Old and New Topics in Conformal Field Theory

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1. Introduction

These notes reflect the structure of the lectures given at the Kathmandu Summer School. They are made of two parts: the first is intended to be an elementary (and standard) introduction to conformal field theory, following the approach of Belavin, Polyakov and Zamolodchikov [1], together with a short and biaised review of some significant results. For the sake of brevity, I shall not provide detailed references in that part. The interested reader is referred to the lecture notes of Cardy and Ginsparg [2] or to the collected reprints of [3]. The second part presents some recent developments on some relations between c.f.t. and classical integrable systems (of KdV type), the so-called W-algebras and related results on the structure of singular vectors.

Conformal field theory has been developed for application to two very different physical situations:

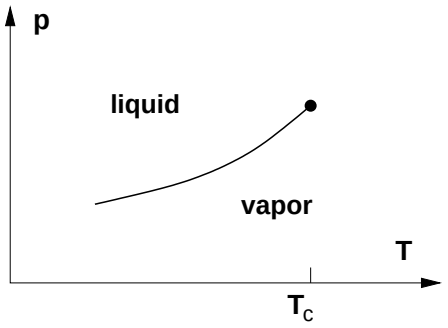
$*$ In string theory, if the metric is written as $g_{ab}(x) = \rho(x)\, g_{ab}^{(0)}(x)$, the conformal factor $\rho(x)$ must not play any role in the dynamics; the theory depends only on conformal classes of metrics. Conformal invariance is then a constraint on the theory.

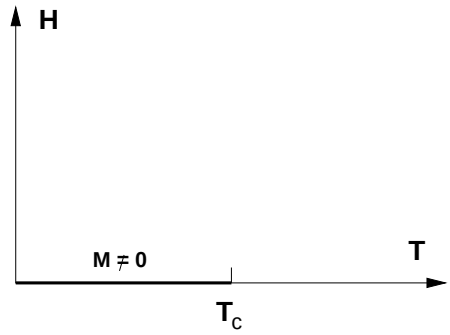
$*$ In critical phenomena in two dimensions: Polyakov (1970) has shown that in arbitrary dimension, the dilatation invariance expected in critical phenomena implies conformal invariance. This remark is particularly fruitful in two dimensions. In the next subsection, we shall recall some basic facts on critical phenomena, and the reader is referred for example to [4] for a thorough treatment of the question.

One should also add that c.f.t. is also now being studied for its own theoretical interest, as a non-perturbative approach to two-dimensional field theory, and for all its remarkable and fascinating connections with various topics in mathematics, topology and knot theory, in particular.

1.1. A little reminder on critical phenomena.

We are interested in systems undergoing a second order i.e. continuous phase transition, for example :





liquid-vapor system
under T_c : two distinct phases;
the order parameter $=\rho_{\rm liq}-\rho_{\rm vap}$
vanishes continuously at T_c.

ferromagnet
under T_c (Curie temperature), as $H \to 0$,
a spontaneous magnetization M appears;
this order parameter M vanishes at T_c.

Such critical systems exhibit a singular behavior at T_c: for example, the specific heat c behaves as $c \sim |T-T_c|^{-\alpha}$, the order parameter as $M \sim (T_c - T)^{\beta}$ etc...; α, β etc... are *critical exponents.*

At some arbitrary temperature, correlation functions of physical quantities decrease exponentially :

$$\langle \varphi(r)\varphi(o)\rangle \sim \mathrm{e}^{-r/\xi} \qquad \xi = \text{correlation length} \tag{1.1}$$

What happens at T_c is that the correlation length diverges. (There may be several ξ for the various physical quantities, and we assume that they all diverge in the same way) :

$$\xi \sim |T-T_c|^{-\nu} \tag{1.2}$$

This is the *fundamental* property of a second order phase transition. It implies that the system has lost a scale ξ (or is massless : mass $\sim \xi^{-1}$). This means that in the critical system, fluctuations occur at all scales : bubbles of one phase inside bubbles of the other inside bubbles ... at all scales ranging between the distance of physical observation and the microscopic scale. One has *scale invariance.* Just at T_c, the correlation functions have a power law fall-off : $\langle \varphi(r)\,\varphi(o)\rangle \sim r^{-\eta}$ and one may regard $\frac{1}{2}\eta$ as the scaling dimension of φ : $\dim\varphi = \frac{1}{2}\eta$.

Following Wilson, one may use renormalization group (R.G.) ideas : if physics at scale a is described by a theory with coupling(s) g , at scale λa, it is described by the same theory with effective couplings $g(\lambda)$. As $\lambda \to \infty$, one assumes that $g(\lambda)$ approaches some fixed point g^*. In general, one may show that this fixed-point theory that describes the critical system is a (massless) scale-invariant Euclidean field theory.

The concept of fixed point of the R.G. also explains the properties of *universality* : modifications of microscopic interactions (that do not affect the symmetries of the problem) do not change the critical behaviour (critical exponents etc...), although they may change the value of the critical temperature. Classes of universality are sets of systems with different microscopic physics but the same critical behaviour. Conformal field theory will describe the universality classes only.

1.2. Example: Ising model.

The partition function of this spin ± 1 ferromagnet reads:

$$Z = \sum_{\{\sigma_i = \pm 1\}} \mathrm{e}^{\beta \sum_{\text{bonds}} \sigma_i \sigma_j} \tag{1.3}$$

In 2 dimensions, $\alpha = 0$ (logarithmic singularity), $\beta = \frac{1}{8}$, $\eta = \frac{1}{4}$, etc... The addition of a term $\gamma \sigma_i \sigma_j \sigma_k \sigma_\ell$ to the 2-spin interaction shouldn't modify the exponents. Actually the Ising model is also appropriate to describe the phase transition of a binary fluid system.

In $d \geq 2$, Wilson and his followers have developed very powerful R.G. improved perturbative techniques to compute critical exponents. The results fit very well with experimental data. In $d = 2$, however, conformal field theory will enable us to go farther and:

- explain why rational exponents or exponents varying continuously occur so frequently in statistical mechanical models,
- possibly achieve a classification of all universality classes,
- compute all the correlation functions at T_c , etc...

2. General features of conformal field theories

2.1. Conformal transformations.

Conformal transformations are point transformations of the d-dimensional Euclidean space which preserve the angles, i.e. they may be regarded as local dilatations. An infinitesimal transformation $x_\mu \to x'_\mu = x_\mu + \epsilon_\mu$ with ϵ_μ satisfying

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\rho \epsilon^\rho \delta_{\mu\nu} \tag{2.1}$$

is such a local dilatation since

$$\left(dx'_\mu\right)^2 = \left(1 + \frac{2}{d}\partial_\rho \epsilon^\rho\right)\left(dx_\mu\right)^2 \tag{2.2}$$

It leaves ratios of lengths i.e. angles unaffected and is thus a conformal transformation. In any dimension $d \geq 3$, the conformal group is made of

$$\text{translations} \ : \delta x_\mu = a_\mu \tag{2.3a}$$

$$\text{rotations} \ : \delta x_\mu = \omega_{\mu\nu} x^\nu, \qquad \omega_{\mu\nu} \text{ antisymmetric} \tag{2.3b}$$

$$\text{dilatations} \ : \delta x_\mu = \gamma x_\mu \tag{2.3c}$$

$$\text{“special conformal transformation”} \ : = \text{inversion} * \text{translation} * \text{inversion} :$$

$$\delta x_\mu = b_\mu x^2 - 2x_\mu \left(b.x\right) \tag{2.3d}$$

In $d = 2$, there are more conformal transformations, as we shall see below.

In a field theory with an action functional $S(\varphi)$, the effect of a change of coordinates is described by the energy-momentum tensor $T_{\mu\nu}$ such that

$$\delta S = \int d^d x \, T_{\mu\nu} \partial^\mu \epsilon^\nu \tag{2.4}$$

Invariance under translations is implied by (2.4), whereas invariance under rotation and dilatation follow from the symmetry and tracelessness of $T_{\mu\nu}$. It is then easy to see that any transformation satisfying (2.1) leaves the action unchanged. Thus

$$\left.\begin{array}{c}\text{translation}\\ \text{rotation}\\ \text{dilatation}\end{array}\right\} \text{ invariances imply conformal invariance.}$$

More precisely, let us write a Ward identity to describe the effect of a change of coordinates. Consider some correlation function

$$\langle A_1(1)...A_n(n)\rangle = \int \mathcal{D}\varphi e^{-S(\varphi)} A_1(\varphi(1))...A_n(\varphi(n)) \tag{2.5}$$

assumed to be given by some functional integral; A is a local function of the basic field(s) φ . Perform a change of coordinates $x \to x' = x + \epsilon(x)$: the form of the function $A(x)$ changes into $A'(x')$ and

$$\delta A(x) = A'(x') - A(x). \tag{2.6}$$

Write that the physical quantity (the l.h.s.) is unaffected by this change of coordinates. In the r.h.s., the measure $\mathcal{D}\varphi e^{-S(\varphi)}$ contributes a term linear in $\partial_\mu \epsilon_\nu(x)$:

$$0 = \sum_i \langle A_1...\delta A_i...A_n\rangle + \int d^d x \langle T_{\mu\nu}(x) A_1...A_n\rangle \partial^\mu \epsilon^\nu(x) \tag{2.7}$$

which defines $T_{\mu\nu}$ (or rather $\langle T_{\mu\nu} A_1...A_n\rangle$). Imposing rotation invariance and dilatation invariance namely :

$$\sum_i \langle A_1...\delta A_i...A_n\rangle = 0 \quad \text{for rotations and dilatations} \tag{2.8}$$

shows that, up to total derivatives,

$$\begin{aligned} T_{\mu\nu}\delta\omega^{\mu\nu} &= 0 \qquad & T_{\mu\nu} \quad &\text{symmetric} \\ T_{\mu\nu}\delta^{\mu\nu} &= 0 \qquad & T_{\mu\nu} \quad &\text{traceless.} \end{aligned}$$

We assume that after possibly a suitable modification ("improvement"), $T_{\mu\nu}$ is symmetric and traceless. (One may show that this improvement is always possible in two dimensions, which is the only case which will concern us here.)

Moreover, the variation δA_i must be local in ϵ and its derivatives: taking a functional derivative of the Ward identity with respect to $\epsilon(x)$ gives

$$\sum_i \left[...\delta(x - x_i) + ...\overline{\nabla}\delta(x - x_i) + ...\right] = \partial_\mu \langle T_{\mu\nu}(x) A_1(1)...A_n(n)\rangle$$

Up to coinciding-point singularities, $T_{\mu\nu}$ is *conserved.*

2.2. In 2 dimensions (Euclidean plane) :

Two new features emerge in two dimensions:

1) One can use complex coordinates $z = x + iy \quad \overline{z} = x - iy$ and write

$$ds^2 = dx^2 + dy^2 = dz d\overline{z} \tag{2.9}$$

as

$$\begin{aligned} ds^2 &= g_{ab} d\xi^a d\xi^b \qquad \xi^a = \{z, \overline{z}\} \\ &= \left(g_{z\overline{z}} + g_{\overline{z}z}\right) dz d\overline{z} \end{aligned} \tag{2.10}$$

hence $g_{z\overline{z}} = g_{\overline{z}z} = \frac{1}{2}$, $g_{zz} = g_{\overline{z}\overline{z}} = 0$ and the inverse tensor is $g^{z\overline{z}} = 2$. The energy momentum tensor satisfies $T_{z\overline{z}} = T_{\overline{z}z} = 0$ as a consequence of its tracelessness.

2) Any analytic (holomorphic) change of coordinates is a conformal transformation

$$z \to z' = f(z) \qquad ds^2 \to |f'(z)|^2 dz d\overline{z}. \tag{2.11}$$

This means that the (infinitesimal) conformal transformations form an infinite dimensional algebra. As one can imagine, the existence of this huge symmetry algebra is very restrictive and enables one to use non-perturbative techniques (mainly Lie algebraic) to study these conformal theories. We give hereafter a brief account of this formalism.

Important : The transformations discussed before: translations, rotations, dilatations, special conformal transformations form the group $SL(2, \mathbb{C})$ of Moebius transformations: $z \to \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C}$, $ad - bc = 1$. Those are the *only* conformal transformations which are one-to-one on the Riemann sphere (completed plane).

2.3. Transformation of fields.

Dealing with complex coordinates enables one to define tensors of non-integer rank: A *primary field* is by definition a $\left(h, \overline{h}\right)$ tensor, i.e. under $z \to z'$:

$$A(z, \overline{z}) = A'(z', \overline{z}') \left(\frac{dz'}{dz}\right)^h \left(\frac{d\overline{z}'}{d\overline{z}}\right)^{\overline{h}} \tag{2.12}$$

where $h, \overline{h}$ are two arbitrary (real) numbers, called the conformal weights of A. We shall soon interpret $h + \overline{h}$ as the scaling dimension of A and $h - \overline{h}$ as its spin. For example, a free

Majorana-Weyl field $\psi(z)$ has $h=\frac{1}{2}$, $\overline{h}=0$ (or vice versa, depending on the chirality). The infinitesimal transformation of a primary field reads:

$$\delta z=\varepsilon(z) \qquad \delta\overline{z}=\overline{\varepsilon}(\overline{z}) \tag{2.13}$$
$$\delta A(z,\overline{z})=\left[\varepsilon(z)\partial_z+h\varepsilon'(z)\right]A(z,\overline{z})+\left[\overline{\varepsilon}(\overline{z})\partial_{\overline{z}}+\overline{h}\overline{\varepsilon}'(z)\right]A(z,\overline{z}).$$

This result is actually quite typical : the contribution of the z and $\overline{z}$ variables decouple, which enables one to treat them as independent variables. In many respects, the conformal field theory behaves as made of two one-dimensional components. Ultimately the "physical world" is recovered by imposing that $\overline{z}=$ complex conjugate of z.

Important warning. Not all fields behave as primary fields. Compute for example the transformation of $\partial_z A(z,\overline{z})$. Another important non primary field is $T(z,\overline{z})$ as we shall see below.

2.4. Two-dimensional conformal field theory

Let us assume $T_{z\overline{z}}=0$ (scale invariance) and return to the Ward identity. In 2 dimensions, it reads, after a little change in the normalization, suited for what follows :

$$\delta\langle A_1(1)...A_n(n)\rangle=\frac{1}{2\pi i}\int dzd\overline{z}\;\partial_{\overline{z}}\varepsilon(z,\overline{z})\langle T_{z,z}(z,\overline{z})A_1(1)...A_n(n)\rangle+c.c. \tag{2.14}$$

For the reason mentioned before (singularities for transformations which are not Moebius), one cannot take $\varepsilon(z,\overline{z})$ analytic right away. By the same argument as presented above, T is conserved everywhere but at the points $1,2,...n$:

$$\partial_{\overline{z}}T_{zz}=0 \qquad \partial_z T_{\overline{z}\overline{z}}=0 \quad \text{i.e.} \quad T_{zz}=T(z) \text{ is analytic and } \quad T_{\overline{z}\overline{z}}=\overline{T}(\overline{z}) \quad \text{is antianalytic} \tag{2.15}$$

except at the points $1,2...n$. Take ε analytic in the vicinity of $z_1,...,z_n$ and vanishing outside a compact domain Δ'. $\langle TA_1...A_n\rangle$ is therefore analytic in a domain Δ excluding the points $z_1,...,z_n$ and using Stokes theorem one may write :

$$\begin{aligned}\delta\langle A_1...A_n\rangle&=-\frac{1}{2\pi i}\int_{\partial\Delta}dz\;\varepsilon(z,\overline{z})\langle T_{zz}(z,\overline{z})A_1...A_n\rangle+c.c.\\&=\sum_i\frac{1}{2\pi i}\oint_{z_i}dz\;\varepsilon(z)\langle T(z)...A(n)\rangle+c.c.\end{aligned} \tag{2.16}$$

because ε vanishes on the external border. One may then identify separately the analytic and antianalytic contributions in ε on both sides of the equation. For a primary field, for example:

$$\sum_i \left[\varepsilon(z_i)\partial_{z_i} + h_i\varepsilon'(z_i)\right] \langle A_1(z_1,\overline{z}_1)\cdots A_n(z_n,\overline{z}_n)\rangle$$
$$= \frac{1}{2\pi i} \oint dz\, \varepsilon(z) \langle T(z)\cdots A(n)\rangle \tag{2.17}$$

where the contour encircles the points $z_1, ..., z_n$. Finally one uses Cauchy theorem to interpret this as :

$$\langle T(z) A_1(z_1,\overline{z}_1) ... A_n(z_n,\overline{z}_n)\rangle = \sum_i \left(\frac{h_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\partial_{z_i}\right) \langle A_1 ... A_n\rangle \tag{2.18}$$

(which is the unique solution vanishing as $z \to \infty$). Notice that this expression of $\langle T(z) ... A_n\rangle$ is indeed holomorphic everywhere except at $z_1, ..., z_n$.

One may also rephrase this result in terms of the short distance expansion :

$$T(z) A_1(z_1,\overline{z}_1) = \frac{h_1}{(z-z_1)^2} A_1(z_1,\overline{z}_1) + \frac{1}{(z-z_1)}\partial_{z_1} A_1(z_1,\overline{z}_1) + \cdots \tag{2.19}$$

where $\cdots$ means regular terms as $z \to z_1$. This expansion is to be understood as valid when inserted into a correlation function, i.e. in the presence of spectator fields.

2.5. Transformation of $T(z)$.

If one wants to consider variations of $\langle T(z) A_1 ... A_n\rangle$ itself under changes of coordinates, one needs the short distance expansion of T with itself. The expression:

$$T(z) T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{(z-w)}\partial_w T(w) + ... \tag{2.20}$$

is the most general form consistent with dimensional counting. The energy momentum tensor is a conserved current, hence has no anomalous dimension: its canonical dimension is 2, and the second and third terms in the expression above are just the normal terms expected for a primary field of dimension $h = 2$, $\overline{h} = 0$. The first term, however, is an anomalous term, allowed by dimensional analysis, hence generically present: c is the "central charge", a denomination to be justified soon.

This short distance expansion of $T.T$ corresponds to an infinitesimal transformation of T of the form

$$\delta T(z) = [\varepsilon(z)\partial_z + 2\varepsilon'(z)]\,T(z) + \frac{c}{12}\varepsilon'''(z) \tag{2.21}$$

It integrates for finite transformations to:

$$T(z) = T'(z')\left(\frac{dz'}{dz}\right)^2 + \frac{c}{12}\mathcal{S}\{z', z\} \tag{2.22}$$

where the "schwarzian derivative" is

$$\mathcal{S}\{z', z\} = \frac{d^3z'}{dz^3}\Big/\frac{dz'}{dz} - \frac{3}{2}\left[\frac{d^2z'}{dz^2}\Big/\frac{dz'}{dz}\right]^2 \tag{2.23}$$

This integration to finite transformations is non trivial. One may show that $\mathcal{S}\{z', z\}$ is the unique term (up to a factor) satisfying :

- antisymmetry i.e. $\mathcal{S}\{z', z\} = -\mathcal{S}\{z, z'\}\left(\frac{dz'}{dz}\right)^2$
- $SL(2, \mathbb{C})$ invariance, i.e. if f is a Moebius transformation :

$$\mathcal{S}\{f(z'), z\} = \mathcal{S}\{z', z\}$$

- cochain condition : changes of coordinates $z \to z' \to z''$ and $z \to z''$ are consistent
- dimension $= 2$ in units of $[length]^{-1}$.

This transformation law will have important consequences when we use conformal transformations to map some domain on some other. For example, map the punctured plane on the cylinder by $z = \mathrm{e}^{2i\pi w}$ and find that the relation between $T_{\text{plane}}(z)$ and $T_{\text{cyl}}(w)$ reads

$$T_{\text{cyl}}(w) = (2i\pi)^2\left[z^2 T_{\text{plane}}(z) - \frac{c}{24}\right] \tag{2.24}$$

2.6. Descendants of a primary field

Only the first two terms in the "operator product expansion" (o.p.e.) of $T(z)$ and of the primary field A have been identified in the expression above. More generally, we write :

$$\begin{aligned} T(z)A(z_1, \overline{z}_1) &= \sum_{k=0}^{\infty}(z - z_1)^{-2+k}\,A^{(-k)}(z_1, \overline{z}_1) \\ A^{(-k)}(z_1, \overline{z}_1) &\equiv (L_{-k}A)(z_1, \overline{z}_1) \\ &= \oint \frac{dz}{2\pi i(z - z_1)^{k-1}} T(z)A(z_1, \overline{z}_1) \end{aligned} \tag{2.25}$$

and by further multiplications by T and/or by $\overline{T}$ and expansions, define also

$$A^{(-k_1,-k_2)}(z_1,\overline{z}_1) = L_{-k_2} A^{(-k_1)}(z_1,\overline{z}_1)$$
$$A^{(-k_1,-\overline{k}_1)}(z_1,\overline{z}_1) = \overline{L}_{-\overline{k}_1} A^{(-k_1)}(z_1,\overline{z}_1) \tag{2.26}$$

etc... The family of fields $A,\ A^{(-k_1)},...,\ A^{(-k_1,...,-\overline{k}_1,...)},...$ constitutes the set of descendants of A, graded by theirs *levels* : $|k| = \sum_\ell \ell k_\ell,\ \left|\overline{k}\right| = \sum_\ell \ell \overline{k}_\ell$.

2.7. Operator algebra

An axiom of conformal field theory is that a product of two primary fields may be expanded on the (finite or infinite) set of primary fields and their descendants:

$$A(z,\overline{z})\,B(w,\overline{w}) = \sum_{\substack{C \\ \{k\},\{\overline{k}\}}} C_{ABC}^{(\{k\},\{\overline{k}\})}\,(z-w)^{h_C-h_A-h_B+|k|} \times$$
$$\times\,(\overline{z}-\overline{w})^{\overline{h}_C-\overline{h}_A-\overline{h}_B+|\overline{k}|}\,C^{(\{-k\},\{-\overline{k}\})}(w,\overline{w}) \tag{2.27}$$

where the sum runs over primary fields C. The coefficient $C_{ABC}^{(\{k\},\{\overline{k}\})}$ may be determined by use of the Ward identity, once the "structure constants" C_{ABC} pertaining to the primary fields are known.

2.8. Two- and three-point functions

Using translation and dilatation invariance, show that the 2-point function of a primary field reads:

$$\langle A(z,\overline{z})\,A(w,\overline{w})\rangle = \frac{1}{(z-w)^{2h_A}\,(\overline{z}-\overline{w})^{2\overline{h}_A}} \tag{2.28}$$

(The residue 1 results from a choice of normalization). Using $SL(2,\mathbb{C})$ invariance to map the three points z_A, z_B and z_C to $0, 1, \infty$ or any permutation thereof, show that (for A, B, C primary):

$$\langle A(z_A,\overline{z}_A)\,B(z_B,\overline{z}_B)\,C(z_C,\overline{z}_C)\rangle = \frac{C_{ABC}}{(z_A-z_B)^{h_A+h_B-h_C}\ \times\ perm.\ \times c.c.} \tag{2.29}$$

and that C_{ABC} is completely symmetric.

The form of the 2-point function above justifies our earlier statement : in radial coordinates $z-w=\rho e^{i\theta}$

$$\langle A(z,\overline{z})\,A(w,\overline{w})\rangle = \frac{e^{-2\left(h_A-\overline{h}_A\right)i\theta}}{\rho^{2\left(h_A+\overline{h}_A\right)}} \tag{2.30}$$

$h_A+\overline{h}_A$ is the scaling dimension of A, $\quad h_A-\overline{h}_A$ its spin.

2.9. Examples:

1) Free boson field.
Its action reads $S = \frac{1}{4\pi}\int \mathrm{d}^2x\,(\partial_\mu\phi)^2$, whence the propagator $\langle\phi(z,\overline{z})\,\phi(0)\rangle = -\frac{1}{2}\log z\overline{z}$.
Caution ! ϕ is NOT a primary field (it has a logarithmic propagator!) but $O_e(z,\overline{z}) = :\exp ie\phi(z,\overline{z}):$ is. This is the vertex operator in the language of string theory, the electric or spin-wave operator in the context of statistical mechanics. Check that $h = \overline{h} = e^2/4$ using Wick theorem. Also using Wick theorem, check that $T(z) = -(\partial_z\phi)^2$ has the right o.p.e. with $O_e(z,\overline{z})$ and from the o.p.e. $T(z)\,T(w)$, determine the value of c.

2) Free Majorana-Weyl fermion.
The action $S = \frac{1}{2\pi}\int \mathrm{d}^2x\;\;\psi\overline{\partial}\psi$ corresponds to the propagator $\langle\psi(z)\,\psi(w)\rangle = \frac{1}{z-w}$, hence $h = 1/2$, $\overline{h} = 0$ as announced above. Check that the o.p.e. with $T(z) = -\frac{1}{2}\psi(z)\,\partial_z\psi(z)$ is what it should be, and determine c.

2.10. Operator language. Virasoro algebra

We want now to reinterpret the previous correlation functions in an operator language, as resulting from the vacuum expectation value of suitably ordered products of field operators:

$$\langle A_1(1)\cdots A_n(n)\rangle = \left\langle 0\left|\mathcal{P}\widehat{A}_1(z_1,\overline{z}_1)\cdots\widehat{A}_n(z_n,\overline{z}_n)\right|0\right\rangle . \tag{2.31}$$

It turns out that the radial quantization, in which fields are ordered from right to left according to growing values of $|z_i|$ is particularly well suited. It corresponds by the exponential mapping to the standard quantization on a cylinder, where the time evolution along the axis of the cylinder describes either the propagation of a closed string, or a statistical mechanical system subject to a periodic boundary condition along the "space" direction.

All the short distance expansions found above must be regarded in this operator formalism as applying to the $\mathcal{P}$-ordered products. Expand $\widehat{T}(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$ hence $L_n = \oint_0 \frac{\mathrm{d}z}{2i\pi} z^{n+1}\widehat{T}(z)$ and compute the commutator:

$$\left[L_n,\widehat{A}(w)\right] = \left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\frac{dz}{2i\pi}z^{n+1}\mathcal{P}\widehat{T}(z)\widehat{A}(w) \tag{2.32}$$

The two $w-$contours may be then deformed into a single one circling around z. For a primary field A or the energy momentum tensor, respectively, this leads to

$$\left[L_n,\widehat{A}(z)\right] = z^{n+1}\partial_z\widehat{A}(z) + h(n+1)z^n\widehat{A}(z) \tag{2.33a}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{2.33b}$$

The latter equation defines the Virasoro algebra. One may regard L_n as the quantum realization of the differential operators $l_n = -z^{n+1}\frac{\partial}{\partial z}$, which generate the diffeomorphisms of the circle, with an algebra :

$$[l_n, l_m] = (n-m)\, l_{n+m}$$

The Virasoro algebra is therefore the "central extension" of the algebra of the diffeomorphisms of the circle by the c-number (Schwinger-term) $\frac{c}{12} n \left(n^2 - 1\right) \delta_{n+m,0}$. Notice that in the plane, L_{-1}, L_0, L_1 generate the $SL\left(2, \mathbb{C}\right)$ transformations $\delta z =$ respectively $\varepsilon,\ \varepsilon z,\ \varepsilon z^2$. Together with their $\overline{z}$ counterparts, generating an isomorphic $\overline{L}$ algebra, they correspond to respectively translations, rotations and dilatations, and special conformal transformations. Notice also that this subalgebra does not "see" the central term: this reflects the fact that the Moebius transformations have a vanishing schwarzian derivative and are the only true symmetries of the theory.

To summarize, in a conformal field theory, there is a natural action of two Virasoro algebras, relative to the z and $\overline{z}$ coordinates (left-moving and right-moving components). It is thus natural to wonder what are the...

2.11. Representations of the Virasoro algebra

In physics, the interesting representations are the so-called highest weight (h.w.) representations: for reasons which will become clear soon, they correspond to systems whose energy is bounded from below. Remember from the discussion of ordinary Lie algebras, $SU\left(2\right)$ say, the construction of h.w. representations. The generators of the algebra are J_+, J_- and J_z. One chooses an eigenvector of J_z of eigenvalue j annihilated by J_+ :

$$J_z \left|j\right\rangle = j \left|j\right\rangle \qquad J_+ \left|j\right\rangle = 0 \tag{2.34}$$

Repeated action of J_- builds a sequence of eigenvectors of J_z

$$J_z \left(J_-\right)^m \left|j\right\rangle = (j-m)(J_-)^m \left|j\right\rangle \tag{2.35}$$

until $\left(J^-\right)^m \left|j\right\rangle = 0$ which occurs for $m = 2j+1$. Likewise, here, we start from a highest weight state, eigenstate of L_0 and annihilated by all $L_n,\quad n > 0$:

$$L_0 \left|h\right\rangle = h \left|h\right\rangle \qquad L_{n>0} \left|h\right\rangle = 0 \tag{2.36}$$

The action of the $L_{n<0}$ then generates a representation space ("Verma module") $V(c,h)$ of the Virasoro algebra: $V(c,h)$ is generated by the combinations $L_{-1}^{\alpha_1}...L_{-k}^{\alpha_k}|h\rangle$. The h.w. representation built on $|h\rangle$ is thus infinite-dimensional, (contrary to the case of $SU(2)$ recalled above). This is an important feature of c.f.t.'s: it means that non trivial theories with an infinite number of degrees of freedom may be accommodated with a *finite* number of representations of the Virasoro algebra. An example will be provided below by the Ising model.

This representation $V(c,h)$ is also *graded* : this means that the eigenvalues of L_0 are integrally spaced; as the Virasoro commutation relations show immediately:

$$L_0 L_{-1}^{\alpha_1}...L_{-k}^{\alpha_k}|h\rangle = \left(h + \sum j\alpha_j\right) L_{-1}^{\alpha_1}...L_{-k}^{\alpha_k}|h\rangle \tag{2.37}$$

and $\sum j\alpha_j$ is the level of the state $|h\rangle$ ("above" the highest weight [2]). The actual states of the theory are made of *tensor products* of representations of the left and right Virasoro algebras:

$$\mathcal{H} = \oplus_h V_h \otimes V_{\overline{h}}$$

Among these h.w. states, one plays a particular role : $h = 0$:

$$L_n|0\rangle = 0 \qquad \text{for all } n \geq -1 \tag{2.38}$$

and thus $L_1|0\rangle = L_0|0\rangle = L_{-1}|0\rangle = 0$, which expresses that $|0\rangle$, the vacuum, is invariant under $SL(2,\mathbb{C})$ transformations.

There is a correspondence between the language of states and that of fields which create these states out of the vacuum :

primary fields $A_{h\overline{h}} \leftrightarrow$ h.w. state $\left|h,\overline{h}\right\rangle$

descendant fields $\leftrightarrow$ descendant states

through :

$$\lim_{z,\overline{z}\to 0} A_{h\overline{h}}(z,\overline{z})|0\rangle = \left|h,\overline{h}\right\rangle. \tag{2.39}$$

Check using the commutator $[L_n, A]$ that this state is indeed a h.w. state; one has similar expressions for the descendants.

The important issue of irreducibility of these representations of the Virasoro algebra will be examined below in sect.6. Suffice it to say here that when the Verma module is

[2] The conventional denomination highest weight is clearly very unfortunate: "lowest state" would be more appropriate.

non irreducible, there is a way to quotient it by its reducible parts and to construct an irreducible representation. We shall assume in the following that this has been done.

The important feature to be remembered is the structure of these representations as infinite "towers" of states with integrally spaced eigenvalues of L_0.

2.12. Conformal data and their consistency.

To summarize, a conformal field theory is fully specified by the following set of conformal data: c, the central charge, $\{(h_i, \overline{h}_i)\}$, the finite or infinite set of conformal weights of the primary fields and $\{C_{ijk}\}$, the set of structure constants of the operator algebra. These data are sufficient in the sense that *in principle and in practice,* they determine everything in the theory: correlation functions, etc... They are not independent, however, which opens the route to a classification of c.f.t.'s. The constraints come either from the representation theory, or from the consistency of the conformal theory (associativity of the operator algebra, etc). We shall see an example of the latter type in sect. 3.2; as an example of the former type, let us quote an important result on the constraints of unitarity.

Unitarity of a c.f.t. means that its Hilbert space may be given a positive definite norm for which the adjoint of the Virasoro generators satisfy

$$L_n^\dagger = L_{-n} \tag{2.40}$$

One proves that a necessary and sufficient condition of unitarity is

either $c \geq 1 \qquad h \geq 0$

or c and h are quantized as follows

$$c = 1 - \frac{6}{m(m+1)} \quad m \in \mathbb{N}, \quad m \geq 3 \tag{2.41}$$

$$h_{rs} = \frac{\big(r(m+1) - sm\big)^2 - 1}{4m(m+1)} \qquad r, s \in \mathbb{N},\ 1 \leq r \leq m-1,\ 1 \leq s \leq m$$

Thus for a given $c < 1$, there is a *finite* number of possible values of h, $\overline{h}$ for the primary fields.

An example is provided by the Ising model which corresponds to $m = 3$ in the previous formulae, thus $c = \frac{1}{2}$, and for which the three spinless fields have $(h, \overline{h}) = (0, 0)$, the identity operator, $(\frac{1}{2}, \frac{1}{2})$, for the energy operator, which describes the response of the model to a departure from the critical temperature, and $(\frac{1}{16}, \frac{1}{16})$, for the spin (in agreement with the value $\eta = \frac{1}{4}$ quoted in sect. 1.2).

Note that the condition of unitarity is not as compulsory in statistical mechanics as it is in string theory: the critical regime of polymers, or the so-called Lee-Yang singularity, i.e. the critical point of a ferromagnet in an *imaginary* magnetic field, are known to be described by non unitary c.f.t.'s.

3. A sample of results.

3.1. Finite size effects.

Consider a critical system in a strip of finite width L. For definiteness, the boundary conditions are assumed to be periodic, and the system is thus living on a cylinder, but the following may be generalized to other types of boundary conditions. One may map the cylinder on the complex plane through the exponential mapping $z = \exp\big(2i\pi w/L\big)$. Using the transformation law (2.12) of a primary field under change of coordinates, it is easy to deduce the two-point function on the cylinder from that on the plane (2.28) :

$$\begin{aligned}\langle A'(w_1)A'(w_2)\rangle_{\rm cyl} =& \left(\frac{\partial z_1}{\partial w_1}\right)^h \left(\frac{\partial z_2}{\partial w_2}\right)^h \frac{1}{(z_1-z_2)^{2h}} \times \text{c.c.} \\ =& \frac{\text{const.}}{\big(\sin\pi\frac{w_1-w_2}{L}\big)^{2h}} \times \text{c.c.}\end{aligned} \tag{3.1}$$

For small separations, $|w_1 - w_2| = r \ll L$, one recovers the universal singular behaviour $\approx r^{-4h}$ (the field A is assumed spinless), but at large separations, $r \gg L$, the 2-point function has an exponential fall-off $\approx \exp -\frac{r}{\xi_L}$, with a correlation length $\xi_L = L/4\pi h$. This relation between h and ξ_L that had been observed empirically is thus justified by simple considerations of conformal invariance.

3.2. Partition function with doubly periodic boundary conditions.

As an example of another finite size effect and a preparation of further considerations, let us consider now our critical system in a box with doubly periodic boundary conditions, i.e. on a domain with the topology of a torus. Alternatively, this torus $\mathbf{T}$ may be regarded as the complex plane in which pairs of points differing by integral combinations of the periods 1 and τ are identified $z \sim z + n1 + p\tau$ or $\mathbf{T} = \mathbb{C}/\Lambda$ where Λ is the lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$. (Thanks to scaling invariance, one of the periods may always be chosen equal to 1, and moreover $Im\tau > 0$: τ is the modular parameter of the torus). Let us also

define $q = \exp 2i\pi\tau$, $\overline{q}$ its complex conjugate. A natural definition of the partition function of the system consists in writing:

$$Z = tr\ \mathrm{e}^{\mathcal{T}\tau + c.c.} \tag{3.2}$$

where $\mathcal{T}$ is the translation operator in the w-variable on the cylinder. Notice that the segment of cylinder is closed into a torus by the trace operation.

Since $\mathcal{T} = L_{-1}^{\mathrm{cyl}} = 2i\pi\left(L_0^{\mathrm{plane}} - \frac{c}{24}\right)$ as follows from the transformation of T_{plane} to T_{cyl} computed in (2.24), we see that $L_0 + \overline{L}_0$ plays the role of Hamiltonian: this justifies our endeavour at restricting our attention to highest weight representations. Returning to the partition function Z we can now write it as

$$Z = tr\ \ q^{L_0 - c/24}\overline{q}^{\overline{L}_0 - c/24}. \tag{3.3}$$

It is important to realize that Z contains all the information about the operator content of the theory: all the pairs of h.w. states, or equivalently all the pairs of conformal weights of primary fields. More precisely, since we have by construction imposed periodic boundary conditions along the "space" direction (period 1), only the states of the periodic (untwisted) sector of the theory are encoded in Z ; stated differently only the integer spin fields of the theory appear in Z . Other sectors of the theory, and the corresponding fields, may be exposed by considering twisted (antiperiodic, etc...) boundary conditions. To summarize, in the context of string theory, the spectrum of energies of the closed string is encoded in Z; in the context of statistical mechanics, it is the set of conformal weights of the theory, i.e. essentially its critical exponents.

To discuss the implications on finite size effects, choose for simplicity $\tau = i\delta$ purely imaginary, hence $q = \exp -2\pi\delta$ real, and let $\delta \to \infty$, $q \to 0$. We can perform an expansion of (3.3) in powers of $q = \overline{q}$. In unitary theories, the leading contribution comes from the identity operator $h = \overline{h} = 0$:

$$Z = (q\overline{q})^{-c/24}\left[1 + \mathcal{O}(q,\overline{q}) + \sum_{h,\overline{h}\neq 0} N_{h\overline{h}} q^h \overline{q}^{\overline{h}} \left(1 + \mathcal{O}(q,\overline{q})\right)\right] \sim \exp + \frac{\pi\delta c}{6}\left[1 + ...\right] \tag{3.4}$$

This has to be compared with the partition function expected for a system in a box $L \times T$:

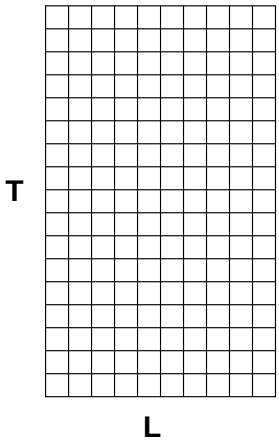
$$Z = \exp LTF(L)\,. \tag{3.5}$$

In the thermodynamic limit $L, T \to \infty$, F approaches a finite limit F_0, the free energy per unit volume. Implicitly, this F_0 has been set equal to zero (at $T = T_c$) in our construction, and the expression above, with $\delta = T/L$, yields the finite size contribution to F :

$$F \sim F_0 + \frac{\pi c}{6L^2} \tag{3.6}$$

We conclude that the finite size contribution to the free energy is proportional to c, the central charge. In other words, c controls the Casimir effect on the cylinder. Other types of boundary conditions may modify the proportionality factor.

The subleading terms in the above expansion also contain an interesting information. Compare with the expression of Z obtained for a lattice model using the transfer matrix $\mathcal{U}$, i.e. the discrete time evolution operator:



$$Z = tr\mathcal{U}^T = \lambda_0^T \left(1 + \left(\frac{\lambda_1}{\lambda_0}\right)^T + \left(\frac{\lambda_2}{\lambda_0}\right)^T + ...\right) \tag{3.7}$$

if $|\lambda_0| > |\lambda_1| > |\lambda_2| ...$ are the ordered eigenvalues of $\mathcal{U}$. In the thermodynamic limit $\lambda_0^T = \mathrm{e}^{LTF}$ is the dominant contribution, but we may identify (for the lattice model close to criticality) the successive ratios $\left(\frac{\lambda_i}{\lambda_0}\right)^T$ with the terms of the q-expansion:

$$\left(\frac{\lambda_i}{\lambda_0}\right)^T = \exp - 2\pi \frac{T}{L} \left(h + \overline{h} + |n| + |\overline{n}|\right) \tag{3.8}$$

for some integers n and $\overline{n}$. This is a method of practical importance. By numerical calculations of the eigenvalues of the transfer matrix, or by studies of the finite size corrections to the Bethe Ansatz results whenever it is possible, one may "read" the values of $\left(h, \overline{h}\right)$ and hence identify the conformal theory which describes a given statistical mechanical model.

Let us now return to the expression (3.3). The trace is to be taken on $\mathcal{H}$, the Hilbert space of states of the theory. The only information we have on $\mathcal{H}$ is that it decomposes

into a (finite or infinite) sum of tensor products of irreducible representations of the left and right Virasoro algebras, characterized by their highest weights h and $\overline{h}$.

$$Z = \sum N_{h\overline{h}} \chi_h(q) \chi_{\overline{h}}(\overline{q}) \tag{3.9}$$

where $N_{h\overline{h}}$ is the multiplicity of $\left(h, \overline{h}\right)$, hence a non-negative integer. (In particular, $N_{0,0}$, the multiplicity of the vacuum, must be equal to one.) In (3.9), $\chi_h(q)$ denotes the "character" of the representation labelled by h, i.e. the expression of $\mathrm{tr} q^{L_0 - \frac{c}{24}}$ evaluated in that representation. Note that this character is essentially, up to an overall power of q, a generating function of the number of states in the tower of states above the highest weight h:

$$\chi_h(q) = q^{h - \frac{c}{24}} \sum_{n=0}^{\infty} (\text{dimension of space of level } n)\, q^n \tag{3.10}$$

These characters have been studied and are explicitly known.

The important observation about Z is that it must be intrinsically attached to the torus, and does not depend on the choice of periods which define Λ . In other words, Z must be invariant under modular transformations:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{array}{l} a, b, cd \in \mathbb{Z} \\ ad - bc = 1 \end{array} \tag{3.11}$$

(The condition $ad - bc = 1$ ensures the invertibility of the change of basis). The modular group is actually generated by the two transformations

$$\tau \to \tau + 1 \qquad \text{and} \qquad \tau \to -1/\tau \tag{3.12}$$

So it is necessary and sufficient to ensure that:

$$Z(\tau) = Z(\tau + 1) = Z(-1/\tau)\,. \tag{3.13}$$

This property of modular invariance, together with the general form (3.9) turns out to be very restrictive and allows a classification of families of conformal field theories. This program has been completed for a few families of c.f.t.'s, in particular for the "minimal theories", i.e. those that contain only a *finite* number of primary fields. (They have necessarily $c < 1$ and contain as a subset the unitary $c < 1$ theories mentioned above). It turns out that this classification may be related to another well-known classification in mathematics, that of simply laced algebras of type A, D, E. Thus this "ADE classification" provides us with a complete list of all universality classes of critical phenomena involving a finite number of primary fields.

3.3. Other results at T_c: miscellaneous.

Let us list some other noticeable achievments of the formalism.

Correlation functions of all kinds are explicitly computed, and expressed through integrable representations. This applies to $n \geq 4$-point functions in the plane, or to all functions in finite or semi-infinite geometries: half-plane, torus, etc...

It is perhaps not too surprising to see that most developments of c.f.t. may be rephrased in the language of free fields: it had been advocated by Kadanoff some thirteen years ago that most two-dimensional critical systems may be expressed in terms of the Gaussian model (or Coulomb gas). This is indeed the case of the 6-vertex, Ashkin-Teller or XY models in their critical regimes, that may be represented by a free $c = 1$ free boson field, compactified on a circle, i.e. regarded as an angular variable, or subject to further identifications. Actually most (all?) known c.f.t.'s may be represented in terms of free fields, with adequate modifications of the boundary conditions. In particular, the $c < 1$ minimal models discussed above may be obtained through the introduction of a "charge at infinity" in the Coulomb gas picture. Generalizations to multicomponent Gaussian fields seem capable to reproduce the existing c.f.t.'s.

Another way to reduce c.f.t.'s to a common ingredient is to emphasize the role played by current (or Kac-Moody) algebras. A current algebra is yet another infinite dimensional algebra based on a finite Lie algebra g. It has generators $J^a(x)$ satisfying the commutation relations of the type

$$[J^a(x), J^b(y)] = \big(f^{abc} J^c(x) + \frac{k}{2}\delta^{ab}\partial_x\big)\delta(x-y) \tag{3.14}$$

where the coefficient k of the central (Schwinger) term is a number. One develops the theory of representations in a way similar to what has been sketched for the Virasoro algebra. One finds that in unitary representations k is quantized as an integer and that for $SU(2)$, for example, the spin j of representations takes integer or half-integer values in the range $0 \leq j \leq k/2$. With such a current algebra, one may construct conformal theories, i.e. the Virasoro generators are quadratic expressions in the J's. Moreover, by various algebraic techniques, (coset construction, Hamiltonian reduction, ...), all "rational" c.f.t.'s (see below) are expressible in terms of these current algebras. A final noteworthy feature is the growing apparent role of these current algebras in condensed matter physics: quantum Hall effect [5][6], Kondo problem [7]...

There is a vast zoo of known c.f.t.'s. The previously discussed minimal or $c = 1$ theories are in many respects the simplest and have to do (in the sense of current algebras) with $SU(2)$ at level 1. It is thus very natural to generalize them using higher rank algebras and/or higher level. Many of the resulting theories appear to be built out of representations of an algebra *larger* than the Virasoro algebra, a so-called *extended algebra*. Examples of such larger algebras are provided by the current algebras themselves, or by superconformal algebras, or also by the so-called W-algebras. The latter are generated by the energy-momentum tensor $T(z)$ together with other fields of integer spin. The first non-trivial example is provided by the critical 3-state Potts model ($c = 4/5$) which exhibits a W_3 algebra generated by T and W of spin 3. The study of these extended W-algebras, of their geometric meaning and of their representations is a very active current domain of research, and we shall see in Sect. 5 how their classical version appears naturally in connection with certain integrable systems. Those c.f.t.'s that involve a finite number of representations of some extended algebra are said to be *rational*: they are the natural generalizations of the minimal theories.

Finally, another interesting feature common to many of these c.f.t.'s is the appearance of parafermions, i.e. of fields with fractional statistics.

3.4. Away from the critical point.

So far, all results derived from conformal theory have concerned the critical point proper. It would, however, be extremely interesting to extract some information about the vicinity of the critical point, where universality still holds, and to derive the values of critical ratios of amplitudes, for example, for the specific heat, $\lim_{t\to 0} c(T_c+t)/c(T_c-t)$, or the expression of the two-point function in the critical regime $T \to T_c$, $r/\xi = x$ fixed. One may develop perturbative techniques, perturbative expansions, R.G. flows, etc... about the conformal theories, but these are still unable to answer these questions.

A remarkable result of non-perturbative nature has been obtained by Zamolodchikov about the possible flow of a theory away from its critical point under the influence of a "relevant" perturbation. Assume that a conformal theory of central charge c_1 is perturbed by a relevant operator. The theory looses its conformal invariance and develops a mass scale; at large distances, this massive theory may reach another fixed point of the R.G., described by another c.f.t. of central charge c_2. Then the theorem asserts that $c_1 > c_2$, and that there exists a function $c(t)$ monotonously decreasing along the flow line interpolating between c_1 and c_2. Somehow, c is a measure of the number of degrees of freedom of the system, and the decrease of c reflects the loss of information in the R.G. flow.

3.5. Connections with integrable models.

One fascinating feature of c.f.t. is the existence of many connections with the a priori distinct area of integrable systems, i.e. systems with an infinite number of conservation laws. In fact, there are relations with various types of such integrable systems:

* lattice integrable models, i.e. solutions of the Yang-Baxter equation. Not only, it seems that every known c.f.t. has a lattice integrable representative, but both families of systems share many algebraic properties: in both, the quantum group and Kac-Moody algebras play prominent roles, although in different guises. We refer the reader to [8] for example for a discussion of the first feature.

* integrable massive field theories: when a c.f.t. is perturbed into a massive theory as explained in the previous section, it may happen that the resulting theory retains some of its integrability. This gives the possibility to compute the S matrix and other scattering data [9], and also possibly off-shell quantities: form factors and correlation functions.

* classical field theories: as will be shown in the next section, there is a precursor of these relations at a classical level: a classical (Poisson bracket) version of the Virasoro algebra is present in classical integrable systems of the KdV type (or sine-Gordon, Toda...). Upon quantization [10], one learns a lot about c.f.t.'s, their free field representation and the role of the quantum group.

4. The KdV equation and hierarchy and their generalization.

We now turn to an apparently very different topic, namely the integrable systems of KdV type. There is a huge literature and lore on the KdV equation, its hierarchy and its generalizations [11][12][13]. I focus on some aspects that are useful in relation with my concern, namely the Hamiltonian structures that are associated with these hierarchies and which turn out to be the classical analogues of the Virasoro algebra and its extensions, the so-called W-algebras (sect. 4.1-3). I also discuss the matrix formalism of Drinfeld and Sokolov (sect. 4.4), which is a natural and useful way to recast differential operators in a matrix form. In sect. 5, covariance properties of differential operators and their possible deformations are studied: this introduces the classical version of W-algebras. Finally in sect. 6, we return to the study of representations of the Virasoro algebra. It appears that there are actually unexpected connections between the two subjects, and a general method to determine singular vectors based on fusion is discussed.

4.1. The KdV equation.

The Korteweg-de Vries equation is the non linear partial differential equation satisfied by a function $u(x,t)$:

$$u_t = u''' + 6uu' \tag{4.1}$$

where the prime denotes differentiation with respect to the variable x and the subscript t stands for $\partial/\partial t$. This equation is known to describe the propagation of waves in shallow water [14]. It possesses solutions of soliton type, i.e. that scatter while conserving their identity (see [11], and [15] for a beautiful visual evidence). This actually follows from the existence of an infinite number of conservation laws. I list hereafter the first conserved quantities, but we shall see later what is the systematic way of generating them [3].

$$\begin{aligned} I_0 &= 2\int u dx \qquad I_1 = \int u^2 dx, \\ I_2 &= \int (u^3 - \frac{u'^2}{2}) dx \qquad I_3 = \frac{1}{4}\int (u''^2 + 5u^2 u'' + 5u^4) dx \qquad \text{etc...} \end{aligned} \tag{4.2}$$

The equation (4.1) is bi-hamiltonian, which means that it admits two hamiltonian descriptions, with two Hamiltonians and two Poisson brackets:

$$u_t(x) = \mathcal{D}^{(i)} \frac{\delta \mathcal{H}^{(i)}}{\delta u(x)} \qquad \text{or equivalently} \quad u_t = \{\mathcal{H}^{(i)}, u\}^{(i)}, \qquad i = 1,2 \tag{4.3}$$

with

$$\{u(x,t), u(y,t)\}^{(i)} = -\mathcal{D}_x^{(i)} \delta(x-y)$$

for two choices of $\mathcal{D}$ and $\mathcal{H}$:

$$\mathcal{D}^{(1)} = \mathrm{d}, \quad \mathcal{H}^{(1)} = I_2, \quad \{u(x), u(y)\}^{(1)} = -\delta'(x-y) \tag{4.4a}$$

$$\mathcal{D}^{(2)} = \frac{1}{2}(\mathrm{d}^3 + 4u\mathrm{d} + 2u'), \quad \mathcal{H}^{(2)} = I_1, \tag{4.4b}$$

$$\{u(x), u(y)\}^{(2)} = -\frac{1}{2}\delta'''(x-y) - (u(x)+u(y))\delta'(x-y)$$

[3] These conserved quantities are also related to the coefficients of the expansion of the resolvent of the operator $\mathrm{d}^2 - f$ [12]:

$$\langle x| \frac{1}{-\mathrm{d}^2 + f + \xi} |x\rangle = \sum_{l=0}^{\infty} \frac{R_l[f]}{\xi^{l+\frac{1}{2}}}$$

by $\delta I_l/\delta u(x) = 2^{l+2} R_l[-u]$. These R_l appear in the generalized Painlevé equations: $x = R_l[f]$ [16].

Here and in the following, d denotes the differential operator $\frac{d}{dx}$. In fact, the conserved quantities are in involution, $\{I_j, I_k\} = 0$, (for either Poisson bracket), and satisfy in general

$$\{I_k, u\}^{(1)} = \{I_{k-1}, u\}^{(2)} \tag{4.5}$$

This enables one to construct them recursively, by one quadrature and one functional integration:

$$\frac{d}{dx}\Big(\frac{\delta}{\delta u(x)} I_k\Big) = \frac{1}{2}\big(\mathrm{d}^3 + 4u(x)\mathrm{d} + 2u'(x)\big)\frac{\delta I_{k-1}}{\delta u(x)} \tag{4.6}$$

The common value (4.5) may be regarded as defining another partial differential equation $\delta_k u \equiv \frac{\partial}{\partial t_k} u = \{I_k, u\}^{(1)} = \{I_{k-1}, u\}^{(2)}$, which is thus also integrable since it has the same set of conservation laws. These new integrable equations are called "higher KdV" and they form the *KdV hierarchy.*

The remarkable feature that we want to explore is that the second hamiltonian structure is nothing but the classical (Poisson) version of the Virasoro algebra [17]. This means that if the function $u(x)$ is a periodic function on $[0, 2\pi]$, say, its Fourier modes u_n

$$u_n = \int_0^{2\pi} \frac{dx}{2\pi} u(x) e^{-inx} - \frac{1}{4}\delta_{n0} \tag{4.7}$$

satisfy

$$-2\pi i\{u_n, u_m\}^{(2)} = -(n-m)u_{n+m} + \frac{1}{2}n(n^2-1)\delta_{n+m,0} \tag{4.8}$$

and upon changing $-2i\pi\frac{6}{c}\{.\,,.\}^{(2)} \to [.\,,.]$, $-\frac{c}{6}u_n \to L_n$, we recover the familiar Virasoro algebra (2.33*b*): in this "quantization", $\frac{6}{c}$ plays the role of $\hbar$ [17].

The KdV equation has one more property important for our purpose: it admits a representation in terms of commutators of differential operators (Lax equation). Let us introduce the differential operator $D = \mathrm{d}^2 + u$. Then, there exists a differential operator Q such that

$$\partial_t D = [Q, D] \tag{4.9}$$

The explicit form of Q reads

$$Q = \mathrm{d}^3 + \frac{3}{2}u\mathrm{d} + \frac{3}{4}u' \tag{4.10}$$

We shall see later what is the way to generate this expression from D, but let's note immediately that the existence of this Q implies that the KdV equation describes an

isospectral deformation (or flow) of the differential operator D. Considered as a function of time, $D(t)$ remains similar to itself:

$$D(t) = S(t)D(0)S^{-1}(t) \tag{4.11}$$

where $S(t)$ satisfies

$$\partial_t S S^{-1} = Q(t), \qquad \text{i.e.} \qquad S(t) = T\int^t \exp Q(t')\mathrm{d}t'. \tag{4.12}$$

The spectrum of the operator D is thus left invariant.

To transform these observations into something more systematic and coherent, we need to introduce the rules of pseudodifferential calculus [12][13].

4.2. Pseudodifferential operators.

The differential operator d is such that for any function $a(x)$, the commutator $[\mathrm{d}, a] = a'$. Let us introduce the formal inverse d^{-1} of d, such that

$$\mathrm{d}\mathrm{d}^{-1} = \mathrm{d}^{-1}\mathrm{d} = 1 \tag{4.13}$$

The commutation properties with a function are easy to derive:

$$\begin{aligned}[a, \mathrm{d}^{-1}] &= a\mathrm{d}^{-1} - \mathrm{d}^{-1}a = \mathrm{d}^{-1}(\mathrm{d}a - a\mathrm{d})\mathrm{d}^{-1} = \mathrm{d}^{-1}a'\mathrm{d}^{-1} \\ &= a'\mathrm{d}^{-2} - [a', \mathrm{d}^{-1}]\mathrm{d}^{-1} = a'\mathrm{d}^{-2} - a''\mathrm{d}^{-3}\cdots\end{aligned}$$

More generally, one has the Leibniz rule:

$$[\mathrm{d}^p, a] = \sum_{k=1}^{\infty}\binom{p}{k}a^{(k)}\mathrm{d}^{p-k} \tag{4.15}$$

with binomial coefficients extended to $p \gtrless 0$

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!} \tag{4.16}$$

We call *valuation* of a differential or pseudodifferential operator the lowest power of d that appears in it, if it is finite, when one chooses an ordering of the operator in which the powers of d are pulled to the right.

This formalism enables us to associate with every linear differential (or even pseudodifferential) operator of order n, normalized so as to have the coefficient of the leading power of d equal to one ("monic" differential operator)

$$\Delta = \mathrm{d}^n + \sum_{j\geq 1} a_j \mathrm{d}^{n-j} \tag{4.17}$$

its inverse Δ^{-1}, its n-th root $L = \Delta^{\frac{1}{n}}$ and more generally, its fractional powers $\Delta^{\frac{k}{n}}$. They are the *unique* pseudodifferential operators satisfying

$$\begin{aligned} \Delta\Delta^{-1} =& \Delta^{-1}\Delta = 1 \\ L^n =& \Delta \\ \Delta^{\frac{k}{n}} =& L^k \\ [\Delta^{\frac{k}{n}}, \Delta^{\frac{\ell}{n}}] =& [L^k, L^\ell] = 0 \end{aligned} \tag{4.18}$$

Indeed if the operator Δ reads as in (4.17) we write

$$L = \Delta^{\frac{1}{n}} = \mathrm{d} + \sum_{k\geq 0} \alpha_k \mathrm{d}^{-k} \tag{4.19}$$

and we determine recursively and uniquely the coefficients α_k by identifying Δ and L^n:

$$n\alpha_k + \text{differential polynomial in } (\alpha_1, \cdots, \alpha_{k-1}) = \text{coefficient of } \mathrm{d}^{n-1-k} \text{ in } \Delta. \tag{4.20}$$

If $a_1 \equiv 0$, the first terms read

$$L = \mathrm{d} + \frac{1}{n} a_2 \mathrm{d}^{-1} + \Big(\frac{a_3}{n} - \frac{n-1}{2n} a_2'\Big)\mathrm{d}^{-2} + \Big(\frac{a_4}{n} + \frac{n^2-1}{12n} a_2'' - \frac{n-1}{2n}(a_2^2 + a_3')\Big)\mathrm{d}^{-3} + \cdots \tag{4.21}$$

The existence and unicity of the inverse Δ^{-1} follow from similar considerations:

$$\Delta^{-1} = \mathrm{d}^{-n} - a_1 \mathrm{d}^{-n-1} + (na_1' - a_2 + a_1^2)\mathrm{d}^{-n-2} + \cdots \tag{4.22}$$

Given a pseudodifferential operator $\Delta = \mathrm{d}^n + \sum_{j>0} a_j \mathrm{d}^{n-j}$, we denote Δ_+ its ordinary differential part

$$\Delta_+ = \mathrm{d}^n + \sum_{j=1}^{n} a_j \mathrm{d}^{n-j} \tag{4.23}$$

and Δ_- the complement

$$\Delta_- = \Delta - \Delta_+. \tag{4.24}$$

The "residue" of Δ is the coefficient of the d^{-1} term

$$\operatorname{res}\Delta = a_{n+1}(x) \tag{4.25}$$

A simple lemma states that the residue of a commutator is a total derivative. This is easily established by linearity, considering two monomials $A = a\mathrm{d}^k$ and $B = b\mathrm{d}^\ell$. Then from (4.15), $\operatorname{res} AB = \binom{k}{k+l+1} ab^{(k+l+1)}$ (i.e. zero if $k+l+1<0$) and

$$\begin{aligned}\operatorname{res}[A,B] =& \frac{k(k-1)\cdots(-l)}{(k+l+1)!}\big(ab^{(k+l+1)} - (-1)^{k+l+1}a^{(k+l+1)}b\big)\\ =& \frac{k(k-1)\cdots(-l)}{(k+l+1)!}\sum_{i=0}^{k+l}(-1)^i\mathrm{d}\big(a^{(i)}b^{(k+l-i)}\big).\end{aligned} \tag{4.26}$$

We shall also make use of the $\mathbb{Z}_2$ involution, a conjugation that leaves the functions unchanged, acts on d as $\mathrm{d}^* = -\mathrm{d}$ and on products of arbitrary pseudodifferential operators according to $(AB)^* = B^*A^*$.

So far we have used only algebraic manipulations and didn't have to make explicit the class of functions on which we are working. To proceed, we need the concept of integration over a non trivial cycle $\mathcal{C}$. We shall thus consider functions $a(x)$ belonging to either smooth periodic functions of one real variable on $\mathcal{C} = [0,1]$ say, or smooth functions of one real variable vanishing fast enough at infinity ($\mathcal{C} = \mathbb{R}$), or else analytic functions of a complex variable in $\mathbb{C} - \{0\}$ (and then $\mathcal{C}$ is any cycle encircling the origin). The trace of a pseudodifferential operator is then defined as the integral over this cycle $\mathcal{C}$ of the residue

$$\operatorname{Tr} A = \oint_{\mathcal{C}} \operatorname{res} A. \tag{4.27}$$

The above lemma on the residue of a commutator guarantees that

$$\operatorname{Tr} AB = \operatorname{Tr} BA. \tag{4.28}$$

4.3. Generalizations of the KdV equation.

We are now well equipped to return to the study of the KdV equation and of its generalizations. We consider a monic differential operator of order n:

$$D = \mathrm{d}^n + a_1\mathrm{d}^{n-1} + \cdots + a_n. \tag{4.29}$$

One can prove the following properties

(i): The set of differential operators Δ such that $[\Delta, D]$ be of order less or equal to $n-1$ is generated as a linear space by the functions and the operators $(D^{k/n})_+$, $k \in \mathbb{N}$.

Elements of proof (for a more detailed discussion, see [13], pages 1984-1985): *One first shows that the space of pseudodifferential operators B that commute with D is generated by the $D^{\frac{k}{n}}$: if $B = b_p \mathrm{d}^p + \cdots$, then $[B, D] = 0$ implies that $b_p' = 0$, thus b_p is a constant and $B^{(1)} = B - b_p D^{\frac{p}{n}}$ also commutes with D but is of a degree one less than B. By iteration, the property follows. Notice that $[B, D] = 0$ implies that $[B_+, D] = -[B_-, D]$ is of degree less or equal to $n-2$. Now if $A = a_p \mathrm{d}^p + \cdots + a_0$ is such that $[A, D]$ is of degree $\leq n-1$, the coefficient of d^{n+p-1} in $[A, D]$ is $n\alpha_p' = 0$, thus α_p is a constant and one repeats again the argument with the operator $A^{(1)} = A - \alpha_p (D^{\frac{p}{n}})_+$. After a finite number of steps, one is led to (i).*

Property (i) means that, for any k not a multiple of n, one may consider the flow

$$\delta_k D = \left[\left(D^{\frac{k}{n}}\right)_+, D\right] \tag{4.30}$$

(Obviously the flows $\left(D^{\frac{k}{n}}\right)_+$ for $k \in n\mathbb{N}$ are trivial.) The flow associated with functions

$$\delta_\phi D = [\phi, D] \tag{4.31}$$

simply corresponds to an infinitesimal change of the functions on which D acts

$$\begin{aligned} f &\to (1-\phi) f \\ D &\to (1+\phi) D (1-\phi) = D + [\phi, D] + \cdots \end{aligned} \tag{4.32}$$

It may be used to bring D to a canonical form with no d^{n-1} term: $\phi = \exp -\frac{1}{n}\int^x a_1(x')dx'$ $\Rightarrow a_1 \to 0$. We shall assume it in the following

$$D = \mathrm{d}^n + a_2 \mathrm{d}^{n-2} + \cdots + a_n. \tag{4.33}$$

The other flows respect this form.

Example

For $n=3$, $k=2$, one gets $D=\mathrm{d}^3+u\mathrm{d}+v$, $\left(D^{\frac{2}{3}}\right)_+=\mathrm{d}^2+u$, and $\delta u=u_t=2v'-u''$, $\delta v=v_t=v''-\frac{2}{3}(uu'+u''')$. Eliminating v one finds the Boussinesq equation

$$u_{tt}=-\frac{1}{3}(4uu''+4u'^2+u'''').\tag{4.34}$$

(ii): All these flows are isospectral and commute.

The first part is a trivial consequence of the commutator form of the flow equation, as shown above in (4.11). For the second part, one first shows (see [13], page 1985) that if $\delta D=[A,D]$, *then* $\delta D^{\frac{k}{n}}=[A,D^{\frac{k}{n}}]$, *hence* $\delta\left(D^{\frac{k}{n}}\right)_+=\left(\left[A,D^{\frac{k}{n}}\right]\right)_+$. *Then using the Jacobi identity, one derives after some algebra*

$$[\delta_k,\delta_\ell]D=\left[[(D^{\frac{k}{n}})_+,(D^{\frac{\ell}{n}})_+],D\right]=0.\tag{4.35}$$

(iii): The traces $I_\ell=\mathrm{Tr}\,(D^{\frac{\ell}{n}})$ form an infinite set of conserved quantities.

As stated above, $\delta_k D^{\frac{\ell}{n}}=\left[\left(D^{\frac{k}{n}}\right)_+,D^{\frac{\ell}{n}}\right]$. *Thus its trace vanishes (see lemma (4.28)).*

Thus with any differential operator D one may associate a hierarchy of integrable flows $\delta_k D$, generalizing the KdV case where $D=\mathrm{d}^2+u$. We shall see now that these hierarchies have also an interesting Hamiltonian interpretation: this will provide generalizations of the (classical) Virasoro algebra encountered for $n=2$.

(iv): There exist two Hamiltonian structures reproducing the flows $\delta_k D$.

The Hamiltonian structures are first defined through their action on *linear* functionals of the coefficients $a_2,\cdots,a_n$ of D and then extended by differentiation to arbitrary polynomial functionals. Let $l_U(D)$ be a linear functional.

$$\begin{aligned}l_U(D)=&\int dx\sum_{i=2}^{n}u_i(x)a_i(x)\\ =&\mathrm{Tr}\,(\mathrm{d}^n+a_2\mathrm{d}^{n-2}+\cdots+a_n)(\mathrm{d}^{1-n}u_2+\mathrm{d}^{2-n}u_3+\cdots+\mathrm{d}^{-1}u_n)\\ =&\mathrm{Tr}\,DU\end{aligned}\tag{4.36}$$

where U denotes the expression in brackets. Since a_1 vanishes one can freely add to U a term of the form $\mathrm{d}^{-n}u_1$. The two Hamiltonian structures discussed in [18][19][12][13] read

$$\{l_U(D),l_V(D)\}^{(1)}=\mathrm{Tr}\left(D[U,V]\right)=l_V(DU-UD)\tag{4.37a}$$

$$\begin{aligned}\{l_U(D),l_V(D)\}^{(2)}=&\mathrm{Tr}\left((DU)_+(DV)-(VD)(UD)_+\right)\\ =&l_V\left((DU)_+D-D(UD)_+\right)\end{aligned}\tag{4.37b}$$

We want to use these Hamiltonian structures to define deformations of the linear functional $l_V(D)$, hence of D itself through $\delta l_V(D) = \{\mathcal{H}, l_V\}$, for some $\mathcal{H}$ (taken for the time being as the functional $l_U(D)$). Some care has to be exercised, however, when using the second Poisson bracket on differential operators D with a vanishing coefficient a_1. It is not generally true that the expression $\big((DU)_+D - D(UD)_+\big)$ respects this property. One may cure this by adding a further term to U [19] :

$$\widehat{U} = U + d^{-n}u_1 \tag{4.38}$$

which does not affect $l_U(D)$ but does modify the second Hamiltonian structure. Adjusting the value of u_1 to

$$u_1 = \frac{1}{n}\int^x \mathrm{Res}\,[U, D] \tag{4.39}$$

is essential to remove the unwanted term of order $n-1$ in $\big((DU)_+D - D(UD)_+\big)$.

One may then extend these formulae to Poisson brackets between a linear functional $l_U(D)$ and an arbitrary polynomial functional $\Psi(D)$ as follows

$$\{\Psi(D), l_U(D)\} = \{l_{V_\Psi}(D), l_U(D)\} \tag{4.40}$$

where

$$V_\Psi = \sum_{i=1}^{n-1} \mathrm{d}^{-i}\frac{\delta\Psi(D)}{\delta a_{n-i+1}} \tag{4.41}$$

and $\frac{\delta}{\delta a_j}$ is a short-hand notation for $\sum_k(-\mathrm{d})^k\frac{\delta}{\delta a_j^{(k)}}$. Note that we are extending the application of (4.37) to the functional $l_{V_\Psi}(D) = \mathrm{Tr}\,(V_\Psi D)$ which is not linear. Finally one may verify [13] that these hamiltonian structures reproduce the flows as

$$\delta_k D = \{I_k, D\}^{(1)} = \{I_{k-1}, D\}^{(2)} \tag{4.42}$$

The form of these two Hamiltonian structures looks at this stage fairly mysterious. That they satisfy the Jacobi identity, (or even antisymmetry for the second one) is far from obvious. (They are also "coordinated", meaning that any linear combination of them is an acceptable Poisson bracket [13], p. 1982). We shall now show that one may reproduce in a natural way these structures, starting from a matrix representation of the differential operators.

4.4. Drinfeld-Sokolov formalism.

Following Drinfeld and Sokolov [13], we substitute for D a $n \times n$ first order differential operator,

$$\widehat{D} = \begin{pmatrix} \mathrm{d} & a_2 & a_3 & \cdots & a_n \\ -1 & \mathrm{d} & 0 & \cdots & 0 \\ 0 & -1 & \mathrm{d} & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & -1 & \mathrm{d} \end{pmatrix} \tag{4.43}$$

i.e. $\widehat{D} = \mathrm{d} + \mathcal{A}$ where the matrix $\mathcal{A}$ may be regarded as belonging to the Lie algebra $A_{n-1} = sl(n)$. This is the basis of an extension of all our considerations to general Lie algebras, a task that we shall not pursue in this presentation (see [13][20][21]). The operator (4.43) is equivalent to the original D in (4.29) in the sense that their kernels are in one-to-one correspondence. If $\underline{f} \in \ker \widehat{D}$, its last component belongs to $\ker D$ and conversely from an element in $\ker D$ we can construct one in $\ker \widehat{D}$.

As emphasized in [13], the above form is by no means unique. There is covariance under gauge transformations of the type

$$\widehat{D} \to N^{-1} \widehat{D} N \tag{4.44}$$

where N is a x-dependent element of $\mathcal{T}$, the group of upper triangular matrices with 1's on the diagonal (its nilpotent Lie algebra of strictly upper triangular matrices is denoted by T). The above transformation does not affect the lowest component of the vector $\underline{f}$ and makes linear combinations of its other components, thus preserves the isomorphism between $\ker D$ and $\ker \widehat{D}$. To be more explicit let

$$J_- = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & 1 & 0 \end{pmatrix} \qquad \mathcal{A} = \begin{pmatrix} 0 & a_2 & a_3 & \cdots & a_n \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$\widehat{D} = -J_- + \mathrm{d} + \mathcal{A} \tag{4.45}$$

Under the action of N

$$\widehat{D} \to -J_- + \mathrm{d} + \left\{ N^{-1} \mathcal{A} N + N^{-1} \big(\mathrm{d}N + [N, J_-] \big) \right\} \tag{4.46}$$

where the matrix in curly brackets is again upper triangular, but not confined to the first row any more. Conversely, given a matrix operator of the form $\mathrm{d} - J_- + \mathcal{A}$, with $\mathcal{A}$ upper triangular, it has a unique representative of the form (4.43) in the orbit under $\mathcal{T}$.

We want to define Poisson brackets on gauge invariant functionals of these matrix differential operators, i.e. functionals $s(\mathcal{A})$ of the $sl(n)$-valued matrix $\mathcal{A}(x)$ invariant under (4.46). (Notice the problem may be regarded as originating from one with an arbitrary differential operator $\mathrm{d}\mathbf{1} + A$, where $A \in sl(n)$ is constrained by the requirements of equ. (4.43). As usual, the constraints have to be accompanied by a gauge fixing procedure. See [22][20] for a further discussion of this "Hamiltonian reduction".) If we were dealing with the case $n = 1$, (ordinary functions $a(x)$), a natural Poisson bracket would be

$$\{a(x), a(y)\} = \partial_x \delta(x - y) \tag{4.47}$$

i.e. for two functionals $s_1(a)$ and $s_2(a)$:

$$\begin{aligned} \{s_1, s_2\} &= \oint dxdy \, \frac{\delta s_1}{\delta a(x)} \frac{\delta s_2}{\delta a(y)} \partial_x \delta(x - y) \\ &= \oint dx \, \frac{\delta s_1}{\delta a(x)} \partial_x \frac{\delta s_2}{\delta a(x)} \end{aligned} \tag{4.48}$$

which is antisymmetric and satisfies the Jacobi identity. In the matrix case $n > 1$, it is very natural to generalize this to

$$\{s_1, s_2\}^{(1)} = \oint dx \, \mathrm{tr}\Big(\frac{\delta s_1}{\delta \mathcal{A}(x)} [\mathcal{B}, \frac{\delta s_2}{\delta \mathcal{A}(x)}]\Big) \tag{4.49a}$$

$$\{s_1, s_2\}^{(2)} = \oint dx \, \mathrm{tr}\Big(\frac{\delta s_1}{\delta \mathcal{A}(x)} [\mathrm{d} + \mathcal{A}(x), \frac{\delta s_2}{\delta \mathcal{A}(x)}]\Big) \tag{4.49b}$$

where $\frac{\delta s}{\delta \mathcal{A}(x)}$ is the functional derivative (denoted $\mathrm{grad}_{\mathcal{A}} s$ in [13]):

$$s(\mathcal{A} + \delta\mathcal{A}) \approx s(\mathcal{A}) + \oint dx \, \mathrm{tr}\Big(\frac{\delta s}{\delta \mathcal{A}(x)} \delta\mathcal{A}(x)\Big). \tag{4.50}$$

Note that $\frac{\delta s}{\delta \mathcal{A}}$ is in fact defined modulo T if the only allowed $\delta\mathcal{A}$ are upper triangular. In (4.49a) $\mathcal{B}$ denotes a constant matrix taken to be

$$\mathcal{B} = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \end{pmatrix} \tag{4.51}$$

This gives rise to two "coordinated" Hamiltonian structures: the antisymmetry is obvious, and it is sufficient to verify the Jacobi identity on linear functionals.

One has actually to verify the consistency of the definition (4.49), namely its independence with respect to the indeterminacy of $\frac{\delta s}{\delta \mathcal{A}} \bmod T$, and its gauge invariance. Both are simple consequences of the gauge invariance of the functionals $s_1(\mathcal{A})$ and $s_2(\mathcal{A})$ and of the commutation of $\mathcal{B}$ with matrices in T. Finally after a painstaking calculation ([13] p. 1996, see also [23]), one finds that these Hamiltonian structures reduce to (4.37) when reexpressed in terms of the original differential operator D. In the following, we shall be only concerned with the second one, because of its remarkable connection with the Virasoro algebra or its extensions.

5. Covariant differential operators and W-algebras.

5.1. Covariance properties.

We return to the linear differential operators of the form

$$D = \mathrm{d}^n + \sum_{j=2}^{n} a_j \, \mathrm{d}^{n-j} \tag{5.1}$$

acting on functions $f(x)$. We are interested to study how D and its coefficients $a_j(x)$, $j = 2, \cdots, n$, transform under changes of variables $x \to \widetilde{x}$. Let $\mathcal{F}_h$ denote the space of functions that transform as h-differentials (conformal weight h), i.e. such that in two coordinates x and $\widetilde{x}$ they are represented by f and $\widetilde{f}$ with

$$\widetilde{f}(\widetilde{x}) \, d\widetilde{x}^h = f(x) dx^h. \tag{5.2}$$

We claim that there exists a natural transformation of the functions $a_2, \cdots, a_n$ such that the operator D maps the space $\mathcal{F}_{-\frac{n-1}{2}}$ into the space $\mathcal{F}_{\frac{n+1}{2}}$. To show this, let $f_1, f_2, \cdots, f_n$ be n linearly independent functions in the kernel of D. Since a_1, the logarithmic derivative of their wronskian W vanishes, W is a constant and by a change of normalization of the $f's$, can be set equal to 1

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1^{(n-1)} & \cdots & f_n^{(n-1)} \\ f_1^{(n-2)} & \cdots & f_n^{(n-2)} \\ \vdots & \ddots & \vdots \\ f_1 & \cdots & f_n \end{vmatrix} = 1. \tag{5.3}$$

The differential operator D may then be defined by its action on the function f according to

$$[Df] = \begin{vmatrix} f^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \\ f^{(n-1)} & f_1^{(n-1)} & \cdots & f_n^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f & f_1 & \cdots & f_n \end{vmatrix}. \tag{5.4}$$

D is clearly of the form (5.1), and it is a simple lemma [24] that if $f_1, f_2, \cdots, f_n$ and f belong to $\mathcal{F}_h$, then $W(f_1, f_2, \cdots, f_n)$ belongs to $\mathcal{F}_{nh+\frac{n(n-1)}{2}}$ and $[Df]$ to $\mathcal{F}_{(n+1)h+\frac{n(n+1)}{2}}$. The choice of $h = -\frac{n-1}{2}$ preserves the condition (5.3). (It also makes fDf a density, i.e. $\int dx f Df$ invariant under changes of coordinate). By identification of the coefficients $a_2, a_3, \cdots, a_n$ with minors of the determinant (5.4), one finds their transformation law, and the previous assertion follows.

In particular, a_2 does not transform as a 2-differential but has an "anomalous term" proportional to the schwarzian derivative $\mathcal{S}\{x, \widetilde{x}\}$ of the change of variable

$$\begin{aligned} \widetilde{a}_2(\widetilde{x}) =& a_2(x)\big(\frac{dx}{d\widetilde{x}}\big)^2 + \frac{n(n^2-1)}{12}\mathcal{S}\{x, \widetilde{x}\} \\ \mathcal{S}\{x, \widetilde{x}\} =& \frac{x'''}{x'} - \frac{3}{2}\Big(\frac{x''}{x'}\Big)^2 \end{aligned} \tag{5.5}$$

where primes on x denote derivatives with respect to $\widetilde{x}$. This is reminiscent of the transformation (2.22) law of the energy-momentum "tensor": it is not an accident, and is related to the Poisson structure of (4.4b) as we shall see below. We recall that under composition of changes of variable, $u \to x \to \widetilde{x}$, the schwarzian derivatives transform according to:

$$\mathcal{S}\{u, \widetilde{x}\} = \mathcal{S}\{u, x\}\big(\frac{dx}{d\widetilde{x}}\big)^2 + \mathcal{S}\{x, \widetilde{x}\} \tag{5.6}$$

which implies the consistency of (5.5) and shows that $a_2(x)$ transforms as $\frac{n(n^2-1)}{12}\mathcal{S}\{u, x\}$, with u a fixed coordinate for which a_2 vanishes. The other coefficients $a_3, \cdots, a_n$ of (5.1) have more complicated transformations involving higher and higher derivatives of the jacobian $\frac{dx}{d\widetilde{x}}$. One may prove, however, that there exists an invertible transformation to differential polynomials $w_k \in \mathcal{F}_k$, $3 \leq k \leq n$, i.e. polynomials of $a_k, a_{k-1}, \cdots, a_2$ and their derivatives that transform as k-differentials. This transformation $a_k \to w_k$ is non unique. The choice made in [21] is to take the w_k, $k \geq 3$ to be linear functionals of a_l, $3 \leq l \leq k$. This is achieved by a splitting of the differential operator in pieces $\Delta_k^{(n)}$ that are separately

covariant i.e. map $\mathcal{F}_{\frac{1-n}{2}}$ into $\mathcal{F}_{\frac{1+n}{2}}$ and depend on w_k $(k \geq 3)$ (and its derivatives) in a linear way:

$$D = \Delta_2^{(n)}(a_2) + \Delta_3^{(n)}(w_3, a_2) + \Delta_4^{(n)}(w_4, a_2) + \cdots \tag{5.7}$$

The key idea of the proof is to choose the special coordinate u in which $a_2(u) \equiv 0$. This requires solving the differential equation $b'(x) - \frac{1}{2}b^2(x) = 12a_2(x)/n(n^2-1)$ where $b(x)$ is the logarithmic derivative ϕ'/ϕ of the jacobian $\phi(x) = du/dx$. In this special coordinate, $\Delta_2^{(n)}(0) = \mathrm{d}_u^n$ and by covariance in the generic variable x it must read

$$\begin{aligned}\Delta_2(a_2) =& \phi^{\frac{n+1}{2}}\left(\phi^{-1}\mathrm{d}\right)^n \phi^{\frac{n-1}{2}} \\ =& (\mathrm{d} - jb)(\mathrm{d} - (j-1)b)\cdots(\mathrm{d} + jb).\end{aligned} \tag{5.8}$$

where we have set $n = 2j+1$. For the consistency of this argument, we have to prove that the expression (5.8) depends upon b only through the schwarzian derivative $s = b' - \frac{1}{2}b^2$ (and its derivatives) proportional to a_2.

Proof: Δ_2 *is a differential operator with coefficients that are polynomials in* b *and its derivatives and may be expressed as polynomials in* b, s *and derivatives of* s. *The proof amounts to showing that these polynomials reduce to their term independent of* b. *To see this, in the expression (5.8) we change* b *into* $b + \delta b$, *keeping* $s = b' - \frac{1}{2}b^2$ *fixed. This implies that* δb *satisfies the equation* $\delta b' - b\delta b = 0$, *or equivalently the commutation relation between differential operators*

$$(\mathrm{d} - (k+1)b)\delta b = \delta b(\mathrm{d} - kb) \tag{5.9}$$

for any k. *The change of* Δ_2 *is thus*

$$\begin{aligned}\delta\Delta_2 =& \sum_{k=-j}^{j} (\mathrm{d} - jb)\cdots(\mathrm{d} - (k+1)b)(-k\delta b)(\mathrm{d} - (k-1)b)\cdots(\mathrm{d} + jb) \\ =& \left(-\sum_{-j}^{j} k\right)\delta b\,(\mathrm{d} - (j-1)b)\cdots(\mathrm{d} + jb) \\ =& 0.\end{aligned} \tag{5.10}$$

Under a change of variable, the operator Δ_2 transforms covariantly, thanks to the transformation properties of the schwarzian derivative (5.6).

The same method applies to the other operators $\Delta_k(w_k, a_2)$, but now in a constructive way. A general expression is written for $\Delta_k(w_k, a_2 = 0)$, $k \geq 3$ of the form

$$\Delta_k(w_k, 0) = \sum_{l=0}^{n-k} \alpha_{kl} w_k^{(l)} \mathrm{d}^{n-k-l}$$

and the coefficients α_{kl} will be determined by requiring that in a generic coordinate Δ_k transforms covariantly and depends only on a_2

$$\begin{aligned}\Delta_k(w_k(x), a_2(x)) =& \phi^{\frac{n+1}{2}} \sum_{l=0}^{n-k} \alpha_{kl} \left[\left(\phi^{-1}\mathrm{d}\right)^l \phi^{-k} w_k \right] \left(\phi^{-1}\mathrm{d}\right)^{n-k-l} \phi^{\frac{n-1}{2}} \\ =& \sum_{l=0}^{n-k} \alpha_{kl} \left[\mathcal{D}^l w_k\right] \mathcal{D}^{n-k-l}. \end{aligned} \tag{5.11}$$

We have introduced the covariant derivative taking h-differentials to $h+1$-differentials:

$$\mathcal{D} f = (\mathrm{d} - hb) f \tag{5.12}$$

thus, $\mathcal{D} w_k = (\mathrm{d} - kb) w_k$, $\mathcal{D}^2 w_k = (\mathrm{d} - (k+1)b)(\mathrm{d} - kb) w_k$, etc... and $\mathcal{D}^{n-k-l}$ in (5.11) maps $\mathcal{F}_{-\frac{n-1}{2}}$ into $\mathcal{F}_{\frac{n+1}{2}-k-l}$. The square brackets in $\left[\mathcal{D}^l w_k\right]$ mean that $\mathcal{D}^l$ does not act further to the right. By differentiating w.r.t. b with a_2 fixed as above, one finds that the coefficients α_{kl} are uniquely determined to be

$$k \geq 3 \qquad \alpha_{kl} = \frac{\binom{k+l-1}{l}\binom{n-k}{l}}{\binom{2k+l-1}{l}} \tag{5.13}$$

(with the normalization $\alpha_{k,0} = 1$). Finally, the identification of the operator D with the sum (5.7) provides the expression of w_k as a (linear) functional of $a_3, \cdots, a_k$. In the coordinate u where $a_2 \equiv 0$

$$w_k(u) = \sum_{l=3}^{k} (-1)^{k-l} \frac{\binom{k-1}{k-l}\binom{n-l}{k-l}}{\binom{2k-2}{k-l}} a_l^{(k-l)}(u) \tag{5.14}$$

and from this the general expression may be restored by the same method.

One may wonder what is the general expression of the polynomials in the a_k's and their derivatives that transform as r-differentials, with r integer larger than 2. Their form may be obtained following the same method as used above to construct the w's: write the expression in the coordinate where $a_2 = 0$ as a differential polynomial in the w's; return to

the generic coordinate, transforming the derivatives into covariant derivatives; derive the conditions on the coefficients that enable one to reconstruct a r-differential depending only on a_2. For example, $kw_l'w_k - lw_lw_k'$ is a (k+l+1)-differential. It is not difficult to compute the dimension $N(r)$ of the space of these r-differentials. One finds [25] that a generating function of the N's is

$$\sum_{r\geq 3} N(r)q^r = q + \frac{1-q}{\prod_{h=3}^{n}\prod_{l\geq 0}\left(1-q^{h+l}\right)}. \tag{5.15}$$

The first $N(r)$ are given in the following Table:

r	3	4	5	6	7	8	9	10
$N(r)$	1	1	1	2	2	4	6	11

Let us return to the matrix form (4.43) of the differential operator (with $a_1 \equiv 0$). We can use the gauge freedom to bring the matrix $\widehat{D}$ into another form, exhibiting another basis of differentials w's. The matrix $-J_- + \mathcal{A} \equiv A$ belongs to the Lie algebra A_{n-1} of traceless matrices graded according to

$$\text{grade}(A_{ij}) = j - i \tag{5.16}$$

so that $\text{grade}(J_-) = -1$. In the nilpotent algebra T of strictly upper triangular matrices graded as in (5.16)

$$T = \oplus_{k=1}^{n-1} T^{(k)}. \tag{5.17}$$

the operator $\text{ad}\, J_-$ induces an injective map

$$\text{ad}\, J_- : \qquad T^{(k)} \to T^{(k-1)} \qquad 2 \leq k \leq n-1 \tag{5.18}$$

and maps $T^{(1)}$ on the Cartan subalgebra (traceless diagonal matrices). Since $T^{(k)}$ has dimension $n-k$, we have

$$\dim\left(T^{(k)}/\text{ad}\, J_-(T^{(k+1)})\right) = 1. \tag{5.19}$$

Thus if one chooses in $T^{(k)}$ a representative R_k of $T^{(k)}$mod $\text{ad}\, J_-(T^{(k+1)})$

$$T^{(k)} \sim \mathbf{C}R_k \oplus \text{ad}\, J_-(T^{(k+1)}) \tag{5.20}$$

one may use the gauge transformations (4.46) to bring $\mathcal{A}$ in the form

$$\mathcal{A}(x) = \sum_{k=1}^{n-1} r_k(x) R_k. \tag{5.21}$$

The initial choice was to take for R_k the matrix which has a unique non zero entry 1 in the first row, column $k+1$. Since an element in $\operatorname{ad} J_-(T^{(k+1)}) \subset T^{(k)}$ is characterized by the fact that the sum of its entries in the grade k principal diagonal vanishes, this choice is perfectly justified.

However there is another appealing possibility. The n-dimensional vector space carries also an irreducible representation of $sl(2)$ of spin $j = \frac{n-1}{2}$. The Lie algebra is spanned by J_- as in (4.45), and J_0, J_+

$$J_+ = \begin{pmatrix} 0 & (n-1).1 & 0 & \cdots & 0 \\ 0 & 0 & (n-2).2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & \cdots & 0 & 1.(n-1) \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$$J_0 = \begin{pmatrix} j & 0 & \cdots & 0 \\ 0 & j-1 & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -j \end{pmatrix} \tag{5.22}$$

satisfying the usual commutation properties:

$$[J_+, J_-] = 2J_0 \qquad\qquad [J_0, J_\pm] = \pm J_\pm. \tag{5.23}$$

Thus

$$J_+^k \in T^{(k)}. \tag{5.24}$$

The sum of entries of J_+^k

$$\begin{aligned} \sigma_k &= \sum_{i=1}^{n-k} i(n-i)(i+1)(n-i-1)\cdots(i+k-1)(n-i-k+1) \\ &= k!^2 \binom{n+k}{2k+1} = \frac{k!^2}{(2k+1)!} n(n^2-1)\cdots(n^2-k^2) \end{aligned} \tag{5.25}$$

is non vanishing; for a proof of this formula, see [26]. Therefore we can choose J_+^k as the element R_k above. We then write, and this defines the W's [20][27]:

$$\widehat{D} = -J_- + \mathrm{d} + \sum_{k=1}^{n-1} W_{k+1} J_+^k. \tag{5.26}$$

Let D be the corresponding n-th order operator. As the notation suggests, the W_k's are an alternative basis of k-differentials ($k > 2$). In other words,

(i) for $3 \leq k \leq n$, $W_k \in \mathcal{F}_k$, i.e. in a change of coordinate $x \to \widetilde{x}$:

$$\widetilde{W}_k(\widetilde{x})\mathrm{d}\widetilde{x}^k = W_k(x)\mathrm{d}x^k$$

(ii) $a_2 = \sigma_1 W_2 = \frac{n(n^2-1)}{6} W_2$, hence

$$\widetilde{W}_2(\widetilde{x})\mathrm{d}\widetilde{x}^2 = W_2(x)\mathrm{d}x^2 + \frac{1}{2}\{x, \widetilde{x}\}\mathrm{d}\widetilde{x}^2$$

(iii) The W_k, $3 \leq k \leq n$ generate the graded ring R of r-differentials, r integral ≥ 3.

To prove this, a simple possibility is to consider the operator

$$\check{D} = -J_- + (\mathrm{d} - bJ_0) + \sum_{k=2}^{n-1} W_{k+1} J_+^k \tag{5.27}$$

where the W_k's are assumed to be k-differentials. We shall show that this $\check{D}$ is gauge equivalent to $\widehat{D}$. Define the n-dimensional vectors $\underline{\check{f}} = (\check{f}_j, \cdots, \check{f}_{-j})^T$ and $\underline{\check{F}} = (\check{F}, 0, \cdots, 0)^T$, where $\check{f}_{-j}$ is a $-j$-differential, and the other $\check{f}_k$ are determined by the relation $\underline{\check{F}} = \check{D}\underline{\check{f}}$. Then one proves by induction that $\check{f}_k$ is a k-differential, and $\mathrm{d} - bJ_0$ acts as covariant derivative on $\underline{\check{f}}$, so that $\underline{\check{F}}$ is a $j+1$-differential. The operator $\widehat{D}$ of eq. (5.26) is then obtained from $\check{D}$ by the gauge transformation

$$\widehat{D} = e^{-\frac{b}{2}J_+} \check{D} e^{\frac{b}{2}J_+} \tag{5.28}$$

provided $W_2 = \frac{b'}{2} - \frac{b^2}{4}$. This gauge transformation does not affect the components $\check{f}_{-j}$ and $\check{F}$ and makes thus D a covariant operator from $\mathcal{F}_{\frac{1-n}{2}}$ to $\mathcal{F}_{\frac{1+n}{2}}$, which completes the proof of the statement.

How are these two bases of k-differentials thus constructed, the w's and the W's, related? The lowest ones differ just by a change of normalization but for k large enough, the relation is non linear:

$$w_k = \sigma_{k-1} W_k \qquad 2 \leq k \leq 5 \tag{5.29}$$

whereas for $k \geq 6$ and for instance for $n = 6$

$$w_6 = 5!^2 (W_6 + \frac{1}{9} W_3^2) \tag{5.30}$$

See [26] for additional formulae.

5.2. Infinitesimal deformations of the differential operator

In its infinitesimal form, the previous analysis is a particular case of the following problem: find two infinitesimal differential operators X and Y mapping $\mathcal{F}_{-\frac{n-1}{2}}$ resp. $\mathcal{F}_{\frac{n+1}{2}}$ onto themselves, such that after a change of functions: $g = (1+X)f$ and $G = (1+Y)F$, the equation $F = Df$ takes the form $G = (D+\delta D)g$, with $D+\delta D$ still of the form (5.1). The variation of D is thus given by:

$$\delta D = YD - DX \tag{5.31}$$

The particular case $X = Y = D^{\frac{k}{n}}$ would correspond to the k-th KdV flow discussed in sect. 2.3, for which the variations δ_k all commute. In the general situation considered here, they do not. And indeed we are dealing here with a different kind of deformations of the differential operator, that preserve its *covariance properties*. The KdV flows, on the other hand, are *isospectral* flows, a concept that is at odds with the one of covariance: since the operator maps a space $\mathcal{F}_{-\frac{n-1}{2}}$ into a *different one* $\mathcal{F}_{\frac{n+1}{2}}$, the concepts of eigenvalue and of isospectrality are non invariant under changes of coordinates.

As mentionned above, the infinitesimal changes of coordinates offer us the first example of such pairs of operators X_1 and Y_1 of order one. The changes of the variable $x \to x + \epsilon(x)$ are generated on $\mathcal{F}_{-\frac{n-1}{2}}$ resp $\mathcal{F}_{\frac{n+1}{2}}$ by:

$$X_1 = \epsilon \mathrm{d} - \frac{n-1}{2}\epsilon' \tag{5.32a}$$

$$Y_1 = \epsilon \mathrm{d} + \frac{n+1}{2}\epsilon' \tag{5.32b}$$

implying:

$$\delta_1 D = Y_1 D - DX_1 \tag{5.33}$$

which summarizes the transformations of the coefficients of D under a change of variable.

More generally we look for deformations (5.31) generated by higher degree differential operators X and Y. Multiplying (5.31) by D^{-1} on the right, one finds $Y = DXD^{-1} + \delta D D^{-1}$. One may take the differential part showing that $Y = (DXD^{-1})_+$ is completely determined by X and of the same order. Also, taking the residue one finds that $\operatorname{res} DXD^{-1} = 0$, hence the term of order zero x_0 in $X = \widetilde{X} + x_0$ is not independent: $\operatorname{res} D\widetilde{X}D^{-1} = -nx_0'$. This may be integrated to

$$X = \widetilde{X} - \frac{1}{n}\int^x \operatorname{res}[D, \widetilde{X}D^{-1}] \tag{5.34}$$

We conclude that the most general variation of the form (5.31) is built from an arbitrary differential operator $\widetilde{X}$. If we add to X any multiple ZD of D and add DZ to Y, δD is unaffected. By use of the Euclidean division algorithm, this implies that there is no loss of generality to restrict X and Y to be of order less or equal to $n-1$. The most general variation δD thus depends on $n-1$ independent functions. Note that the number $n-1$ is nothing else than the rank of $sl(n)$. We shall denote $\delta_X D$ the variation (5.31) acting on D.

The commutators of δ's follow from the definition (5.31):

$$[\delta_X, \delta_{X'}] = \delta_{[X,X']+\delta_{X'}X-\delta_X X'} \tag{5.35}$$

where $\delta_X X'$ for example denotes the variation δ_X of X' which may be a functional of D. We now want to define a basis $\delta_k(\eta) = \delta_{X_k(\eta)}$ such that:

$$[\delta_1(\epsilon), \delta_k(\eta)] = \delta_k(\epsilon\eta' - k\eta\epsilon') \tag{5.36}$$

which amounts to saying that η transforms as a $-k$-differential under changes of variable. The corresponding X_k and Y_k are covariant operators mapping $\mathcal{F}_{-\frac{n-1}{2}}$, resp. $\mathcal{F}_{\frac{n+1}{2}}$ into themselves, and could be constructed by a method similar to that of the previous section. We shall rather determine them by using a Hamiltonian language. The variations δ_k may be generated by Hamiltonians of the form $H_k = \int dx\, \eta(x) w_{k+1}(x)$, through the second Poisson structure studied in sect. 2. (Notice that the fact that η is a $-k$-differential guarantees the invariance of the former integral.)

5.3. Explicit formulae for the X_k and Y_k.

The comparison of (5.31) with (4.37*b*) suggests that we can identify the expressions of X_k and Y_k from the one of w_{k+1}. Taking $\Psi(D) = H_k = \int dx\, \epsilon w_{k+1}$, we get:

$$\begin{aligned} \delta_k l_U(D) =& \left\{ \int dx\, \epsilon w_{k+1}, l_U(D) \right\}^{(2)} \\ =& l_U(\delta_k D) \\ \delta_k D =& \big(DV_k\big)_+ D - D\big(V_k D\big)_+ \end{aligned} \tag{5.37}$$

where $V_k \equiv \widehat{V}_{H_k}$ with the notations of (4.38)-(4.41). It is now easy to identify $X_k = (V_k D)_+$ and $Y_k = (DV_k)_+$. Knowing w_{k+1}, this gives compact expressions for the parts $\widetilde{X}_k$ and $\widetilde{Y}_k$ of valuation one of X_k and Y_k,

$$\widetilde{X}_k = \Big(\big(\sum_{i=1}^{n-1} \mathrm{d}^{-i} \frac{\delta H_k}{\delta a_{n-i+1}} \big) \big(\mathrm{d}^n + \sum_{j=2}^{n} a_j \mathrm{d}^{n-j} \big) \Big)_{++} \tag{5.38a}$$

$$\widetilde{Y}_k = \Big(\big(\mathrm{d}^n + \sum_{j=2}^{n} a_j \mathrm{d}^{n-j} \big) \big(\sum_{i=1}^{n-1} \mathrm{d}^{-i} \frac{\Delta H_k}{\Delta a_{n-i+1}} \big) \Big)_{++} \tag{5.38b}$$

The ++ subscript means that we keep only the contribution of valuation one. From (5.38) the full expression of X_k and Y_k may be reconstructed as explained in (5.34).

Table II: The generators X_k and Y_k, $SU(n)$ case

$$\begin{aligned}
X_1 =& \epsilon \mathrm{d} - \frac{n-1}{2}\epsilon' \\
X_2 =& \epsilon \mathrm{d}^2 - \frac{n-2}{2}\epsilon' \mathrm{d} + \big\{ \frac{2}{n}\epsilon a_2 + \frac{1}{12}(n-1)(n-2)\epsilon'' \big\} \\
X_3 =& \epsilon \mathrm{d}^3 - \frac{n-3}{2}\epsilon' \mathrm{d}^2 + \big\{ \frac{(n-2)(n-3)}{10}\epsilon'' + \frac{6}{5}\frac{3n^2-7}{n(n^2-1)} a_2 \epsilon \big\} \mathrm{d} \\
&+ \big\{ \frac{3}{n} w_3 \epsilon - \frac{3(n+2)(n-7)}{10n(n+1)} a_2' \epsilon - \frac{(n-3)(4n+7)}{5n(n+1)} a_2 \epsilon' \\
&\qquad - \frac{(n-1)(n-2)(n-3)}{5!}\epsilon''' \big\}
\end{aligned}$$

A sample of the first X_k is displayed in Table II; the expression of the corresponding Y's is simply obtained by conjugation: $Y_k = (-1)^k X_k^*$. For ϵ x-independent and $k = 1, 2$, $X_k = Y_k$ coincide with $D^{\frac{k}{n}}$ of the KdV flows (and accordingly, the generators w_{k+1}, $k = 1, 2$ coincide with $\mathrm{res}\left(D^{\frac{k}{n}}\right)_+$). It would thus appear that the W-flows are a sort of local (x-dependent) extension of the KdV flows. For general reasons explained at the beginning of this section, this is not so in general.

5.4. $W-$algebra.

With the explicit expressions of the w's, the X's and the Y's at our disposal, we can now form the Poisson brackets of the w's among themselves. In general $\{w_k(x), w_l(y)\}$ is by

construction a sum of monomials in the w's and their derivatives times a derivative of $\delta(x-y)$. The set of Poisson brackets $\{w_k, w_l\}$, $k,l = 2,\cdots,n$ defines the W-algebra (of type A_{n-1}). It always contains the (classical) Virasoro algebra generated by a_2 and the relations expressing that the w_k, $k \geq 3$, transform as k-differentials:

$$\{a_2(y), a_2(x)\} = \big(a_2'(x) + 2a_2(x)\mathrm{d} + c_n \mathrm{d}^3\big)\delta(x-y) \tag{5.39a}$$

$$\{a_2(y), w_k(x)\} = \big(w_k'(x) + k w_k(x)\mathrm{d}\big)\delta(x-y) \tag{5.39b}$$

As for the other brackets $\{w_k, w_l\}$, $k,l \geq 3$, it is easier to compute and tabulate them again in the coordinate u where a_2 vanishes. According to an argument used repeatedly in these notes, if

$$\{w_k(u), w_l(v)\}|_{a_2=0} = \Delta(w_j, \mathrm{d}_u)\delta(u-v) \tag{5.40}$$

with Δ some differential operator, then in the generic coordinate

$$\{w_k(x), w_l(y)\} = \phi^k \Delta(\phi^{-j} w_j, \phi^{-1}\mathrm{d})\phi^{l-1}\delta(x-y). \tag{5.41}$$

(The δ-function has contributed an extra ϕ^{-1}). The operator Δ must satisfy certain constraints in order that the r.h.s. of (5.41) depends only on the schwarzian derivative of the change of coordinate. Let us illustrate these considerations on the set of Poisson brackets $\{w_k, w_l\}$, $k,l = 3,4$, for generic n.

$$\begin{aligned}
\{w_3(v),w_3(u)\}|_{a_2=0} &= \Bigg(2[w_4, \mathrm{d}]_+ \\
&\quad - \frac{(n-2)(n-1)n(n+1)(n+2)}{6!}\mathrm{d}^5\Bigg)\delta(u-v) \\
\{w_3(v),w_4(u)\}|_{a_2=0} &= \Bigg(5w_5\mathrm{d} + 2w_5' \\
&\quad - \frac{(n-3)(n+3)}{70}(14w_3\mathrm{d}^3 + 14w_3'\mathrm{d}^2 + 6w_3''\mathrm{d} + w_3''')\Bigg)\delta(u-v) \\
\{w_4(v),w_4(u)\}|_{a_2=0} &= \Bigg(3[w_6, \mathrm{d}]_+ - \frac{n^2-19}{30}(3[w_4, \mathrm{d}^3]_+ - 2[w_4'', \mathrm{d}]_+) \\
&\quad -3\frac{n-3}{n}w_3\mathrm{d}w_3 + \frac{(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)}{20.7!}\mathrm{d}^7\Bigg)\delta(u-v)
\end{aligned} \tag{5.42}$$

where all the w's on the r.h.s. are evaluated at u and d stands for d/du. Notice that even for $a_2 = 0$, non linearities in the w's appear.

One general feature of the W-algebra which is provided by our approach is the form of the central term. One finds [21]

$$\{w_k(y), w_l(x)\}|_{\text{central term}} = \tag{5.43}$$
$$(n-k+1)(n-k+2)\cdots(n+k-1)\frac{(-1)^k\big((k-1)!\big)^2}{(2k-2)!(2k-1)!}\delta_{k,l}\mathrm{d}^{k+l-1}\delta(x-y).$$

All these formulae have been written explicitly in the basis provided by the w's. One could of course also list the corresponding Poisson brackets obtained using the basis W of sect. 3.1 [27]. The formulae would look slightly different, but the intrinsic algebra of deformations of the differential operators D generated by this other basis would be the same.

Let us finish this section with two comments on W-algebras.
W-algebras were originally introduced by Zamolodchikov [28] as extensions of the Virasoro algebra by higher spin generators. It was later suggested that the classical (Poisson bracket) version of these algebras must have to do with the KdV or Liouville-Toda hierarchies and may be obtained by the procedure of Hamiltonian reduction (see [20]-[22] for references). What we have been doing here was to identify precisely the generators of these classical algebras that transform as primary fields. There is another way to generate these W-algebras from Kac-Moody algebras, for values of the level opposite to the Coxeter number [29]. That these two constructions produce equivalent results is quite non trivial.
Another point that remains elusive is the geometric meaning of these W-transformations. We have seen that they may be regarded as describing deformations of differential operators that preserve their covariance, and the simplest case corresponds simply to changes of coordinates, but what is the meaning in more geometric terms? I refer the reader to [30][23][31] for attempts in this direction.

6. Singular vectors in representations of the Virasoro algebra.

6.1. Some basic properties of Verma modules of the Virasoro algebra.

We now return to the "quantum" Virasoro algebra (2.33b). We change slightly the notations of sect. 2 and denote f a highest weight vector:

$$\begin{aligned} L_0 f =& hf \\ L_m f =& 0 \qquad m>0 \end{aligned} \tag{6.1}$$

and use a shorthand for its descendants

$$f_{\{p\}} = L_{-\{p\}} f \equiv L_{-p_1} L_{-p_2} \cdots L_{-p_k} f \tag{6.2}$$

where $\{p\}$ is the multi-index $(p_1, p_2, \cdots, p_k)$, $1 \leq p_1 \leq p_2 \leq \cdots \leq p_k$. Let $|\{p\}| = \sum p_i$ be the "level" of this vector. If $L_{\{q\}} \equiv L_{q_1} L_{q_2} \cdots L_{q_l}$, with $|\{p\}| = |\{q\}|$, the vector $L_{\{q\}} L_{-\{p\}} f$ is of level zero, hence proportional to the highest weight vector f. This defines the matrix elements of the contravariant form:

$$\mathcal{M}_{\{q\},\{p\}} f = L_{\{q\}} L_{-\{p\}} f \tag{6.3}$$

on the $p(n)$-dimensional vector space of level n ($p(n)$ denotes the number of partitions of the integer n).

An important issue is the reducibility or irreducibility of this representation of the Virasoro algebra. One proves that $V(c,h)$ is reducible if and only if one of the three equivalent properties is satisfied:
(i) there exists in $V(c,h)$ at some level n a "singular" (or "null") vector F

$$F = \sum_{|\{p\}|=n} C_{\{p\}} f_{\{p\}} = \phi(L_{-1}, L_{-2}, \cdots) f \tag{6.4}$$

satisfying the same properties as a highest weight vector:

$$\begin{aligned} L_0 F =& (h+n) F \\ L_m F =& 0 \qquad m > 0 \end{aligned} \tag{6.5}$$

It is indeed clear that such a vector, if it exists, carries its own Verma module $V(c, h+n)$ which is an invariant submodule of $V(c,h)$.
(ii) the form $\mathcal{M}$ at level n is degenerate: $\det \mathcal{M} = 0$.
(iii) if one parametrizes c, h as follows

$$\begin{aligned} c =& 1 - \frac{6}{m(m+1)} \qquad\qquad m \in \mathbb{C} \\ h =& \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)} \qquad\qquad r, s \in \mathbb{N} \end{aligned} \tag{6.6}$$

then there exists a singular vector at level $n = rs$. Beware! In contrast with the proposition on unitary representations in (2.41), here $m \in \mathbb{C}$ and there is no restriction on the positive integers r and s.

Only the third part is (highly) non-trivial [32][24][33]. For a given c it asserts that the set of h giving reducible modules is *discrete*. Also for a given h there may be several pairs (r, s) such that one may write (6.6). The theorem implies the existence of a singular vector for each such pair. (Notice that this may happen only for $m \in \mathbb{Q}$).

In the following we shall find it useful to write the labels $r = 2j' + 1$ and $s = 2j + 1$, in a way suggesting a connection with spins of $SU(2)$, and to introduce the notations

$$t = -\frac{m}{m+1} = \theta^2. \tag{6.7}$$

The above formulae then read

$$\begin{aligned} c =& 1 + 6(\theta + \theta^{-1})^2 \\ h =& h_{j',j} = -(j\theta + j'\theta^{-1})\big((j+1)\theta + (j'+1)\theta^{-1}\big) \\ n =& (2j+1)(2j'+1) \end{aligned} \tag{6.8}$$

If $V(c, h)$ turns out to be reducible, then the irreducible part of V is constructed by modding out the singular vectors and their descendants:

$$M(c, h) = V(c, h)/\text{maximal invariant submodule}$$

This is what is usually done in conformal field theory [1] where the singular fields and their descendants are set equal to zero (whence the name "null"). It has important consequences as it leads to differential equations satisfied by the correlation functions of the theory.

It has been a long-standing problem in the representation theory of the Virasoro algebra to give an explicit expression of singular vectors, namely of the polynomial $\phi(L_{-1}, L_{-2}, \cdots)$ in (6.4), normalized by $\phi = L_{-1}^n + \cdots$. For small values of j, j', one can obtain such expressions by direct calculation. For example, for $n = 2$, $r = 1$, $s = 2$,

$$F = \big(L_{-1}^2 - \frac{m}{m+1} L_{-2}\big) f. \tag{6.9}$$

The goal is however to obtain universal formulas. Let us recall that, beside being important for the study of Verma modules, these null states plays an important role in conformal field theory. The condition that their correlation function $\langle F f_1 \cdots \rangle$ with other highest weight (primary) fields vanishes is transformed through the conformal Ward identities into partial differential equations satisfied by $\langle f f_1 \cdots \rangle$. Their explicit form is thus important in connection with the determination of correlation functions. In fact explicit expressions for these correlation functions have been obtained by the "Coulomb gas" (or free field) representation [34][35]. One should in principle be able to recover the differential equation, hence the null vector, from this information. We shall, however, propose a more direct route.

In fact there is a subfamily of singular vectors that have been known for some time [36]. They correspond to either r or s equal to one, i.e. in our alternative notations, to either j' or j vanishing. As we shall see, they have a (not totally unexpected) connection with the considerations of the previous sections.

The formula given in [36] reads for $r = 1$, $s = 2j + 1 = n$

$$\phi = \sum_{\substack{\text{partitions of } n \\ n=p_1+\cdots+p_r, p_i\geq 1}} t^{n-r} \frac{(n-1)!^2}{\prod_{i=1}^{r-1}(p_1+\cdots+p_i)(n-p_1-\cdots-p_i)} L_{-p_1}\cdots L_{-p_r} \tag{6.10}$$

where t has been defined in (6.7). There is an analogous formula in the case $r = n$, $s = 1$, obtained by changing t into t^{-1}.

The case $j' = 0$ has the virtue that the h_{0j} are the only values in (6.8) to have a limit as $m \to 0$ (hence $c \to \infty$) ("classical limit", see above in (4.8)). This limiting value is $h = -(n-1)/2$, i.e. the conformal weight of functions f studied in sect. 3. The operator ϕ in this limit is expected [17] to reduce to an ordinary differential operator with covariance properties, since it maps a highest weight (i.e. a h-differential) onto another one (a $h+n$-differential). The matching is provided by the following identification

$$\begin{aligned} L_{-1} &\to \mathrm{d} \\ -mL_{-k} &\to \frac{a_2^{(k-2)}}{(k-2)!\sigma_1} \end{aligned} \tag{6.11}$$

where $\sigma_1 = n(n^2-1)/6$ is the coefficient introduced in section 3. The form of the operator ϕ is thus known in this limit: $\phi = \Delta_2^{(n)}(a_2)$ and this suggests to recast the form of ϕ using the matrix formalism of section 3. Unexpectedly, but very fortunately, this matrix turns out to embody the whole form of ϕ_{0j} — beyond the limit $m \to 0$ — and enables one to reproduce the results of [36].

Let us concentrate on the case $r = 1, n = s = 2j + 1$ for definiteness. Then (6.8) reduces to

$$\begin{aligned} c &= 13 + 6(t + t^{-1}) \\ h &= -j - tj(j+1) \\ n &= 2j + 1 \end{aligned} \tag{6.12}$$

where $j = \frac{1}{2}, 1, \frac{3}{2}, \cdots$. In $V(c,h)$ we introduce a sequence of elements denoted

$$f = f_{-j}, f_{-j+1}, \cdots, f_j, f_{j+1} = F \tag{6.13}$$

where f_{-j} is the highest weight state f, f_{j+1} is the singular vector F and f_M ($j+M$ a non negative integer) satisfies

$$L_0 f_M = (h + j + M) f_M \tag{6.14}$$

We define the n-dimensional vectors

$$\begin{aligned} \underline{f} =& (f_j, f_{j-1}, \cdots, f_{-j})^T \\ \underline{F} =& (F, 0, \cdots, 0)^T \end{aligned} \tag{6.15}$$

We also use the notations $J_\pm, J_0$ introduced in section 3. Now the claim is that the set of equations embodied in the linear system

$$\underline{F} = \big(- J_- + \sum_{k=0}^{n-1} L_{-k-1} (tJ_+)^k \big) \underline{f} \tag{6.16}$$

defines $F = f_{j+1}$ as a non vanishing singular state at level n in $V(c,h)$. Moreover the successive components of $\underline{f}$ satisfy the relations

$$p > 0 \qquad L_p \underline{f} = \big[\big(J_0 - \frac{3p+1}{2} \big) - t^{-1} \frac{3p+1}{4} \big] (tJ_+)^p \underline{f}. \tag{6.17}$$

proved by induction on the label M of the components of $\underline{f}$ starting from the last one, for which both sides of (6.17) vanish as a consequence of the highest weight property. Equation (6.17) for $p = 1, 2$ implies it for higher p by commutation. It also extends to $M = j+1$ for which it means that F is annihilated by the L_p, $p > 0$, i.e. F satisfies the axioms of a singular vector, *q.e.d.*

Remarks.

(1) The case $r = 2j' + 1, s = 1$ is simply obtained by changing $j \to j'$, $t \to t^{-1}$.

(2) Eliminating all components f_j, $-j+1 \leq k \leq j$ one finds the explicit form of the operator $\phi : \quad f \equiv f_{-j} \to F$ given in (6.10).

What is very striking, and remains slightly mysterious today, is the similarity between the operator ϕ found here in a matrix form in (6.16) and the matrix differential operator (5.26). The substitution

$$L_{-1} \to \frac{d}{dx} \qquad\qquad t^k L_{-k-1} \to W_{k+1} \tag{6.18}$$

in (6.16) reproduces the $n \times n$ differential operator (5.26).

To obtain the general form of the singular vectors for both j' and j different from zero, we shall now appeal to the operation of fusion. It is indeed part of the standard lore of c.f.t. (and it will be justified below) that fusion of $(j', 0)$ and $(0, j)$ highest weight representations yields the one labelled by (j', j). It is thus expected that the relevant information about the general singular vectors is somehow encoded in the previous cases. In order to transform this qualitative remark into something operative, we need reexamine first the precise meaning and use of fusion.

6.3. Fusion revisited.

We now adopt the language of conformal field theory. To each point x of the Riemann sphere, we attach a "primary" field $f_h(x)$ and its descendants $L_{-\{p\}} f_h(x)$ which form a highest weight module of a copy of the Virasoro algebra at this point. Moreover the interpretation of the Virasoro generator L_n is that it carries out the infinitesimal changes of coordinates: $\delta x = -x^{n+1}$. In this respect the highest weight vector $f_h(x)$ is a $h-$ differential: $f_h \in \mathcal{F}_h$, and this is the meaning of "primary".

$$L_n f_h(x) = x^{n+1} \frac{d}{dx} f_h(x) + h(n+1) f_h(x) \tag{6.19}$$

The action of the Virasoro algebra on f_h is also nicely encoded in the short distance expansion of $f_h(x)$ with the energy-momentum tensor

$$T(\xi) = \sum_{p \in \mathbb{Z}} (\xi - x)^{-p-2} L_p^{(x)} \tag{6.20a}$$

$$\text{i.e.} \quad T(\xi) f(x) = \sum_{p \geq 0} (\xi - x)^{p-2} \big(L_{-p} f\big)(x) \quad etc... \tag{6.20b}$$

$$\text{or conversely } L_{-p} f_h(x) = \oint \frac{d\xi}{2\pi i} (\xi - x)^{-p+1} T(\xi) f_h(x) \tag{6.20c}$$

In conformal field theory, we have seen in sect. 2.7 that an important axiom is that a product of two fields may be expressed in a short distance expansion in terms of other conformal fields. In the present context, we write

$$f_0(x_0) f_1(x_1) \sim \sum_h \left\{ \frac{g(h_0, h_1, h)}{z^{h_0 + h_1 - h}} \sum_{n \geq 0} z^n f_h^{(n)}(x_1) \right\}. \tag{6.21}$$

The notations are as follows: $z = x_0 - x_1$ is the formal expansion parameter, $g(h_0, h_1, h)$ is the coupling between the three fields f_0, f_1 and f_h, and $f_h^{(n)}$ denotes a descendant of level

n of f_h. Whenever a coupling $g(h_0, h_1, h)$ is non vanishing, we say that there is possible fusion of $f_0 \times f_1 \to f_h$. We shall find later a necessary condition for this to occur, in the cases relevant for our discussion. As the notations suggest and as we shall justify below, the leading term $f_h^{(0)}$ in (6.21) is a primary field, a property of crucial importance in the our forthcoming construction [4].

We can now use equ. (6.20c) to derive explicit formulae for the fusion of descendant fields of f_0 with a primary field f_1. We start from

$$\Big(L_{-p} f_0(x_0)\Big) f_1(x_1) = \oint_{C_0} \frac{d\xi}{2\pi i} (\xi - x_0)^{-p+1} T(\xi) f_0(x_0) f_1(x_1)$$
$$= \Big\{ \oint_C - \oint_{C_1} \Big\} \Big[\frac{d\xi}{2\pi i} (\xi - x_0)^{-p+1} T(\xi) f_0(x_0) f_1(x_1) \Big] \tag{6.22}$$

the contours C_0, C_1 and C are depicted on the Figure. The contribution of C_1 is readily evaluated using (6.20b)

$$\oint_{C_1} \frac{d\xi}{2\pi i} (\xi - x_0)^{-p+1} T(\xi) f_0(x_0) f_1(x_1)$$
$$= \oint_{C_1} \frac{d\xi}{2\pi i} (\xi - x_0)^{-p+1} \Big(\frac{h_1}{(\xi - x_1)^2} + \frac{\partial_{x_1}}{(\xi - x_1)} \Big) f_0(x_0) f_1(x_1)$$
$$= -\Big(-\frac{1}{z} \Big)^p \big((p-1) h_1 + z \partial_{x_1} \big) f_0(x_0) f_1(x_1) \tag{6.23}$$

whereas the contribution of the other contour is expanded for $x_0 \sim x_1$ in the form

$$\oint_C \frac{d\xi}{2\pi i} (\xi - x_0)^{-p+1} T(\xi) f_0(x_0) f_1(x_1)$$
$$= \oint_C \frac{d\xi}{2\pi i} (\xi - x_1 - z)^{-p+1} T(\xi) f_0(x_0) f_1(x_1)$$
$$= \sum_{k \geq 0} z^k \binom{p+k-2}{k} \oint_C \frac{d\xi}{2\pi i} \ (\xi - x_1)^{-p+1} \Big(T(\xi) f_0(x_0) f_1(x_1) \Big)$$
$$= \sum_{k \geq 0} z^k \binom{p+k-2}{k} L_{-p-k} \Big(f_0(x_0) f_1(x_1) \Big) \tag{6.24}$$

Thus the action of $L_{-p} \otimes \mathbf{1}$ on $f_0(x_0) f_1(x_1)$ may be reexpressed in terms on the action of the L's on the field f_h and its descendants

$$L_{-p} \otimes \mathbf{1} = \frac{(-1)^p}{z^p} \big[h_1 (p-1) + z L_{-1} - z \frac{d}{dz} \big] + \sum_{k \geq 0} z^k \binom{k+p-2}{k} L_{-p-k}. \tag{6.25}$$

[4] This is to be contrasted with the case of extended W-algebras where this property is not generally true; for example in the 3-state Potts model, the fusion of the "energy operator" ϵ ((2,1) in the Kac table) with itself produces the (3,1) operator, which is a W_3-descendant of ϵ [37]

The reader should not be misled by the slightly abusive notations: in the l.h.s., the Virasoro generator L acts in the module $V(c,h_0)$, on the r.h.s., the same symbol denotes the action in $V(c,h)$. In this derivation, we have used the fact that f_1, not f_0 is a primary field: it may thus be iterated for products of descendants of f_0 with f_1. There exist of course analogous formulae for the action of the Virasoro generators on f_1, i.e. expressing $\mathbf{1}\otimes L$ in terms of L's acting in $V(c,h)$.

In the literature, the precise definition of fusion has been the object of much attention. In particular, the action of Virasoro on the fused field has been given as a "coproduct" expressing the action of L in terms of linear combinations $L\otimes\mathbf{1}$ and $\mathbf{1}\otimes L$ [38][39]. (Note that the naïve coproduct $L_n = L_n\otimes\mathbf{1}+\mathbf{1}\otimes L_n$ cannot be correct as it would lead to a central charge $2c$ for the fused representation.) We shall not elaborate on this subject but just observe that the previous equations are inverse formulae expressing $L_n\otimes\mathbf{1}$ in terms of the coproducts L_p.

Let us now turn to the determination of the descendants that appear in (6.21). We require that the two sides of that equation transform in a consistent way under a change of variable $x\to\widetilde{x}$.

$$\begin{aligned}&\frac{1}{z^{h_0+h_1-h}}\sum_n z^n f^{(n)}(x_1) = \\ &= \left(\frac{d\widetilde{x}_0}{dx_0}\right)^{h_0}\left(\frac{d\widetilde{x}_1}{dx_1}\right)^{h_1}\frac{1}{(\widetilde{x}_0-\widetilde{x}_1)^{h_0+h_1-h}}\sum_n(\widetilde{x}_0-\widetilde{x}_1)^n\widetilde{f}^{(n)}(\widetilde{x}_1)\end{aligned} \tag{6.26}$$

The easiest way is to apply the above formula in infinitesimal form. Set

$$\widetilde{x} = x-\epsilon(x-x_1)^{k+1} \tag{6.27}$$

so that

$$\delta_\epsilon f^{(n)}(x) = \epsilon L_k f^{(n)}(x) \tag{6.28}$$

The covariance condition becomes, with $k\geq -1$

$$\Big[L_k-\Big(h_0(k+1)z^k+z^{k+1}\partial_z\Big)\Big]\frac{1}{z^{h_0+h_1-h}}\sum_{p\geq 0}z^p f^{(p)} = 0. \tag{6.29}$$

Since L_1 and L_2 generate by commutators the complete algebra of L_k's, $k\geq 1$, it is sufficient to impose the two relations pertaining to $k=1$ and $k=2$. The above translates into the conditions

$$L_1 f^{(n)} = (n-1+h+h_0-h_1)f^{(n-1)} \tag{6.30a}$$

$$L_2 f^{(n)} = (n-2+h+2h_0-h_1)f^{(n-2)} \tag{6.30b}$$

which imply

$$L_p f^{(n)} = (n - p + h + ph_0 - h_1) f^{(n-p)}. \tag{6.31}$$

It is understood that $f^{(q)}$ vanishes if $q < 0$. As a consequence we recover the fact that $f^{(0)} \equiv f$ is a highest weight state (or primary field) as already claimed. Let us exemplify on the first few values of n how these equations determine recursively the $f^{(q)}$'s.

With $f^{(1)} = \beta_1 L_{-1} f$ the first equation gives

$$2h\beta_1 = h + h_0 - h_1 \Rightarrow \beta_1 = \frac{1}{2h}(h + h_0 - h_1) \qquad \text{if } h \neq 0. \tag{6.32}$$

For $n = 2$,

$$f^{(2)} = \Big(\beta_{1,1} L_{-1}^2 + \beta_2 L_{-2}\Big) f \tag{6.33}$$

we get from (6.30a)

$$4h(2h+1)\beta_{1,1} + 6h\beta_2 = (h + 1 + h_0 - h_1)(h + h_0 - h_1) \tag{6.34}$$

while (6.30b) yields

$$6h\beta_{1,1} + \big(4h + \frac{c}{2}\big)\beta_2 = h + 2h_0 - h_1 \tag{6.35}$$

Provided the determinant

$$K_2 = h\Big(16h^2 + 2h(c-5) + c\Big) \tag{6.36}$$

is different from zero, we obtain

$$\beta_{1,1} = \frac{(h + 1 + h_0 - h_1)(h + h_0 - h_1)(8h + c) - 12h(h + 2h_0 - h_1)}{4h[16h^2 + 2h(c-5) + c]} \tag{6.37a}$$

$$\beta_2 = \frac{h^2 + h\big(2(h_0 + h_1) - 1\big) + h_0 + h_1 - 3(h_0 - h_1)^2}{16h^2 + 2h(c-5) + c} \tag{6.37b}$$

The moral of these sample calculations is clear: the linear system (6.30) has a unique solution if and only if there is no singular vector at level n. The determinant of this system is indeed a polynomial, factor of the Kac determinant at that level. Thus the operation of fusion that we are using is actually only well defined when restricted to the irreducible target module $M(c, h)$.

6.4. General singular vectors.

We now want to obtain singular vectors in a Verma module $V\big(c, h_{j',j}\big)$ from fusion, using the explicit expressions of sect. 4.2 for the particular cases $h_{j',0}$ and $h_{0,j}$. Let us look at the implications in the fusion process when $V(c,h_0)$ or $V(c,h_1)$ or both possess singular vectors. Suppose we started by "fusing" $f_0(x_0)$ and $f_1(x_1)$ and assume that $F_0(x_0) = \phi_0 f_0(x_0)$ is again a primary field of weight $h_0 + n_0$. Starting from $(f^{(0)} \equiv f)$

$$f_0(x_0)f_1(x_1) \to \frac{1}{z^{h_0+h_1-h}} \sum_r z^r f^{(r)}(x_1) \tag{6.38}$$

where the arrow denotes the fusion map, we would derive

$$F_0(x_0)f_1(x_1) = \big(\phi_0 f_0\big)(x_0)f_1(x_1) \to \frac{1}{z^{h_0+h_1-h+n_0}} \sum_r z^r \psi^{(r)}(x_1) \tag{6.39}$$

with $\psi^{(r)}(x)$ some descendant of level r of f. I shall first present a qualitative sketch of the method. I want to argue that the coefficient of the leading term vanishes. This must be indeed be so if fusion is consistent with the quotient of $V(c,h_0)$ by its singular vectors: in that operation, the left hand side of (6.39) vanishes, and so must do the r.h.s. On the other hand we have seen above that the first non vanishing term in the fusion of the primary fields F_0 and f_1 must be a primary: this ensures that together with $\psi^{(0)}$, all the further $\psi^{(r)}$ must vanish until we reach another primary field among the descendants, which may be nothing else than the desired singular field in $V(c,h)$!

Let us now give a rigorous argument to the effect that $\psi^{(0)}$ vanishes. It is easy to see that the coefficient of the leading term is given by

$$\psi^{(0)} = \Big\{(-1)^{n_0} z^{h_0+h_1-h+n_0} \phi_0\Big(L_{-k} \to \frac{1}{z^k}\big[h_1(k-1) - z\frac{d}{dz}\big]\Big)\frac{1}{z^{h_0+h_1-h}}\Big\} f \tag{6.40}$$

where in ϕ_0, ordinary differential operators acting on $\frac{1}{z^{h_0+h_1-h}}$ have been substituted for the Virasoro operators. It turns out that the result of this action is known and takes a quite explicit form [40]. In general, let us consider

$$l_{-k} = -\frac{1}{z^k}\Big[\lambda(k-1) + z\frac{d}{dz}\Big] \tag{6.41}$$

acting on the basis $z^{-p-\mu}$. The effect of substituting l_{-k} for L_{-k} in $\phi_{j'_0,j_0} \equiv \phi_0$ is described by the following formulae

$$\phi_{j',j}(l_{-1}, l_{-2}, \cdots)\frac{1}{z^\mu} = \varphi_{j',j}(\lambda,\mu)\frac{1}{z^{\mu+n_0}} \qquad \lambda = -h_1, \qquad \mu = h_0 + h_1 - h, \tag{6.42}$$

$$\varphi^2_{j',j}(\lambda,\mu) = \prod_{\substack{-j\le M\le j\\ -j'\le M'\le j'}} [\big(\mu + A(M,M')\big)\big(\mu + A(-M,-M')\big) - 4\lambda(M\theta + M'\theta^{-1})^2] \quad (6.43)$$

where

$$A(M,M') = [(j+M)\theta + (j'+M')\theta^{-1}][(j+1-M)\theta + (j'+1-M')\theta^{-1}]. \quad (6.44)$$

Note that the right hand side of (6.43) is a perfect square: if the pair $(M,M') = (0,0)$ is allowed it corresponds to a square. Otherwise (M,M') and $(-M,-M')$ give identical factors. For a direct proof of formula (6.43) in the case where j or j' vanishes, (which is sufficient for our purpose), see [26], end of sect. 3.

Using these explicit formulae, it is easy to see that $\psi^{(0)}$ indeed vanishes whenever fusion $f_0 f_1 \to f$ takes place, in particular for $h_0 = h_{j',0}$, $h_1 = h_{0,j}$ and $h = h_{j',j}$. Conversely, this vanishing gives us a necessary condition for fusion.

To illustrate the procedure, let us show how to recover the singular vectors of type $(0,j)$ studied in sect. 4.2. Let $h_0 \equiv h_{\frac12,0} = -\frac12 - \frac{3}{4t}$, $h = h_{0,j} = -j - tj(j+1)$ and $\lambda = -h_1$, $\mu = h_0 + h_1 - h$ as above. Then

$$\varphi_{\frac12,0}(\lambda,\mu) = \mu\Big(\mu + 1 + \frac{1}{t}\Big) - \frac{\lambda}{t}$$

is quadratic in h_1 and vanishes for

$$h_1 \equiv h_{\frac12,j} = -\frac{3}{4t} - \frac12 - 2j - j(j+1)t \quad (6.45)$$

(and the Verma module is degenerate at level $2(2j+1)$) but also for

$$h_1 = \frac12 + \frac{1}{4t} - j(j+1)t. \quad (6.46)$$

Using the general formula (6.8) for singular vectors, the latter value can be interpreted as $h_{-\frac12,j}$, which has *a negative label,* i.e. for generic t, *is not* the weight of a singular vector. Let us choose this second value and consider the fusion symbolically written as

$$\big(\tfrac12,0\big)\otimes \text{“}\big(-\tfrac12,j\big)\text{”} \to (0,j)$$

We know that on f_0:

$$\phi_{\frac12,0} = L_{-1}^2 + \frac{1}{t}L_{-2} \quad (6.47)$$

In the fusion, each of these L's is to be interpreted as $L\otimes\mathbf{1}$, i.e. using (6.25)

$$\begin{aligned} L_{-1}\otimes\mathbf{1} =& \partial_z \\ L_{-2}\otimes\mathbf{1} =& \frac{h_1}{z^2} - \frac{1}{z}\partial_z + \sum_{k=1}^{\infty} z^{k-2}L_{-k} \end{aligned} \quad (6.48)$$

Thus imposing the vanishing of the singular vector F_0 in $V(c,h_0)$, we find that

$$\Big\{\partial_z^2+\frac{1}{t}\Big(\frac{h_1}{z^2}-\frac{1}{z}\partial_z+\sum_{k=1}^{\infty}z^{k-2}L_{-k}\Big)\Big\}\frac{1}{z^{h_0+h_1-h}}\sum_{p\geq 0}z^p f^{(p)}=0 \tag{6.49}$$

which gives a recursive procedure to compute the sequence $f^{(p)}$. Since $h_0+h_1-h=j-\frac{1}{2t}$, (6.49) reads

$$p\big((2j+1)-p\big)f^{(p)}=\frac{1}{t}\sum_{k\geq 1}L_{-k}f^{(p-k)} \tag{6.50}$$

The $f^{(p)}$ computed from this relation agree with the values obtained from conditions (6.30a,b). The above recursion relation is nothing else than our previous matrix equation (6.16) for the singular vector if we relabel and rescale the intermediate components according to

$$f^{(j+M)}\equiv t^{-j-M}f_M \qquad\qquad f_{-j}\equiv f$$

and make use of a representation where the generators of angular momentum read (with J_0 unchanged, $n=2j+1$)

$$J_-=\begin{pmatrix} 0 & & & & \\ 1(n-1) & 0 & & & \\ & 2(n-2) & 0 & & \\ & & & \ddots & \\ & & & (n-1)1 & 0 \end{pmatrix}$$

$$J_+=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix} \tag{6.51}$$

We have thus recovered by fusion the special case of Benoit and Saint-Aubin. Moreover the "descent equations" (6.17) find a natural explanation as they coincide with the covariance equations (6.30).

In a similar way, we now study the general case (j',j) by the fusion:

$$\big(j'+\tfrac{1}{2},0\big)\otimes\ \text{“}\big(-\tfrac{1}{2},j\big)\text{”}\rightarrow(j',j)$$

where

$$h_0=-\big(j'+\frac{1}{2}\big)\Big[1+\frac{j'+1+\frac{1}{2}}{t}\Big] \tag{6.52}$$

and we use the same abusive notation for

$$h_1=h_{-\frac{1}{2},j}=\frac{1}{4t}+\frac{1}{2}-tj(j+1) \tag{6.53}$$

which again for generic t corresponds to an irreducible Verma module, while

$$h \equiv -\Big[tj(j+1) + j + j' + 2jj' + \frac{j'(j'+1)}{t}\Big]. \tag{6.54}$$

Using the formula (6.43) above, we check that the factor φ indeed vanishes. Therefore we obtain the singular vector in the module $V(c,h)$ by requiring that $\phi_0 f_0 \times f_1$ vanishes. Let

$$\overline{\phi} = \phi_{j'+\frac{1}{2},0}\big(L_{-p} \to \frac{(-1)^p}{z^p}[h_1(p-1) + z(L_{-1} - \partial_z)] + \sum_{k\geq 0} z^k \binom{k+p-2}{k} L_{-p-k}\big)$$

then the equation

$$\overline{\phi}\frac{1}{z^{h_0+h_1-h}} \sum_k z^k f^{(k)} = 0$$

determines recursively $f^{(k)}$ for $0 < k < n = (2j+1)(2j'+1)$ in terms of f and yields at level n

$$\phi_{j',j} f = 0$$

up to a non vanishing factor. Moreover the intermediate coefficients $f^{(k)}$, $0 \leq k \leq n$ satisfy "descent equations" given in $(6.30a, b)$.

The interested reader will find an ample collection of examples and detailed calculations in [26].

To conclude, it is expected that the methods used here for the Virasoro algebra will also be useful in the case of other infinite dimensional algebras, and one may also hope that the connections between classical integrable systems and conformal theories will appear in other contexts.

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