CONFORMAL INVARIANT THEORIES ON A TORUS AND THEIR MODULAR INVARIANCE

J.-B. ZUBER
Service de Physique Théorique
CEN-SACLAY
91191 Gif-sur-Yvette Cedex, France

1. Introduction

As we have heard at this meeting /1/, conformal field theory is an essential piece in the construction of string theory. It also applies to two-dimensional critical phenomena; indeed, a critical model is not only dilatation but also conformally invariant /2/.

In a two-dimensional conformal invariant theory, two copies of the Virasoro algebra act: $V \otimes \bar{V}$, with the same central charge c, corresponding to analytic changes of the independent variables z and $\bar{z}$. A conformal field theory /3/ is thus specified by giving the value of c and of the (finite or infinite) set of conformal weights $(h_i, \bar{h}_i)$ of the primary fields. Primary fields /4/ transform as $(h,\bar{h})$-tensors under changes of coordinates $z' = z'(z)$:

$$\varphi_{h\bar{h}}(z',\bar{z}')\,dz'^{h}\,d\bar{z}'^{\bar{h}} = \varphi_{h\bar{h}}(z,\bar{z})\,dz^{h}d\bar{z}^{\bar{h}} \tag{1}$$

with $h+\bar{h}$ interpreted as the scaling dimension, and $h-\bar{h}$ as the spin of φ. These fields are in one-to-one correspondence with the highest weights of $V \otimes \bar{V}$:

$$\lim_{z,\bar{z}\to 0} \varphi_{h\bar{h}}(z,\bar{z})\,|0\rangle = |h,\bar{h}\rangle \tag{2}$$

Among the non-primary fields, the energy-momentum tensor $T_{zz}=T(z)$, $T_{\bar{z}\bar{z}}=\bar{T}(\bar{z})$ plays a fundamental role. It is the generator of infinitesimal changes of variables:

$$T(z) = \sum_{-\infty}^{\infty} z^{-n-2}\, L_n \qquad\qquad \bar{T}(\bar{z}) = \sum_{-\infty}^{\infty} \bar{z}^{n-2}\, \bar{L}_n \tag{3}$$

with L_n, $\bar{L}_n$ the Virasoro generators. Under the change $z \to z'(z)$:

$$T(z) = T'(z')\left(\frac{dz'}{dz}\right)^2 + \frac{c}{12}\,\{z',z\} \tag{4}$$

where the second, anomalous, term involves the product of the central charge by the schwarzian derivative:

$$\{z',z\} = \frac{d^3z'/dz^3}{dz'/dz} - \frac{3}{2}\left(\frac{d^2z'/dz^2}{dz'/dz}\right)^2 \tag{5}$$

In the following, I shall discuss mainly the case of the $c < 1$ unitary conformal theories /5/. They come in a discrete series, of central charge $c = 1-6/m(m+1)$, m an integer $\geqslant 3$ and the possible values of h and $\bar{h}$ are zeros of Kac' determinant /6/

$$h_{rs} = h_{m-r\ m+1-s} = \frac{[(m+1)r-ms]^2-1}{4m(m+1)} \qquad 1 \leqslant s \leqslant r \leqslant m-1 \qquad (6)$$

All the following discussion may, however, be generalized to the class of minimal $c < 1$ degenerate theories /3/, corresponding to a rational value of m, where unitarity is lost but where a finite set of conformal fields is still consistent.

Minimal and in particular unitary $c < 1$ theories seem to play a central role in statistical mechanical models. The cases m=3,4 have been identified as respectively the critical and tricritical Ising models /7/, m=5 and 6 as the critical and tricritical Potts models /8/, while the generic m theory seems to describe the so-called RSOS model /9/. These identifications, however, are empirical (beside the well-known case of the c = 1/2 Ising model = free fermion theory), and are based on coïncidences of critical exponents.

One would like therefore to have a general classification of the consistent conformal theories, and also to have some a priori knowledge of their global properties, in particular of their symmetries. One might also like to find representatives of these various theories, namely either a Lagrangian field theory, or a discrete statistical mechanical problem, whose critical regime is described by that particular theory. Two strategies have been followed for this purpose. The first one makes use of the conformal algebra /3/. The "structure constants" appearing in the operator product expansions satisfy non linear relations due to the associability of the algebra. This approach has been brillantly illustrated by the investigation of various conformal theories /8,10/ and the discovery of new families of such theories /11/. The second, more recent, approach has been initiated by Cardy /12/ who noticed that if a conformal theory is put on a torus, the modular invariance of the partition function imposes stringent constraints on the operator content of the theory. This is this approach, as pursued in collaboration with C.Itzykson, H.Saleur and A.Capelli /13-15/, that I want to present here.

2. The partition function on a torus

Consider a torus defined by two of its complex periods ω_1 and ω_2, chosen such that $\tau = \omega_2/\omega_1$ lies in the upper half-plane, $\mathrm{Im}\tau > 0$, and let $q = e^{2\pi i\tau}$. In order to define the theory on a torus by means of the formalism developed for the infinite plane /3/, we cut the torus along one of its geodesics, say ω_1, and map the resulting bit of cylinder in the plane by the exponential mapping: $w \to z = \exp 2\pi i\ w/\omega_1$. According to Eq.(4), the stress energy tensor on the cylinder and in the plane are related by:

$$T_{pl}(z) = T_{cyl}(w) \left(\frac{dw}{dz}\right)^2 + \frac{c}{12}\{w,z\} \qquad (7)$$

After projection on the moments of their Laurent expansions (3):

$$L_{-1}^{cyl} = \frac{2\pi i}{\omega_1}\left(L_o^{pl} - \frac{c}{24}\right) \qquad (8)$$

It is then natural to define the partition function on the torus as the trace of the exponential of the translation operator along ω_2:

$$Z = \mathrm{Tr}\exp\left(\omega_2 L_{-1}^{cyl} + \bar\omega_2 \bar L_{-1}^{cyl}\right)$$

$$= \mathrm{Tr}\exp 2\pi i\left[\tau\left(L_o^{pl} - \frac{c}{24}\right) - \bar\tau\left(\bar L_o^{pl} - \frac{c}{24}\right)\right] \qquad (9)$$

$$= (q\bar q)^{-c/24}\ \mathrm{Tr}\left(q^{L_o}\bar q^{\bar L_o}\right)$$

In this procedure, the two periods have been treated in a very asymmetric way. The content of the theory is hidden in what is meant by "Tr": the trace is to be taken on the (Hilbert) space of states, which depends on which particular theory we are considering. This is most clearly exposed by decomposing this space into a sum of irreducible representations of $V \otimes \bar V$:

$$Z = (q\bar q)^{-c/24} \sum_{h,\bar h} N_{h,\bar h}\, \chi_h(q)\, \chi_{\bar h}(\bar q) \qquad (10)$$

Here, $\chi_h(q) = \mathrm{tr}\ q^{L_o}$ denotes the Virasoro character in the irreducible representation of highest weight h, and $N_{h\bar h}$ are multiplicities, i.e. are positive integers, that encode the operator content of the theory.

Equation (10) is the master formula of this analysis. To make use of it, let us examine the structure of the characters χ_h. Recall that representations of the Virasoro algebra ("Verma modules") are built by the action of the L_n, $n < 0$ on a highest weight vector $|h\rangle$; if h is a zero of Kac' determinant, this representation is reducible and its irreducible part is obtained by factoring out the reducible submodules. As the eigenvalues of L_o in the irrep.$\{h\}$ are integer spaced,

$$\chi_h(q) = q^h \sum_{n=0}^{\infty} (\text{number of states at level } n).q^n$$

This is easy to evaluate if h is not a zero of the Kac determinant, and hence the number of states at level n is p(n), the number of partitions of n:

$$\chi_h(q) = q^h \sum p(n)\, q^n = \frac{q^h}{\prod_1^{\infty}(1-q^n)} \qquad (11)$$

When h is a zero of Kac' determinant, we have to subtract from this the contribution of the reducible submodules. The analysis of the embeddings of these submodules has been carried out in detail by Feigin and Fuchs /6/. I only quote the resulting formula for the unitary representations of Eq.(6) /17/:

$$\chi_h = \chi_{rs} = \frac{1}{\prod_1^{\infty}(1-q^n)} \sum_{n=-\infty}^{\infty}\left\{ q^{\frac{[2nm(m+1)+(m+1)r-ms]^2}{4m(m+1)}} - (s \to -s)\right\} \qquad (12)$$

Before discussing modular invariance, let us briefly comment on other practical applications of the master equation (10). Take $\omega_2 = iT$, $\omega_1 = L$, hence $q = e^{-2\pi T/L}$ and let $T \to \infty$. For small q, if all dimensions $h, \bar h$ are non negative, which is true in particular in unitary theories /14/, the identity operator dominates the sum in (10). This exhibits the interpretation of c as a finite width (L) contribution to the free energy per unit length of the system /18/

$$\lim_{T\to\infty} \frac{\ln Z}{T} = \frac{\pi c}{6L} \tag{13}$$

Moreover, the characters may be expanded in powers of q, and the resulting expansion for Z confronted with the trace of the transfer matrix of a discrete but critical model expressed in terms of its eigenvalues. In connection with numerical calculations on finite strips, this has enabled one to identify the central charge and (some of) the conformal weights of several interesting models /18,19/.

3. Modular invariance

As noticed above, in the master formula (10), the two periods of the torus are not treated on the same footing. We now want to repair this, by imposing that Z be invariant by unimodular redefinitions of ω_1 and ω_2:

$$\begin{pmatrix}\omega_2'\\ \omega_2'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \qquad \begin{matrix}a,b,c,d \in \mathbb{Z}\\ ad-bc = 1\end{matrix} \tag{14}$$

Demanding that $Z(\tau)$ be invariant under the modular transformations $\tau \to (a\tau+b)/c\tau+d)$ turns out to be a very strong constraint /12/. Let us try to construct modular invariants of the form (10).

It is sufficient to ensure invariance under two generators of the modular group Γ, such as:

$$T : \tau \to \tau+1 \qquad\qquad S : \tau \to -\frac{1}{\tau} \tag{15}$$

Under T, $q \to e^{2\pi i}q$, $\chi_h(q) \to e^{2\pi i h}\,\chi_h(q)$ and hence invariance of (10) is satisfied if

$$N_{h\bar{h}} \neq 0 \qquad \text{only for} \qquad h-\bar{h} \in \mathbb{Z} \tag{16}$$

i.e. if only integer spin operators contribute to Z.
Under S, $\tau \to -1/\tau = \tau'$, $q \to q'$, the Poisson summation formula applied to (12) yields /12,13/:

$$q'^{-c/24}\,\chi_h(q') = \sum_{h'} A_{hh'}\, q^{-c/24}\,\chi_{h'}(q) \tag{17}$$

where A is a $\frac{m(m-1)}{2} \times \frac{m(m-1)}{2}$ matrix, whose generic entry:

$$A_{h\bar{h}'} = A_{rs;r's'} = [8/m(m+1)]^{1/2}\,(-1)^{(r+s)(r'+s')}\sin\frac{\pi rr'}{m}\,\sin\frac{\pi ss'}{m+1} \tag{18}$$

is an algebraic number. In (17), the sum over h' runs over $1\leqslant s'\leqslant r'\leqslant m-1$. The matrix A is symmetric and of square one, and the invariance of (10) may thus be expressed in a matrix form as:

$$A.N = N.A \tag{19}$$

i.e. as a system of diophantine equations. Two general classes of solutions have been found /13/.

i) main series:

$$N_{h\bar{h}} = \delta_{h\bar{h}} \quad \text{hence} \quad Z = (q\bar{q})^{-c/24} \sum_h |\chi_h|^2 \tag{20}$$

All the spinless operators contribute to this solution, which is appropriate to describe the Ising model and its multicritical extensions /9/.

ii) complementary series: if m=1 mod 4, there exist other modular invariants, with the general form:

$$z = (q\bar{q})^{-c/24} \sum_r \sum_{s\ \mathrm{odd}} |\chi_{rs} + \chi_{r\ m+1-s}|^2 \tag{21}$$

(and similarly, if m=2 mod 4, with the roles of r and s interchanged; see /13/ for precise expressions and normalizations). The distinctive feature of this new solution is that only a restricted set of spinless operators contribute, along with non zero (but integer) spin operators. This solution includes the critical and tricritical 3-state Potts models (m=5 and 6). For m=0 and 3 (mod 4), there are also modular invariants containing the contribution of non zero spin operators(*). For m=4k+3, they read:

$$\sum_{\substack{s\ \mathrm{odd}\\ 1\leqslant s\leqslant r\leqslant m-1}} |\chi_{rs}|^2 + \sum_{\substack{1\leqslant r\leqslant 2k+1\\ 2\leqslant s\ \mathrm{even}\ \leqslant 2k}} (\chi_{rs}\chi^*_{r,m+1-s} + \mathrm{c.c.}) + \sum_{2k+2\leqslant r\leqslant m-1} |\chi_{r,2k+2}|^2$$

and for m=4k:

$$\sum_{\substack{r\ \mathrm{odd}\\ 1\leqslant s\leqslant r\leqslant m-1}} |\chi_{rs}|^2 + \sum_{\substack{2\leqslant r\ \mathrm{even}\ \leqslant 2k-2\\ 1\leqslant s\leqslant 2k}} (\chi_{rs}\chi^*_{m-r,s} + \mathrm{c.c.}) + \sum_{1\leqslant s\leqslant 2k} |\chi_{2k,s}|^2$$

There are also other invariants /13/ in which the identity operator $|\chi_{11}|^2$ comes with a factor 2. This means that the vacuum is twice degenerate and these solutions are rejected as unphysical.

In addition to these two series of solutions, new solutions have been found recently /20,15/. For instance for m=11,

$$\sum_{1\leqslant r_{\mathrm{odd}}\leqslant 9} |\chi_{r1}+\chi_{r7}|^2 + |\chi_{r4}+\chi_{r8}|^2 + |\chi_{r5}+\chi_{r,11}|^2 \tag{22a}$$

or for m=17

$$\sum_{1\leqslant r\leqslant 8} |\chi_{r3}+\chi_{r15}-\chi_{r9}|^2 \tag{22b}$$

are modular invariants. ((22b) should be subtracted from (21) to ensure positivity). In general, the number of solutions depends on arithmetic properties of m, in particular, on the decomposition into prime factors of both m and m+1. Although a general classification of all solutions may be difficult to achieve, it must be clear that the problem of finding all modular invariants

----

(*) These invariants were omitted in ref./13/.

for a given m is a finite problem.

To summarize this discussion, any modular invariant of the form (10), with non-negative integer coefficients $N_{h\bar{h}}$ and $N_{oo}=1$ is a candidate for a the partition function of a consistent conformal theory.

4. Symmetries /21,22/

It is well known /5/ that unitary $c < 1$ conformal theories cannot admit any continuous symmetry. This is because $h=1$, $\bar{h}=0$ (or vice versa) corresponding to a conserved current does not appear in the conformal table. On the other hand, examples of discrete symmetries are known: $\mathbb{Z}_2$ for the Ising or RSOS model, $\mathbb{Z}_3$ for the 3 state Potts model. Let us examine how this fits in the previous framework and if the discrete symmetries of any given conformal theory can be predicted.

Let us return for a while to a discrete case; a lattice model lives on a (discrete) torus, and has a certain G-invariant Hamiltonian. For expliciteness, take a nearest-neighbour interaction:

$$\mathcal{H} = \sum_{i=1}^{N}\sum_{j=1}^{M} [h(S_{i,j},S_{i+1,j})+h(S_{i,j},S_{i,j+1})] \tag{23}$$

$$h({}^{g}S_i,\ {}^{g}S_j) = h(S_i,S_j) \tag{24}$$

where ${}^{g}S$ represents the action of the group on the degrees of freedom S. The important point is that there is a direct relation between the symmetry group G and the type of boundary conditions that may be imposed on the system and that respect the torus geometry /23/. Suppose some type of G-twisted boundary condition is chosen, namely:

$$S_{N+1,j}={}^{g_1}S_{1,j}\ ,\ \text{all } j \text{ and } S_{i,M+1}={}^{g_2}S_{i,1}\ ,\ \text{all } i \tag{25}$$

This may be also regarded as the introduction of two "frustration" lines in the Hamiltonian (23). Using the G-symmetry of the interaction and of the integration measure on S, one can, by changes of variables, shift these frustration lines along the torus. The translation invariances of the torus are respected, and the frustrated partition function does not depend on the location of the lines.

In the critical, conformal theory, some of the primary fields must transform under non trivial representations of G. For the sake of simplicity, we restrict ourselves from now on to the cyclic group $\mathbb{Z}_N$; fields carry some charge p: if $g = e^{2\pi i t/N}$, with t, p, defined mod N,

$$\varphi \to {}^{g}\varphi = \exp\left(\frac{2\pi i}{N}\,tp\right)\varphi \tag{26}$$

According to the previous considerations, consider $\mathbb{Z}_N$-twisted boundary conditions on these fields:

$$\varphi(z+n_1\omega_1+n_2\omega_2) = e^{i\frac{2\pi}{N}(n_1t_1+n_2t_2)p}\ \varphi(z) \tag{27}$$

t_1 and t_2 are the twists of the two frustration lines, and N must be prime to the set of non-zero p's: $(N,p_1,\dots,p_l)=1$, otherwise, we would be effectively dealing with a smaller N'. As translations along ω_1 are related by (8) to rotations in the plane:

$$\varphi_{h\bar{h}}(z+n_1\omega_1) = e^{2\pi i(h-\bar{h})n_1}\varphi(z) = e^{\frac{2\pi i}{N}n_1t_1p}\ \varphi(z) \tag{28}$$

The spins $(h-\bar{h})$ are rational numbers, with as a denominator a divisor of $4m(m+1)$. Hence N must be chosen among the divisors of these denominators, i.e. among the divisors of $4m(m+1)$. This is the first constraint on the possible values of N, such that the theory admits a $\mathbb{Z}_N$ symmetry.

Consider the frustrated partition function Z_{t_1,t_2} in the presence of the boundary conditions (27). It is not invariant under the full modular group Γ, but only under the subgroup that preserves (28) /24/. For instance, for any t_1 prime to N, $Z_{t_1,o}$ is invariant under the transformation (14) such that:

$$e^{2\pi\frac{i}{N}(n_1't_1p)} = e^{2\pi\frac{it_1p}{N}(dn_1+bn_2)} = e^{\pm 2\pi\frac{in_1t_1p}{N}} \tag{29}$$

for all n_1, n_2 and all fields (all p). This implies $b=0$ mod N, $d=\pm1$ mod N (hence $a=\pm1$ mod N). The reason for the $\pm$ sign is that we look for a *real* partition function $Z_{t_1,t_2} = Z_{-t_1,-t_2}$. The subgroup $\hat{\Gamma}^o(N)$ of invariance of $Z_{t_1,o}$, for $(t_1,N)=1$, is thus the set of unimodular matrices $\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} \pm1 & 0\\ \dots & \pm1\end{pmatrix}$ mod N.

Given a $Z(\tau)$ invariant under $\hat{\Gamma}^o(N)$, the full modular group Γ modifies the boundary conditions and acts on Z, defining new functions $(Z^{[g]})(\tau)=Z(g\tau)$ which satisfy $(Z^{[g_1]})^{[g_2]} = Z^{[g_1g_2]}$. $Z^{[g]}$ is invariant under the conjugate group $g^{-1}\hat{\Gamma}^og$, and if g_1 and g_2 belong to the same class of the left coset $\Gamma/\hat{\Gamma}^o(N)$: $g_1=\gamma g_2$, $\gamma\in\hat{\Gamma}^o$, then $Z^{[g_1]} = Z^{[g_2]}$. The number of these functions is thus the index of $\hat{\Gamma}^o$ in Γ /25/

$$I_N = \frac{N^2}{2}\prod_{p/N}\left(1-\frac{1}{p^2}\right)\ ,\ \text{p prime divisors of N}$$

$$I_N = 3$$

This is indeed the number of independent real $\mathbb{Z}_N$-frustrated partition functions Z_{t_1,t_2}, with twists prime to N: $(t_1,t_2,N)=1$. Starting from $Z=Z_{1,o}$, one may generate them all, using the rules: $Z^S_{t_1,t_2}= Z_{t_2,-t_1}$, $Z^{T^k}_{t_1,t_2} = Z_{t_1,t_2+kt_1}$. In particular, if α is prime to N, $(\alpha,N)=1$, there exists α^*: $\alpha.\alpha^*=1$ mod N, and $(Z_{1,o})^{T^{\alpha}ST^{\alpha^*}} = Z_{\alpha,o}$ (check that $Z_{\alpha,o}$ is invariant under $\hat{\Gamma}^o$!)

All these I_N functions are invariant under $\Gamma(N)$, the largest subgroup common to all $g^{-1}\hat{\Gamma}^og$: it is the principal congruence subgroup of level N: $a=d=\pm1$ mod N, $b=c=0$ mod N; it is an invariant subgroup of Γ and $\Gamma/\Gamma(N)=PSL(2,\mathbb{Z}_N)$ is a finite subgroup, which permutes the functions Z_{t_1,t_2}. In fact, each function $Z_{t_1,t_2} = Z^{[g]}$ is invariant under a group $g^{-1}\hat{\Gamma}^o g/\Gamma(N)$ isomorphic to $\mathbb{Z}_N$: this reflects the fact that each frustration sector is still $\mathbb{Z}_N$ invariant.

As each $Z_{t_1,o}$ may be regarded as the critical limit of the trace $\mathrm{tr}\, e^{\mathcal{H}_{t_1}}$ of the exponential of some frustrated Hamiltonian, to which the arguments of § 2 apply, we expect $Z_{t_1,o}$ to have an expansion of the form (10) with non negative coefficients. On the contrary, the other partition functions Z_{t_1,t_2}, $t_2 \neq 0$, should be regarded as $\mathrm{tr}(e^{\mathcal{H}_{t_1}}\Sigma_{t_2})$ with Σ_{t_2} a frustration operator, and their coefficients in (10) are not constrained to be either integral or positive.

To summarize, the search for a $\mathbb{Z}_N$ symmetry goes through the following steps:

i) pick a value of N among the divisors of $4m(m+1)$;

ii) construct the most general invariant under $\hat{\Gamma}^o(N)$, made of Virasoro characters: $\sum N_{h\bar{h}}(q\bar{q})^{-c/24} \chi_h \bar{\chi}_{\bar{h}}$, consider it as $Z_{1,o}$ and compute the other $Z_{t_1,o}$.

iii) impose that all $N_{h\bar{h}}$ are non negative, for all the $Z_{t_1,o}$.

If N is a prime, one may conclude that the system has a $\mathbb{Z}_N$ symmetry; if N is not a prime, one should also construct the other partition functions Z_{t_1,t_2}, $(N,t_1,t_2)=\alpha$, invariant under a smaller group ($\hat{\Gamma}^o(N/\alpha)$ and its conjugates). Finally, comparing $Z_{o,1}$ to $Z_{o,o}$ (the ordinary partition function with periodic b.c.), or more generally $Z_{t,1}$ to $Z_{t,o}$ tells us how the frustration operator Σ acts on the various operators of the sector of twist t /21/.

This program has been successfully applied to the unitary cases m=3,4,... in the main series (20), and to the Potts cases m=5,6 in the complementary series (21), and agrees nicely with the known $\mathbb{Z}_2$ and $\mathbb{Z}_3$ symmetries of these models. The expression of the corresponding invariants has been given in /22/. On the other hand, the analysis of the general model of the complementary series (m=1 or 2 mod 4) is more elusive. A first difficulty is to find a system of generators of the group $\hat{\Gamma}^o(N)$. For $N \leq 4$, it is easy to see that T^N and STS are two generators. For $N \geq 5$, more generators may be needed. For N=5 and 7, they were found by constructing the fundamental domain of $\hat{\Gamma}^o$ using its decomposition into classes /25/, and by patiently identifying its edges. Clearly, higher values of N would require a more elaborate method! In the models (21), there are $\left[\frac{N}{2}\right]$ independent invariants under STS and T^N (not including Z_{oo} invariant under Γ), where $N=\left[\frac{m+1}{2}\right]$: this has been achieved for m=9,10,13,14,17,18,21,22, and is likely to be true in general. For m=9,10,13,14, they are actually invariant under $\hat{\Gamma}^o(5)$ and $\hat{\Gamma}^o(7)$ respectively. However, it seems impossible to satisfy condition iii) above for all the $Z_{t,o}$. For instance, for m=9, let:

$$I_o = \sum_{r=1}^{4} \left\{ |\chi_{r1}+\chi_{r9}|^2 + |\chi_{r3}+\chi_{r7}|^2 + 2|\chi_{r5}|^2 \right\} : \Gamma \text{ invariant}$$

$$I_1 = \sum_{r=1}^{4} \left\{ |\chi_{r3}+\chi_{r7}|^2 + (\chi_{r1}+\chi_{r3}+\chi_{r7}+\chi_{r9})\,\bar{\chi}_{r5} + \text{c.c.} + |\chi_{r5}|^2 \right\}$$

$$I_2 = \sum_{r=1}^{4} \left\{ 3|\chi_{r3}+\chi_{r7}|^2 + (\chi_{r1}+\chi_{r9})(\bar{\chi}_{r3}+\bar{\chi}_{r7}) + \text{c.c.} \right.$$
$$\left. + 2(\chi_{r1}+2\chi_{r3}+2\chi_{r7}+\chi_{r9})\bar{\chi}_{r5} + \text{c.c.} + 4|\chi_{r5}|^2 \right\}$$

If $Z_{1o} = N_o I_o + N_1 I_1 + N_2 I_2$

then $Z_{2o} = (N_o+N_1+3N_2) I_o - (N_1+6N_2) I_1 + N_2 I_2$

and the positivity of the coefficients of $\chi_{r1}\bar{\chi}_{r5}$, $\chi_{r3}\bar{\chi}_{r5}$ and $\chi_{r1}\bar{\chi}_{r3}$ cannot be achieved but for the Γ-invariant solution $N_1=N_2=0$. I can see three possible ways out of this problem:

- other invariants exist, that might reinstate positivity; this is not ruled out, in view of the difficulty of exhausting all the solutions of diophantine systems like (19).
- invariants with negative coefficients might be given a physical interpretation.
- there is no $Z_{\left[\frac{m+1}{2}\right]}$ symmetry in these models!

At any rate, these new submodular invariants await a good interpretation. All the previous analysis should also be generalized to the newly discovered models presented in § 2 and to other discrete groups.

Acknowledgements

Most of the results reported here have been obtained in collaboration and/or in discussions with A.Capelli, C.Itzykson and H.Saleur. I am happy to express them my gratitude. I want also to thank D.Friedan and V.Kac for interesting discussions and communications.

REFERENCES

/1/ D.Friedan and E.Martinec, contributions to this Symposium

/2/ A.M.Polyakov, Zh.E.T.F.Lett. 12 (1970),538 (J.E.T.P.Lett. 12 (1970) 381)

/3/ A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl.Phys. B241 (1984) 333

/4/ J.L.Gervais and B.Sakita, Nucl.Phys. B34 (1971), 477

/5/ D.Friedan, Z.Qiu and S.Shenker, Phys.Rev.Lett. 52 (1984), 1575 and in "Vertex Operators in Mathematics and Physics", edited by J.Lepowsky, S.Mandelstam and I.Singer, Springer Verlag N.Y. 1985

/6/ V.G.Kac, Lecture Notes in Physics 94, 441 (1979), B.L.Feigin and D.B.Fuchs, Functs.Anal.Priloshen 16 (1982),47 (Funct.Anal and Appl. 16, (1982) 114)

/7/ D.Friedan, Z.Qiu and S.Shenker, Phys.Lett. 151B (1985), 37 Z.Qiu, Nucl.Phys. B270 [FS 16], (1986) 205

/8/ V.S.Dotsenko, Nucl.Phys. B235 [FS 11] (1984) 54 V.S.Dotsenko and V.A.Fateev, Nucl.Phys. B240 [FS 12] (1984) 312; B251 [FS 13] (1985) 691.

/9/ G.E.Andrews, R.J.Baxter and P.J.Forrester, J.Stat.Phys. 35 (1984) 193 D.A.Huse, Phys.Rev. B30 (1984) 3908

/10/ V.Knizhnik and A.B.Zamolodchikov, Nucl.Phys. B247 (1984) 83

/11/ V.A.Fateev and A.B.Zamolodchikov, Landau Institute preprints

/12/ J.L.Cardy, Nucl.Phys. B270 [FS 16] (1986) 186

/13/ C.Itzykson and J.B.Zuber, Nucl.Phys. to appear

/14/ C.Itzykson, H.Saleur and J.B.Zuber, Europhysics Journal, to appear

/15/ A.Capelli, C.Itzykson and J.B.Zuber, in preparation

/16/ B.L.Feigin and D.B.Fuchs, Functs. Anal.Prilozhen 17 (1983) 91 (Funct.Anal. and Appl. 17 (1983) 241) and Moscow preprint, 1984

/17/ A.Rocha-Caridi in "Vertex Operators in Mathematics and Physics", op.cit.

/18/ H.W.Blöte, J.L.Cardy, M.P.Nightingale, Phys.Rev.Lett. 56 (1986) 742 I.Affleck, Phys.Rev.Lett. 56 (1986) 746

/19/ G.V.Gehlen, V.Rittenberg and H.Ruegg, Univ.of Genève preprint UGVA/85/04.460; H.Saleur, Saclay preprint SPhT 86/022

/20/ V.G.Kac, private communication; see also V.G.Kac and M.Wakimoto (Bures I.H.E.S.preprint, to appear in Lecture Notes in Physics, Proceedings of Clausthal 1985) for an alternative approach to the modular invariance of subsets of characters

/21/ J.L.Cardy, Nucl.Phys. to appear

/22/ J.B.Zuber, Phys.Lett. to appear

/23/ G.v.Gehlen and V.Rittenberg, Bonn preprint HE-86-03

/24/ W.Nahm, Nucl.Phys. B114 (1976) 174; N.Seiberg and E.Witten, Princeton preprint, Feb.1986 C.Imbimbo and A.Schwimmer, to appear

/25/ R.C.Gunnings, Lectures on Modular Forms, Princeton University Press 1962; M.I.Knopp, Modular Functions in Analytic Number Theory, Markham Publishing Co, Chicago 1970.