lecture notes at the 1988 Les Houches Summer School.