# Mathematical Harmony and the Quantum World

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1985–1990 : Meudon– Princeton:
affine Kac–Moody algebras, WZW models, etc

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**STRING CHARACTERS FROM KAC-MOODY AUTOMORPHISMS**

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We construct modular invariant partition functions for strings which propagate on a group manifold. These string characters are built upon automorphisms of affine Kac-Moody algebras. Namely, to each element of the center of the group we associate infinite series of modular invariant string characters.

Prehistory : 1985, Meudon: QFT in curved space

1985–1990 : Meudon– Princeton:
affine Kac–Moody algebras, WZW models, etc

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DENIS BERNARD

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# 1985–1990 : Meudon– Princeton:
# affine Kac–Moody algebras, WZW models, etc

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## ON THE WESS-ZUMINO-WITTEN MODELS ON THE TORUS*

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We discuss the Ward identities of the Wess-Zumino-Witten models on Riemann surfaces and point out some ambiguities in the description of the zero modes of the currents. In the case of the torus, we show how to describe them and we write the Ward identities in such a way that they become complete. We examine in detail how the Ward identities are related to the Kubo-Martin-Schwinger condition. As an illustration of this formulation, we present a new proof of the Weyl-Kac character formula. The proof essentially relies on the mixed Virasoro × Kac-Moody Ward identities and explains the relation of the heat equation on the group manifold to the Weyl-Kac character formula.

Nuclear Physics B309 (1988) 145–174
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## ON THE WESS–ZUMINO–WITTEN MODELS ON RIEMANN SURFACES*

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(Revised 2 May 1988)

We give a formulation of the Wess–Zumino–Witten models on Riemann surfaces of arbitrary genus in which the Ward identities for the current algebras become complete. It requires twisting the models in a non-abelian way by Lie group elements. The Ward identities are written in terms of twisted Poincaré series for Schottky groups and the zero modes of the currents are defined by a Lie derivation acting on the twists. Furthermore, we identify the denominator of the chiral partition function and we argue that both the numerator and the denominator of the partition function satisfy a heat equation on the moduli space.

CFT, WZW ...

[DB , J. Thierry-Mieg, A. LeClair, G. Felder] (1985–1990)

CFT, WZW ...

integrability, symmetries, yangians

(1990–1997)

[DB, O. Babelon, G. Felder, A. LeClair, V. Pasquier, F. Smirnov, . . . ]

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# Yangian Symmetry of Integrable Quantum Chains with Long-Range Interactions and a New Description of States in Conformal Field Theory

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(Received 6 July 1992)

The SU($n$) quantum chains with inverse-square exchange exhibit a novel form of Yangian symmetry compatible with periodic boundary conditions, allowing states to be countable. We characterize the "supermultiplets" of the spectrum in terms of generalized "occupation numbers." We embed the model in the $k$=1 SU($n$) Kac-Moody algebra and obtain a new classification of the states of conformal field theory, adapted to particlelike elementary excitations obeying fractional statistics.

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In this Letter, we introduce a new description of the states of non-Abelian conformal field theories that is in some sense a generalization of the Fock-space occupation-number description to describe excitations of an ideal gas with fractional statistics [1]. This is applicable to, e.g., the "spinon" excitations in a gapless Fermi fluid (Luttinger liquid) with spin-charge separation, in one or possibly higher spatial dimension. The results also appear to shed new light on the algebraic structures of integrable models, placing the Bethe-ansatz-solvable models in the context of a larger family of models, and identifying the inverse-square interaction models as possibly the simplest example of Yang-Baxter integrability, where the excitations have purely statistical interactions.

These results have emerged from an extensive study by three of us [2] of the remarkable underlying symmetry algebra of the $S = \frac{1}{2}$ Heisenberg spin chains with inverse-square interactions [3,4], and an embedding of this model and its SU($n$) generalizations in conformal field theory. As that study reached completion, it became apparent that the algebraic structures under investigation were a novel presentation of the Yangian algebra [5] that has been emphasized by another of us [6] as the key algebraic structure in integrable models with non-Abelian symmetry. The novel feature is that in contrast to the usual presentation (e.g., in Bethe-ansatz models [7]), this form of the Yangian can coexist with periodic boundary conditions that make states countable.

We will first describe the integrable spin-chain Hamiltonians

$$H_2 = \sum_{ij}{}' \left(\frac{z_i z_j}{z_{ij} z_{ji}}\right)(P_{ij} - 1)\,, \tag{1}$$

where $P_{ij}$ exchanges the states on sites $i$ and $j$. The primed sum omits equal values of the summation variables, and $z_{ij} \equiv z_i - z_j$. The distinct complex numbers $z_i$ parametrize the lattice sites. Translational invariance is present if $\{z_i\} = \{\omega^n\}$; inversion symmetry ($z_i \to 1/z_i$) means we can choose $|\omega| \leq 1$.

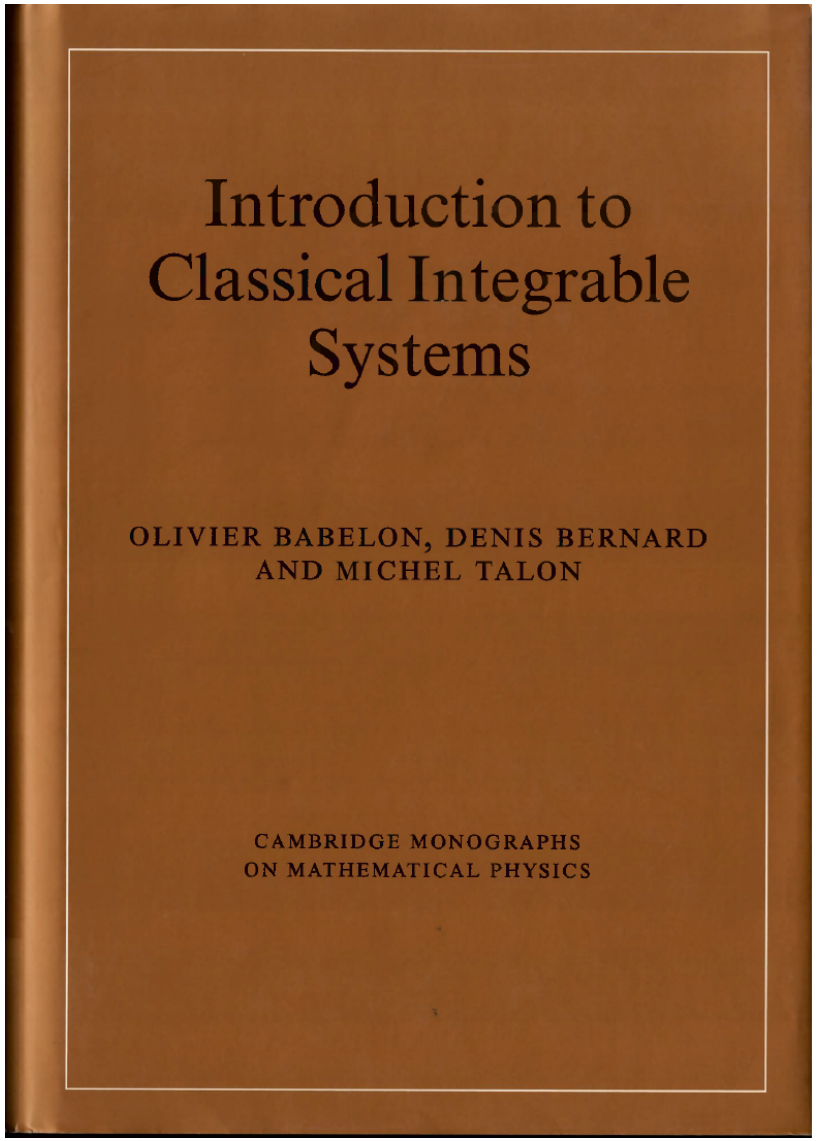
There are two families of models where $H_2$ is Hermitian and translationally invariant. The *trigonometric* models have $\omega = \exp(2\pi i/N)$, and $N$ distinct sites on a circle, with exchange between sites proportional to the inverse-square of their chord distance. With two states per site [SU(2) or $S = \frac{1}{2}$ Heisenberg chain] this model was independently introduced in [3] and [4], and has a straightforward SU($n$) generalization [1, 8] to $n$ states per site. The *hyperbolic* family has $N = \infty$, and real $\omega$ in the ranges $[-1, 1]$. Rescaling $H_2$ in the singular limit $\omega \to 0$ gives the familiar nearest-neighbor-exchange Bethe-ansatz (BA) model [9, 10]. The rescaled limits of the hyperbolic and trigonometric models as $\omega \to 1$ co-

CFT, WZW ...

integrability, symmetries, yangians

(1990–1997)

[DB, O.Babelon, G.Felder, A.LeClair, V.Pasquier, F.Smirnov, . . . ] and (1993) [D. Serban]


Introduction to
Classical Integrable
Systems
OLIVIER BABELON, DENIS BERNARD
AND MICHEL TALON
CAMBRIDGE MONOGRAPHS
ON MATHEMATICAL PHYSICS

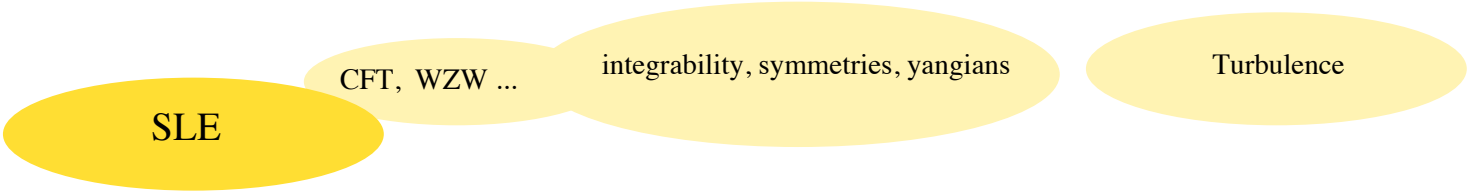
CFT, WZW ...

integrability, symmetries, yangians

Turbulence

(1996–2007)

[DB, K. Gawedzki, A. Kupiainen, M.Bauer, G. Falkovich, . . . ]




(2002–2010)

[DB, M. Bauer, K. Kytölä, L. Cantini, C. Hagendorf, J. Houdayer, . . . ]


(2011–. . . )

[DB, M. Bauer, B. Doyon, A. Tilloy, T. Jin, J. De Nardis, A. De Luca, J. Viti, P. Le Doussal, T. Benoist, F. Essler, . . . ]

and also, "ancillary" activities in

– 2D reduced gravity [DB, N. Regnault, B. Julia, . . . ]

– quantum Hall effect [DB, N. Regnault, D. Serban, A. Leclair]

– random walks, random networks, disordered systems [DB, M. Bauer, J.-M. Luck]

– topological insulators [DB, A. LeClair, . . . ]

. . .


(2011–. . . )

# What next ??

Congratulations, Denis, many happy returns and . . .

Congratulations, many happy returns and . . .
many new lampposts !