Horn's problem,
from classical to quantum

LPTHE, 5 October 2017

J.-B. Z., *"Horn's problem and Harish-Chandra integrals. Probability density functions"*, arxiv:1705.01186,

Robert Coquereaux and J.-B. Z., *"From orbital measures to Littlewood–Richardson coefficients and hive polytopes"*, arxiv:1706.02793

What is Horn's problem ?

Given two Hermitian $n \times n$ matrices A and B, of known spectrum

$$\alpha = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$$

and $\beta = \{\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\}$, what can be said on the spectrum $\gamma = \{\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}$ of their sum $C = A + B$?

An old problem, with a rich history

Obviously, $\operatorname{tr} C = \operatorname{tr} A + \operatorname{tr} B$, *i.e.*,, $\sum_{k=1}^{n} \gamma_k - \alpha_k - \beta_k = 0$, thus the scene is in $\mathbb{R}^{n-1}$.

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For example, $\gamma_1 \leq \alpha_1 + \beta_1$, or Weyl's inequality (1912)

$i + j - 1 \leq n \;\Rightarrow\; \gamma_{i+j-1} \leq \alpha_i + \beta_j$, etc.

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In general, set of *linear* inequalities between the α's, β's, γ's.
Thus the γ's belong to a *convex polytope* in $\mathbb{R}^{n-1}$.
Horn (1962) conjectures the form of a (necessary and sufficient) set of inequalities

$$\sum_{k\in K} \gamma_k \leq \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j$$

for some subsets I, J, K of $\{1, \cdots, n\}$.

⋮

Klyachko (1998) and Knutson and Tao (1999) prove Horn's conjecture.

Problem interesting by its many facets and ramifications, symplectic geometry, algebraic geometry (Schubert calculus), representation theory . . .

Outline of this talk

1. The classical Horn problem revisited

2. Extension to other matrix sets

3. Connection with representation theory

1. The classical Horn problem revisited

Rephrase problem as follows:
Let $\mathcal{O}_\alpha$ be the *orbit* of $\mathrm{diag}\,(\alpha_1, \alpha_2, \cdots, \alpha_n)$ under adjoint action of $\mathsf{U}(n)$,

$$\mathcal{O}_\alpha = \{U \mathrm{diag}\,(\alpha_1, \alpha_2, \cdots, \alpha_n) U^\dagger \,|\, U \in \mathsf{U}(n)\}$$

and likewise $\mathcal{O}_\beta$. Which orbits $\mathcal{O}_\gamma$?

In the present work, I take A *uniformly distributed* on $\mathcal{O}_\alpha$ (for the Haar measure), and likewise B on $\mathcal{O}_\beta$, and *independent* of A, and determine the PDF (probability distribution function) of γ.

A central role is played by the Harish-Chandra *orbital integral*

$$\mathcal{H}(\alpha, \mathrm{i}\, x) = \int_{\mathrm{U}(n)} \exp(\mathrm{i}\,\mathrm{tr}\,(xU\alpha U^\dagger))\, dU$$

aka the Fourier transform of the orbital measure.

In the present case, explicit formula known for long

[Harish-Chandra 1957, Itzykson–Z 1980]

$$\mathcal{H}(\alpha, \mathrm{i}\, x) = \prod_{p=1}^{n-1} p!\, \frac{(\det e^{\mathrm{i}\, x_i \alpha_j})_{1\le i,j\le n}}{\Delta(ix)\Delta(\alpha)}$$

where $\Delta(x) = \Pi_{i<j}(x_i - x_j)$ is the Vandermonde determinant of the x's.

Main result: for A and B, independent and uniformly distributed on their orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$,
(and A and B "regular", *i.e.*, $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$),
PDF of γ is

$$p(\gamma|\alpha,\beta) = \frac{1}{(2\pi)^n}\left(\frac{\Delta(\gamma)}{\prod_{p=1}^n p!}\right)^2 \int_{\mathbb{R}^n} d^n x\, \Delta(x)^2\, \mathcal{H}(\alpha, \mathrm{i}\, x)\mathcal{H}(\beta, \mathrm{i}\, x)\mathcal{H}(\gamma, \mathrm{i}\, x)^*\,.$$

For low values of n, x-integration may be carried out, resulting in explicit formulae.

Proof is elementary:

1. Introduce characteristic function of random variable $A \in \mathcal{O}_\alpha$

$$\varphi_A(X) := \mathbb{E}(e^{\mathrm{i}\,\mathrm{tr}\, XA}) = \int_{\mathrm{U}(n)} DU \exp(\mathrm{i}\,\mathrm{tr}\, XU\alpha U^\dagger) = \mathcal{H}(\alpha, \mathrm{i}\, x)$$

($X \in H_n$, space of Hermitian $n \times n$ matrices, $x =$ eigenvalues of X) and likewise for B.

2. Since A and B are independent, characteristic function of the sum $C = A + B$ is the product

$$\mathbb{E}(e^{\mathrm{i}\,\mathrm{tr}\, XC}) = \varphi_A(X)\varphi_B(X)\,.$$

3. The PDF of C then recovered by inverse Fourier transform

$$p(C|\alpha,\beta) = \frac{1}{(2\pi)^{n^2}} \int DX e^{-\mathrm{i}\,\mathrm{tr}\, XC} \varphi_A(X)\varphi_B(X)\,.$$

4. $\varphi_A(X)$ depends only of eigenvalues x of X, and $p(C|\alpha,\beta)$ only on eigenvalues γ of C, hence define

$$\begin{aligned} p(\gamma|\alpha,\beta) &= \frac{(2\pi)^{n(n-1)/2}}{\prod_{p=1}^n p!} \Delta(\gamma)^2 p(C|\alpha,\beta) \\ &= \frac{1}{(2\pi)^n (\prod_{p=1}^n p!)^2} \Delta(\gamma)^2 \int_{\mathbb{R}^n} \prod_{i=1}^n dx_i\, \Delta(x)^2\, \mathcal{H}(\alpha, \mathrm{i}\, x)\mathcal{H}(\beta, \mathrm{i}\, x)\mathcal{H}(\gamma, \mathrm{i}\, x)^*\,. \qquad \square \end{aligned}$$

Now, make use of explicit expression of HCIZ integral

$$p(\gamma|\alpha,\beta) = \text{const.}\,\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}\int\frac{d^n x}{\Delta(x)}\,\det e^{\mathrm{i}\,x_i\alpha_j}\,\det e^{\mathrm{i}\,x_i\beta_j}\,\det e^{-\mathrm{i}\,x_i\gamma_j}$$

and in each determinant, separate the "barycenter" $\frac{1}{n}\sum_{j=1}^{n} x_j$ from the relative coordinates $u_j := x_j - x_{j+1}$:

$$\begin{aligned}\det e^{\mathrm{i}\,x_i\alpha_j} &= e^{\mathrm{i}\,\frac{1}{n}\sum_{j=1}^{n} x_j \sum_{k=1}^{n}\alpha_k}\,\det e^{\mathrm{i}\,(x_i-\frac{1}{n}\sum x_k)\alpha_j}\\ &= e^{\mathrm{i}\,\frac{1}{n}\sum_{j=1}^{n} x_j \sum_{k=1}^{n}\alpha_k}\sum_{P\in S_n}\varepsilon_P\prod_{j=1}^{n-1} e^{\mathrm{i}\,u_j(\sum_{k=1}^{j}\alpha_{P(k)}-\frac{j}{n}\sum_{k=1}^{n}\alpha_k)}\,.\end{aligned}$$

Final result

$$\begin{aligned}p(\gamma|\alpha,\beta) &= \frac{\prod_1^{n-1} p!}{n!}\delta(\sum_k(\gamma_k-\alpha_k-\beta_k))\,\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}\,\mathcal{J}_n(\alpha,\beta;\gamma)\\ \mathcal{J}_n(\alpha,\beta;\gamma) &= \frac{\mathrm{i}^{-n(n-1)/2}}{2^{n-1}\pi^{n-1}}\sum_{P,P'\in S_n}\varepsilon_P\,\varepsilon_{P'}\int\frac{d^{n-1}u}{\widetilde{\Delta}(u)}\prod_{j=1}^{n-1} e^{\mathrm{i}\,u_j A_j(P,P',I)}\,,\end{aligned}$$

with $A_j(P,P',P'') = \sum_{k=1}^{j}(\alpha_{P(k)}+\beta_{P'(k)}-\gamma_{P''(k)}) - \frac{j}{n}\sum_{k=1}^{n}(\alpha_k+\beta_k-\gamma_k)\,.$

Linear combination of integrals over $u \in \mathbb{R}^{n-1}$ of the form $\int \frac{d^{n-1}u}{\widetilde{\Delta}(u)} e^{\mathrm{i}\, A_j u_j}$, generalizing Dirichlet integral

$$\mathcal{P} \int_{\mathbb{R}} \frac{du}{u} e^{\mathrm{i}\, uA} = \mathrm{i}\, \pi \epsilon(A)\,, \qquad \text{if } A \neq 0,$$

with ϵ the sign function and $\mathcal{P}$, Cauchy's principal value.

In general, $\mathcal{J}_n(\gamma)$ is a distribution (generalized function), in fact
– a function of class C^{n-3},
and
– a piece-wise polynomial of degree $(n-1)(n-2)/2$.

Explicit results for $n = 2, 3$

$n = 2$. A classroom exercise . . .

Take $A = \mathrm{diag}\,(\alpha_1, \alpha_2)$, $B = \mathrm{diag}\,(\beta_1, \beta_2)$, $U = \exp -\imath\sigma_2\psi$, $\psi \in (0, \pi)$.
Eigenvalues of $A + UBU^\dagger$ are

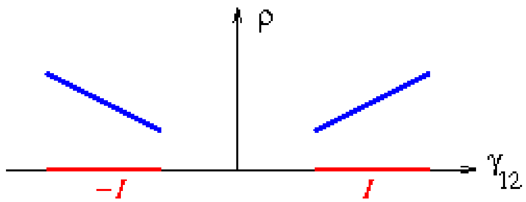
$$\gamma_{1,2} = \frac{1}{2}\left[\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \pm \sqrt{\alpha_{12}^2 + \beta_{12}^2 + 2\alpha_{12}\beta_{12}\cos\psi}\right]$$

where $\alpha_{12} := \alpha_1 - \alpha_2$, etc.

$$\gamma_{12} = \gamma_1 - \gamma_2 = \pm\sqrt{\alpha_{12}^2 + \beta_{12}^2 + 2\alpha_{12}\beta_{12}\cos\psi}$$

with density

$$\rho(\gamma_{12}) = -\frac{1}{4}\sin\psi\frac{d\psi}{d\gamma_{12}} = \frac{1}{2}\frac{|\gamma_{12}|}{\alpha_{12}\beta_{12}}\,,$$



on support $-I \cup I$

$$-(\alpha_{12} + \beta_{12}) \le \gamma_{12} \le -|\alpha_{12} - \beta_{12}| \;\cup\; |\alpha_{12} - \beta_{12}| \le \gamma_{12} \le \alpha_{12} + \beta_{12}\ .$$

(rings a bell ?)

Alternatively
Horn's inequalities

$$\begin{aligned} \max(\alpha_1+\beta_2, \alpha_2+\beta_1) &\le \gamma_1 \le \quad \alpha_1+\beta_1 \\ \alpha_2+\beta_2 &\le \gamma_2 \le \quad \min(\alpha_1+\beta_2, \alpha_2+\beta_1) \end{aligned}$$

PDF

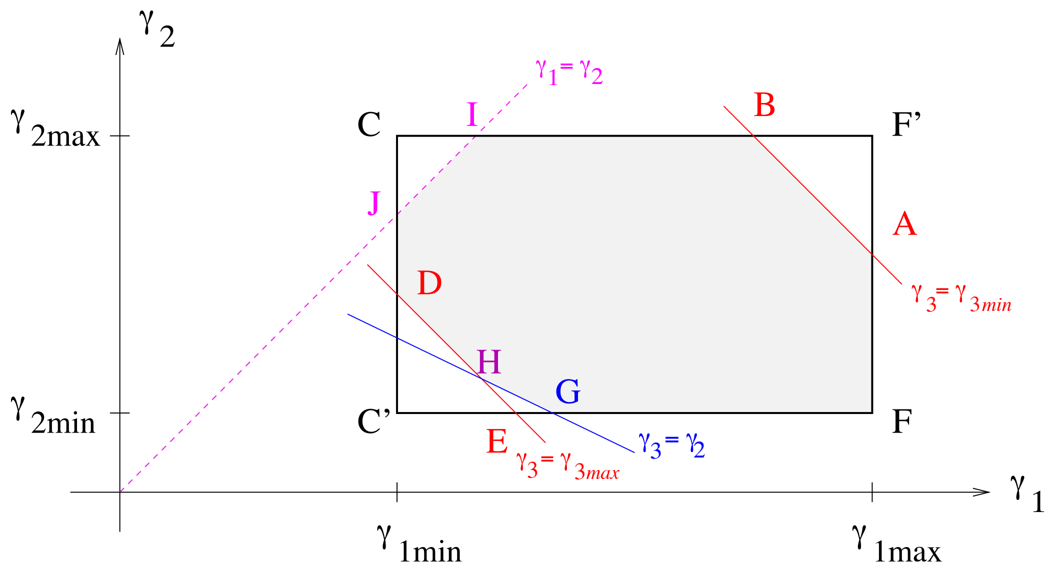
$$\begin{aligned} \mathcal{J}_2(\alpha,\beta;\gamma) &= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{du}{u} \sum_{P,P'\in S_2} \varepsilon_P \varepsilon_{P'} e^{\mathrm{i}\, uA(P,P',I)} \\ &= \frac{1}{2\pi \mathrm{i}} \sum_{P,P'\in S_2} \varepsilon_P \varepsilon_{P'} \, \mathcal{P}\!\int_{\mathbb{R}} \frac{du}{u} e^{\mathrm{i}\, uA(P,P',I)} \qquad \text{Cauchy principal value} \\ &= \frac{1}{2}(\epsilon(\gamma_{12}-\alpha_{12}+\beta_{12})+\epsilon(\gamma_{12}+\alpha_{12}-\beta_{12})-\epsilon(\gamma_{12}-\alpha_{12}-\beta_{12})-\epsilon(\gamma_{12}+\alpha_{12}+\beta_{12})) \\ &= (\mathbf{1}_I(\gamma_{12})-\mathbf{1}_{-I}(\gamma_{12})) \qquad (\mathbf{1}_I = \text{indicator function of interval } I) \\ p(\gamma|\alpha,\beta) &= \underbrace{\frac{(\gamma_1-\gamma_2)}{2(\alpha_1-\alpha_2)(\beta_1-\beta_2)}\Big(\mathbf{1}_I(\gamma_1-\gamma_2)-\mathbf{1}_{-I}(\gamma_1-\gamma_2)\Big)}_{\rho(\gamma_{12})} \delta(\gamma_1+\gamma_2-\alpha_1-\alpha_2-\beta_1-\beta_2) \end{aligned}$$

$n = 3$

Horn's inequalities read

$$\begin{aligned}
\gamma_{3min} := \alpha_3 + \beta_3 \le \quad & \gamma_3 \quad \le \min(\alpha_1+\beta_3, \alpha_2+\beta_2, \alpha_3+\beta_1) =: \gamma_{3max} \\
\gamma_{2min} := \max(\alpha_2+\beta_3, \alpha_3+\beta_2) \le \quad & \gamma_2 \quad \le \min(\alpha_1+\beta_2, \alpha_2+\beta_1) =: \gamma_{2max} \\
\gamma_{1min} := \max(\alpha_1+\beta_3, \alpha_2+\beta_2, \alpha_3+\beta_1) \le \quad & \gamma_1 \quad \le \alpha_1+\beta_1 =: \gamma_{1max}\,.
\end{aligned}$$

They express that γ belongs to the interior of a polygon in the plane γ_1, γ_2 (with $\gamma_3 = \sum_1^3(\alpha_i+\beta_i) - \gamma_1 - \gamma_2$), at most an octagon,



Which PDF in that domain ?

$$p(\gamma|\alpha,\beta) = \frac{1}{3}\delta(\sum \gamma_i - \alpha_i - \beta_i)\,\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}\,\mathcal{J}_3(\alpha,\beta;\gamma)$$

$$\begin{aligned}\mathcal{J}_3(\alpha,\beta;\gamma) &= \frac{\mathrm{i}}{4\pi^2}\int_{\mathbb{R}^2}\frac{du_1du_2}{u_1u_2(u_1+u_2)}\sum_{P,P'\in S_3}\varepsilon_P\varepsilon_{P'}\,e^{\mathrm{i}\,(u_1A_1+u_2A_2)}\\ A_1 &= \alpha_{P(1)}+\beta_{P'(1)}-\gamma_1 \qquad A_2 = -\alpha_{P(3)}-\beta_{P'(3)}+\gamma_3\end{aligned}$$

Integrating once again term by term by principal value and contour integrals, we find

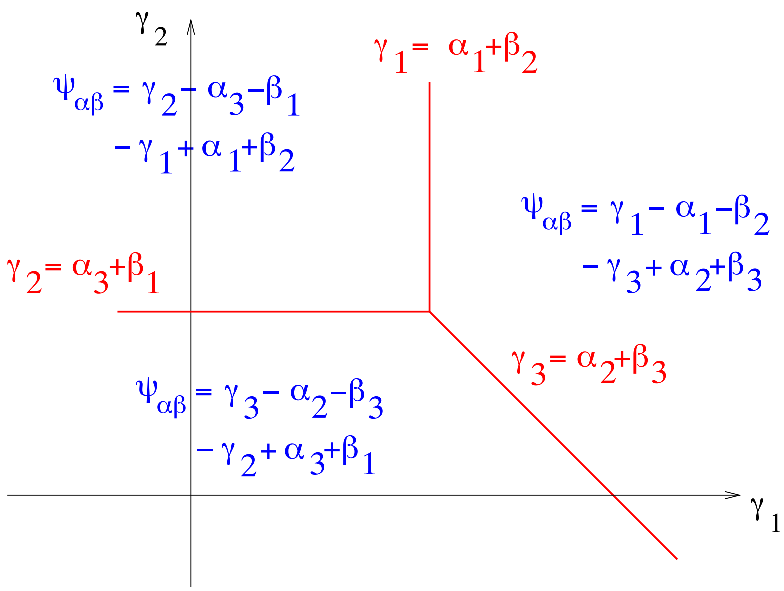
$$\mathcal{J}_3(\alpha,\beta;\gamma) = \frac{1}{4}\sum_{P,P'\in S_3}\varepsilon_P\varepsilon_{P'}\,\epsilon(A_1)\,(|A_2|-|A_2-A_1|)\,,$$

a sum of $3!^2\times 2 = 72$ terms, which for $\gamma_3\le\gamma_2\le\gamma_1$, vanishes out of the Horn polygon. Inside that polygon, may be recast as a sum of 4 terms

$$\mathcal{J}_3(\alpha,\beta;\gamma) = \frac{1}{6}(\alpha_1-\alpha_3+\beta_1-\beta_3+\gamma_1-\gamma_3) - \frac{1}{2}|\alpha_2+\beta_2-\gamma_2| - \frac{1}{3}\psi_{\alpha\beta}(\gamma) - \frac{1}{3}\psi_{\beta\alpha}(\gamma)$$

where

$$\psi_{\alpha\beta}(\gamma) = \begin{cases} (\gamma_2-\alpha_3-\beta_1)-(\gamma_1-\alpha_1-\beta_2) & \text{if } \gamma_2-\alpha_3-\beta_1 \geq 0 \text{ and } \gamma_1-\alpha_1-\beta_2 < 0 \\ (\gamma_3-\alpha_2-\beta_3)-(\gamma_2-\alpha_3-\beta_1) & \text{if } \gamma_3-\alpha_2-\beta_3 \geq 0 \text{ and } \gamma_2-\alpha_3-\beta_1 < 0 \\ (\gamma_1-\alpha_1-\beta_2)-(\gamma_3-\alpha_2-\beta_3) & \text{if } \gamma_1-\alpha_1-\beta_2 \geq 0 \text{ and } \gamma_3-\alpha_2-\beta_3 < 0 \end{cases}$$



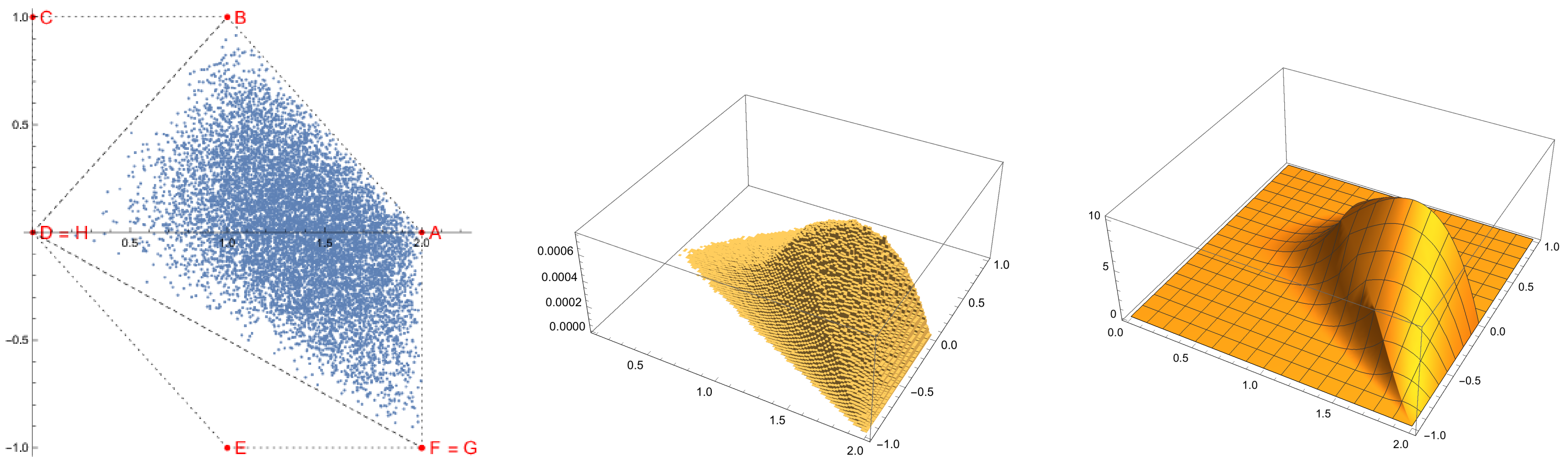
a continuous function, piece-wise linear, non differentiable across the red lines.

(Also non differentiability of $\mathcal{J}_3$ across $\alpha_2+\beta_2-\gamma_2 = 0$.)

Remark: Yet another alternative expression

$$\frac{1}{24}\mathcal{J}_3(\alpha,\beta;\gamma) = \min(\alpha_1, -\beta_3+\gamma_2, \alpha_1+\alpha_2+\beta_1-\gamma_1)$$
$$- \quad \max(\alpha_1-\gamma_1+\gamma_2, \gamma_3-\beta_3, \alpha_2, -\beta_2+\gamma_2, \alpha_1+\alpha_3+\beta_1-\gamma_1, \alpha_1+\alpha_2+\beta_2-\gamma_1)$$

Example: $\alpha = \beta = (1, 0, -1)$.

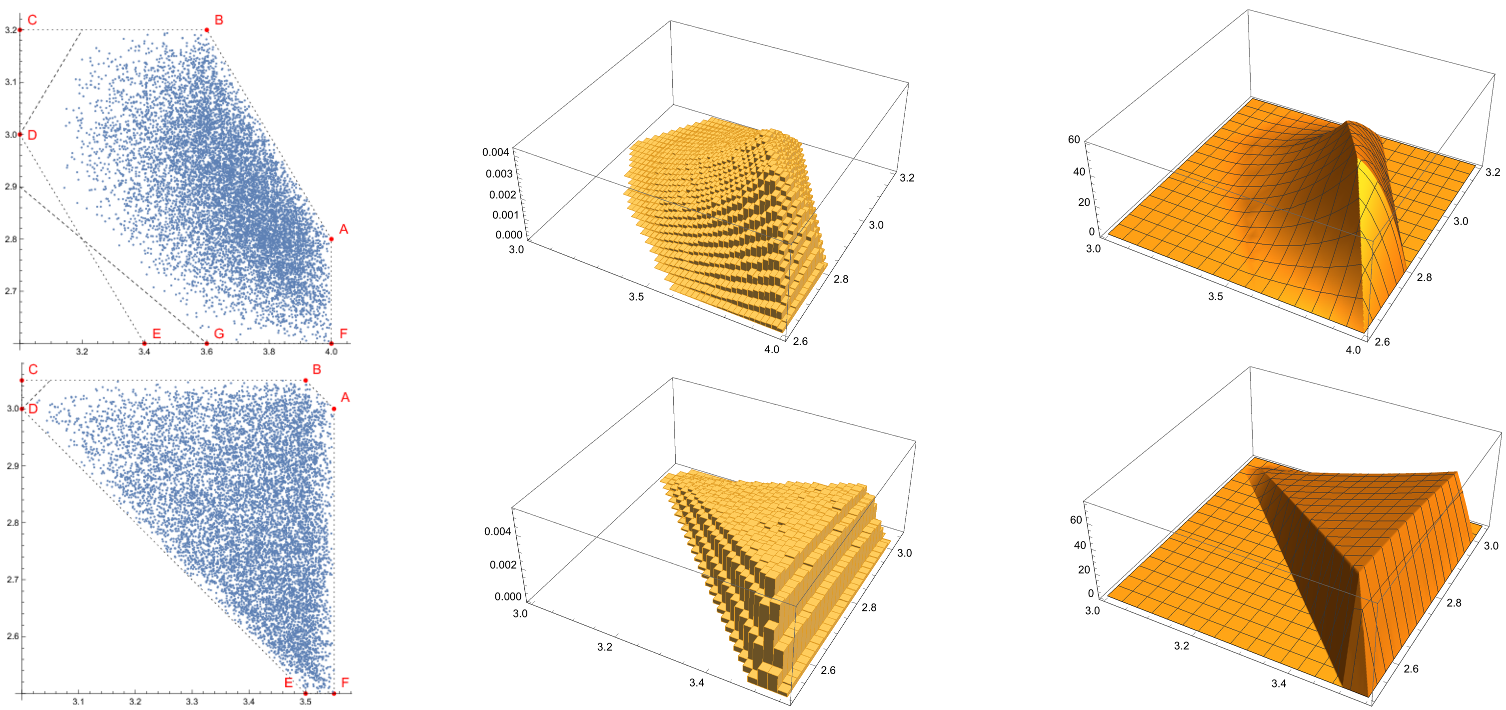


left: distribution of 10,000 eigenvalues in the γ_1, γ_2 plane; middle: histogram of 5×10^6 eigenvalues; right: plot of the PDF as computed above

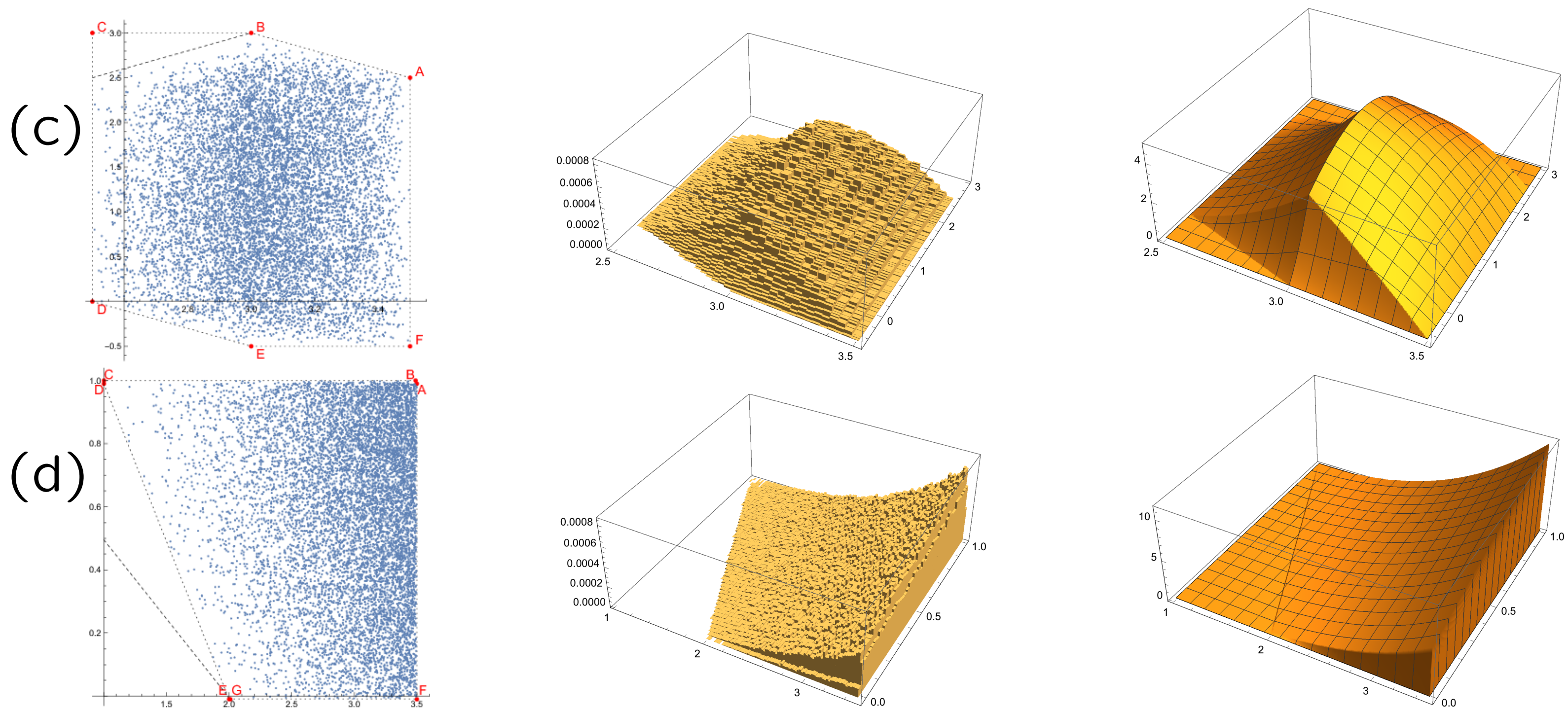
Other examples

(a)

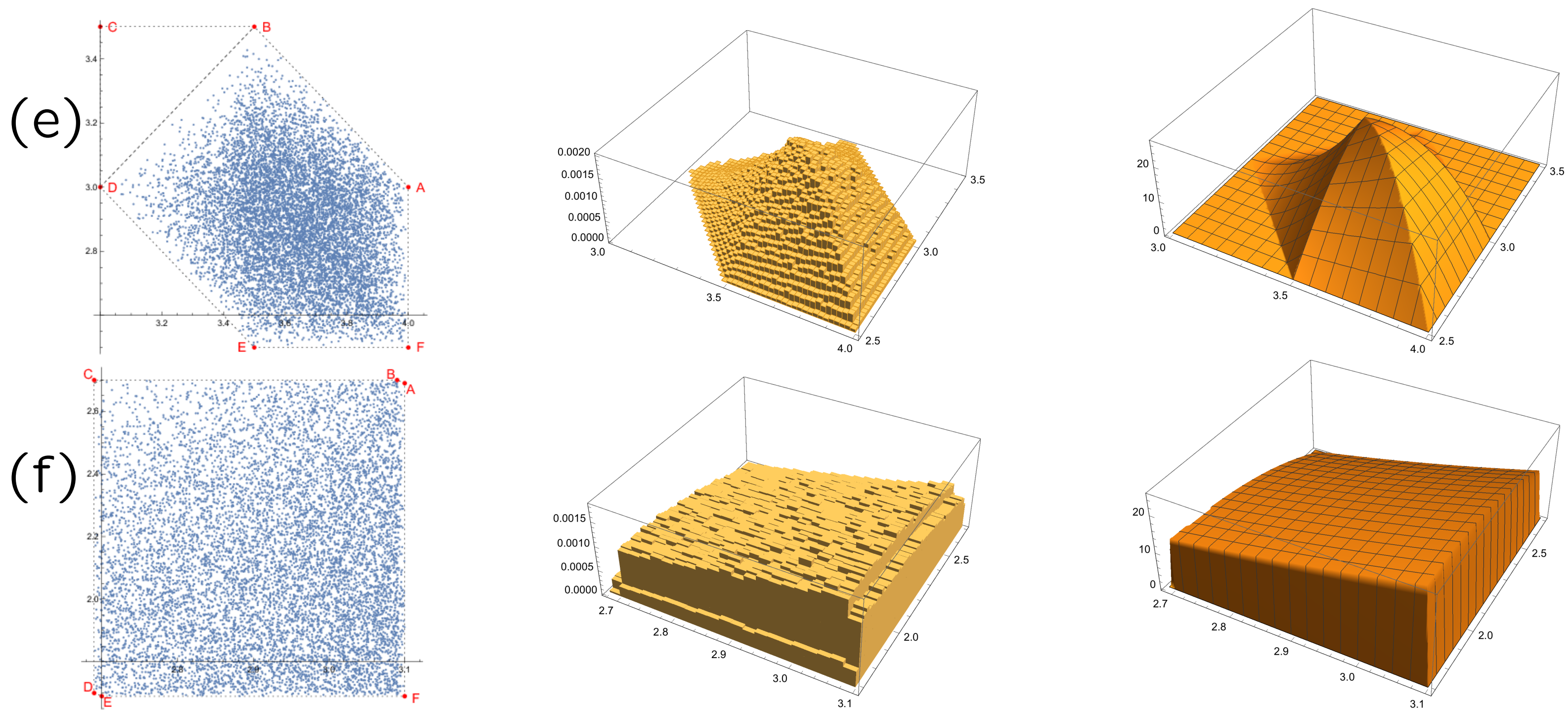
(b)



(a) $\alpha = (2, 1.2, 1), \beta = (2, 1.6, 1)$; (b) $\alpha = (1.55, 1.5, 1), \beta = (2, 1.5, -3.5)$



(c) $\alpha = (1.5, 1, -2), \beta = (2, 1.5, -3.5)$; (d) $\alpha = (2, 1.99, -0.5), \beta = (1.5, -1, -2)$;



(e) $\alpha = (2, 1.5, 1), \beta = (2, 1.5, -4)$; (f) $\alpha = (1.5, 1.49, -3), \beta = (1.6, 1.2, 0.2)$.

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∘ O(n) or SO(n) orbits of *skew-symmetric matrices*. (A Harish-Chandra formula exists).

⋆ O(n) or SO(n) orbits of *real symmetric matrices*. (No Harish-Chandra formula for $n > 2$!!).

Eigenvalues γ still in a convex polytope (the same as for Hermitian matrices with the same α and β [Fulton]), but PDF quite different !

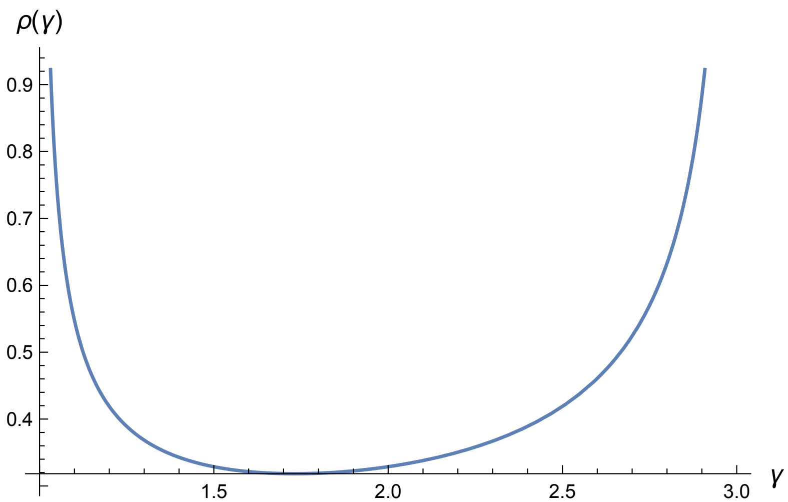
$n = 2$ another classroom exercise !

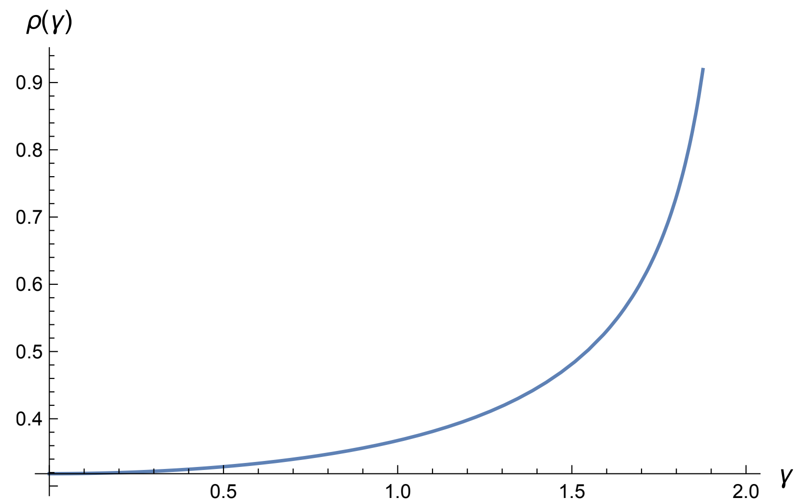
For $A + OBO^T$, $\gamma_{12} = \sqrt{\alpha_{12}^2 + \beta_{12}^2 + 2\alpha_{12}\beta_{12}\cos(2\theta)}$,

$\rho(\gamma_{12}) = -\frac{2}{\pi}\frac{d\theta}{d\gamma_{12}}$, equal to

$$\rho(\gamma) = \begin{cases} \frac{2}{\pi}\frac{\gamma}{\sqrt{(\gamma_{12max}^2-\gamma^2)(\gamma^2-\gamma_{12min}^2)}} & \gamma_{12min} \leq \gamma \leq \gamma_{12max} \\ 0 & \text{otherwise} \end{cases}$$

with $\gamma_{12min} = |\alpha_{12} - \beta_{12}|$, $\gamma_{12max} = \alpha_{12} + \beta_{12}$

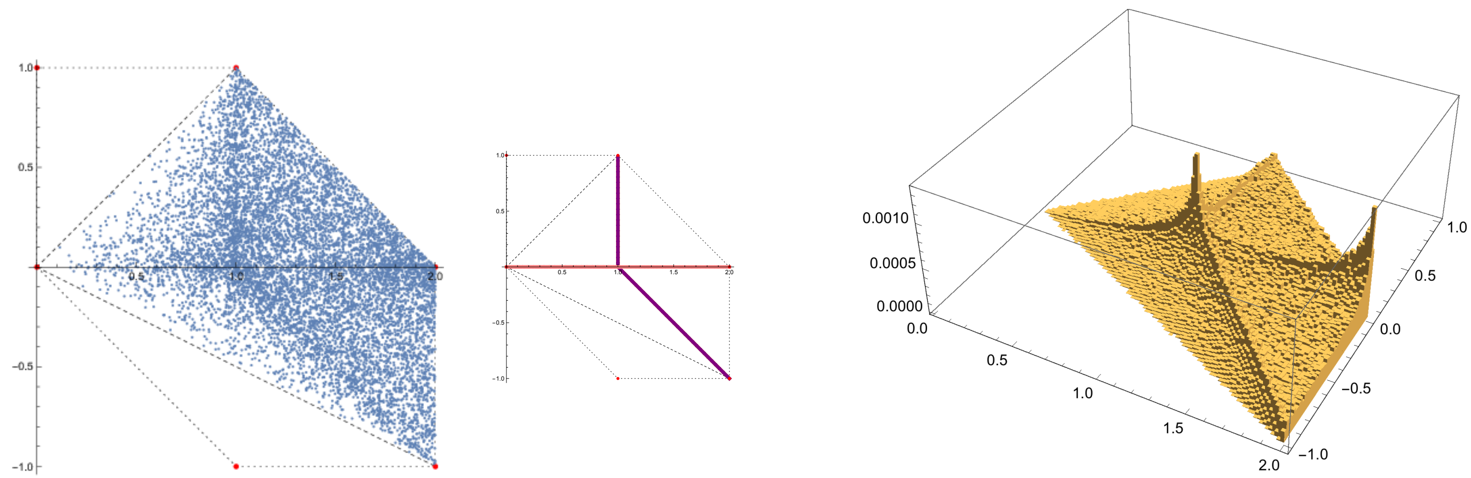




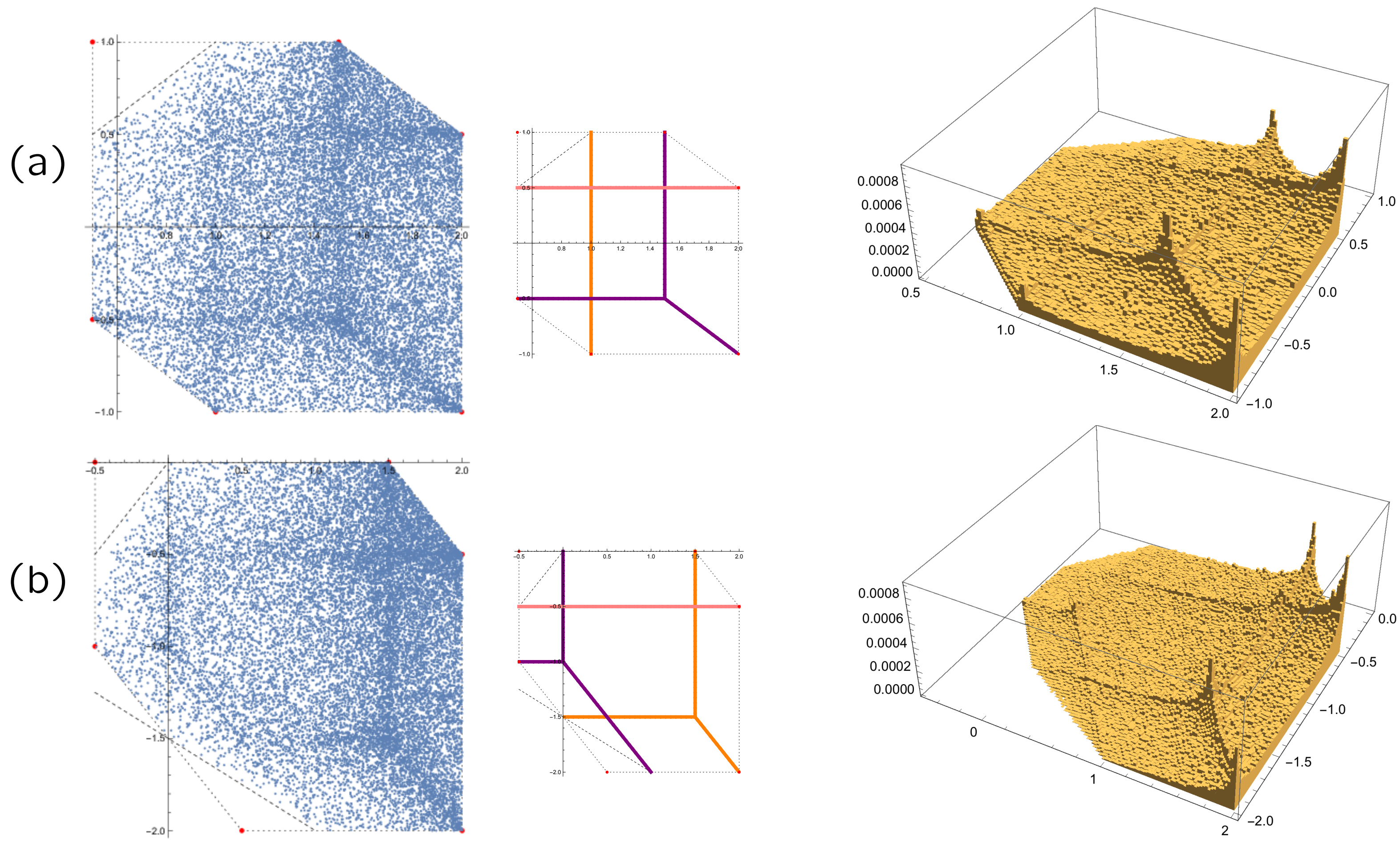
The density ρ: left, for $\alpha_{12} = 1, \beta_{12} = 2$ and right, $\alpha_{12} = \beta_{12} = 1$.
(Integrable) edge singularity !

$n = 3$. A few examples

$A = B = \mathrm{diag}\,(1, 0, -1)$



Plot (left) and histogram (right) of respectively 10^4 and 10^6 eigenvalues γ_1, γ_2 for the sum of 3 by 3 symmetric matrices of eigenvalues $\alpha = \beta = (1, 0, -1)$. The density appears to be enhanced along the lines (middle) $\gamma_1 = 1, \gamma_2 = 0$ and $\gamma_3 = -\gamma_1 - \gamma_2 = -1$.



Same with (a) $\alpha = (1, 0.5, -2.5)$, $\beta = (1, 0, -1.5)$ and (b) $\alpha = (1, -1, -2.5)$, $\beta = (1, 0.5, -2)$.

Questions:

Interpretation, location and nature of these singularities/enhancements?
Same lines as those of non differentiability in Hermitian case ?
Why ?

Physically observable ?

3. Another interesting direction: connection with representation theory

Expression of PDF : $p \propto \int_{\mathbb{R}^n} \prod_{i=1}^n dx_i \, \Delta(x)^2 \, \mathcal{H}(\alpha, \mathrm{i}\, x)\mathcal{H}(\beta, \mathrm{i}\, x)\mathcal{H}(\gamma, \mathrm{i}\, x)^*$ or $\mathcal{J}_n(\alpha, \beta; \gamma) = \mathrm{const.} \sum_{P,P',P'' \in S_n} \varepsilon_P \, \varepsilon_{P'} \, \varepsilon_{P''} \int \frac{d^{n-1}u}{\widetilde{\Delta}(u)} \prod_{j=1}^{n-1} e^{\mathrm{i}\, u_j A_j(P,P',P'')}$, quite reminiscent of expressions of multiplicities in tensor products, aka Littlewood–Richardson (LR) coefficients, (or of Verlinde formula for fusion coefficients).

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or $\mathcal{J}_n(\alpha, \beta; \gamma) = \mathrm{const.} \sum_{P,P',P''\in S_n} \varepsilon_P\, \varepsilon_{P'}\, \varepsilon_{P''} \int \frac{d^{n-1}u}{\widetilde{\Delta}(u)} \prod_{j=1}^{n-1} e^{\mathrm{i}\, u_j A_j(P,P',P'')}$, quite reminiscent of expressions of multiplicities in tensor products, aka Littlewood–Richardson (LR) coefficients, (or of Verlinde formula for fusion coefficients).

For a triplet (λ, μ, ν) of (highest weights) of three irreps of SU(n)

$$N_{\lambda\mu}^{\nu} = \int_{\mathrm{SU}(n)} du\, \chi_\lambda(u)\chi_\mu(u)\chi_\nu^*(u) \quad \text{or} \quad N_{\lambda\mu}^{\nu} = \int_{\mathbb{T}_n} dT\, \chi_\lambda(T)\chi_\mu(T)\chi_\nu^*(T)$$

with $\mathbb{T}_n = U(1)^{n-1}$ the Cartan torus,

$$T = \mathrm{diag}\,(e^{\mathrm{i}\, t_j})_{j=1,\cdots,n} \quad \text{with} \ \sum_{j=1}^{n} t_j = 0\,, \qquad dT = \frac{1}{(2\pi)^{n-1} n!} |\Delta(e^{\mathrm{i}\, t})|^2 \prod_{i=1}^{n-1} dt_i\,.$$

3. Another interesting direction: connection with representation theory

Expression of PDF : $p \propto \int_{\mathbb{R}^n} \prod_{i=1}^n dx_i \, \Delta(x)^2 \, \mathcal{H}(\alpha, \mathrm{i}\, x)\mathcal{H}(\beta, \mathrm{i}\, x)\mathcal{H}(\gamma, \mathrm{i}\, x)^*$
or $\mathcal{J}_n(\alpha, \beta; \gamma) = \mathrm{const.} \sum_{P,P',P'' \in S_n} \varepsilon_P \, \varepsilon_{P'} \, \varepsilon_{P''} \int \frac{d^{n-1}u}{\widetilde{\Delta}(u)} \prod_{j=1}^{n-1} e^{\mathrm{i}\, u_j A_j(P,P',P'')}$, quite reminiscent of expressions of multiplicities in tensor products, aka Littlewood–Richardson (LR) coefficients, (or of Verlinde formula for fusion coefficients).

For a triplet (λ, μ, ν) of (highest weights) of three irreps of SU(n)

$$N_{\lambda\mu}^{\nu} = \int_{\mathrm{SU}(n)} du \, \chi_\lambda(u)\chi_\mu(u)\chi_\nu^*(u) \quad \text{or} \quad N_{\lambda\mu}^{\nu} = \int_{\mathbb{T}_n} dT \, \chi_\lambda(T)\chi_\mu(T)\chi_\nu^*(T)$$

with $\mathbb{T}_n = U(1)^{n-1}$ the Cartan torus,

$$T = \mathrm{diag}\,(e^{\mathrm{i}\, t_j})_{j=1,\cdots,n} \quad \text{with} \sum_{j=1}^{n} t_j = 0\,, \qquad dT = \frac{1}{(2\pi)^{n-1} n!} |\Delta(e^{\mathrm{i}\, t})|^2 \prod_{i=1}^{n-1} dt_i\,.$$

This is not a coincidence: L-R coefficients Horn problem

"quantum" classical

Asymptotics of LR for large irreps (large λ, μ, ν) given by Horn's problem

A few facts:

1. Kirillov orbit theory: natural correspondence between ("classical") adjoint orbits of Hermitian matrices and ("quantum") irreps of SU(n)

2. For "large" representations, $N_{\lambda\mu}^{\nu}$ has a semi-classical description [Heckman '82]:
$N_{s\lambda\, s\mu}^{s\nu} \sim s^d \mathcal{V}$, $d = (n-1)(n-2)/2$, $\mathcal{V}$ the volume of a symplectic manifold

3. The same combinatorial objects ("pictographs") describe both the classical Horn problem and the SU(n) L-R coefficients, (for ex Knutson–Tao Honeycombs). They depend on $d = (n-1)(n-2)/2$ parameters, subject to linear inequalities, thus defining a d'-dim polytope, $d' \leq d$.

Orbit–Irrep Correspondence:

Irrep V_λ of h.w. $\lambda \mapsto$ Young diagram Y_λ. Lengths of rows $\alpha_i = \ell_i(\lambda)$, $i = 1, \cdots, n$, $\alpha_n = 0$.
In fact, shift λ by Weyl vector ρ, so $\alpha' = \ell(\lambda + \rho)$, *i.e.*, $\alpha'_i = \sum_{j=i}^{n} \lambda_i + n - i$

$$\text{dimension of irrep} \qquad \dim V_\lambda = \frac{\Delta(\alpha')}{\prod_{p=1}^{n-1} p!}$$

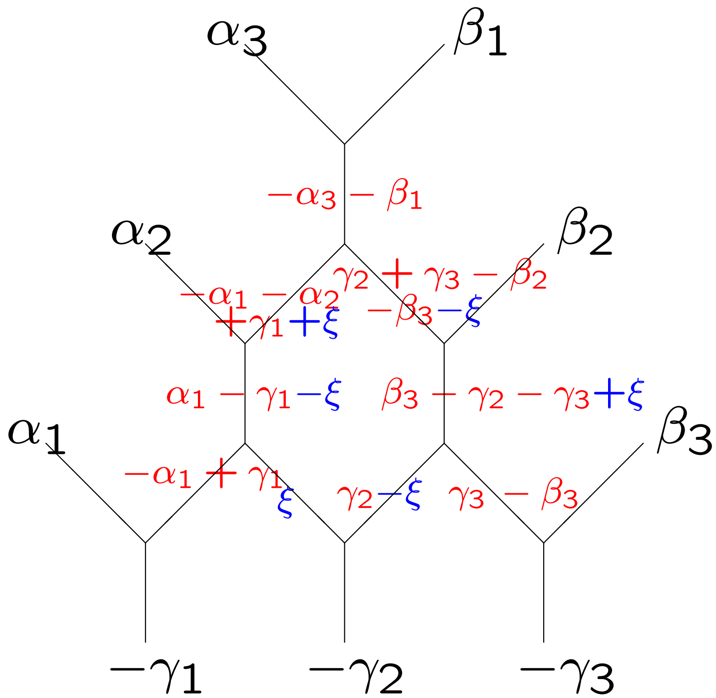
Weyl character formula, for $T = \operatorname{diag}(e^{\mathrm{i}\, t_j})_{j=1,\cdots,n} \in \mathbb{T}_n$

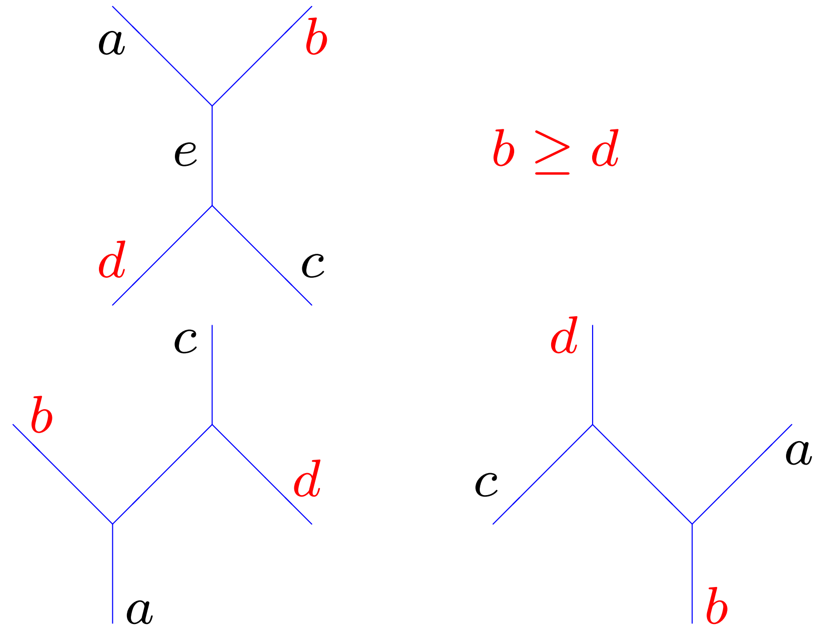
$$\chi_\lambda(T) := \operatorname{tr}_{V_\lambda}(T) = \frac{\det e^{\mathrm{i}\, t_i \alpha'_j}}{\Delta(e^{\mathrm{i}\, t})} \quad \text{with} \quad \Delta(e^{\mathrm{i}\, t}) = \prod_{1 \le i < j \le n} (e^{\mathrm{i}\, t_i} - e^{\mathrm{i}\, t_j}),$$

$$\chi_\lambda(T) = \frac{\Delta(\alpha')}{\prod_{p=1}^{n-1} p!} \left(\prod_{1 \le i < j \le n} \frac{\mathrm{i}\,(t_i - t_j)}{(e^{\mathrm{i}\, t_i} - e^{\mathrm{i}\, t_j})} \right) \mathcal{H}(\alpha', \mathrm{i}\, t)$$

$$\frac{\chi_\lambda(T)}{\dim V_\lambda} = \frac{\Delta(\mathrm{i}\, t)}{\Delta(e^{\mathrm{i}\, t})} \mathcal{H}(\alpha', \mathrm{i}\, t) \qquad \text{[Kirillov]}$$

Knutson–Tao Honeycombs





A Knutson–Tao honeycomb, here for $n = 3$, depending on $d = (n-1)(n-2)/2$ parameter(s)

Linear inequalities on these parameters $\Rightarrow$ convex polytope $\mathcal{H}_{\alpha\beta}^{\gamma}$ in $\mathbb{R}^d$.

"Classical" Horn problem: $\gamma \in \mathrm{Spec}(C = A + B)$ if $\mathrm{Vol}(\mathcal{H}_{\alpha\beta}^{\gamma}) \neq 0$.

"Quantum" problem: $N_{\lambda\mu}^{\nu} = \#(\mathcal{H}_{\alpha\beta}^{\gamma} \cap \mathbb{Z}^{d-1})$ with $\alpha = \ell(\lambda)$ etc.

Thus expect semi-classically for large λ, μ, ν, $N_{\lambda\mu}^{\nu} \approx \mathrm{Vol}(\mathcal{H}_{\alpha\beta}^{\gamma})$.

A direct connection between $\mathcal{J}_n$ and LR coefficients [Coquereaux–Z]

"Compactify" integral in $\mathcal{J}_n$ (assuming that $\sum_{k=1}^n(\lambda_k+\mu_k-\nu_k)=0$)

$$\int_{\mathbb{R}^{n-1}} \frac{\prod_{j=1}^{n-1} du_j e^{\mathrm{i}\, u_j A_j}}{\widetilde{\Delta}(u)} = \mathrm{i}^{\,n(n-1)/2} \int_{(-\pi,\pi)^{n-1}} \prod_{j=1}^{n-1} du_j\, e^{\mathrm{i}\, u_j A_j} \frac{R_n(T)}{\Delta(e^{\mathrm{i}\, t_i})}$$

where $u_j = t_j - t_{j+1}$, $R_n(T) = \sum_{\kappa\in\mathcal{K}} r_\kappa \chi_\kappa(T)$, $\mathcal{K}$ a finite (n-dependent) set of weights. ($R_n(T)$ computed for $n \le 6$)

Then with $\alpha' = \ell(\lambda+\rho)$, $\beta' = \ell(\mu+\rho)$, $\gamma' = \ell(\nu+\rho)$,

$$\mathcal{J}_n(\alpha',\beta';\gamma') = \int_{\mathbb{T}_n} dT\, \chi_\lambda(T)\chi_\mu(T)\chi_\nu^*(T)\, R_n(T)$$

$$\mathcal{J}_n(\alpha',\beta';\gamma') = \sum_{\substack{\kappa\in\mathcal{K}\\ \nu'}} r_\kappa N_{\lambda\mu}^{\nu'} N_{\kappa\nu}^{\nu'} = \sum_{\nu'} c_{\nu'}^{(\nu)} N_{\lambda\mu}^{\nu'}$$

where the sum runs over the finite set of irreps ν' obtained in the decomposition of $\oplus_{\kappa\in\mathcal{K}}(\nu\otimes\kappa)$, with rational coefficients $c_{\nu'}^{(\nu)} = \sum_{\kappa\in\mathcal{K}} N_{\kappa\nu}^{\nu'} r_\kappa$.

Likewise for unshifted weights, $\mathcal{J}_n(\alpha,\beta;\gamma) = \sum_{\substack{\kappa\in\hat{\mathcal{K}}\\ \nu'}} \hat{r}_\kappa N_{\lambda\mu}^{\nu'} N_{\kappa\nu}^{\nu'} = \sum_{\nu'} \hat{c}_{\nu'}^{(\nu)} N_{\lambda-\rho\,\mu-\rho}^{\nu'}$.

For large λ, μ, ν, recover asymptotics: $N_{\kappa\nu}^{\nu'} \approx N_{\kappa\nu}^{\nu}$ (piece-wise polynomial), and $\sum c_{\nu'}^{(\nu)} = 1$, hence

$$\mathcal{J}_n(\alpha', \beta'; \gamma') \approx \mathcal{J}_n(\alpha, \beta; \gamma) \approx N_{\lambda\mu}^{\ \nu} .$$

Thus $\mathcal{J}_n$ identified as the volume of the d-dimensional polytope

$$\mathcal{J}_n = \mathrm{Vol}(\mathcal{H}_{\alpha\beta}^{\gamma})$$

[find a direct proof ?]

Conclusions, Prospect, Open issues, etc

– general piecewise polynomial expression of $\mathcal{J}_n$ for $n > 3$?

– find a direct proof of $\mathcal{J}_n =$ volume of polytope

– invert the relation $\mathcal{J}_n = \sum N_{\lambda\mu}^{\nu'}$

– use those relations to study the *stretching* (Ehrhart) polynomials $N_{s\lambda\, s\mu}^{s\nu} = P_{\lambda\mu}^{\nu}(s)$

– etc.

⋆