RECENT PROGRESS in CONFORMAL FIELD THEORY

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Abstract
I review some recent work on conformal invariant theories, their classification, free field and lattice representations, and calculations of correlation functions.

1. CLASSIFICATION

Much progress has been accomplished over the last months concerning the general construction of conformal invariant theories and their classification. What is at stake is a unified description of all universality classes of 2-D critical phenomena in a statistical mechanical context, or of all possible ground states in string theory (see Dan Friedan's lecture at this meeting).

To start with, let me recall that such a theory is completely characterized [1] by the value of the common central charge c of the two commuting Virasoro algebras Vir_L and Vir_R, the spectrum of dimensions $h, \bar{h}$ of its primary fields, and the operator product algebra that these fields satisfy. At first sight, it seems that one may distinguish discrete theories for which the possible values of c, h, $\bar{h}$ belong to a discrete set from the continuous ones. We will see later that this distinction is only partially relevant. Several families of discrete conformal theories have been discovered by finding consistent solutions to the operator algebra. Among the most noticeable ones, let me quote:

a) the "minimal" theories [1], characterized by two mutually prime positive integers p and p', involve only a finite number of primary fields. Their central charge is given by:

$$c = 1 - 6(p-p')^2/pp' \qquad (1.1)$$

and their possible conformal dimensions take their value in the set:

$$h, \bar{h} \in \{ h_{r,s} = ((r\,p - s\,p')^2 - (p - p')^2) / 4pp' \} \qquad r,\ s \text{ integers: } 1 \le r \le p'-1 \quad 1 \le s \le p-1 \qquad (1.2)$$

It is known that "minimality", i.e. a finite number of Virasoro primaries, requires $c < 1$ (see below) and it is likely that these are the only such theories. A most interesting subset of these minimal theories are the $c<1$ unitary ones, for which p and p' have to be consecutive integers denoted k+2 and k+3 [2] and therefore c reads:

$$c = 1 - 6 / (k+2)(k+3) \qquad k \ge 1 \qquad (1.3)$$

Recall that there may be, in a statistical mechanical context, physically interesting non-unitary theories [3].

b) the N=1 superconformal unitary theories [4] with central charge

$$c = 3/2\ (\ 1 - 8 / (k+2)(k+4)\ ) \qquad k \ge 1 \qquad (1.4)$$

and a spectrum of h, $\bar{h}$ given by a formula analogous to (1.2);

c) theories with a Z_k symmetry and parafermions of spin $(k-1)/k$, for which [5]

$$c = 2\ (k-1) / (k+2) \qquad k \ge 2 \qquad (1.5)$$

d) theories with a S_3 symmetry and spin 4/3 parafermions [6]:

$$c = 2\ (1 - 12 / (k+2)(k+6)) \qquad k \ge 1 \qquad (1.6)$$

e) theories with a Z_3 symmetry and a spin 3 chiral current [7]:

$$c = 2\ (1 - 12 / (k+3)(k+4)) \qquad k \ge 1 \qquad (1.7)$$

etc,etc... It is interesting to notice that the above list is somewhat redundant: some models may belong to several families. Such is the case, for instance, of the 3-state Potts model which belongs simultaneously to families a, c and e. Such redundancies are sometimes quite fruitful as they may offer different standpoints concerning a definite model and lead to interesting identities (see below). Another remark that one makes in view of this list is that in all the cases beside the minimal one, an algebra larger than the Virasoro algebra is at work: either the N=1 NSR superconformal algebra, or some kind of "parafermionic algebra", or (in case e) an algebra generated by the energy-momentum tensor and the spin 3 chiral field. This feature, first emphasized in [8], will reappear in the forthcoming discussion.

It has been realized only recently that all these discrete series of conformal theories have a common ancestor: the theories with a $g_L x g_R$ current algebra, alias the WZW theories over a group G [9]. "Current algebra" means that a Kac-Moody algebra $g^{(k)}$ is present, (actually a product $g_L x g_R$), for which the value of the (Kac-Moody) central charge k is given and denoted here by a superscript. The Virasoro algebra appears in the enveloping algebra, i.e. the energy-momentum tensor is constructed by the Sugawara formula in terms of the currents:

$$T_G(z) = \text{const.} \sum :J^a(z)\ J_a(z): \qquad (1.8)$$

with a suitable normalization and a normal ordering prescription. The corresponding value of c reads (for g simple):

$$c_G = k \dim(G) / (k+g) \qquad (1.9)$$

where g is the dual Coxeter number of the g Lie algebra (g=N for su(N)...). If the algebra is semi simple, $g = g_1 \oplus g_2$, the energy momentum tensors decouple and the central charges add up: $c=c_1+c_2$ while taking a subalgebra h of g, the difference $T_G - T_H$ satisfies the Virasoro algebra with a central charge $c_G - c_H$. This is the so-called G/H coset construction [10]. Choosing the algebra $g=su(2)^{(k)} \oplus su(2)^{(1)}$ and $h=su(2)^{(k+1)}$ leads to values of c that reproduce the unitary minimal series (1.3). Likewise, the superconformal series is obtained by taking $g=su(2)^{(k)} \oplus su(2)^{(2)}$ and $h=su(2)^{(k+2)}$. It has been realized recently that the same procedure may account for all the known discrete cases. For example taking $g/h = su(2)^{(k)} / u(1)$, $su(2)^{(k)} \oplus su(2)^{(4)} / su(2)^{(k+4)}$, $su(3)^{(k)} \oplus su(3)^{(1)} / su(3)^{(k+1)}$ yields respectively the cases c)[11], d)[12] and e)[13] listed above. There has been a lot of activity lately generalizing this to $su(N)^{(k)} \oplus su(N)^{(m)} / su(N)^{(k+m)}$ [14], or even to general g/h [15,16]. It is now an interesting question to know if this method exhausts all the discrete cases.

At this stage, the method, however, only gives the possible values of c, h, and h that may appear in a G/H theory. Can one go further and ascertain what are the consistent field theories viz what are the possible operator contents and operator algebras? As we will see, there may be several possibilities for a given choice of G/H. As for the operator content, i.e. the set of pairs (h,h) for the primary fields, it has been shown [17] that it is conveniently coded in the partition function of the system in a box with periodic boundary conditions, hence on a torus of periods ω_1, ω_2. Denoting $\tau = \omega_2/\omega_1$ the modular ratio of the torus, $q = \exp 2i\pi\tau$, this "one-loop" partition function reads

$$Z = \mathrm{tr}\,(\, q^{L_0 - c/24}\ \bar{q}^{\bar{L}_0 - c/24}\,) \qquad (1.10)$$

where the trace is to be taken in the Hilbert space of states. As the latter decomposes into a sum of irreducible representations of $Vir_L \times Vir_R$ labelled by (h,h), one may write:

$$Z = \sum N_{h,\bar{h}}\ \chi_h(q)\ \chi_{\bar{h}}(\bar{q}) \qquad (1.11)$$

where χ_h is the Virasoro character, i.e. $\mathrm{tr}\ q^{L_0 - c/24}$ evaluated in the irrep $\{h\}$. It counts the number d_n of states at level n above the highest weight state $|h\rangle$

$$\chi_h = (\mathrm{tr}\ q^{L - c/24})_h = \sum d_n\ q^{h+n-c/24} \qquad (1.12)$$

Quite explicit expressions have been given in the literature for these characters [18]. In equ.(1.11), the $N_{h,\bar{h}}$ are multiplicities, hence non negative integers. Formula (1.11) encodes the operator content of the theory in the sense that to each term $\chi_h\chi_{\bar{h}}$ corresponds a primary field in the (untwisted sector of the) theory. Moreover, modular invariance, i.e. the condition that Z should be intrinsically attached to the torus, not to a choice of basis (ω_1, ω_2) of its periods, turns out to be a very strong constraint.

For example, it implies that for $c \geq 1$, there must be an infinite number of primary fields [17]. The distinguished feature of the "minimal" theories (1.1) is that the corresponding Virasoro characters χ_h transform as a finite dimensional representation under the modular group. This does not mean that for $c \geq 1$ the sum becomes untractable. As mentioned above, in all known discrete cases, an algebra A larger than Vir may be identified, and it turns out that the field content may be recast in a finite number of irreducible representations of A. An irreducible representation of A may decompose into an infinite number of irreps of Vir, and some primary fields of Vir may appear as secondaries of A. In such a case, Z can be written as a finite sum:

$$Z = \sum N_{\mu\bar{\mu}}\ \chi_\mu(q)\ \chi_{\bar{\mu}}(q) \qquad (1.13)$$

where the $\chi_\mu(q)$ is now $\mathrm{tr}(q^{L_0 - c/24})$ evaluated in the μ-representation of A. The importance of such "rational" theories in which the modular group acts through a finite dimensional representation, has been emphasized at this meeting by D. Friedan. We have just seen that all known discrete theories have a large algebra and are rational. Conversely there are arguments [19,15] that in any rational theory, there should be a large algebra. The definition of this extended algebra is not unambiguous, though: one may think of the (integral or fractional spin) chiral fields coupled to the identity operator in a block $\chi_0(q)$ as generators of this algebra [16]. This may be studied more explicitly in the coset models derived from su(N) where these new currents are constructed from the J^a using higher Casimir invariants [13,14], giving rise to a large associative algebra [7,20].

It is possible to classify all the modular invariants of the form (1.11). Some exploratory work in the case of minimal [17,21,22], of affine su(2) [23,22], and of N=1 superconformal [24] theories has led to a conjecture [25], which has then be proved in two steps [11,26]. There too, the basic problem [22] is to understand the nature of the affine $su(2)^{(k)}$ invariants, of the form (1.13), with χ standing for the affine characters of level k [27]. While there is a huge number of modular invariants with indefinite coefficients, positivity reduces it to two infinite series plus three exceptional cases, in one-to-one correspondence with simply laced Lie algebras. This unexpected correspondence is most visible on the operator content of the theory: the primary operators (for the KM algebra $su(2)^{(k)}$) have an "isospin" l (integer or half-integer) satisfying $0 \leq l \leq k/2$; in all the admissible solutions, the spinless ($\bar{h}$=h) operators that contribute to Z have an "isospin" l such that $2l+1$ reproduces with their multiplicities all the exponents of a simply laced algebra of Coxeter number k+2. The deep reason behind this A-D-E classification of su(2) modular invariants is still unclear. Explicit field-theoretic realizations of all these partition functions have been constructed [28a].

From the set of affine invariants, one may derive modular invariant partition functions for the G/H models. Following the coset construction, the character of G factorizes as

$$\chi_G = \sum b_{G/H}\ \chi_H \qquad (1.14)$$

where the branching functions $b_{G/H}$ are combinations of Virasoro characters for the G/H model. Given a pair of modular invariants for the G and H algebras, one then construct another one for G/H [28b,29,13,14,15]. It is likely that this procedure exhausts all G/H invariants, but to the best of my knowledge, this has so far been proven in detail only for the minimal and N=1 superconformal theories, for which the different models are therefore classified by a pair of Lie algebras A, D, E [25,26]. It would also be interesting to know the classification for current algebras other than su(2). Although the general form of the invariants has been derived [30], the classification of the positive ones is still missing.

Finally one may wonder whether the knowledge of the operator content suffices to uniquely determine the operator algebra. Classification of operator algebras is harder, and although existing solutions are consistent with the operator classification [31], a full proof of this unicity would be desirable.

Let us now discuss the case of "continuous" theories, for which the spectrum of $\{h,\bar{h}\}$ depends continuously on one or several parameters. We shall consider first the case of one massless free (Gaussian) field Φ compactified on a circle S^1 of radius r. This theory has c=1 and is a prototype of the continuous case. The primary operators are vertex operators $e^{ie\Phi}$ ("electric" operators), solitons ("magnetic" operators) and mixed operators: the spectrum of $\{h,\bar{h}\}$ reads

$$\{h, \bar{h}\} = \{p_L^2/2,\ p_R^2/2\} = \{(\ell/2r + mr)^2/2, (\ell/2r - mr)^2/2\} \qquad (1.15)$$

with ℓ, m integers, and is known in the statistical mechanics language as the Coulomb gas spectrum of exponents [32]. It enjoys the duality symmetry : $\ell \to m,\ r \to 1/2r$, and it is easy to construct its modular invariant partition function on a torus:

$$Z_G(r) = 1/\eta\bar{\eta} \sum_{\ell,m} q^h \bar{q}^{\bar{h}} = \sum_{m,m'} Z_{m,m'}(r) \qquad (1.16)$$

(see for example [33], [34]; in the notations of the latter reference, $g=2r^2$). Here η is Dedekind 's eta function $\eta(q) = q^{1/24} \prod(1-q^n)$ and $Z_{m,m'}$ denotes the partition function of the free field subject to shifted boudary conditions:

$$\Phi(z+1) = \Phi(z) + 2\pi m \quad , \quad \Phi(z+\tau) = \Phi(z) + 2\pi m' \qquad (1.17)$$

i.e. in the sector of winding numbers m, m'. $Z_{m,m'}$ which reads explicitly:

$$Z_{m,m'} = r\sqrt{2}\ (\mathrm{Im}^{1/2}\tau\ \eta\bar{\eta})^{-1} \exp-(2\pi r^2\ |m'-m\tau|^2/\mathrm{Im}\tau) \qquad (1.18)$$

enjoys the following modular properties:

$$Z_{m,m'}((a\tau+b)/(c\tau+d)) = Z_{cm'+dm,am'+bm}(\tau) \qquad (1.19)$$

These c=1 models may be regarded as U(1) current algebra theories [35].

One may also consider the case of a field Φ living on the orbifold S^1/Z_2, which amounts to identifying the angles Φ and $-\Phi$. Taking this identification into account gives three additional "twisted" sectors where the field may be antiperiodic along either period of the torus , and thus changes the partition function to

$$Z_{orb}(r) = Z_G(r) + Z_{PA} + Z_{AP} + Z_{AA} \qquad (1.20)$$

This has been recognized to describe the critical regime of [36]) by various people using various approaches [37].

At this stage, we thus have two lines of continuous c=1 conformal theories. The two lines have a common point, the self-dual point of the orbifold, which describes the Kosterlitz-Thouless point of the XY model [38,39]. The uncompactified gaussian field may be regarded as the $r\to\infty$ limit of the circle compactification. More remarkably, all known discrete theories having c=1 fall on one of these lines, with three exceptions namely theories associated with Dynkin diagrams of extended $\hat{E}_6$, $\hat{E}_7$ and $\hat{E}_8$ algebras ([40] and see below) which have been recognized by Ginsparg to be isolated [41]. One may indeed study the nature and number of the operators M responsible for the continuous variations of the conformal dimensions, the "marginal" operators in the jargon of statistical mechanics (see [42] and the talk of R. Dijkgraaf at this meeting).

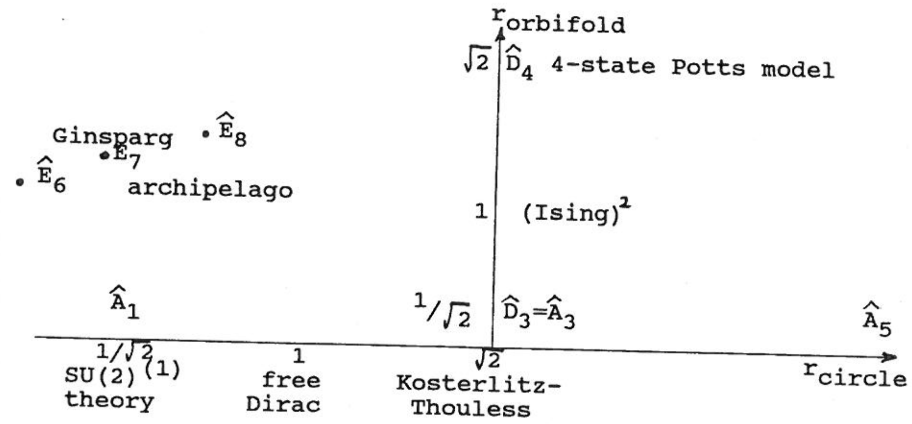

Fig. 1: Known c=1 theories

At the present time, the chart of known c=1 theories looks as on Fig.1 [39,41,42]. Notice that the two lines are <u>not</u> coordinate axis, only points on them and the three additional points make sense. Are there other terrae incognitae to be discovered? There are arguments [42] that the chart is actually complete.

Let us recall that such theories are not only a prototype but that c=1 is also a natural value for continuous theories [43]: if there is no spin-2 conserved current beside T(z) and if $\langle M\, A\, A\rangle = 0$ for any scalar quasi-primary field A and the marginal operator M, i.e. roughly speaking in the absence of extra symmetries, then necessarily c is equal to 1. At any rate, one may summarize what has been learnt in the case c=1 and expect that it has some degree of generality.

Along the two continous lines of Fig.1, at all rational values of the squared radius, the theory is rational in the sense defined above. To be explicit if $r^2 = P/Q$, the dimension of the modular group is (at most), P.Q+1 (see for example [34]). It would be interesting to find a coset construction, if any, of all these rational theories (see [41,35] for such a realization of some gaussian points). If all rational theories are associated with a large algebra, it would also be interesting to study how the latter gets deformed as the continuous parameter is changed. Thus at c=1, rational theories are dense within the continuous ones. If this is general, one sees that the distinction between discrete and continuous theories becomes slightly blurred. Also at this meeting, we have heard from D. Gepner [44] the exciting news that by appropriately tensoring and projecting discrete (N=2 superconformal) theories, one may obtain special points of Calabi-Yau compactifications. Could this procedure give a dense set of points? In other words, could discrete theories, all obtained by some coset construction ?, all associated with some extended algebra?, be dense in all conformal theories ? By the way, this also shows that the traditional qualitative distinction between discrete theories good for statistical mechanics and continuous ones, appropriate for string theory, is too superficial.

2. FREE FIELD and LATTICE REALIZATIONS.

1) Coulomb gas with a charge at infinity.

It has been known for long [45-47] that one may represent minimal conformal theories in terms of a free field ϕ of propagator $\langle\phi(z,\bar z)\ \phi(w,\bar w)\rangle = -2\ \mathrm{Log}\ (z-w)(\bar z-\bar w)$. Primary fields are vertex operators $e^{ie\phi}$, and a "charge at infinity" is added in the functional measure in the form $e^{-2ie_0\phi(\infty)}$. This charge e_0 modifies the balance of charge, hence the conformal dimensions, the energy-momentum tensor and its central charge according to:

$$\begin{aligned} h &= e^2 &&\rightarrow & h &= e(e-2e_0)\\ T(z) &= -1/4\ (\partial\phi)^2 &&\rightarrow & T(z) &= -1/4(\partial\phi)^2 + ie_0\,\partial^2\phi \\ c &= 1 &&\rightarrow & c &= 1 - 24\ e_0^{\ 2}\end{aligned} \qquad (2.1)$$

Moreover, through the introduction of "screening operators" [46-47], one may determine the null vectors of the Virasoro representations, recover a spectrum of $\{h,\bar h\}$ given by Kac's formula (1.2), and compute correlation functions. This has been generalized to N=1 [4,48] and to N=2 [49] superconformal theories, and more recently to the various coset models [7,50,14,15]. The method, however, works well in the plane but on a higher topology, e.g. on a torus, it is not clear where to put the charge e_0.

2) Magnetic defect on a torus

In the case of a torus, the dual magnetic picture answers (partially) the question. One may show starting from the lattice version of the theory and check on the explicit expressions that the torus partition functions of minimal models may be expressed as a linear combination of Gaussian partition functions $Z_G(r)$ for different radii. For the minimal model classified as $A_{p-1}, G_{p'-1}$, where $G_{p'-1}$ is an A-D-E algebra of Coxeter number p', Kostov [51] has reexpressed former results [38,40] by the following strange formula

$$\begin{aligned} Z_{A_{p-1},G_{p'-1}} &= 1/2 \sum\nolimits_{m,m'} \sum\nolimits_n \cos[2\pi n \langle m,m'\rangle/p']\ Z_{m,m'}(r=\sqrt{p/2p'}) \\ &= \sum\nolimits_i k_i\ Z_G(r_i) \end{aligned} \qquad (2.2)$$

where the sum over n runs over the exponents of the algebra $G_{p'-1}$, and $\langle m,m'\rangle$ denotes the greatest common divisor of m and m'. In the second expression, the k_i are integers of either sign: indeed $\sum_i k_i = 0$ is required if we want to recover for small q a behaviour given by c<1 [52]. Explicit expressions for these coefficients may be found in [38].

The form of (2.2) is consistent with general arguments on modular invariance [11]. Suppose that some partition function may be written as a superposition of $Z_{m,m'}$ weighted by some function of the winding numbers:

$$Z = \sum\nolimits_{m,m'} f(m,m')\ Z_{m,m'}(r) \qquad (2.3)$$

In all known cases, only a finite subgroup of order N of the modular group acts effectively on $Z_{m,m'}$ [22,25,11] (is this a priori obvious for any rational theory?); one may thus rewrite Z as:

$$\begin{aligned} Z &= N^{-1} \sum\nolimits_g \sum\nolimits_{m,m'} f(m,m')\ Z_{gm,gm'} \\ &= N^{-1} \sum\nolimits_{m,m'} \sum\nolimits_g f(gm,gm')\ Z_{m,m'} \\ &= \sum\nolimits_{\text{orbits}} F[\text{orbit}] \sum\nolimits_{m,m' \in \text{orbit}} Z_{m,m'} \end{aligned} \qquad (2.4)$$

where the action of the modular subgroup has been transferred from the variable τ to the shifts m,m' using Eq.(1.19). The orbits of such an action have been shown, however, to be characterized by the g.c.d. $\langle m,m'\rangle$ [11]. The partition function has therefore the general form:

$$Z = \sum_{m,m'} F(\langle m,m'\rangle)\; Z_{m,m'}(r) \qquad (2.5)$$

Similar expressions may also be written for the various $su(2)^{(k)}$ partition functions

$$Z = (k+2)/4\pi\; Z_1{}^2 \sum_i k_i\; (\partial Z_G/r\partial r)\big|_{r_i} \qquad (2.6)$$

(where Z_1 denotes the free uncompactified partition function) or for $su(N)^{(k)} \oplus su(N)^{(1)}/su(N)^{(k+1)}$ coset models for which it involves N-1 Gaussian fields compactified on circles of various radii [50,51]. The physical meaning of expressions like (2.6), however, is not clear to me. Does it point to another kind of bosonization of these theories?

Before leaving this discussion of torus partition functions, let us illustrate the equivalences between alternative descriptions of a given model mentioned in the first section. The critical Ising model may be described either as a free Majorana field, or as a free boson field with a magnetic defect, or, if duplicated, as a special point of the orbifold (Ashkin-Teller) model. Accordingly, its partition function may take the forms:

$$Z = \sum_{\text{spin structures}} (\det - \Delta\,)^{1/2} = \sum_{i=2,3,4} \left|\theta_i / \eta\right|$$

$$= {}^1/_2\; (\; Z_G(\sqrt{6}) - Z_G(\sqrt{2/3})\;) \qquad (2.7)$$

$$Z^2 = {}^1/_2\;\; Z_G(1) + \sum_{i=2,3,4} \left|\eta / \theta_i\right|$$

Consistency between these expressions result from identities between Jacobi θ functions. On a higher genus Riemann surface, the analogs of (2.7a) and (2.7c) exist and agree thanks to non trivial identities [53].

It would be nice to have expressions similar to (2.2-6) for higher genus

$$Z = \sum f(\vec{m},\vec{m}')\; Z_{\vec{m},\vec{m}'} \qquad ?? \qquad (2.8)$$

where now $\vec{m}$, $\vec{m}'$ would be g-vectors denoting the winding numbers in a canonical homology basis . By the same arguments as in Equ. (2.3-5), one could write

$$Z = \sum F[\text{orbit}]\; Z_{\vec{m},\vec{m}'} \qquad (2.9)$$

The orbit, however, is again completely characterized by the greatest common divisor or the 2g integers $m_1,\ldots,m'_g$, and a function F of this g.c.d. can factorize on degenerate surfaces:

$$F(\langle m_1,m_2,m'_1,m'_2\rangle) = F(\langle\langle m_1,m_2\rangle,\langle m'_1,m'_2\rangle\rangle)$$

$$\rightarrow F_1(\langle m_1,m_2\rangle)\; F_2(\langle m'_1,m'_2\rangle)$$

only if it is constant. For the Ansatz (2.8), there is a clash between modular invariance and factorization. It might be that some ingredient is still missing, such as the screening operators of the construction of ref [46]. As a matter of a fact, no partition function in genus greater than one is explicitly known, but for free (fermion or boson) theories [33,54].

3) Lattice realizations

In parallel to the study of conformal field theories, some very interesting work is taking place in the realm of completely integrable lattice models. By different routes different groups are constructing generalizations of Baxter's vertex models whose critical behaviour is described by the conformal theories just discussed. It had been recognized some time ago [55] that the generic unitary minimal model, now labelled as (A_{k+1},A_{k+2}), correspond to a critical regime of the "restricted solid-on-solid" (RSOS) model [56]; in the latter a fluctuating surface may take integral heights h_i ranging between 1 and k+2, and neighbouring sites have to be at heights differing by 1: $|h_i-h_j|=1$. Boltzmann's weights are attached to these configurations so as to ensure integrability. In Pasquier's generalization [57,40], the states of the surface are in one-to-one correspondence with nodes of the Dynkin diagram of a simply laced algebra G with neighbouring sites occupying neighbouring states on the Dynkin diagram. The original RSOS model corresponds of course to the A algebras. By studying the algebra of the microscopic transfer matrices (Temperley-Lieb-Jones algebra), and its unitary representations, one finds the Boltzmann's weights of the system in terms of the eigenspectrum of the Cartan matrix of G. Moreover, on a torus, the model may be mapped on a discrete Gaussian model and in the continuous limit, expressions of the form (2.2) are derived, therefore completing the identification with the conformal theory (A_{p-1},G_p). The other unitary theories (G_p,A_{p+1}) and non-unitary models may be also constructed in the same way, as well as models attached to extended Dynkin diagrams [40]. The latter, which have a central charge c=1, appear on the chart of Fig.1. On the other hand, the Kyoto group [58] is considering a large class of lattice models, whose critical regime is described by some coset conformal theories. Quite remarkably, the probabilities for the system to be in a certain state are given in terms of a variable measuring the separation from criticality by the same functions (modular forms) as the conformal characters in terms of the torus modular ratio. A good understanding of this property would certainly give much insight on the connection between integrable and conformal theories.

3. NON-ZERO GENUS CORRELATION FUNCTIONS

Correlation functions in non zero genus are interesting in statistical mechanics (genus one especially) where they describe the critical system in the presence of periodic boundary conditions, and in string theory. They are also interesting from a technical point of view, as they are a testing ground of many important ideas on bosonization, orbifolds or singular limits of Riemann surfaces. As pointed out in [59], there are relationships between n-point functions in genus g and n- and n+2-point functions in genus g-1, obtained by pinching the Riemann surface along a homology cycle. It is instructive to work out in detail this relation in the case of a torus. For some 2-point function on a torus,

$$\langle A(z)\ A(0)\rangle \ = Z^{-1}\ \mathrm{tr}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\ \hat{A}(z)\ \hat{A}(0)) \qquad (3.1)$$

we may write an expansion as the torus degenerates to a long cylinder: $\tau \rightarrow i\infty$, i.e. $q = \exp 2i\pi\tau \rightarrow 0$

$$\langle A(z)\ A(0)\rangle \ = Z^{-1} \sum q^{h+|n|-c/24}\ \bar{q}^{\bar{h}+|\bar{n}|-c/24} \langle h\{n\};\bar{h}\{\bar{n}\} \mid A(z)\ A(0) \mid h\{n\};\bar{h}\{\bar{n}\}\rangle_{cylinder} \qquad (3.2)$$

$$= Z^{-1}\ (2\pi)^{2(h_A+\bar{h}_A)}\ e^{2i\pi(h_A z-\bar{h}_A\bar{z})}\ \ q^{h+|n|-c/24}\ \bar{q}^{\bar{h}+|\bar{n}|-c/24}$$
$$\times \lim_{w_1\to\infty} w_1^{\ 2(h+|n|)}\bar{w}_1^{\ 2(\bar{h}+|\bar{n}|)} \langle \phi_{hn\bar{h}\bar{n}}(w_1)A(w)A(1)\phi_{hn\bar{h}\bar{n}}(0)\rangle_{plane}$$

Here the sum runs over all primary fields $\phi_{h\bar{h}}$ in the untwisted sector of the theory and all their descendants labelled by the integers $\{n_i\},\{\bar{n}_i\}$ and the correlators on the cylinder have been mapped on the plane by $w = \exp 2i\pi z$. By the same token, this two-point function in genus 1 should be obtained as a limit of a genus 2 partition function. This shows that there is more or less the same information and the same complexity in the (n,g) functions as in the (n+2,g-1) ones.

In a minimal theory, all primary fields A are "degenerate", which means that a linear combination of $L_{-1}^{\alpha_1}\ldots L_{-n}^{\alpha_n}$ A of a given level $L=\sum k\alpha_k$ may be consistently set equal to zero. This results in partial differential equations of order L satisfied by the correlation functions of A, which have been written down explicitly in the plane in [1] and on a higher genus Riemann surface in [60]. In the plane, these equations may be either solved explicitly in terms of hypergeometric functions in the case of level 2 (actually algebraic functions for the Ising model), or given an integral representation [46-47]. On higher genus, much activity has been devoted over the last year to the calculation of various types of correlation functions [61-65,53,42]. So far, however, this concerns only free fields

of various kinds: torus and orbifold compactifications, Ising model... One finds quite explicit expressions in terms of theta functions . In the limit where the torus degenerates to a cylinder or a plane, this reduces to algebraic functions. By the argument given above (equ.(3.1)), one may expect more complicated results for other theories, where the correlations in the plane are known to be non algebraic.

My purpose is not to discuss in detail this highly technical field but only to illustrate the state of the art on the case of the Ising model. As already pointed out, it may be regarded in several different ways (cp. Equ (2.7)). Unfortunately, we do not know how to calculate non-zero-genus correlation functions using the idea of charge at infinity or of magnetic defects; this is a pity, because this would generalize to all minimal theories. Again what seems to be missing here is the appropriate generalization of screening operator. This leaves us with two other methods for handling the Ising model and this leads to remarkable identities on its correlators.

1) In the first approach, one uses the fermionic representation of the model. This requires working in a given fermionic sector, in which the fermion is assigned definite boundary conditions (spin structure). The critical Ising model is a conformal theory with two primary spinless fields (beside the identity): the energy operator ε and the spin σ (or its dual, the disorder operator μ); both are non-chiral, i.e. depend on both z and $\bar{z}$. In terms of the 2-component free Majorana field, ε is easy to express

$$\varepsilon(z,\bar{z}) = \psi(z)\ \bar{\psi}(\bar{z}) \qquad (3.3)$$

and its correlators are combinations of fermion propagators (theta functions); σ, however is non local in terms of ψ. This makes the calculation of the spin correlation functions non trivial. One may either use the monodromy properties of the spin with respect to the fermion, or bosonize the model. Monodromy tells you that a spin field creates a square branch point in a fermion correlator: circling with the spin argument $(z,\bar{z})$ around the fermion location changes the sign of the function. The program [66] is then to write an Ansatz for the correlator $\langle\sigma(z_1,z_1)\ \sigma(z_2,z_2)\ \psi(z)\ \ \psi(w)\rangle$ for a given spin structure, with the right short distance singularities, periodicity and monodromy properties, prove its uniqueness, take the limit $w \rightarrow z$ to construct the energy-momentum correlator $\langle\sigma(z_1,z_1)\ \ \sigma(z_2,z_2)\ \ T(z)\rangle$, use the conformal Ward identity to transform this into a differential equation for $\langle\sigma(z_1,z_1)\ \sigma(z_2,z_2)\rangle$, and solve it!! This has been used in the case of complex (Dirac) field and their chiral "spin fields" [61], and may also be applied to the Ising case [62]. The other route, which is easier to extend to multipoint correlation functions, uses chiral bosonization [67]. The latter usually deals with complex fermions: it is better to duplicate the theory i.e. to square the correlation functions (for a given spin structure), which enables one to reexpress things in terms of Dirac fields $\psi(z)$, $\bar{\psi}(\bar{z})$, bosonized in the form:

$$\psi(z) = e^{i\varphi(z)} \qquad \bar{\psi}(\bar{z}) = e^{i\bar{\varphi}(\bar{z})} \tag{3.4}$$

The chiral boson fields φ and $\bar{\varphi}$ are assigned computation rules (depending on the spin structure) [67] designed to reproduce the fermionic functions; this is consistent thanks to non trivial identities between Θ functions (Fay's identities). It turns out that the following general formula holds for the square of the Ising correlators in a given spin structure denoted α:

$$Z_\alpha^2 \langle \Pi\varepsilon_i \, \Pi\sigma_j \, \Pi\mu_k \rangle_\alpha^2 = Z_\alpha^2 \langle \Pi(\vec{\nabla}\phi_i)^2 \, \Pi\cos\phi_j \, \Pi\sin\phi_k \rangle_\alpha \tag{3.5}$$

where the right-hand side is computed using chiral bosonization prescriptions for $\phi = 1/2(\varphi - \bar{\varphi})$. This leads to algebraic combinations of theta functions and one may check (at least in the simplest cases!) that these expressions satisfy the appropriate differential equations. In general, these equations amount to non trivial non-holomorphic identities between Θ functions, i.e. identities involving $\Theta(z_i,\tau)$ and $\Theta(\bar{z}_j,\bar{\tau})$. From the formulae (3.3), one must reconstruct the physical correlator as

$$\langle \Pi\varepsilon_i \, \Pi\sigma_j \, \Pi\mu_k \rangle = Z^{-1} \sum Z_\alpha \langle \Pi\varepsilon_i \, \Pi\sigma_j \, \Pi\mu_k \rangle_\alpha \tag{3.6}$$

and the bosonization rules are such that some sectors in which Z_α vanishes may actually contribute to the r.h.s. [61-62].

As a variation on the same theme, one may also try to bosonize the real (Majorana) fermion of the Ising model in the form [68,42,69]:

$$\psi(z) = \cos\varphi(z) \tag{3.7}$$

One may verify the consistency of the previous rules [67], thanks again to Fay's identities (fermion correlators are now Pfaffians). This bosonization is fine for the computation of energy correlators but leaves open the question of spin correlators.

2) In the second approach, one duplicate the whole Ising model, irrespective of the fermion spin structures. Use is made of the orbifold nature of the $(\text{Ising})^2$ model: in the Ashkin-Teller model, of which two decoupled Ising models are a special point [37,70,38], it is known that the product of the two copies of the energy operator, resp. of the spin, may be represented by the following bosonic expressions:

$$\varepsilon^{(1)} \varepsilon^{(2)} = (\vec{\nabla}\phi)^2 \qquad \sigma^{(1)} \sigma^{(2)} = \cos\phi \tag{3.8}$$

where now ϕ is the field living on the orbifold. This leads to formulae

$$\langle \Pi\varepsilon_i \, \Pi\sigma_j \, \Pi\mu_k \rangle^2 = \langle \Pi(\vec{\nabla}\phi_i)^2 \, \Pi\cos\phi_j \, \Pi\sin\phi_k \rangle \tag{3.9}$$

which look like (3.3), but involve quite different rules! Here the r.h.s. must be split into sectors corresponding to different types of shifted or antiperiodic boundary conditions on ϕ (compare equ.(1.20)). Consistency of the two sets of formulae (3.3) and (3.6) is non trivial, has been checked rather extensively in genus one in [62], and is likely to remain true in arbitrary genus.

Such calculations on Z_2 or even Z_n orbifolds have been carried out in great detail and/or generality in ref.[53,42,63-65]. Of special interest is the "twist field" responsible for the change of the boundary conditions of ϕ: it plays with ϕ the same role as the spin field discussed above with the fermion field . This suggests to use similar monodromy considerations to calculate its correlation functions: this has been done in the plane in [66] and extended to higher topologies in [63,65]. Another possibility is to duplicate (or to n-plicate for Z_n) the Riemann surface, after cutting it along the cycles where the field is twisted [53,42,64]. The twist field correlators can then be obtained from the untwisted partition function on the new Riemann surface by pinching in an appropriate way some cycles. Comparison between the two methods yields again identities! As a physical application, let us notice that the Ising spin field can also be regarded as the twist field of the orbifold field ϕ [37a]. This offers yet another route to the calculation of its correlation functions [42,65]. For those of you who think that too many different methods are no good and must hide some vice, I also mention that the two- and four-point spin correlation functions on a torus have also been tested against ...experiment, i.e. numerical (transfer matrix) calculations, needless to say with an excellent agreement [72]. When shall we see a similar test in string theory???

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