ADE and all that

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Harvard, 9 October 2018

Thirty years after. . .

Personal Recollections

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The Truth, The Whole Truth and Nothing But The Truth . . .

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The Truth, The Whole Truth and Nothing But The Truth . . .

+ a few anecdotes and pictures. . .

2D-Conformal Field Theories (CFT): Quantum Field Theories covariant under conformal transfos. In 2d: analytical changes of the variable $z = x_1 + \mathrm{i}\, x_2$, enforced by action of Virasoro algebra (or some "extended chiral algebra" $\mathcal{A} \supset \mathrm{Vir}$)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)$$

(a quantum realization of $\ell_z = -z^{n+1}\frac{\partial}{\partial z}$), $c =$ "central charge".

In fact two copies of Vir, acting on variables z and $\bar{z}$.
L_0, $\bar{L}_0$, L_{-1}, $\bar{L}_{-1}$ generators of rotation/dilatation, translations.

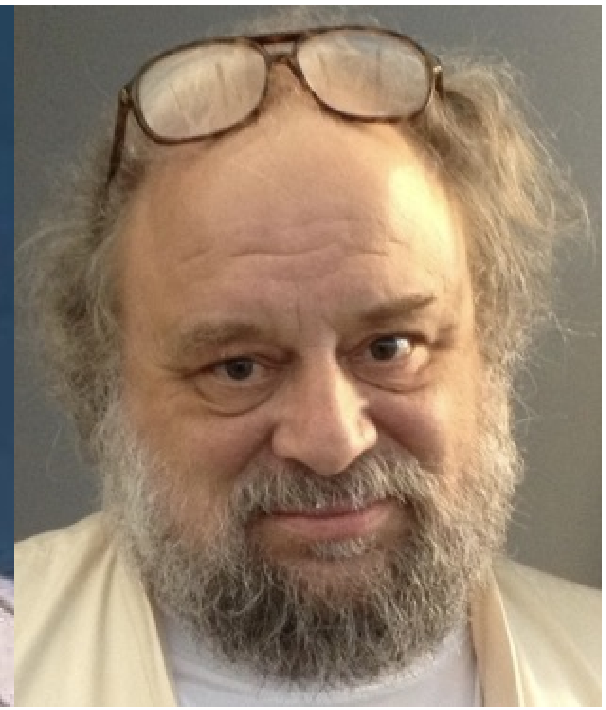
States fall into representations $V_h \otimes V_{\bar{h}}$ of $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$:

Hilbert space : $\mathcal{H} = \oplus_{h,\bar{h}} Z_{h,\bar{h}} V_h \otimes V_{\bar{h}}$ $\qquad Z_{h,\bar{h}} \in \mathbb{N}$.

Problem: Determine the possible multiplicities $Z_{h,\bar{h}}$.

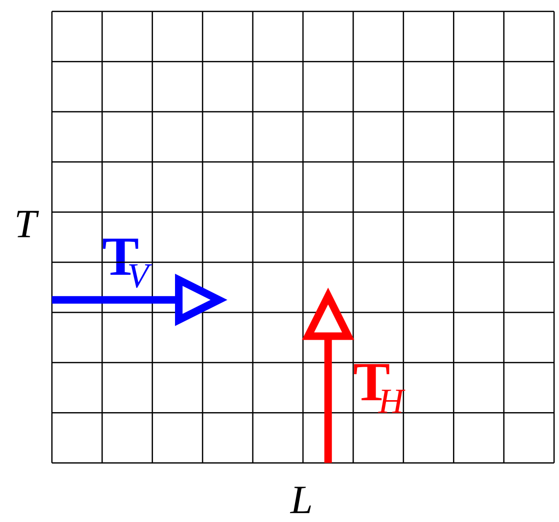

Alexander Belavin


Alexander Polyakov


Alexander Zamolodchikov

Problem: Determine the possible multiplicities $Z_{h,\bar{h}}$.
Cardy '86 Compute the partition function on a *torus*, *i.e.*, with doubly periodic boundary conditions, and impose *modular invariance*. Transfer matrix in statistical mechanics $Z = \mathrm{tr}\, \mathcal{T}^T$.

Z must be invariant under $L \leftrightarrow T$:
$Z = \mathrm{tr}\, \mathcal{T}_{\mathcal{H}}{}^T = \mathrm{tr}\, \mathcal{T}_{\mathcal{V}}{}^L$



Problem: Determine the possible multiplicities $Z_{h,\bar{h}}$.

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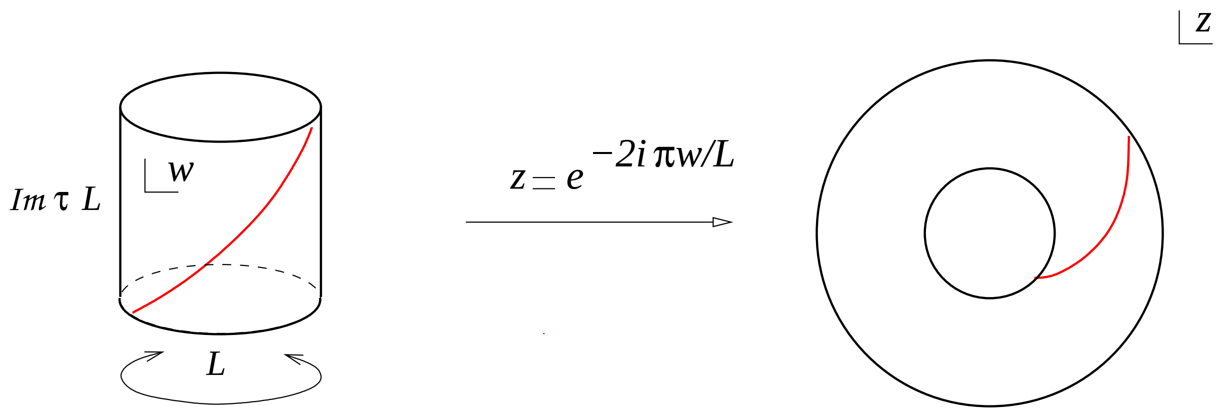

In CFT, translation operator on cylinder $L_{-1}^{cyl} \mapsto \frac{-2i\pi}{L}(L_0^{plane} - \frac{c}{24})$, $q := \exp 2i\pi\tau$, $\Im m\,\tau > 0$

$$Z = \mathrm{tr}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} = \sum_{h,\bar{h}} Z_{h,\bar{h}} \chi_h(q) \bar{\chi}_{\bar{h}}(\bar{q})$$

with $\chi_h(q) = \mathrm{tr}_{V_h} q^{L_0 - c/24}$. Z must be modular invariant, *i.e.*,invariant under $q = \exp 2i\pi\tau \mapsto \exp -2i\pi/\tau$ and $q \mapsto \exp 2i\pi(\tau + 1)$.

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Characters $\chi_h(q)$ transform by a linear, unitary, transformation under these modular transformations, $\chi_i(q) = \sum_j S_{ij} \chi_j(\tilde{q})$. Thus $Z_{h,\bar{h}} = \delta_{h\bar{h}}$ always a solution (diagonal).
Are there other solutions (with $N_{00} = 1$)?
Search in **rational** CFT's, those that have a *finite* collection of irreps of the chiral algebra (Vir or $\mathcal{A}$).

Two classes of RCFT's related to SU(2)

Minimal CFT's, with $c = 1 - \frac{6(p-p')^2}{pp'}$, p, p' two coprime integers [Belavin–Polyakov–Zamolodchikov], finite set of possible h-values, $h_{rs} = \frac{(rp-sp')^2-(p-p')^2}{4pp'}$, $1 \le r \le p'-1, 1 \le s \le p-1$; [Kac]

CFT's with a current (aka affine Kac–Moody) algebra $\widehat{su}(2)$

[Zamolodchikov] $[J^a_n, J^b_m] = \mathrm{i}\,\epsilon_{abc} J^c_{n+m} + kn\delta_{n+m,0}\delta_{ab}$, $k \in \mathbb{N}$, irreps labelled by $\lambda = 1, \cdots, k+1$, $c = 3k/(k+2)$

Non diagonal modular invariant solutions found gradually in the spring 1986,

[Cardy], [Itzykson–Z], [Gepner–Witten], [Kac], [Gepner], [Z],

until complete solution was conjectured [Cappelli–Itzyskon–Z], with parallel work on integrable lattice models [Pasquier].

level	Z	diagram
$k \geq 0$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	A_{k+1}
$k = 4\rho \geq 4$	$\sum_{\lambda \text{ odd } =1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$D_{2\rho+2}$
$k = 4\rho - 2 \geq 6$	$\sum_{\lambda \text{ odd } =1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda \text{ even}=2}^{2\rho-2} \left(\chi_\lambda \bar{\chi}_{4\rho-\lambda} + \text{c. c.}\right)$	$D_{2\rho+1}$
$k = 10$	$\lvert\chi_1 + \chi_7\rvert^2 + \lvert\chi_4 + \chi_8\rvert^2 + \lvert\chi_5 + \chi_{11}\rvert^2$	E_6
$k = 16$	$\lvert\chi_1 + \chi_{17}\rvert^2 + \lvert\chi_5 + \chi_{13}\rvert^2 + \lvert\chi_7 + \chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + \left[(\chi_3 + \chi_{15})\bar{\chi}_9 + \text{c. c.}\right]$	E_7
$k = 28$	$\lvert\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}\rvert^2 + \lvert\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}\rvert^2$	E_8

Table 1: List of modular invariant partition functions of $\widehat{sl}(2)$ RCFTs

χ_λ

are characters of representations of the affine algebra at level k . The last column shows the associated ADE Dynkin diagram.

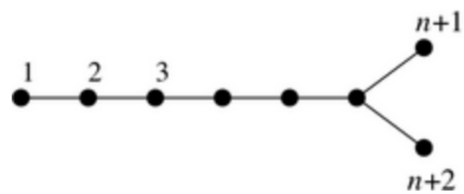
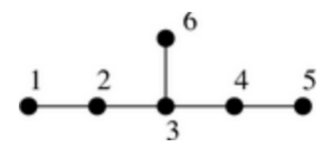
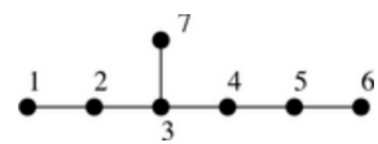
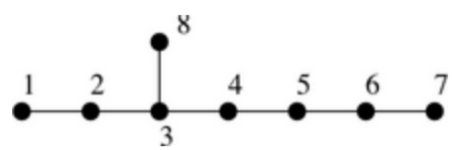
G	diagram	h	exponents ℓ_n
A_n		$n+1$	$1, 2, \cdots, n$
D_{n+2}		$2(n+1)$	$1, 3, \cdots, 2n+1, n+1$
E_6		12	$1, 4, 5, 7, 8, 11$
E_7		18	$1, 5, 7, 9, 11, 13, 17$
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$

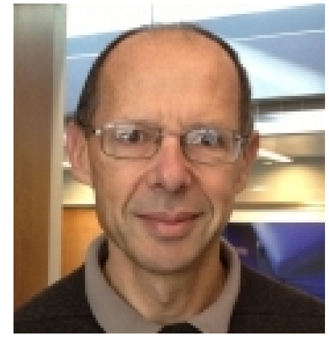
Table 2: ADE Dynkin diagrams with Coxeter numbers h and exponents ℓ_n .

Eigenvalues of adjacency matrix of diagram $= 2\cos \pi \ell_n / h$

p'	Z	diagrams
	$\frac{1}{2}\sum_{r=1}^{p'-1}\sum_{s=1}^{p-1}\lvert\chi_{rs}\rvert^2$	$(A_{p'-1}, A_{p-1})$
$p' = 4\rho+2$ $\rho \geq 1$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}\ =1}^{2\rho-1}\lvert\chi_{r,s}+\chi_{4\rho+2-r,s}\rvert^2 + 2\lvert\chi_{2\rho+1,s}\rvert^2\right\}$	$(D_{2\rho+2}, A_{p-1})$
$p' = 4\rho$ $\rho \geq 2$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}\ =1}^{4\rho-1}\lvert\chi_{r,s}\rvert^2 + \lvert\chi_{2\rho,s}\rvert^2 + \sum_{r\ \mathrm{even}=2}^{2\rho-2}\left(\chi_{r,s}\overline{\chi}_{4\rho-r,s} + \mathrm{c.c.}\right)\right\}$	$(D_{2\rho+1}, A_{p-1})$
$p' = 12$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\lvert\chi_{1,s}+\chi_{7,s}\rvert^2 + \lvert\chi_{4,s}+\chi_{8,s}\rvert^2 + \lvert\chi_{5,s}+\chi_{11,s}\rvert^2\right\}$	(E_6, A_{p-1})
$p' = 18$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\lvert\chi_{1,s}+\chi_{17,s}\rvert^2 + \lvert\chi_{5,s}+\chi_{13,s}\rvert^2 + \lvert\chi_{7,s}+\chi_{11,s}\rvert^2 + \lvert\chi_{9,s}\rvert^2\right.$ $\left. + \left[\left(\chi_{3,s}+\chi_{15,s}\right)\overline{\chi}_{9,s} + \mathrm{c.c.}\right]\right\}$	(E_7, A_{p-1})
$p' = 30$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\lvert\chi_{1,s}+\chi_{11,s}+\chi_{19,s}+\chi_{29.s}\rvert^2 + \lvert\chi_{7,s}+\chi_{13,s}+\chi_{17,s}+\chi_{23,s}\rvert^2\right\}$	(E_8, A_{p-1})

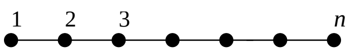
Table 3: List of modular invariant partition functions of minimal models with $c(p,p') < 1$: $\chi_{r,s}$ are characters of Virasoro representations with highest weight h_{rs} in (15). Each invariant, but the first, also occurs for $p \leftrightarrow p'$.

Lattice height models



Vincent Pasquier

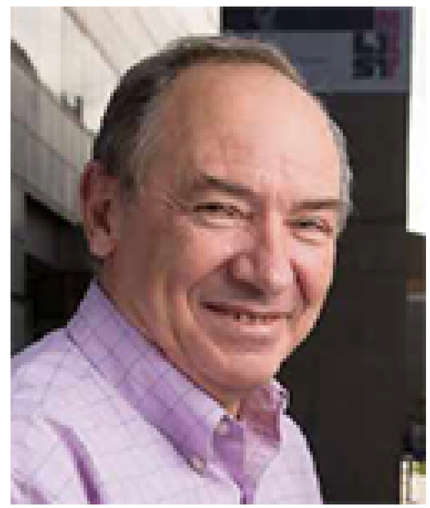
[Andrews–Baxter–Forrester], [Huse] had studied integrable lattice height "RSOS" models, in which at each lattice site i, a "height" variable takes an integer value $h_i \in \{1, \cdots, n\}$, and neighbouring sites : heights $|h_i - h_{i+1}| = 1$. One of their critical regimes was described by a minimal $c < 1$ CFT.

Pasquier reinterpreted that as follows: h_i is a node of diagram 

and neighbouring sites on the lattice are assigned neighbouring heights on the diagram.

Then generalise to arbitrary diagram. Yang–Baxter integrability condition (realized through the Temperley–Lieb algebra) and criticality require that the eigenvalues of the adjacency matrix of the diagram be in $]-2, 2[$. Hence ADE Dynkin diagram! ☺

John Cardy



Victor Kac

Proof of the ADE classification of modular invariants went in two steps:

– characterization of all invariants, irrespective of the condition $Z_{h\bar{h}} \in \mathbb{N}$, [Gepner–Qiu], [Cappelli–Itzykson–Z]

– imposing $Z_{h\bar{h}} \in \mathbb{N}$ [CIZ], [Kato], later simpler proof by [Gannon]

Claude Itzykson

Andrea Cappelli

Three natural questions:

1. Which of the items previously classified by ADE is that classification related to ?
2. Manifestations and/or implications of this ADE scheme ?
3. Extensions/generalizations to other RCFT ?

Three natural questions:

1. **Which of the items previously classified by ADE is that classification related to?**

Long list of mathematical objects classified by ADE:

simply laced Dynkin diagrams, *i.e.*, simply laced root systems [Killing], [Cartan]

finite simply laced crystallographic Coxeter (reflection) groups

finite subgroups of SU(2): McKay correspondence

Kleinian or simple singularities [Klein], [Arnold]

$\mathbb{N}$-valued symmetric matrices with eigenvalues in $]-2,2[$;

finite index subfactors [Jones]

triplets of integers (p,q,r) such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$

etc etc

2. Manifestations and/or implications of this ADE scheme ?
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Three natural questions:

1. Which of the items previously classified by ADE is that classification related to?

2. **Manifestations and/or implications of this ADE scheme?**

Other occurrences of ADE in related theories or issues:

– $N = 2$ superCFT's: chiral sector described by a "superpotential" in the list of simple singularities [Martinec '89], [Lerche–Vafa–Warner '89]

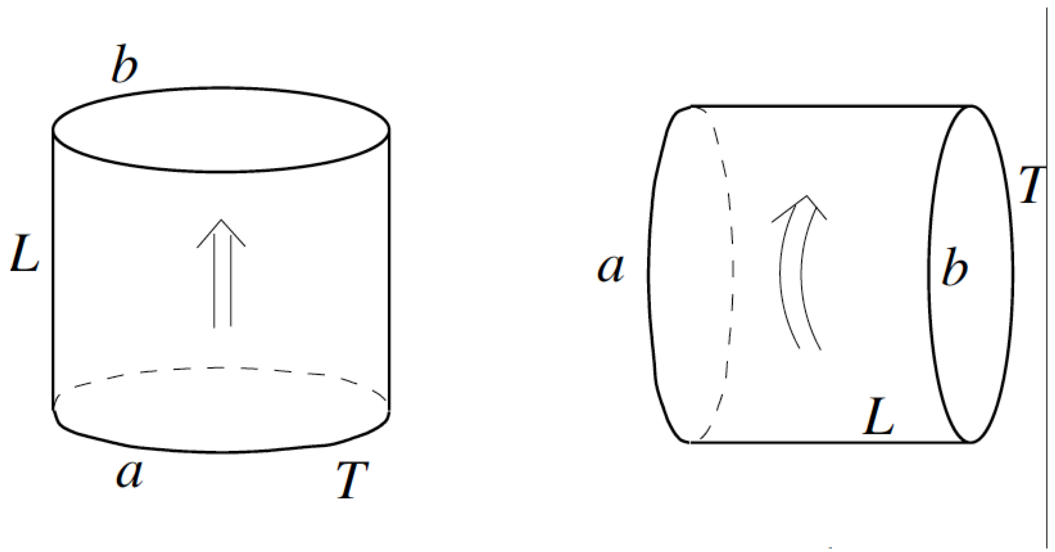
– Graph algebras [Ocneanu], [Pasquier '87] encode information about the Operator Product Algebra of the CFT [Pasquier '87] [Petkova–Z '94]

– Reflection group as a monodromy group of equations in TFT's (twisted $N = 2$ SCFT's) [Dubrovin '92]

– Boundary conditions are determined by the ADE graph [. . .]

3. Extensions/generalizations to other RCFT ?

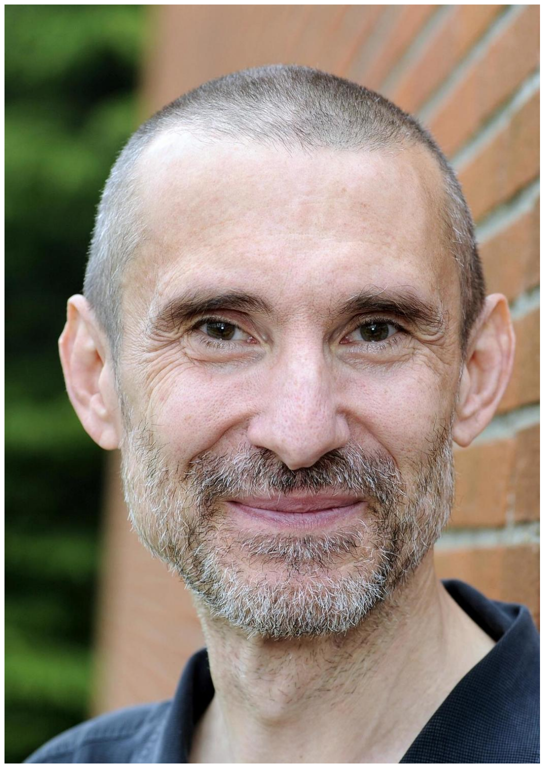
Boundary CFT: Cardy consistency condition



$$Z_{ab} = \sum_{j\in\mathcal{E}} \psi_a^{(j)}\psi_b^{(j)*}\frac{\chi_j(\mathrm{e}^{-4\pi L/T})}{S_{j1}} = \sum_i n_{ia}{}^{b}\chi_i(\mathrm{e}^{-\pi T/L})$$

Cardy: diagonal theory, $\psi_a^{(j)} = S_{aj} \Rightarrow$ Verlinde fusion algebra $N_{ij}^{k} = \sum_\ell \frac{S_{i\ell}S_{j\ell}S^*_{k\ell}}{S_{1\ell}}$.

[Cardy '89], [Saleur–Bauer '89],[Cardy–Llewellen '92]



Michel Bauer

Hubert Saleur

BCFT in non-diagonal theories

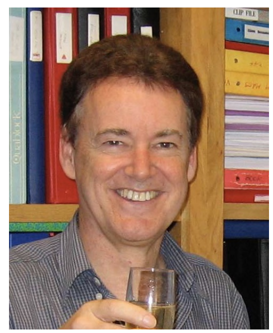
[Pradisi–Sagnotti–Stanev '95], [Affleck–Oshikawa–Saleur '98]

[Fuchs-Schweigert '98], [Runkel-Schomerus '98], [Watts], [Behrend-Pearce-Petkova-Z '98]

$$Z_{ab} = \sum_i n_{ia}{}^b \chi_i(\mathrm{e}^{-\pi T/L}) = \sum_{j \in \mathcal{E}} \psi_a^{(j)} \psi_b^{(j)*} \frac{\chi_j(\mathrm{e}^{-4\pi L/T})}{S_{j1}}$$

The index $j \in \mathcal{E}$ runs over the labels of the diagonal matrix elements of Z (Coxeter exponents and generalization).

Roger Behrend



Paul Pearce

Valentina Petkova

$n_{ia}{}^b = \sum_{j \in \mathcal{E}} \frac{S_{ij}}{S_{i1}} \psi_a^{(j)} \psi_b^{(j)*}$ form a *nim-rep* (non-negative integer valued representation) of the fusion (Verlinde) algebra: $n_i n_j = \sum_k N_{ij}^k n_k \quad N_{ij}^k = \sum_\ell \frac{S_{i\ell} S_{j\ell} S^*_{k\ell}}{S_{1\ell}}$. For $\widehat{su}(2)$ theories and minimal models, the matrix n_2 should have its eigenvalues in $]-2,2[$, hence be the adjacency matrix of an ADE Dynkin diagram!

Return to question 1:

1. **Which of the items previously classified by ADE is that classification related to?**

– simply laced Dynkin diagrams, *i.e.*, simply laced root systems ?

– finite simply laced crystallographic Coxeter (reflection) groups [Dubrovin] ✓

– finite subgroups of SU(2): McKay correspondence: the n_i matrices are related to Kostant polynomials. . . [DiFrancesco],[Ocneanu],[Dorey],[Z] ✓

– Kleinian or simple singularities : $N = 2$ SCFT's [Martinec '89], [Lerche–Vafa–Warner '89] ✓

– $\mathbb{N}$-valued symmetric matrices with eigenvalues in $]-2, 2[$: [Pasquier, BPPZ] ✓

– subfactors of finite index [Pasquier] ✓

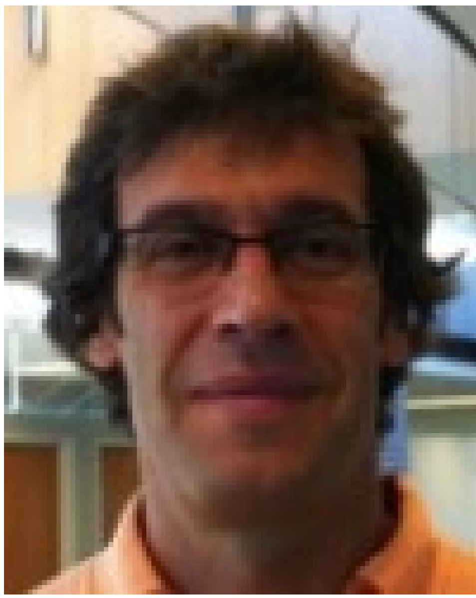
– triplets of integers (p, q, r) such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$?

Three natural questions:

1. Which of the previous items classified by ADE is that classification related to ?
2. Manifestations and/or implications of this ADE scheme ?
3. **Extensions/generalizations to other RCFT ?**

Go over to $\widehat{su}(3)$ current algebra and associated lattice models.
– List of modular invariants [Bernard '87] . . . completed by Gannon '94
– Are there corresponding graphs ? Guiding principle: the labels of diagonal matrix elements of Z must also parametrize the eigenvalues of the adjacency matrix. Graph= deformation/truncation of SU(3) weight lattice. Possibly, graphs are deformations/truncations of graphs of finite subgroups of SU(3) (generalized McKay ?).

First examples due to [Kostov '88]. Then [Di Francesco-Z '89] with "computed-assisted-flair": finite list of exceptional graphs. List finally completed in 2000 [Ocneanu], [Pugh–Evans]

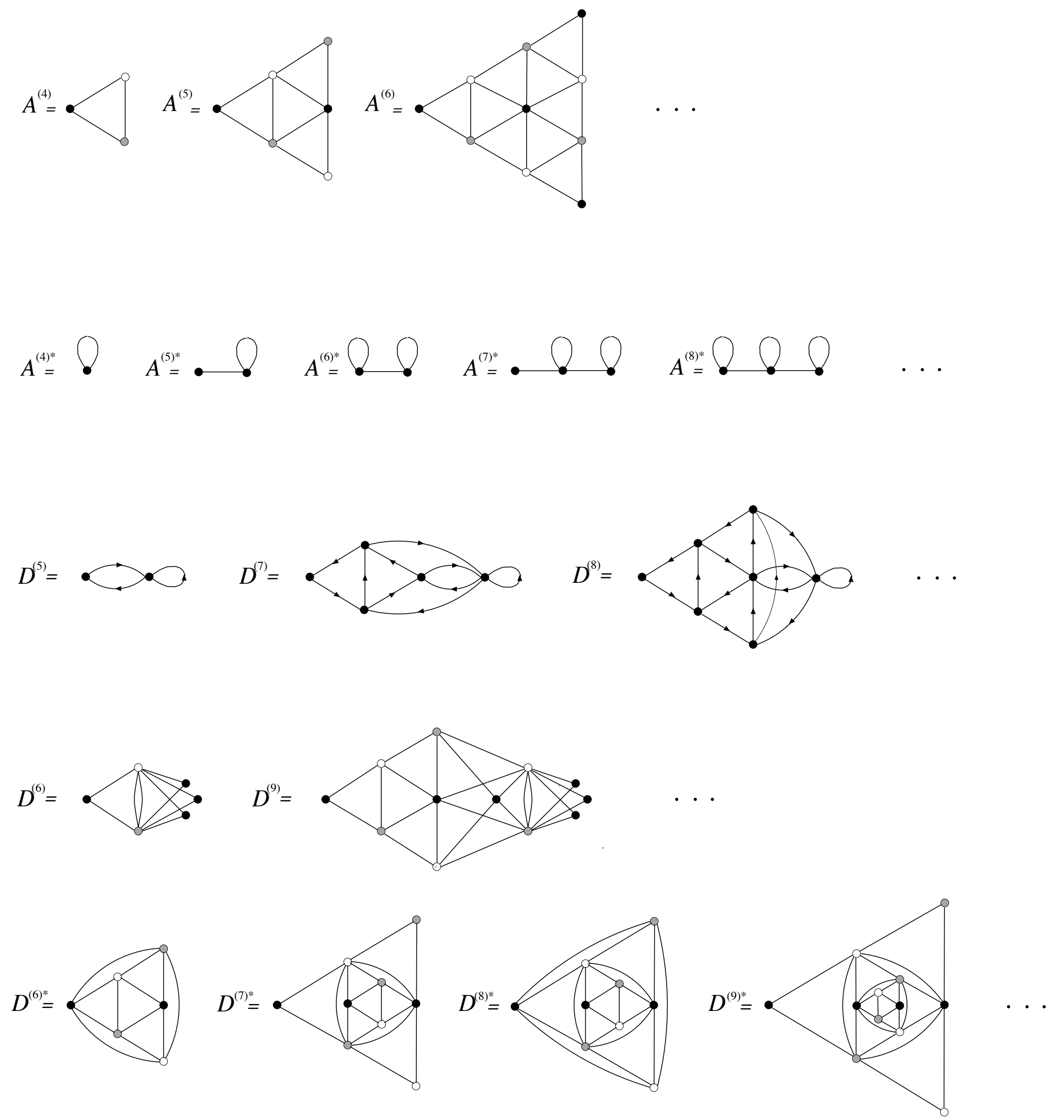


Denis Bernard

Ivan Kostov

Philippe Di Francesco

Adrian Ocneanu and JBZ


$A^{(4)}=$
$A^{(5)}=$
$A^{(6)}=$
. . .
$A^{(4)*}=$
$A^{(5)*}=$
$A^{(6)*}=$
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$A^{(8)*}=$
. . .
$D^{(5)}=$
$D^{(7)}=$
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$D^{(6)}=$
$D^{(9)}=$
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$D^{(6)*}=$
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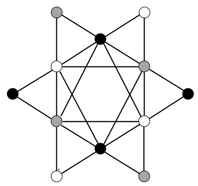
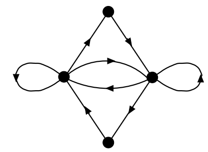
. . .

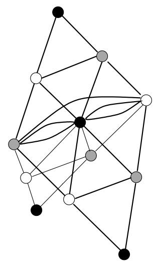
$E^{(8)}=$



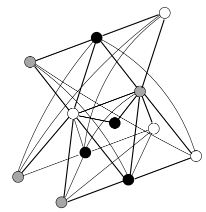
$E^{(8)*}=$



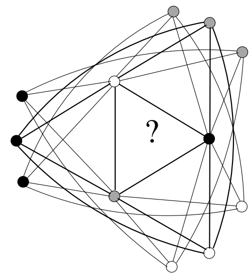
$E_1^{(12)}=$



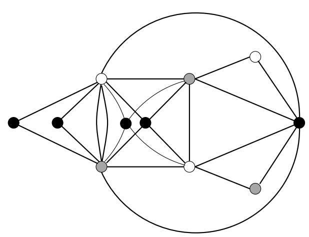
$E_2^{(12)}=$



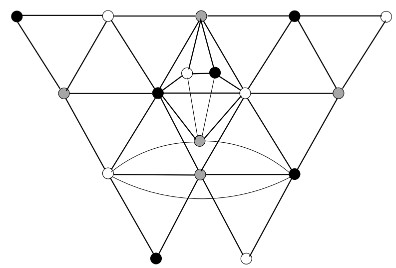
$E_3^{(12)}=$



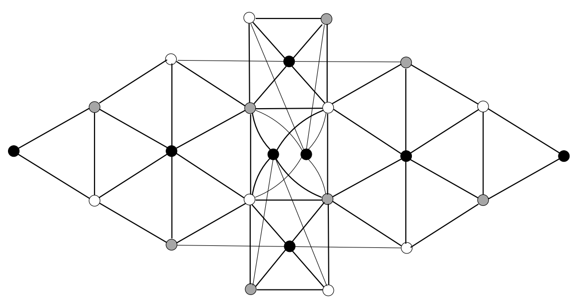
$E_4^{(12)}=$



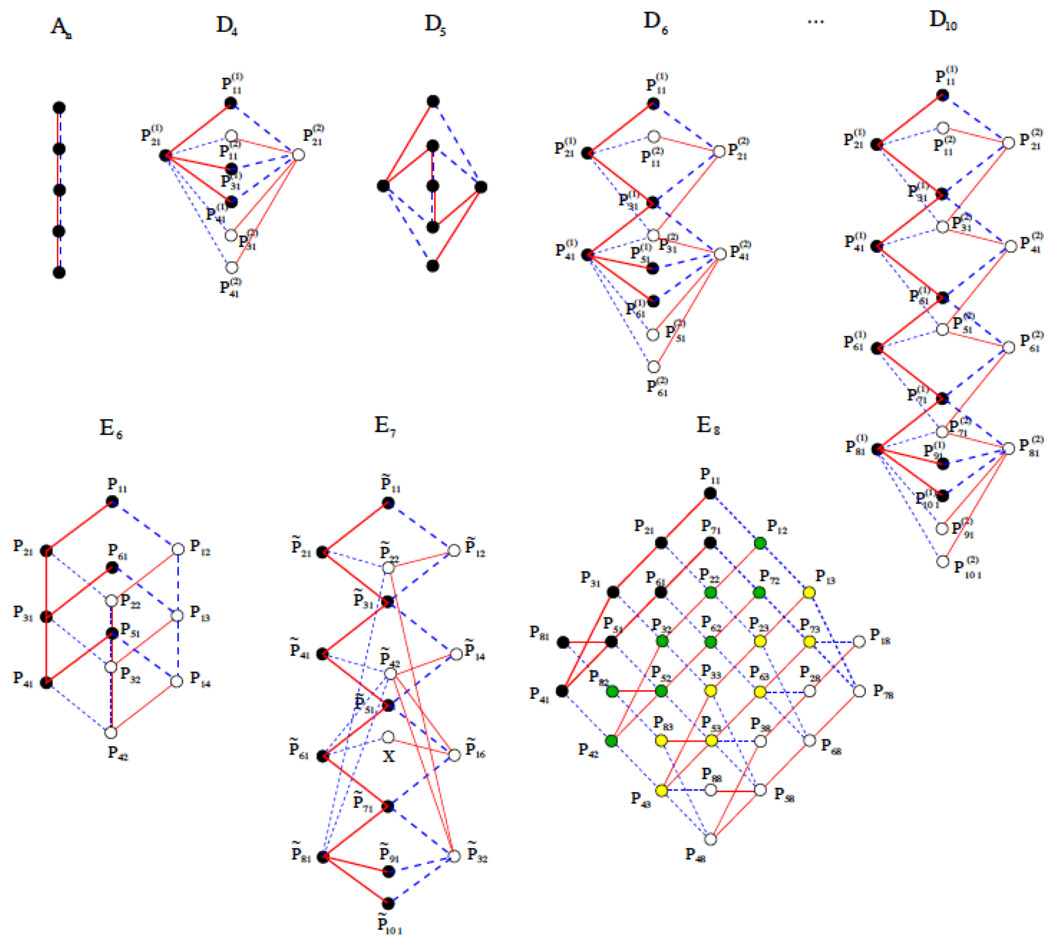
$E_5^{(12)}=$



$E^{(24)}=$



Ocneanu's graphs and defects in CFT's



These graphs encode the partition functions $Z_{x|y}$ in the presence of defect lines [Petkova–Z'00]

Generalized to higher rank [Coquereaux et al] [Evans–Kawahigashi et al]

Big pattern and several approaches:
– fusion algebras of fields (CVO), of boundaries, of defects
– "Ocneanu's cells"
$\rightarrow$ 3-j and 6-j symbols of CVO algebra and
$\rightarrow$ Boltzmann weights of integrable lattice models
– Ocneanu's "double triangle algebra" as a weak Hopf algebra [Böhm–Szlachányi], [Ostrik] . . .
– operator algebra picture, subfactors, α-induction etc. [Xu, Böckenhauer–Evans–Kawahigashi–Pugh]
– 2D-CFT's $\leftrightarrow$ 3D-TFT's, [Fröhlich–Fuchs–Runkel–Schweigert]

One last natural question: What do these new graphs encode, from a geometric point of view?

Generalized root system and corresponding reflection group ?

[Z. '97] : reflection group relevant in Topological FT's à la Dubrovin??

Higher Lie theory ? [Ocneanu] . . .

Thank you