COURSE 3

CONFORMAL FIELD THEORIES, COULOMB GAS PICTURE AND INTEGRABLE MODELS

J.-B. ZUBER

Service de Physique Théorique, CEN Saclay
F-91191, Gif-sur-Yvette Cedex, France

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1. Introduction

These lectures aim at presenting links between some recent results of conformal field theory, the more conventional Coulomb gas picture in statistical mechanics and the approach of integrable models. What is at stake is an interpretation of a large class of currently known conformal theories in terms of suitably modified free fields: the idea that many two-dimensional critical systems derive from free fields was advocated some ten years ago by Kadanoff [1]. The other important issue concerns the connection between conformal theories and integrable systems. Beyond the dictionary that we shall sketch in these lectures, it is clear that each field has much to learn from the other's techniques.

As it has been abundantly explained by the previous lecturers, the operator content of a conformal theory is conveniently encoded in the genus-one (torus) partition function

$$Z = \sum N_{i\bar{i}} \chi_i(q)\, \chi_{\bar{i}}(\bar{q})\,. \tag{1.1}$$

$\chi(q)$ and $\chi(\bar{q})$ are characters of the left and right Virasoro algebras, of argument $q = \exp\, 2i\pi\tau$, $\bar{q}$ its complex conjugate, where $\tau = \omega_2/\omega_1$ is the modular ratio of the torus. The $N_{i\bar{i}}$ are the multiplicities, hence positive integers. A fundamental property of Z is its modular invariance (see Cardy's lectures).

2. $c = 1$ conformal theories and the 6-vertex model

The simplest example of a conformal theory is provided by a free massless boson field: its Euclidean action reads

$$A = \frac{g}{4\pi} \int d^2x\, (\partial_\mu \phi)^2 \tag{2.1}$$

whence the propagator $\langle \phi(x)\, \phi(0) \rangle = -\frac{1}{g} \log |x|$. The Virasoro algebra generated by the energy-momentum tensor $T(z) = -g\,(\partial_z \phi)^2$ has a

central charge $c = 1$, and primary fields are simply given by exponentials of ϕ ("vertex operators"):

$$\mathcal{O}_{eo}(x) = e^{ie\phi(x)} \qquad h = \bar{h} = e^2/4g. \tag{2.2}$$

To give some more flavor to this simple case, one may assume that ϕ has an angular nature: $\phi \equiv \phi + 2\pi$, in other words, one may compactify it on a circle. New scaling fields are then provided by the vortex operators $\mathcal{O}_{om}(x)$, which create a discontinuity of ϕ of $2\pi m$ around the point x. The calculation of the correlator $\langle \mathcal{O}_{om}(z,\bar{z})\, \mathcal{O}_{o,-m}(0) \rangle$ thus amounts to computing the ratio of partition functions in the plane $\tilde{Z}/Z$, both equal to $\int \mathcal{D}\phi\, e^{-A}$, but where in $\tilde{Z}$, the field ϕ has a discontinuity $2\pi m$ between 0 and z, whereas it is regular in Z. A classical solution with such a discontinuity is $\phi_{cl}(\zeta) = m\,[\mathrm{Arg}\,(\zeta - z) - \mathrm{Arg}\,\zeta]$, the action of which is easily computed to be

$$A_{cl} = -g\; m^2 \mathrm{Log}\,|z|\,. \tag{2.3}$$

The fluctuations about ϕ_{cl} contribute a factor Z, hence the correlator reads:

$$\langle \mathcal{O}_{om}(x)\, \mathcal{O}_{o,-m}(y) \rangle = \frac{1}{|x-y|^{m^2 g}} \tag{2.4}$$

hence one assigns the conformal weight $h = \bar{h} = m^2 g/4$ to $\mathcal{O}_{om}$. The mixed electric-magnetic operator $\mathcal{O}_{em}$ has

$$\left.\begin{matrix} h_{em} \\ \bar{h}_{em} \end{matrix}\right\} = \frac{1}{4}\left(\frac{e}{\sqrt{g}} \pm m\sqrt{g}\right)^2 . \tag{2.5}$$

In the language of string compactification, this corresponds to $\frac{1}{2}p_{L,R}^2$, and $r = \sqrt{g/2}$ (see Ginsparg's lectures).

On a torus, the partition function receives contributions from the various topological sectors in which ϕ winds a certain number of times around the circle as z circles along one of the periods ω_1 or ω_2:

$$\delta_1\phi = 2\pi m \qquad \delta_2\phi = 2\pi m'$$

$$Z_{mm'}(\tau, g) = \frac{\sqrt{g}}{\eta(q)\,\eta(\bar{q})\,\sqrt{\mathrm{Im}\,\tau}}\; \exp -\pi g \frac{|m' - m\tau|}{\mathrm{Im}\,\tau} \tag{2.6}$$

where $\eta(q) = q^{1/24}\prod_1^\infty(1-q^n)$. $Z_{mm'}(g)$ by itself is not modular invariant:

$$Z_{mm'}(\tau) = Z_{cm'+dm,am'+bm}\left(\frac{a\tau+b}{c\tau+d}\right) \tag{2.7}$$

and the most natural way to construct an invariant is to sum over all the sectors:

$$Z_C = \sum_{m,m'\in\mathbb{Z}} Z_{mm'}(\tau, g)\,. \tag{2.8}$$

After a Poisson transformation,

$$Z_C = \frac{1}{\eta\bar\eta}\sum_{e,m\in\mathbb{Z}} q^{h_{em}}\bar q^{\bar h_{em}} \tag{2.9}$$

displays the spectrum of conformal weights of the electro-magnetic operators (2.5). Referring to Ginsparg's lectures for a more complete treatment, let us only recall that this $c=1$ "Coulomb gas" system admits a self duality by $g \to 1/g$ which exchanges the roles of electric and magnetic excitations

$$Z_C(g) = Z_C(1/g)\,. \tag{2.10}$$

The self-dual point $g=1$ is nothing but the level 1 SU(2) Wess-Zumino-Witten model.

For arbitrary g, which critical systems of statistical mechanics are described by this simple conformal model? As the angular nature of ϕ suggests, the XY model [2], with its spin wave and vortex excitations, is a natural candidate. Actually the whole critical line of the XY model (low temperature phase) maps onto $4 \le g \le \infty$. For $g \ge 4$, the vortex operators of vorticity $|m| = 1$ have a dimension $h + \bar h \ge 2$ and become irrelevant. A duality transformation leads to a lattice model, the degrees of freedom of which are integers, with an action:

$$A = \sum_{i:j} f(|n_i - n_j|)\,. \tag{2.11}$$

Here f is a smooth function, whose precise form does not affect the critical behaviour. This is an instance of a class of solid-on-solid (SOS) models, in which the integers n_i are considered as heights of a fluctuating connected two-dimensional surface. In the continuum limit, the discrete nature of these variables is washed out and one describes the system by (2.1) for some g.

There is an integrable model behind all these variations: the six-vertex model. Its states live on the links of a square lattice and may take two values: one may represent them as arrows (fig. 1) or as states of a spin 1/2 system.

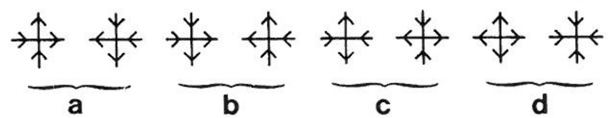

Fig. 1. The configurations of the 6- and 8-vertex models

The only allowed configurations (the first six of fig. 1) conserve the flow of arrows. Up to an overall factor the Boltzmann weights of fig. 1 may be parametrized by:

$$\begin{aligned} a &= \sin\frac{\lambda}{2}(\pi-\alpha)\\ b &= \sin\frac{\lambda}{2}(\pi+\alpha)\\ c &= \sin\lambda\pi\\ d &= 0 \end{aligned} \tag{2.12}$$

where α is a spectral parameter, which introduces an anisotropy of the weights. The model is completely integrable [3]: transfer matrices with the same value of λ and different values of α commute. The model has a critical regime with power-law decay of the correlations, when λ satisfies $0 \le \lambda \le 1$. One may argue that the continuum limit is again the Gaussian model (2.1). This may be seen by introducing a solid-on-solid model on the dual lattice according to the assignments of fig. 2. We use conventions in which the heights ϕ are integral multiples of π; ϕ is single valued on the plane, thanks to the aforementioned conservation of flow. On a doubly-periodic lattice with even numbers of rows and columns, the total winding number of ϕ may be an arbitrary integral multiple of 2π . Renormalization group arguments lead to a continuum description by (2.1), and one may identify the coupling g in the following way.

Fig. 2. SOS configuration associated with a vertex

In the 8-vertex model (which possesses all the vertices of fig. 1), the free energy is singular as

$$F \sim |d|^{1/\lambda} \tag{2.13}$$

as $d \to 0$; d plays the role of a temperature variable of scaling dimension 2λ, and the dimension of the energy operator ε coupled to it is

$$d_\varepsilon = 2 - 2\lambda. \tag{2.14}$$

On the other hand, the two right most vertices of fig. 1 create vorticity 2 with the conventions of fig. 2, and the energy operator is identified with $\mathcal{O}_{om}$. Hence

$$d_\varepsilon = \left.\frac{gm^2}{2}\right|_{m=2} \tag{2.15}$$

whence

$$g = 1 - \lambda. \tag{2.16}$$

3. $c < 1$ conformal theories

3.1. *Classification of minimal theories*

It follows from the modular invariance of the partition function Z that a finite number of Virasoro primary fields —a finite number of terms in (1.1)— is only possible when the central charge is smaller than one (see [4] or section 4.3 in Cardy's lectures). It is likely the only such theories are the so-called minimal theories [5], which have a central charge

$$c = 1 - \frac{6\,(p-p')^2}{pp'} \tag{3.1}$$

with p and p' two coprime integers, and a finite number of possible conformal weights given by Kac' formula:

$$h_{rs} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'} \qquad 1 \le r \le p'-1, \;\; 1 \le s \le p-1. \tag{3.2}$$

Only when $|p - p'| = 1$ are these theories unitary [6]. The non–unitary instances, however, may have some interest in statistical mechanics. An example is provided by the theory with $p = 2$, $p' = 5$ which describes [7] the Lee–Yang singularity, i.e. the critical behaviour of an antiferromagnetic system in an imaginary magnetic field.

The possible operator contents, i.e. the consistent choices of $(h_i, \bar{h}_i)$ among the values (3.2), may be found by a classification of all modular invariants of the form (1.1), where the explicit expression of the characters may be found in [8] and in Cardy's lectures. Because of the underlying coset construction of the $c < 1$ representations (see [9] and Ginsparg's lectures[1]), these characters have the same modular transformations as the tensor product of SU(2) Kac-Moody (KM) characters of levels p' and p. It is better to first examine the modular invariants built out of KM characters of a given level [11]. For a reason which remains mysterious, the SU(2) KM modular invariants are in one-to-one correspondence with the Dynkin diagrams of simply laced algebras: A_n, D_n, E_6, E_7, E_8 (simply laced Lie algebras have all their roots of the same length). This result follows from a two step analysis [12]. First, relaxing the condition of positivity of the coefficients N_{ij}, the general expression of the modular invariants of the form (1.1) made of KM characters of level k is derived: it is shown that there is an independent invariant associated with each factorization of the integer $k+2$ into a product of two integers. In a second step, the constraint that the coefficients N_{ij} should be non negative integers and that $N_{00} = 1$, expressing the unicity of the ground state, is proved to reduce this huge set of invariants to two infinite series and three exceptional cases, labelled by the simply laced Lie algebras.

This correspondence is most visible on the operator content of the theory (see Table 1): the primary operators for the KM algebra SU(2) of level k have an isospin j integer or half-integer satisfying $0 \le j \le k/2$. In all the modular invariant solutions, the spinless $h = \bar{h} = j\,(j+1)\,/\,(k+2)$ operators are such that the values of $\lambda = 2j+1$ re-

[1] The coset construction has been extended to non-unitary representations in [10].

Table 1

List of modular invariant partition functions in terms of SU(2) Kac–Moody characters χ_λ with $\lambda = 2j+1$.

$k \geq 0$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	(A_{k+1})
$k = 4\rho \geq 4$	$\sum_{\lambda\ \mathrm{odd}=1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$(D_{2\rho+2})$
$k = 4\rho - 2 \geq 6$	$\sum_{\lambda\ \mathrm{odd}=1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda\ \mathrm{even}=2}^{2\rho-2} (\chi_\lambda \chi^*_{4\rho-\lambda} + \mathrm{c.c.})$	$(D_{2\rho+1})$
$k = 10$	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	(E_6)
$k = 16$	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + [(\chi_3+\chi_{15})\chi_9^* + \mathrm{c.c.}]$	(E_7)
$k = 28$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	(E_8)

produce with their multiplicities all the exponents of the simply laced algebra. In particular, the Coxeter number h is equal to $k+2$. (I recall that adding 1 to the exponents gives the degrees of the independent invariant polynomials of the algebra; the exponents n_i also yield the eigenvalues of the Cartan matrix as $4\sin^2(\pi n_i/2h)$, $h =$ Coxeter number $=$ largest $n+1$: see Table 2). As the Coxeter exponents of the D_n and E algebras are even, the only modular invariant for odd k is labelled by A. It must be emphasized that in this $A-D-E$ classification, the corresponding Lie algebra is *not* a symmetry algebra of the conformal theory.

From this classification of SU(2) KM partition functions, follows that of the minimal models. The latter are classified by a pair of algebras of Coxeter numbers p and p', and since either p or p' is odd, one of the two algebras is necessarily of A type. At the possible price of interchanging p and p', and indexing (unconventionally) a Lie algebra by its largest exponent, we conclude that the set of pairs $(A_{p-1}, G_{p'-1})$ classifies all the minimal conformal theories, i.e. all the universality classes of critical phenomena with a finite number of primary fields. Moreover in the

Table 2

The Dynkin diagrams of simply laced Lie algebras, their Coxeter number and exponents.

Name of algebra	Diagram	Coxeter number	Exponents
A_N		$N+1$	$1, 2, \ldots, N$
D_N		$2(N-1)$	$1, 3, \ldots, 2N-3; N-1$
E_6		12	1,4,5,7,8,11
E_7		18	1,5,7,9,11, 13,17
E_8		30	1,7,11,13,17, 19,23,29

$(A_{p-1}, G_{p'-1})$ model, the spinless operators have conformal dimensions $h = \bar{h}$ given by (3.2) with r (resp. s) taking its values among the Coxeter exponents of G (resp. A) (see Table 3).

Strictly speaking, this classification of operator contents (the spectra) should be supplemented by a discussion of the structure constants (the couplings). It may be proved that for minimal theories, the C's are unique [13].

A detailed inspection of the minimal partition functions reveals that they may be expressed as linear combinations of Gaussian partition functions $Z_c(g)$ evaluated for different g [14]:

$$Z_{A_{p-1}, G_{p'-1}} = \frac{1}{2} \sum_\nu \epsilon_\nu Z_C \left(g = \frac{p\nu^2}{p'} \right) \tag{3.3}$$

where the sum runs over certain divisors ν of p' weighted by signs $\epsilon_\nu = \pm 1$: for $A_{p'-1}$, $\nu = p'$ or 1, with respectively $\epsilon = 1$ and -1; for $D_{p'/2+1}$, $\nu = p'$, $p'/2$, 2 or 1, with respectively $\epsilon = 1, -1, 1$ and -1; in the exceptional cases E_6, E_7 and E_8, ν runs over all the divisors of p' (=12, 18, 30) and $\epsilon_\nu = (-1)^{n_\nu}$, where n_ν counts the number of prime factors in p'/ν. In all the cases, $\sum \epsilon_\nu = 0$, which is needed to cancel the leading

Table 3
List of minimal modular invariant partition functions in terms of Virasoro characters $\chi_{r\,s}$.

	$\frac{1}{2}\sum_{s=1}^{p-1}\sum_{r=1}^{p'-1} \lvert\chi_{r\,s}\rvert^2$	$(A_{p-1}, A_{p'-1})$
$p' = 4\rho+2 \geq 6$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}=1}^{2\rho-1} \lvert\chi_{r\,s}+\chi_{4\rho+2-r\,s}\rvert^2 + 2\lvert\chi_{2\rho+1\,s}\rvert^2\right\}$	$(A_{p-1}, D_{2\rho+2})$
$p' = 4\rho \geq 8$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}=1}^{4\rho-1} \lvert\chi_{r\,s}\rvert^2 + \lvert\chi_{2\rho\,s}\rvert^2 + \sum_{r\ \mathrm{even}=2}^{2\rho-2}(\chi_{r\,s}\chi^*_{4\rho-r\,s}+\mathrm{c.c.})\right\}$	$(A_{p-1}, D_{2\rho+1})$
$p' = 12$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1\,s}+\chi_{7\,s}\rvert^2+\lvert\chi_{4\,s}+\chi_{8\,s}\rvert^2 + \lvert\chi_{5\,s}+\chi_{11\,s}\rvert^2\}$	(A_{p-1}, E_6)
$p' = 18$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1\,s}+\chi_{17\,s}\rvert^2+\lvert\chi_{5\,s}+\chi_{13\,s}\rvert^2+\lvert\chi_{7\,s}+\chi_{11\,s}\rvert^2 + \lvert\chi_{9\,s}\rvert^2+[(\chi_{3\,s}+\chi_{15\,s})\chi^*_{9\,s}+\mathrm{c.c.}]\}$	(A_{p-1}, E_7)
$p' = 30$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1\,s}+\chi_{11\,s}+\chi_{19\,s}+\chi_{29\,s}\rvert^2 + \lvert\chi_{7\,s}+\chi_{13\,s}+\chi_{17\,s}+\chi_{23\,s}\rvert^2\}$	(A_{p-1}, E_8)

singularity of (3.3) as $q \to 0$ and to expose the $c<1$ central charge. There is a remarkable identity satisfied by the exponents:

$$\sum_n \cos\frac{2\pi kn}{p'} = \sum_\nu \epsilon_\nu\, \nu\, \delta_{k,0 \bmod \nu} \tag{3.4}$$

where n is summed over the exponents of the $G_{p'-1}$ algebra. This enables us to recast (3.3) in the form [15 – 17]:

$$Z_{A_{p-1},G_{p'-1}} = \frac{1}{2}\sum_{m,m'\in Z}\sum_n \cos\frac{2\pi\langle m,m'\rangle n}{p'} Z_{mm'} \quad \left(g=\frac{p}{p'}\right). \tag{3.5}$$

There $\langle m,m'\rangle$ denotes the greatest common divisor of m and m' (by convention, $\langle m,0\rangle = m$). The different winding sectors which were summed over uniformally in (2.8), have now been given a non-trivial weight. In the following, we shall see that this strange formula may be interpreted as reflecting the introduction of defects (electric charges) related to the exponents. As this is an elaboration of earlier considerations [18 – 21], let us first review how $c<1$ theories *in the plane* may be generated from the free $c=1$ theory through the introduction of a charge at infinity.

3.2. *Feigin–Fuchs representation*

We return to the free field (2.1), add to the action the contribution of a "charge at infinity":

$$A = \frac{g}{4\pi}\int \mathrm{d}^2x\,(\partial\phi)^2 + 2ie_0\phi(R) \qquad R\to\infty \tag{3.6}$$

and compute the correlation functions of $\mathcal{O}_{e0}$ operators by:

$$\left\langle \prod_j \mathrm{e}^{ie_j\phi(x_j)}\right\rangle_{e_0} = \lim_{R\to\infty} R^{4e_0^2/g}\left\langle\prod_j \mathrm{e}^{ie_j\phi(x_j)}\mathrm{e}^{-2ie_0\phi(R)}\right\rangle_0 . \tag{3.7}$$

The only non-vanishing correlation functions have $\sum e_j = 2e_0$. In the following, the subscript e_0 will be implicit. For example, the two-point function reads:

$$\left\langle \mathrm{e}^{ie\phi(x)}\mathrm{e}^{i(2e_0-e)\phi(0)}\right\rangle = \frac{1}{|x|^{e(e-2e_0)/g}} \tag{3.8}$$

and this is interpreted as a shift of the conformal weight

$$h=\frac{e^2}{4g} \;\to\; h=\frac{e(e-2e_0)}{4g}=\frac{(e-e_0)^2-e_0^2}{4g}. \tag{3.9}$$

Correspondingly, the energy-momentum tensor and the central charge are modified according to:

$$\begin{aligned} T(z) = -g(\partial_z\phi)^2 \;&\to\; -g(\partial_z\phi)^2 + ie_0\partial_z^2\phi \\ c=1 \;&\to\; c=1-\frac{6e_0^2}{g} \end{aligned} \tag{3.10}$$

The additional term in $T(z)$ may be regarded as an "improvement" term, required by the introduction of the charge at infinity. (See Friedan's lectures for a discussion of a similar term in the BRS quantization of the bosonic string). For the minimal theories (3.1–2), one takes $e_0 = p - p'/p'$, $g = p/p'$. Moreover, upon the introduction of screening operators [19–20], it has been shown that the values of e are naturally quantized, leading to Kac' formula (3.2)

There is, however, a little subtlety in the case of non-unitary theories. They have (at least) one negative conformal weight, and the effective central charge observed in finite size effects [22] is not c but rather [23]

$$c_{\text{eff}} = c - 24h_0 \tag{3.11}$$

where h_0 is the lowest (negative) conformal weight. For the minimal theories, $h_0 = (1-(p-p')^2)/4pp'$, and

$$c_{\text{eff}} = 1 - 6/pp'. \tag{3.12}$$

This value may be interpreted as associated with an effective charge at infinity $e_{0\,\text{eff}} = 1/p'$ (keeping $g = p/p'$).

3.3. Coulomb gas picture in Statistical Mechanics

All the previous discussion has its analogue in the study of lattice models [24]. For simplicity, consider first the $O(n)$–model: the spin variables are n–vectors normalized by $\mathbf{S}^2 = n$ with nearest-neighbor interactions, and the partition function of the system reads

$$Z = \prod_i \int \mathrm{d}\mathbf{S}_i \textstyle\prod_{i:j} (1 + \beta\, \mathbf{S}_i \cdot \mathbf{S}_j)\,. \tag{3.13}$$

The lack of positivity of the "Boltzmann weight" for large β does not cause any problem in the discussion of the critical behaviour, thanks to universality. The values of $n=1$ and 2 describe the Ising and (the universality class of) the XY models. Non integer values of n, or $n=0$, obtained by extrapolation, may also be of interest [25].

In a high temperature (low β) expansion, repeated use is made of

$$\int \mathrm{d}\mathbf{S} = 1 \;\; ; \;\; \int \mathrm{d}\mathbf{S}\; S^\alpha S^\beta = \delta^{\alpha\beta}\,. \tag{3.14}$$

On a hexagonal lattice (fig. 3), these are the only relevant non vanishing integrals, in such a way that the expansion is expressed as a sum over all closed non–intersecting loops on the lattice:

$$Z = \sum_{\text{closed loops}} \beta^{\#\ \text{bonds}} n^{\#\ \text{loops}}\,. \tag{3.15}$$

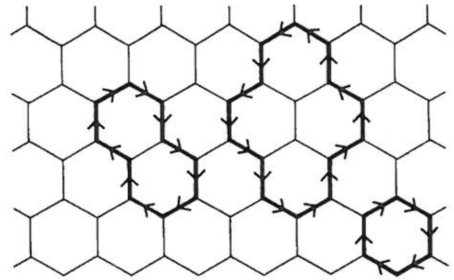

Fig. 3. High temperature expansion of the $O(n)$ model

In order to reexpress this in terms of local discrete heights, the loops are oriented in an arbitrary way. Then heights ϕ_a multiple of π are defined on the dual lattice in such a way that across a selected bond, ϕ changes by $\pm\pi$, according to the orientation of the bond. The number of bonds of the original diagram is nothing but $\frac{1}{\pi}\sum_{a:b} |\phi_a - \phi_b|$, but each loop is given the incorrect weight 2 (1 for each orientation) instead of n. This is corrected by the additional prescription of attaching the weight e^{iu} (resp. e^{-iu}) to each left (right) turn along the oriented loop, which may be easily rephrased in terms of the ϕ–configuration. Because of the simple topological property that *in the plane*, for each loop:

$$t = |\#\ \text{left turns} - \#\ \text{right turns}| = 6 \tag{3.16}$$

the correct weight n is reproduced provided

$$2\ \cos 6u = n. \tag{3.17}$$

Finally, in the critical limit, this SOS model is expected by renormalization group arguments to flow to the Gaussian model. A more careful analysis yields the value of g, as a function of n [24]:

$$n = -2\ \cos \pi g \qquad g \in [1,2]\,. \tag{3.18}$$

If we want to repeat this discussion on a cylinder (which is a good way to get access to critical exponents through the study of finite size effects: cf. Cardy's lectures), we immediately encounter a problem: the relation (3.16) does not hold in general; for a non intersecting non contractible loop we find $t = 0$. This is why adding charges at infinity is required. We imagine that at the two ends of the cylinder, denoted $\pm\infty$, charges

$\pm e_0$ have been added. Computations should be now carried out with a functional integral of the form:

$$\int \mathcal{D}\phi \ \mathrm{e}^{-A(\phi)-ie_0(\phi(\infty)-\phi(-\infty))}. \tag{3.19}$$

As any non-contractible loop contributes $\phi(\infty)-\phi(-\infty)=\pm\pi$ depending on its orientation, the correct weight n is recovered provided

$$2\cos\pi e_0 = n. \tag{3.20}$$

For an arbitrary number $\mathcal{N}$ of non-contractible loops of orientation specified by $\epsilon_i=\pm1$, $i=1,\ldots,\mathcal{N}$, one gets a contribution $\cos\pi e_0\sum\epsilon_i$ and after summing over the orientations, one recovers the correct weight

$$\sum_{\epsilon_i}\cos\pi e_0\sum\epsilon_i = n^{\mathcal{N}}. \tag{3.21}$$

It may be shown [22] that this introduction of charges at $\pm\infty$ shifts the ground state energy of the system, and this "Casimir effect" reflects the change in the central charge

$$c=1 \quad \rightarrow \quad c=1-\frac{6e_0^2}{g} \tag{3.22}$$

where e_0 denotes the solution of (3.20) of the form

$$e_0 = g-1. \tag{3.23}$$

For example, for the $n=1$ (Ising) model, $g=4/3$, $e_0=1/3, c=1/2$.

3.4. *Floating charges on a torus*

If the cylinder is closed into a torus by identification of the two ends, we have again to ensure that non-contractible loops are correctly weighted. Consider such a loop $\mathcal{C}$, which is wrapped n_1 times around the period ω_1, n_2 times around ω_2, and thus imposes on ϕ the discontinuities $\delta_1\phi=\pi n_1$, $\delta_2\phi=\pi n_2$. (Remember that at this stage, the ϕ's are integral multiples of π). By a unimodular change of basis, it is always possible to change these discontinuities to $\delta_1\phi=\langle n_1,n_2\rangle\pi$. Indeed let $n_1=\langle n_1,n_2\rangle\nu_1$, $n_2=\langle n_1,n_2\rangle\nu_2$; as ν_1 and ν_2 are coprimes, there exist two integers a and b such that $a\nu_1+b\nu_2=1$, hence $an_1+bn_2=\langle n_1,n_2\rangle$ and one may construct the desired modular transformation as

$$\begin{pmatrix} a & b \\ -\nu_2 & \nu_1 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \langle n_1,n_2\rangle \\ 0 \end{pmatrix}. \tag{3.24}$$

In this new basis, the fact that $\mathcal{C}$ must be non-intersecting implies $\langle n_1,n_2\rangle=\pm1$. Moreover, any other non-contractible loop $\mathcal{C}'$ does not intersect $\mathcal{C}$ and must be homotopic to it. For $\mathcal{N}$ such homotopic loops of orientations $\epsilon_i=\pm1$, the total discontinuity of ϕ in the new basis is $(\sum\epsilon_i\pi,0)$, and these are the configurations which have been given the weight $\cos\pi e_0\sum\epsilon_i$ in (3.21). Returning to the original basis, we see that a configuration with discontinuities (n_1,n_2) must be given the weight $\cos\pi e_0\langle n_1,n_2\rangle$. If we change the normalization of the field ϕ to make its discontinuities integral multiples of 2π as in section 2, we have to change g into $g/4$ and e_0 into $e_0/2$. The partition function is the sum of contributions of winding sectors $Z_{m,m'}$ weighted by $\cos\pi e_0\langle m,m'\rangle$

$$Z=\sum_{m,m'\in\mathcal{Z}}\cos\left(2\pi\frac{e_0}{2}\langle m,m'\rangle\right)Z_{m,m'}(g/4). \tag{3.25}$$

Together with eqs. (3.20), (3.23), this gives the expression of the partition function of the $O(n)$-model. On the torus, the identification of e_0 as the charge of a localized additional charge is lost, and we shall refer to e_0 as a "floating charge". The important feature is that e_0 is coupled to dual magnetic variables m,m' by the interaction $\cos\ 2\pi\frac{e_0}{2}\langle m,m'\rangle$.

For the Ising model, which is the only minimal $O(n)$ model, we have the alternative expressions:

$$\begin{aligned} Z &= \sum_{m,m'}\cos\left(\frac{2\pi}{6}\langle m,m'\rangle\right)Z_{m,m'}\left(\frac{1}{3}\right) \\ &= \sum_{m,m'}\cos\left(\frac{2\pi}{3}\langle m,m'\rangle\right)Z_{m,m'}\left(\frac{4}{3}\right) \\ &= \frac{1}{2}\sum_{m,m'}\sum_{n=1,2}\cos\left(\frac{2\pi n}{3}\langle m,m'\rangle\right)Z_{m,m'}\left(\frac{4}{3}\right) \end{aligned} \quad . \tag{3.26}$$

Going from the first to the second form requires some algebra and makes use of the particular value of e_0 and g. In the later form, it generalizes to all minimal models: this is eq. (3.5). The interpretation of the exponents of $G_{p'-1}$ is that they give the system of "floating charges" n/p' to be coupled to the system. The smallest one, $e_0 = 1/p'$, is the only one needed on the cylinder, or on the plane for the correlation functions.

More generally, let us consider a partition function of the form

$$Z = \sum_{m,m' \in Z} f(m,m')\, Z_{mm'}(\tau, g) \tag{3.27}$$

with some weight f and examine the constraints on f to make Z modular invariant. For a large class of conformal theories (all rational theories?) only a subgroup Γ' of finite order $\mathcal{N}$ of the modular group acts non trivially on sesquilinear forms in the characters. This is indeed true for the Kac-Moody characters [26] and hence for all Virasoro characters obtained by some G/H coset construction. This is also the case for the Gaussian model, at any rational value of g (see for example [27]). In such a case, we may write, using (2.7):

$$\begin{aligned} Z &= \frac{1}{\mathcal{N}} \sum_{\gamma\in\Gamma'} \sum_{m,m'} f(m,m')\, Z_{m,m'}({}^{\gamma}\tau, g) \\ &= \frac{1}{\mathcal{N}} \sum_{\gamma\in\Gamma'} \sum_{m,m'} f(m,m')\, Z_{\gamma m,\gamma m'}(\tau, g) \\ &= \frac{1}{\mathcal{N}} \sum_{\gamma\in\Gamma'} \sum_{m,m'} f(\gamma m,\gamma m')\, Z_{m,m'}(\tau, g) \\ &= \sum_{\text{orbits of } \Gamma} f(\text{orbit}) \sum_{m,m'\in\text{orbit}} Z_{m,m'}(\tau, g) \end{aligned} \tag{3.28}$$

By the same argument as in (3.24), one may show [28] that the orbits of the pairs (m,m') of winding numbers under the modular group are characterized by $\langle m,m' \rangle$. Therefore

$$Z = \sum_{m,m' \in Z} f(\langle m,m' \rangle)\, Z_{m,m'}(\tau, g) \tag{3.29}$$

is the most general partition function of the form (3.27) consistent with modular invariance. Equations (3.26) and (3.5) are thus particular realizations of this form. Let us finally mention that the straightforward generalization of (3.29) to higher genus is inconsistent with the expected property of factorization. For example in genus $g = 2$, one would introduce in (3.27) a weight f depending on the winding numbers

m_1, m_2, m_1', m_2' in some homology basis. By the same argument as before one may show that f should depend only on the greatest common divisor of m_1, m_2, m_1', m_2'. However the only function f consistent with factorization properties when the surface pinches:

$$f(\langle m_1, m_2, m_1', m_2' \rangle) \to f_1(\langle m_1, m_1' \rangle) f_2(\langle m_2, m_2' \rangle)$$

is trivial: f = constant. What is presumably missing is the correct incorporation of "screening operators "[20] in that picture.

3.5. Restricted Solid-on-Solid models

There is again an integrable model behind each minimal model. Baxter [29] has shown that the 8-vertex model can be reformulated as a solid-on-solid model on the dual lattice. Heights, which vary by ± 1 from site to site, have "interactions round a face" (IRF). Their Boltzmann weights are parametrized in terms of elliptic Θ–functions. This model has a critical regime in which it reduces to the 6-vertex model, (but the latter SOS formulation differs from that of fig. 2). When λ takes the rational value $\lambda = 1/(N+1)$, the heights may be consistently restricted to the finite set $\ell = 1, 2, \ldots, N$ [30]. One may also regard the states of this restricted solid-on-solid (RSOS) system as attached to the nodes of the A_N Dynkin diagram, neighbouring sites of the lattice occupying neighbouring states on the diagram. Depending on the value of N, other restrictions may be possible, in which the states of the system are attached to a D or E Dynkin diagram of Coxeter number $N+1$ [31]. This will be described in more detail in the next chapter. The continuum limit of these models is what was denoted above as the (A_{N-1}, G_N) minimal theory. By allowing vacancies in the RSOS system, one may also generate the (G_N, A_{N+1}) theory. For example, the two models attached to the A_3 Dynkin diagrams are the Ising (A_2, A_3) and tricritical Ising (A_3, A_4) models. As every second site (even site, say) of the lattice occupies necessarily the middle state of the A_3 Dynkin diagram, the configurations of the odd sites are indeed of Ising type.

By choosing other rational values of $\lambda = R/(N+1)$, the non-unitary minimal models (at least some of them) may also be given an integrable RSOS realization. The partition functions of all these integrable models on a torus have been computed and identified with (3.5) [15].

4. Generalized height models

4.1. Yang-Baxter equation and Temperley-Lieb algebra

We consider a model on a square lattice, the degrees of freedom of which live on sites i. In the simple RSOS model defined in section (3.5), these "heights" ℓ_i are bounded integers, but we consider more generally that the ℓ's belong to some finite set $\mathcal{L}$. We shall later impose suitable restrictions on the allowed configurations $i \to \ell_i$. These heights are assumed to interact through interactions–round–a–face, around each plaquette of the lattice. The statistical model is thus defined by the Boltzmann weights

$$w\left(\ell_1, \ell_2, \ell_3, \ell_4\right) = {}^{\ell_4}_{\ell_1}\square^{\ell_3}_{\ell_2}$$

In particular, the partition function reads

$$Z = \sum_{\substack{\text{configurations} \\ \{\ell_i\}}} \prod_{\substack{\text{plaquettes} \\ (ijk\ell)}} w\left(\ell_i, \ell_j, \ell_k, \ell_\ell\right) \tag{4.1}$$

with some boundary conditions, etc... As usual in this context, it is convenient to reformulate things in terms of a row-to-row transfer matrix:

$$V_{\ell,\ell'} = \prod_{i+1}^{N} w\left(\ell_i, \ell_{i+1}, \ell'_{i+1}, \ell'_i\right) \tag{4.2}$$

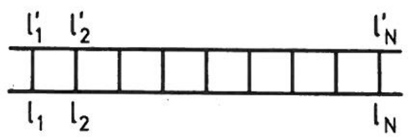

Fig. 4. Row-to-row transfer matrix

We use for definiteness periodic boundary conditions : $\ell_{N+1} \equiv \ell_1$. Then for a M–row lattice with periodic boundary conditions along the "time" direction:

$$Z = \operatorname{tr} V^M . \tag{4.3}$$

It is very important to actually find a one-parameter family of weights $w\left(\ell_1, \ell_2, \ell_3, \ell_4 \mid u\right)$ and corresponding transfer matrices $V(u)$, in such a way that

$$V(u)\, V(u') = V(u')\, V(u) . \tag{4.4}$$

I only recall here that these commutation properties enable one to use a Bethe Ansatz or the quantum inverse scattering method to diagonalize V and solve the model [3,32].

In turn, a sufficient condition for (4.4) to hold is the celebrated Yang-Baxter equation on the $w's$. To see this, rewrite $V(u)\, V(u')$ in a matrix form:

$$\begin{aligned}(VV')_{\ell,\ell'} &= \sum_{\ell''} \prod_{i=1}^{N} w\left(\ell_i, \ell_{i+1}, \ell''_{i+1}, \ell''_i\right) w'\left(\ell''_i, \ell''_{i+1}, \ell'_{i+1}, \ell'_i\right) \\ &= \operatorname{tr} \prod_{1}^{N} T\left(\ell_i, \ell'_i; \ell_{i+1}, \ell'_{i+1}\right)\end{aligned} \tag{4.5}$$

with the matrix $T\left(\ell_i, \ell'_i; \ell_{i+1}, \ell'_{i+1}\right)$ defined by:

$$\left[T\left(\ell_i, \ell'_i; \ell_{i+1}, \ell'_{i+1}\right)\right]_{\ell''_i \ell''_{i+1}} = w\left(\ell_i, \ell_{i+1}, \ell''_{i+1}, \ell''_i\right) w'\left(\ell''_i, \ell''_{i+1}, \ell'_{i+1}, \ell'_i\right) . \tag{4.6}$$

Call T' the similar matrix associated with $V'V$. Then the desired commutation holds provided there exists a matrix $U(\ell, \ell')$ such that:

$$T(\ell, \ell'; k, k') = U(\ell, \ell')\, T'(\ell, \ell'; k, k')\, U^{-1}(k, k') \tag{4.7}$$

or

$$TU = UT'$$

for any allowed configuration. It suffices to take $U(\ell, \ell')$ as given itself by the Boltzmann weight $w(\ldots| u'') = w''$ for some other value of the spectral parameter u'':

$$U(\ell, \ell')_{\ell'' k''} = w\left(\ell'', \ell, k'', \ell' \,|u''\right) \equiv w''\left(\ell'', \ell, k'', \ell'\right) \tag{4.8}$$

provided the $w's$ satisfy the Yang-Baxter equation:

$$\begin{aligned}&\sum_{k''} w\left(\ell_i, \ell_{i+1}, k'', \ell''_i\right) w'\left(\ell''_i, k'', \ell'_{i+1}, \ell'_i\right) w''\left(k'', \ell_{i+1}, \ell''_{i+1}, \ell'_{i+1}\right) \\ &= \sum_{k''} w''\left(\ell''_i, \ell_i, k'', \ell'_i\right) w'\left(\ell_i, \ell_{i+1}, \ell''_{i+1}, k''\right) w\left(k'', \ell''_{i+1}, \ell'_{i+1}, \ell'_i\right) .\end{aligned} \tag{4.9}$$

Diagrammatically, this may be represented as:

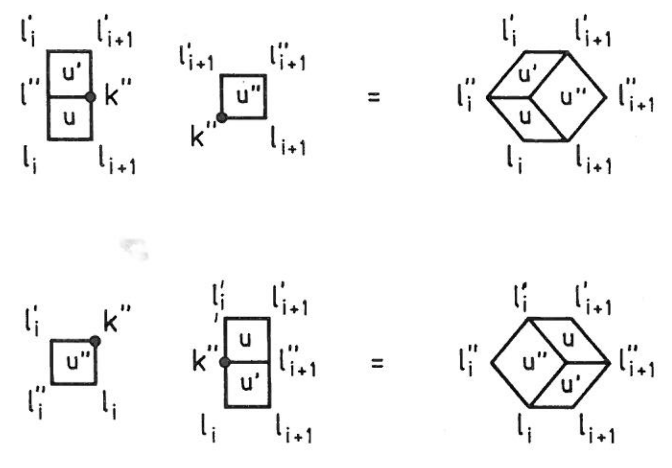

Fig. 5. The Yang-Baxter equation

The spectral parameter u'' depends on u and u' and we assume that there exists a parametrization such that it takes the form

$$u'' = u' - u. \tag{4.10}$$

For the next step, it is useful to change our standpoint and to consider "fixed time" states attached to the first diagonal of the lattice. Alternatively, one rotates the lattice by 45°, attach configurations to the first "row" and define the matrix X_i ("face transfer matrix") between adjacent configurations (see fig. 6):

$$(X_i)_{\ell,\ell'} = w\,(\ell_i, \ell_{i+1}, \ell'_i, \ell_{i-1}) \prod_{j\neq i} \delta_{\ell_j,\ell'_j}. \tag{4.11}$$

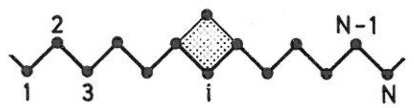

Fig. 6. Face transfer matrix

It is easy to see that the Yang-Baxter condition amounts to (fig. 7)

$$X_{i+1}(u'-u)\, X_i(u')\, X_{i+1}(u) = X_i(u)\, X_{i+1}(u')\, X_i(u'-u)\,. \tag{4.12}$$

Fig. 7. Commutation of face transfer matrices

In the q–state Potts model, one finds a realization of these matrices in the form [3]

$$X_i(u) = \rho\,[\sin\pi(\lambda-u)\,\mathbf{1} + \sin\pi u\; U_i] \tag{4.13}$$

where ρ is some overall normalization, λ depends on q by $q^{1/2} = 2\cos(\lambda\pi)$ and the U's are explicit matrices. This suggests to look for general solutions of Y.B. (4.12) of the previous form. As a simple exercice of trigonometry reveals, (4.12) is satisfied provided the $U's$ satisfy the Temperley-Lieb algebra:

$$U_i^2 = q^{1/2} U_i \tag{4.14a}$$

$$U_i U_{i\pm1} U_i = U_i \tag{4.14b}$$

$$U_i U_j = U_j U_i \qquad \text{if} \quad |i-j| \geq 2. \tag{4.14c}$$

Moreover the calculation of the partition function requires the knowledge of traces. One imposes again the same expression as in the Potts model

$$\mathrm{tr}\,(U_{i_1} U_{i_2} \cdots U_{i_n}) = q^{-n/2}\,\mathrm{tr}\mathbf{1} \qquad i_1 < i_2 < \cdots < i_n. \tag{4.15}$$

4.2. Models attached to graphs

We now restrict ourselves to height models in which the allowed configurations are specified by nearest-neighbour relations on a graph G: each height $\ell \in \mathcal{L}$ is associated with a node of the graph, and neighbouring sites of the lattice must occupy states which are neighbours on the graph. In other words, the incidence (or adjacency) matrix of the graph:

$$A_{ab} \quad \begin{aligned} &= 1 \quad \text{if nodes } a \text{ and } b \text{ are linked} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

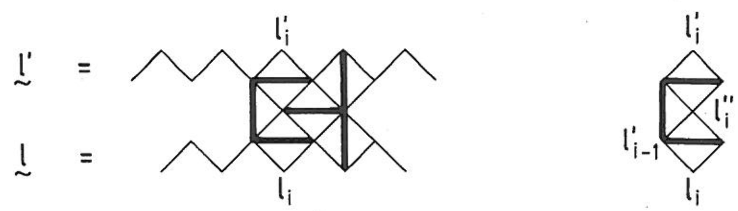

Fig. 8. Graphical interpretation of eqs. (4.14b) and (4.14a)

describes the hopping from site to site. For simplicity we consider only connected unoriented graphs (A symmetric), with no multiple lines ($A_{ij}=0$ or 1) nor tadpole ($A_{ii}=0$) and assume that the graph may be coloured with 2 colours. This means that sublattices of even (resp. odd) sites occupy states of a given colour. Note that in this language, the RSOS model ($1 \le \ell \le N$) is described by the A_N Dynkin diagram.

Let us look for realizations of the algebra (4.14) of the following form:

$$(U_i)_{\ell,\ell'} = \prod_{j\neq i} \delta_{\ell_j,\ell'_j} \left(\frac{S_{\ell_i}S_{\ell'_i}}{S_{\ell_{i-1}}S_{\ell_{i+1}}}\right)^{1/2} \delta_{\ell_{i-1},\ell_{i+1}} \tag{4.16}$$

for allowed configurations. The S_ℓ remain to be determined. Notice the last Kronecker delta function in (4.16). It implies that in a cluster expansion, choosing **1** (resp. U_i) in (4.13) identifies $\ell_i = \ell'_i$ (resp. $\ell_{i-1} = \ell_{i+1}$):

$$X_i \propto \sin(\lambda - u)\pi \quad \cdots \quad + \sin \pi u \quad \cdots \quad . \tag{4.17}$$

Equation (4.14c) is automatically satisfied by (4.16), eq. (4.14b) may be most easily verified by a graphical method (fig. 8):

$U_iU_{i+1}U_i$ identifies $\ell_{i-1} = \ell'_{i-1} = \ell_{i+1} = \ell'_{i+1}$ and $\ell_j = \ell'_j$ *for* $j \neq i$

Finally, (4.14a) is satisfied provided

$$\sum_{\ell''_i} \frac{\left(S_{\ell_i}S_{\ell''_i}\right)^{1/2}}{S_{\ell_{i-1}}} \frac{\left(S_{\ell''_i}S_{\ell'_i}\right)^{1/2}}{S_{\ell_{i-1}}} = q^{1/2} \frac{\left(S_{\ell_i}S_{\ell'_i}\right)^{1/2}}{S_{\ell_{i-1}}}$$

$$\text{i.e.} \qquad \sum_{\ell''_i} S_{\ell''_i} = q^{1/2} S_{\ell_{i-1}} . \tag{4.18}$$

The sum runs over states ℓ''_i which are neighbours of ℓ_{i-1} on the graph. Condition (4.18) means that S_ℓ is an eigenvector of the incidence matrix of eigenvalue $q^{1/2}$.

Restrictions on this incidence matrix and on the height model come from two requirements:

1) the model defined by (4.13 – 16) may be argued to have the same thermodynamic limit as the q-state Potts model. As is well known, the latter has a second-order phase transition only for $q \le 4$ and we impose that condition.

2) Boltzmann weights should be positive, as least for unitary theories. This implies that the $S's$ must be all positive and determines $\{S_a\}$ as the eigenvector of A of largest eigenvalue $q^{1/2}$ (Perron-Frobenius theorem). Therefore the graph should be such that its incidence matrix has its largest eigenvalue $q^{1/2}$ less or equal to 2. It is well known that this leads to Dynkin diagrams of ordinary $(q^{1/2}<2)$ or extended $(q^{1/2}=2)$ simply-laced Lie algebras. In the conformal field theoretic language, the former correspond to $c<1$ unitary theories, the latter associated with $\hat{A}$, $\hat{D}$, $\hat{E}$ diagrams correspond to special $c=1$ theories respectively on the Gaussian line, on the orbifold line or isolated [33]. This is established [15] by computing the partition functions of these models and making contact with eq. (3.5) (the latter equation also applies to the $\hat{A}$, $\hat{D}$, $\hat{E}$ theories). One performs a cluster expansion of the lattice, following the observation made in (4.17). Adjacent clusters correspond to adjacent heights on the Dynkin diagram G. On the plane, the adjacency graph *of these clusters* is a tree, while it contains exactly one cycle on a toroïdal lattice. This enables one to show that on the plane any configuration contributes $q^{1/2(\#\text{clusters})}$, just as the q-state Potts model and hence justifies the above statement that the new models have the same thermodynamic limit as the q-state Potts model. On the torus, on the other hand, a cycle of N clusters ($N \ge 1$) contributes $\mathrm{tr}A^N$ and as the eigenvalues of A are $2\cos n\pi/h$ where n runs over the exponents and h is the Coxeter number, this is nothing but:

$$\mathrm{tr}\, A^N = \sum_n \left(2\cos\frac{n\pi}{h}\right)^N . \tag{4.19}$$

The idea is to match this expression against that corresponding to the $\hat{A}_{2h-1}$ extended algebra, the Dynkin diagram of which is a cycle of $2h$ points, corresponding to a periodic SOS model of period $2h$. We denote $\hat{A}$ the adjacency matrix of $\hat{A}_{2h-1}$: its eigenvectors are

$$\psi_k^{(\nu)} = e^{i2\pi k\nu/2h} \qquad \nu = 0,1,\dots,2h-1,$$

of eigenvalue $2\cos 2\pi\nu/2h$ and we have

$$\mathrm{tr}\ \hat{A}^N = \sum_{\nu=0}^{2h-1}\left(2\cos\frac{\nu\pi}{h}\right)^N = 2\sum_{\nu=0}^{h-1}\left(2\cos\frac{\nu\pi}{h}\right)^N . \tag{4.20}$$

This cannot match (4.19) for all even N unless one introduces a projector of ν over the appropriate exponents:

$$\left(\hat{A}^N\right)_{\ell k} = \left(\hat{A}^N\right)_{k\ell} = \frac{1}{2h}\sum_{\nu=0}^{2h-1}\left(2\cos\frac{\nu\pi}{h}\right)^N \psi_\ell^{(\nu)}\psi_k^{(\nu)*} \tag{4.21}$$

$$P_{k\ell} = \frac{1}{2h}\sum_{\substack{\nu\ exponent\\ of\ G}} \psi_k^{(\nu)}\psi_\ell^{(\nu)*} = \frac{1}{2h}\sum_\nu e^{2i\pi\frac{\nu(k-\ell)}{2h}} . \tag{4.22}$$

In the D_{even} case, the sums run over both $\nu = h/2$ and $3h/2$, to account for the multiplicity of the exponent $h/2$. Then

$$trA^N = tr\hat{A}^N P = \frac{1}{2h}\sum_{n\,\mathrm{exponent}}\ \sum_{k,l}\left(\hat{A}^N\right)_{lk} e^{2i\pi\frac{n(k-\ell)}{2h}} . \tag{4.23}$$

This formula is the discrete equivalent of (3.5). The difference $(k-l)$ in (4.23) represents the discontinuity of the height of the periodic SOS model along some non contractible loop. In the continuum limit, each term in the sum (4.23) contributes to $Z_{m,m'}$ with $\langle m,m'\rangle \propto k-\ell$, and the projector P leads to $\sum_n \cos 2\pi n\,\langle m,m'\rangle/h$. Notice that in the form (4.23), the interpretation of the exponents as background charges coupled to the discontinuity of the height is made clear.

It remains to determine the value of g in $Z_{mm'}(g)$ as $g = 1 \pm \frac{1}{h}$ depending whether one takes the critical or tricritical version of the model (or $g=1$ for $c=1$ $\hat{A},\hat{D},\hat{E}$ models). In that way, one recovers the formula (3.5) for unitary theories ($|p-p'|=1$). The non unitary minimal models —at least those with $|p-p'|<p'$— may also be constructed if one uses for S an eigenvector of the matrix A relative to the $G_{p'-1}$ algebra different from the Perron-Frobenius eigenvector: some of the components are then necessarily negative and yield imaginary Boltzmann weights in (4.16).

5. Higher spin models

We have described so far the simplest possible situation, made of a single free field in the conformal language, or of a 2-state system (i.e. a spin $\frac{1}{2}$ SU(2) representation) in the 6-vertex picture. The chain of equivalences:

minimal conformal theories $\leftrightarrow$ Coulomb gas with

electric floating charges $\leftrightarrow$ RSOS models,

extends to systems with more degrees of freedom. It may actually be generalized in two directions:
- higher spin representations of SU(2)
- representations of higher rank simple algebras

although the results are not as complete and systematic as in the former case. We shall describe briefly only the first generalization.

5.1. Higher vertex models and their critical line

In the same way as we discussed the free Gaussian model, its critical line and its 6-vertex realizations, let us first introduce the unrestricted higher spin models. From the 6-vertex model, considered as a spin 1/2 system, higher order integrable models may be generated by the so-called fusion procedure, i.e. by tensoring and projecting onto an irreducible spin $\ell/2$ representation. Only configurations compatible with a current conservation at each lattice node are allowed: this leaves us with $\Gamma_\ell = 19$, 44,... vertices for $\ell = 2,3,\dots$. Their Boltzmann weights have been computed [34]; as in (2.12), they depend on a spectral parameter and a variable λ.

For the description of their critical limit, one has to appeal to new ingredients, namely the $\mathcal{Z}_\ell$ parafermion theories [35]. The latter, which include for $\ell=2$ and 3 the Ising and 3-state Potts model have a central charge equal to

$$c_\ell = \frac{2(\ell-1)}{\ell+2} . \tag{5.1}$$

In general, these models possess $\ell-1$ independent order parameters σ_a, transforming as the a–th representation of $\mathcal{Z}_\ell$. One may therefore impose on σ_1 $\mathcal{Z}_\ell$–twisted boundary conditions on a torus (and the boundary conditions on the others follow: $\sigma_a \sim \sigma_1^{\otimes a}$)

$$\begin{aligned}\sigma_1(z+1) &= e^{2i\pi r/\ell}\sigma_1(z)\\ \sigma_1(z+\tau) &= e^{2i\pi s/\ell}\sigma_1(z)\end{aligned}\qquad r,s = 1,2,\dots,\ell-1 . \tag{5.2}$$

Let $Z_\ell(r,s)$ denote the partition function in the corresponding sector.

As the relevant representations of the Virasoro algebra are derived from the coset $\mathrm{SU}(2)_\ell/U(1)$, their characters are nothing but the SU(2) Kac-Moody string functions c^ℓ_m times the Dedekind function η, and the possible partition functions $\mathcal{Z}^{(G)}_\ell(r,s)$ are classified in the same way as the SU(2)-current algebra theories, and labelled by a simply laced algebra $G_{\ell+1}$[28].

We form the modular invariant partition function:

$$Z^{(G)}_{\ell.C}(g) = \sum_{r,s\in Z_\ell} Z^{(G)}_\ell(r,s) \sum_{\substack{m=r \bmod \ell \\ m'=s \bmod \ell}} Z_{mm'}(g)\,. \qquad (5.3)$$

It describes this parafermionic theory coupled to a free boson through boundary conditions. The central charge is

$$c = 1 + c_\ell = \frac{3\ell}{\ell+2}. \qquad (5.4)$$

Moreover, $Z_{\ell.C}$ enjoys a self-duality property:

$$Z_{\ell.C}(g) = Z_{\ell.C}\left(\frac{1}{\ell^2 g}\right) \qquad (5.5)$$

and one may show that at the self-dual point $g = 1/\ell$, it reduces to the level ℓ SU(2) current algebra theory

$$Z^{(G)}_{\mathrm{SU}(2)_\ell} = Z^{(G)}_{\ell.C}\left(g = \frac{1}{\ell}\right). \qquad (5.6)$$

The claim [17] is that the continuum limit of the spin $\frac{\ell}{2}$ integrable model is described by $Z^{(A)}_{\ell.C}(g)$, with the identification

$$0 \le \lambda = \frac{1}{\ell} - g \le \frac{1}{\ell}. \qquad (5.7)$$

This is supported by

1) non-abelian bosonization arguments which relate the isotropic point $\lambda = 0$ with the level ℓ SU(2) Wess–Zumino –Witten theory [36].
2) numerical transfer matrix calculation (for $\ell = 2$)[17]
3) Bethe Ansatz computations [37].

Presumably, the other $Z^{(G)}_{\ell.C}$, $G = D$ or E, partition functions are relevant for other choices of boundary conditions of the spin $\ell/2$ model. For $l = 2$, these models have $c = 3/2$ and are superconformal: the pattern of the corresponding continuous lines and isolated points is discussed in [38].

5.2. *Higher* SU(2) × SU(2)/SU(2) *coset models*

The Virasoro representations that occurred in minimal unitary theories may be obtained by the coset construction $\mathrm{SU}(2)_k \times \mathrm{SU}(2)_1/\mathrm{SU}(2)_{k+1}$, with the identification $p = k+2$, $p' = k+3$. This suggests to consider the conformal field theories built with representations derived from the coset

$$\mathrm{SU}(2)_k \times \mathrm{SU}(2)_\ell \;/\; \mathrm{SU}(2)_{k+\ell}$$

Their central charge reads

$$\begin{aligned} c &= \frac{3k}{k+2} + \frac{3\ell}{\ell+2} - \frac{3(k+\ell)}{k+\ell+2} \\ &= \frac{2(\ell-1)}{\ell+2} + 1 - \frac{6\ell}{(k+2)(k+\ell+2)} \end{aligned} \,. \qquad (5.8)$$

When compared to (5.1), expression (5.8) suggests that the system may be described by the coupling of a parafermionic and a free boson theory, in the presence of some floating charges [39].

This appears clearly on the expressions of the partition functions of these models. With any triplet of simply laced algebras G, G', G'' of respective Coxeter numbers $k+2$, $\ell+2$, $k+\ell+2$ one may associate a modular invariant partition function built out of the branching functions of the coset $\mathrm{SU}(2)_k \times \mathrm{SU}(2)_\ell/\mathrm{SU}(2)_{k+\ell}$, contracted with the matrices relative to G, G', G'' [40]

$$Z = \sum b^p_{ij}(q)\, b^{\bar p}_{\bar i \bar j}(\bar q)\, N^{(G)}_{i\bar i} N^{(G')}_{j\bar j} N^{(G'')}_{p\bar p}\,. \qquad (5.9)$$

Notice, however, that this labelling has not been proved to be exhaustive, but for $\ell = 1$ (minimal) and $\ell = 2$ (super minimal) [41]. It may also be redundant: for example, for $\ell = 2$, G' is always of the A–type and may be dropped, and the only pairs (G, G'') which occur are up to permutation (A,A), (A,D), (A,E), and (D,E_6).

If G is of the A type, the partition function of the coset theory may be recast as:

$$Z = \sum_{r,s\in\mathcal{Z}_\ell} Z_\ell^{(G')}(r,s) \sum_{\substack{m=r\,(\mathrm{mod}\,\ell)\\ m'=s\,(\mathrm{mod}\,\ell)}} Z_{mm'}\left(g=\frac{k+2}{\ell(k+\ell+2)}\right) \sum_{n=\text{exponent of } G''} \cos 2\pi\frac{n}{k+\ell+2}\langle m,m'\rangle \tag{5.10}$$

Equation (5.10) shows that the new coset models are obtained from the continuous set of theories (5.3) by the same modification as the minimal theories from the Gaussian one.

This reflects once again similar patterns in the underlying integrable models. The latter have been constructed by fusion of the $c<1$ RSOS models [42]. The previous discussion shows that they may also be obtained by restriction of the fused vertex model at a rational value of $\lambda = 1/(k+2)$. The fact that some modular invariant partition functions escape this framework (e.g. for $\ell = 2$, the superminimal models (D, E_6)) shows that there are still a few subtleties in these restriction and fusion operations which await to be mastered.

6. Conclusion

We have seen that families of conformal theories related by the coset construction to the SU(2) KM algebra may be regarded as obtained from some free field (possibly supplemented by parafermions) modified by the coupling of its winding numbers to floating charges. This representation has been shown to reflect the procedure of restriction of the corresponding integrable lattice models. The whole discussion may be generalized to models based on the coset construction with higher rank algebras. The corresponding integrable models have been identified [43]. In the conformal field description, generalized parafermions appear [44], and are coupled to free fields living on a higher-dimensional torus [16]. Notice, however, that the analysis is not as exhaustive as in the SU(2) case: all the various restrictions have not been identified, nor the modular invariants completely classified.

As for the parafermions of eq. (5.1), their own bosonization is possible using $(\ell-1)$–component free fields in the vertex operator construction of [45]. Alternatively, they may be obtained by the $\mathrm{SU}(\ell)_1\times\mathrm{SU}(\ell)_1/\mathrm{SU}(\ell)_2$ coset construction, which admits itself a Coulomb gas description, as just mentioned.

At the term of this discussion, it seems that we can reproduce (essentially) all the known discrete series in terms of free fields with appropriate modifications and couplings of their boundary conditions. Two questions arise naturally:

- how to generalize this picture to higher genus partition functions or (more or less equivalently) to genus $g \geq 1$ correlation functions?
- is this a general feature or have we seen so far only the tip of the iceberg of the two-dimensional conformal field theories?

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