CLASSIFICATION OF MODULAR INVARIANT PARTITION FUNCTIONS

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ABSTRACT

Two-dimensional minimal conformal invariant theories are classified, using the constraint of modular invariance of their partition function on a torus.

1. INTRODUCTION

Conformal field theory is an essential piece in the construction of string theory[1]. It is also of importance in the study of two-dimensional critical systems, since dilatation invariance implies conformal invariance[2].

A conformal invariant theory in the Euclidean plane contains fields that transform covariantly under independent conformal transformations of the variables $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$:

$$z \to z' = f(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = g(\bar{z}) \tag{1}$$

In the quantum theory, these transformations are realized by two sets of Virasoro generators L_n and $\bar{L}_n$ satisfying

$$\begin{aligned} [L_n, L_m] &= (n-m)\, L_{n+m} + \frac{c}{12}\, n\left(n^2-1\right) \delta_{n+m,0} \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\, \bar{L}_{n+m} + \frac{c}{12}\, n\left(n^2-1\right) \delta_{n+m,0} \\ [L_n, \bar{L}_m] &= 0 \end{aligned} \tag{2}$$

The c-number "central charge" c is the coefficient of an anomalous term, that arises in the quantum theory.

A conformal theory[3] in the complex plane is specified by giving

i) the value of c

ii) the finite or infinite set of "primary" fields which transform[4] as $(h,\bar{h})$-tensors under (1):

$$\varphi_{h,\bar{h}}(z,\bar{z})\, dz^h\, d\bar{z}^{\bar{h}} = \varphi'_{h,\bar{h}}(z',\bar{z}')\, dz'^h\, d\bar{z}'^{\bar{h}} \tag{3}$$

or of the corresponding highest weight states of the two Virasoro algebras $V \otimes \bar{V}$:

$$\left|h,\bar{h}\right\rangle = \text{"}\lim_{z,\bar{z}\to 0}\text{"}\ \varphi_{h,\bar{h}}\ |0\rangle$$

(see [5] for a detailed discussion of the limit involved).

iii) the operator algebra[3]

$$\varphi^{(a)}(z,\bar{z})\ \varphi^{(b)}(0) = \sum_c C_{abc}\ z^{\Delta_{abc}}\ \bar{z}^{\bar{\Delta}_{abc}}\ \varphi^{(c)}(0) \tag{4}$$

where Δ and $\bar{\Delta}$ are simple combinations of the dimensions of the (primary or non primary) fields $\varphi^{(a,b,c)}$. The "structure constants" C_{abc} determine this algebra.

What are the consistent conformal theories? Two approaches have been followed to tackle this question. The first ones makes use of the crossing symmetry constraint i.e. of the associativity of the operator algebra: the structure constants satisfy non-linear relations. Classes of solutions have been discovered in this way[6], but a general classification of solutions seems difficult. On the other hand, Cardy [7] has suggested to consider a finite geometry, namely to put the conformal theory in a box with periodic boundary conditions, i.e. on a torus, and to use the modular invariance of the partition function to constraint the operator content of the theory. This approach, that we developped with A. Cappelli, C. Itzykson and H. Saleur[8-10] is quite effective and seems to lead to an exhaustive classification, at least of the minimal conformal theories to be defined soon.

Let us examine how a conformal theory may be defined on a torus. The torus is defined by two of its complex periods ω_1 and ω_2, chosen such that $\tau = \omega_2/\omega_1$ lies in the upper half-plane. If we want to make use of the formalism developped for the complex plane, we have to open the torus by cutting it along one of its geodesics, say ω_1. The resulting piece of cylinder may then be mapped onto an annulus in the plane by the exponential mapping:

$$w \to z = \exp 2\pi i\ w/\omega_1 \tag{5}$$

In this mapping, the translation operator along ω_2, i.e. the "Hamiltonian", is mapped on a certain combination of dilatation and rotation operators in the plane. To make things precise, we have to remember[3] that the stress-energy "tensors"

$$T(z) = \sum_{-\infty}^{\infty} z^{-n-2}\ L_n \qquad \bar{T}(\bar{z}) = \sum_{-\infty}^{\infty} \bar{z}^{-n-2}\ \bar{L}_n \tag{6}$$

are *not* primary fields but transform under (1) as:

$$T(z) = T'(z')\left(\frac{dz'}{dz}\right)^2 + \frac{c}{12}\left\{\frac{d^3z'/dz^3}{dz'/dz} - \frac{3}{2}\left(\frac{d^2z'/dz^2}{dz'/dz}\right)^2\right\} \tag{7}$$

Applied to the mapping (5), this leads to the following relations between the Virasoro generators on the cylinder and in the plane

$$\begin{aligned} L_{-1}^{cyl} &= \frac{1}{2\pi i}\oint_{\text{geodesic } \omega_1} T^{cyl}(w)\ dw \\ &= \frac{1}{\omega_1}\oint_0 \frac{dz}{z}\left(T^{pl}(z)\ z^2 - \frac{c}{24}\right) \\ &= \frac{2\pi i}{\omega_1}\left(L_0^{pl} - \frac{c}{24}\right) \end{aligned} \tag{8}$$

The translation operator along ω_2 is therefore

$$H = \omega_2\ L_{-1}^{cyl} + \bar{\omega}_2\ \bar{L}_{-1}^{cyl} \tag{9}$$

$$= 2\pi i\left[\tau\left(L_0^{P1} - \frac{c}{24}\right) + \bar\tau\left(\bar L_0^{P1} - \frac{c}{24}\right)\right]$$

It is now natural to define the partition function of the system on *the torus* by

$$\begin{aligned} Z &= \mathrm{Tr}\ e^{H} \\ &= \left(q\bar q\right)^{-c/24}\ \mathrm{Tr}\left(q^{L_0}\bar q^{\bar L_0}\right) \end{aligned} \tag{10}$$

where $q \equiv \exp 2\pi i\tau$. The expression (10) is very general but quite implicit, since all the content of the theory is hidden in the symbol "Tr": the trace must be taken in the Hilbert space of states, which depends on the theory. As this Hilbert space carries a representation of the two Virasoro algebras $V \otimes \bar V$, we decompose it into a sum of irreducible representations labelled by highest weights $\left(h,\bar h\right)$, of multiplicities $\mathcal{N}_{h\bar h}$ (non negative integers). Thus

$$Z = \sum_{h,\bar h} \mathcal{N}_{h\bar h}\ \chi_h(q)\ \chi_{\bar h}^{*}(q) \tag{11}$$

where

$$\chi_h(q) = q^{-c/24}\left(\mathrm{tr}\ q^{L_0}\right)_{\mathrm{irrep.}\{h\}} \tag{12}$$

The Virasoro characters (defined here with an unconventional prefactor $q^{-c/24}$) may be regarded as generating functions of the number of states in the irreducible representation of highest weight h:

$$\chi_L(q) = q^{h-c/24}\sum_{n=0}^{\infty} d_n\ q^n \tag{13}$$

Here d_n is the dimension of the space of eigenstates of L_0 of eigenvalue h+n.

I recall that explicit formulae exist for all the possible cases [10].

In addition to the positivity of its entries, $\mathcal{N}_{h\bar h}$ must be a symmetric matrix (reality of Z), and $\mathcal{N}_{00}$ must be equal to 1, to express the unicity of the "vacuum" $h = \bar h = 0$). (In non-unitary theories, there may be states with negative h's, and the interpretation of this condition may be questionable).

Finally, one notices that in the calculation of the partition function on the torus, the two periods have not been treated on the same footing: the periodicity along ω_2 has been enforced by the trace operation. The partition function Z, however, should not depend on this choice. More generally, Z should be attached to the lattice generated by ω_1 and ω_2 but not depend on the particular choice of the periods ω_1, ω_2. In other words Z should be invariant under unimodular changes of the periods:

$$\left.\begin{aligned} \omega_2' &= a\ \omega_2 + b\ \omega_1 \\ \omega_1' &= c\ \omega_2 + d\ \omega_1 \end{aligned}\right\}\quad \begin{aligned} &a,\ b,\ c\ ,\ d \in \mathbb{Z} \\ &ad - bc = 1 \end{aligned} \tag{14}$$

Since Z is a function of the dimensionless ratio $\tau = \frac{\omega_2}{\omega_1}$, it should be invariant under modular transformations of τ:

$$\tau = \frac{a\tau+b}{c\tau+d} \tag{15}$$

These modular transformations form an infinite discrete group, the modular group, denoted Γ; this group is generated for example by the two transformations $\tau \to \tau+1$ and $\tau \to -1/\tau$. The fact that a function of the form (11) must be a modular invariant turns out to be a very strong constraint. We shall examine its consequences for the simplest conformal theories: the "minimal" theories[3].

2. MINIMAL CONFORMAL AND A_1^1 AFFINE THEORIES

Minimal conformal theories are those containing a finite number of *primary* fields. As shown by Cardy [7], the modular invariance of (11) forces c to be less than one. Belavin, Polyakov and Zamolodchikov have introduced the set of theories of central charge

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{16}$$

where p and p' are coprime integers, and shown that the operator algebra closes for the finite number of primary fields $\varphi_{h,\bar{h}}$, where h and $\bar{h}$ take their values in the "Kac'table"[11]:

$$h_{rs} = h_{p'-r,p-s} = \frac{(pr-p's)^2-(p-p')^2}{4pp'} \tag{17}$$

with the restriction

$$1 \le r \le p'-1 \qquad\qquad 1 \le s \le p-1 \tag{18}$$

It is likely that those are the only theories with a finite number of primary fields. An important subset of the minimal theories is made of the $c < 1$ unitary theories[12] for which p and p' must be consecutive integers: $|p-p'| = 1$.

It is convenient to replace the pair of labels (r,s) by a single variable λ defined modulo $N = 2pp'$

$$\lambda = rp - sp' \tag{19}$$

That r,s may be recovered from λ results from the existence of ω_0, also defined mod N, and satisfying

$$\begin{aligned} rp + sp' &= \omega_0\,\lambda \quad \text{mod } N \\ \omega_0^2 &= 1 \qquad\ \text{mod } 2N \end{aligned} \tag{20}$$

This number is easily constructed; given the pair of coprimes p and p', there exist a and b such that

$$ap - bp' = 1$$

Then $\omega_0 = ap + bp'$ does the job.

We then introduce the set of functions:

$$K_\lambda(\tau) = \frac{1}{\eta(\tau)} \sum_{-\infty}^{\infty} \exp\left[2i\pi\tau(Nn+\lambda)^2/2N\right] \tag{21}$$

where η is Dedekind's function:

$$\eta(\tau) = q^{1/24} \prod_1^{\infty} (1-q^n), \qquad q = e^{2\pi i\tau} \tag{22}$$

In terms of these functions, the conformal characters read

$$\chi_\lambda(\tau) = K_\lambda(\tau) - K_{\omega_0\lambda}(\tau) \tag{23}$$

They satisfy periodicity and reflection properties:

$$\chi_\lambda(\tau) = \chi_{\lambda+N}(\tau) = \chi_{-\lambda}(\tau) = -\chi_{\omega_0\lambda}(\tau) \tag{24}$$

These properties enable us to restrict ultimately the values of λ to a fundamental domain (represented by (18)) but for the time being, we only use the periodicity in $\lambda \to \lambda+N$.

It is not difficult to see that these characters transform under the two generators $T: \tau \to \tau+1$ and $S: \tau \to -\frac{1}{\tau}$ of the modular group Γ according to

$$\begin{aligned} \chi_\lambda(\tau+1) &= \exp 2i\pi\left(\frac{\lambda^2}{2N} - \frac{1}{24}\right) \chi_\lambda(\tau) \\ \chi_\lambda(-1/\tau) &= \frac{1}{\sqrt{N}} \sum_{\lambda' \in \mathbb{Z}/N\mathbb{Z}} \exp 2i\pi\, \lambda\lambda'/N \quad \chi_{\lambda'}(\tau) \end{aligned} \tag{25}$$

i.e. they form a unitary representation of Γ.

There is an unexpected connection between the construction of conformal modular invariants and that of modular invariants made of A_1^1 Kac-Moody characters. In their study of the string compactification on a SU(2) group manifold, Gepner and Witten[13] have shown that the one-loop partition function of the group degrees of freedom takes the form

$$Z = \sum \mathcal{N}_{\lambda\bar\lambda}\, \chi_\lambda(\tau)\, \chi^*_{\bar\lambda}(\tau) \tag{26}$$

Here $\lambda = 2l+1$, where l is the integer or half-integer spin labelling the integrable representations of $A_1^{(1)}$ of central charge k ($\lambda \le k+1$); χ_λ are the affine characters[14], up to a certain power of q; $\mathcal{N}_{\lambda\bar\lambda}$ are non negative integers, with again $\mathcal{N}_{11} = 1$, and Z must be modular invariant. As we shall see there is a direct relation between the conformal and the affine invariants[15]. That there are connections between the minimal conformal and A_1^1 theories was actually already noticed or used in Ref.[10c,16]. At any rate, it is natural to treat both problems in parallel, and to introduce convenient notations for this purpose. In the A_1^1 case, we denote

$$N = 2(k+2)$$

The variable λ takes its values in a fundamental domain

$$1 \le \lambda \le k+1 \tag{27}$$

but it is actually suitable to extend it to a variable of $\mathbb{Z}/N\mathbb{Z}$. The characters read[13-15]

$$\chi_\lambda(\tau) = \frac{1}{\eta^3(\tau)} \sum_{-\infty}^{\infty} (nN+\lambda)\, e^{2i\pi(nN+\lambda)^2/N} \tag{28}$$

They satisfy

$$\chi_\lambda = \chi_{\lambda+N} = -\chi_{-\lambda} \tag{29}$$

and they have the following modular transformations

$$\chi_\lambda(\tau+1) = \exp 2i\pi\left(\frac{\lambda^2}{2N} - \frac{1}{8}\right) \chi_\lambda(\tau)$$

$$\chi_\lambda\left(-\frac{1}{\tau}\right) = \frac{-i}{\sqrt{N}} \sum_{\lambda' \in \mathbb{Z}/N\mathbb{Z}} \exp 2i\pi\, \lambda\lambda'/N \;\; \chi_{\lambda'}(\tau) \tag{30}$$

and form also a finite dimensional unitary representation of Γ.

In both the conformal and the affine cases, one may show that the infinite subgroup Γ_{2N} of Γ, made of the transformations (15) with

$$a = d = \pm 1 \mod 2N, \qquad b = c = 0 \mod 2N$$

changes the characters only by a τ- and λ-independent phase. This is particularly clear for the transformation T^{2N} for which a=d=1, b=2N, c=0 and $\chi_\lambda(\tau+2N) = \exp -2i\pi N/12\;\; \chi_\lambda(\tau)$. The subgroup Γ_{2N}, however, is *not* generated by T^{2N} and its conjugates, for $2N \ge 6$, and this property requires therefore an explicit proof[9]. It implies that only the finite coset group

$$\Gamma/\Gamma_{2N} = PSL(2, \mathbb{Z}/2N\mathbb{Z})$$

acts non trivially on characters through a projective unitary representation U. Modular invariance of (11) or (26) is thus equivalent to

$$U_{\lambda\lambda'}(A)\, \mathcal{N}_{\lambda'\bar\lambda'}\, U^\dagger_{\bar\lambda'\bar\lambda}(A) = \mathcal{N}_{\lambda\bar\lambda}$$

or

$$U(A)\mathcal{N} = \mathcal{N}\, U(A) \tag{31}$$

for any $A \in \Gamma/\Gamma_{2N}$. The forthcoming discussion is just tantamount to a study of the commutant of this projective representation.

3. GENERAL FORM OF THE INVARIANTS

Let us first consider the implications of the invariance under T: $T\mathcal{N}T^\dagger = \mathcal{N}$. From Eqs.(25) or (30), we see that $\mathcal{N}_{\lambda\bar\lambda} \ne 0$ only if

$$\lambda^2 = \bar\lambda^2 \mod 2N \tag{32}$$

It is not difficult[9] to show that this condition is equivalent to the following set of conditions:

i) there exists an integer α, divisor of λ and $\bar\lambda$, such that α^2 divides $\frac{N}{2}$

ii) there exists μ defined modulo N/α^2, such that $\mu^2 = 1 \mod 2N/\alpha^2$

iii) $\quad \dfrac{\bar\lambda}{\alpha} = \mu \dfrac{\lambda}{\alpha} \mod \dfrac{N}{\alpha^2}$

or equivalently, there exists ξ defined mod α, such that

$$\bar\lambda = \mu\,\lambda + \xi\,\frac{N}{\alpha} \mod N$$

In other words, invariance under T forces $\mathcal{N}$ to have the general form:

$$\mathcal{N}_{\lambda\bar\lambda} = \sum_{\alpha,\mu,\xi} C_{\alpha,\mu,\xi,\lambda}\, \delta_{\bar\lambda,\mu\lambda+\xi\frac{N}{\alpha}} \tag{33}$$

with α,μ,ξ satisfying the conditions above.

It seems that invariance under S forces $C_{\alpha,\mu,\xi,\lambda}$ in (33) to be independent of ξ and λ. This has been proved in the case where $\frac{N}{2}$ has no square divisor ($\alpha = 1$ is the only possibility) but is likely to be general. We thus conjecture that the general solution is

$$\mathcal{N}_{\lambda\bar\lambda} = \sum_{\substack{\alpha,\mu\\ \text{as above}}} C_{\alpha,\mu} \sum_{\xi\in\mathbb{Z}/\alpha\mathbb{Z}} \delta_{\bar\lambda,\mu\lambda+\xi\frac{N}{\alpha}} \tag{34}$$

This conjecture has two immediate consequences:

i) If

$$N = \prod_{i=1}^{n} p_i^{r_i} \tag{35}$$

is the decomposition of N into a product of powers of primes p_i, the number of linearly independent invariants in the A_1^1 case is simply

$$\psi\left(\frac{N}{2}\right) = \frac{1}{2}\left\{\prod_{i=1}^{n} (1+r_i) - \delta\right\} \tag{36}$$

with $\delta = 1$ if $\frac{N}{2}$ is a square, 0 otherwise. For large N, on the average, $\psi(N/2)$ grows as $\frac{1}{2}\ln N$.

ii) For the conformal invariants, one has first to reexpress the invariant associated with (34) in terms of independent characters satisfying the condition (18). It turns out that the matrix $\mathcal{N}$ then factorizes as:

$$\mathcal{N}^{conf}_{rs,r's'} = \sum \mathcal{N}_{rr'}\left(\frac{N}{2} = p'\right) \cdot \mathcal{N}_{ss'}\left(\frac{N}{2} = p\right) \tag{37}$$

i.e. as a sum of tensor products of invariants of the form (34) relative to the indices $r,\bar r$ and $s,\bar s$. This is the remarkable connection between affine and conformal invariants alluded to above.

The number of independent conformal invariants is thus $\psi(p')\,\psi(p)$.

4. POSITIVE MODULAR INVARIANTS

We now enforce the two constraints $\mathcal{N}_{\lambda\bar\lambda} \geq 0$ for $\lambda, \bar\lambda$ in the fundamental domain (18) or (27) and $\mathcal{N}_{00} = 1$. The invariants listed in Table 1 and 2 satisfy these constraints. We conjecture that these lists are exhaustive! We are led to this conjecture by:

i) the inspection of all affine invariants up to $\frac{N}{2} = 100$

ii) The observation of a totally unexpected connection with the classification of simply laced Lie algebras. Not only our affine solutions come in two infinite series plus three exceptional cases, but the values of the indices $\lambda = \bar\lambda$ carried by the *diagonal* terms in Table 1 are the exponents of the Lie algebras A, D, E[17]. For an algebra of rank r, there are r exponents e_i, such that $1 + e_i$ give the degrees of a set of independent generators of invariant polynomials (or "higher Casimir invariants"). In particular, the algebra $D_{2\rho+2}$ has two independent invariants of degree $(2\rho+1)+1$, and this agrees nicely with the factor 2 in the corresponding invariant of Table 1.

The affine invariants associated with the algebra A are nothing but the diagonal sum of all $|\chi_\lambda|^2$. Other solutions exist only for even $\frac{N}{2}$. The conformal invariants are then obtained by tensor products of affine invariants pertaining to the values p' and p of $k+2 = \frac{N}{2}$. As p and p' are coprimes, they cannot be simultaneously even, and conformal invariants are thus labelled by pairs of simply laced algebras (A,A), (A,D), (A,E),... with at least one A entry. (see Table 2).

To summarize, we believe we now have a full classification of all minimal conformal theories, i.e. of all universality classes of critical theories with a finite number of primary fields.

What do these new theories describe?

In the affine case, the A and D series were interpreted[10] as the Wess-Zumino theory on SU(2) and SO(3) respectively, while the E_6 and E_8 cases are associated with embeddings of SU(2) into SU(3) and G_2 [18]; the case E_7 remains unexplained.

In the conformal case, and for unitary theories ($p' = m$, $p = m+1$ or vice versa), the (A,A) series describes the critical Ising model or its RSOS generalizations[19]. The (A_4,D_4) and (D_4,A_6) theories represent the critical and tricritical 3-state Potts models. What about the others? In recent papers[20], Pasquier has introduced a new family of completely integrable lattice models, in which Dynkin diagrams of simply laced algebras encode the selection rules between states at neighbouring sites. He suggested that the critical behaviour of these theories is described by the unitary conformal theories just described. Computation of critical exponents in these models would confirm this suggestion.

Finally, I want to return to the yet mysterious classification of invariants in terms of simply laced Lie algebras. It does not mean that the corresponding algebra is a symmetry of the theory but rather gives one more example that the (A,D,E) classification appears in a great variety of apparently unrelated problems[21].

We have seen how powerful the constraint of modular invariance is, in the simplest case of minimal conformal theories. It seems natural to apply it to superconformal[22] and to non minimal theories[23].

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TABLE 1

List of known partition functions in terms of $A_1^{(1)}$ characters

$k \geq 1$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	A_{k+1}
$k = 4\rho,\ \rho \geq 1$	$\sum_{\substack{\lambda\ \mathrm{odd}\,=\,1 \\ \lambda \neq 2\rho+1}}^{4\rho+1} \lvert\chi_\lambda\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2 + \sum_{\lambda\ \mathrm{odd}\,=\,1}^{2\rho-1} (\chi_\lambda \chi^*_{4\rho+2-\lambda} + \mathrm{c.c.})$ $= \sum_{\lambda\ \mathrm{odd}\,=\,1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$D_{2\rho+2}$
$k = 4\rho-2,\ \rho \geq 2$	$\sum_{\lambda\ \mathrm{odd}\,=\,1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda\ \mathrm{even}\,=\,2}^{2\rho-2} (\chi_\lambda \chi^*_{4\rho-\lambda} + \mathrm{c.c.})$	$D_{2\rho+1}$
$k+2 = 12$	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	E_6
$k+2 = 18$	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + [(\chi_3+\chi_{15})\chi_9^* + \mathrm{c.c.}]$	E_7
$k+2 = 30$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	E_8

TABLE 2

List of known partition functions in terms of conformal characters. The unitary series corresponds to $p' = m+1$, $p = m$ or $p = m+1$, $p' = m$, $m = 3,4,\ldots$.

$$\frac{1}{2}\sum_{r=1}^{p'-1}\sum_{s=1}^{p-1} |\chi_{rs}|^2 \qquad (A_{p'-1}, A_{p-1})$$

$$p' = 4\rho+2,\ \rho\geq 1 \quad \frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{\substack{r\ \mathrm{odd}\ =1 \\ r\neq 2\rho+1}}^{4\rho+1} |\chi_{rs}|^2 + 2|\chi_{2\rho+1\ s}|^2 + \sum_{r\ \mathrm{odd}\ =1}^{2\rho-1} (\chi_{rs}\chi^*_{p'-r\ s} + \mathrm{c.c.})\right\} \qquad (D_{2\rho+2}, A_{p-1})$$

$$p' = 4\rho,\ \rho\geq 2 \quad \frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}\ =1}^{4\rho-1} |\chi_{rs}|^2 + |\chi_{2\rho\ s}|^2 + \sum_{r\ \mathrm{even}\ =1}^{2\rho-2} (\chi_{rs}\chi^*_{p'-r\ s} + \mathrm{c.c.})\right\} \qquad (D_{2\rho+1}, A_{p-1})$$

$$p' = 12 \quad \frac{1}{2}\sum_{s=1}^{p-1}\left\{|\chi_{1s}+\chi_{7s}|^2 + |\chi_{4s}+\chi_{8s}|^2 + |\chi_{5s}+\chi_{11s}|^2\right\} \qquad (E_6, A_{p-1})$$

$$p' = 18 \quad \frac{1}{2}\sum_{s=1}^{p-1}\left\{|\chi_{1s}+\chi_{17s}|^2 + |\chi_{5s}+\chi_{13s}|^2 + |\chi_{7s}+\chi_{11s}|^2 + |\chi_{9s}|^2 + [(\chi_{3s}+\chi_{15s})\chi^*_{9s} + \mathrm{c.c.}]\right\} \qquad (E_7, A_{p-1})$$

$$p' = 30 \quad \frac{1}{2}\sum_{s=1}^{p-1}\left\{|\chi_{1s}+\chi_{11s}+\chi_{19s}+\chi_{29s}|^2 + |\chi_{7s}+\chi_{13s}+\chi_{17s}+\chi_{23s}|^2\right\} \qquad (E_8, A_{p-1})$$