Chapter 5

Gauge theories. Standard model

Transformations considered so far were global, space-time independent, transformations. Another type of symmetry, which is restricting the dynamics of the system in a much more stringent way, considers *local* transformations. At each point of space–time, acts a distinct copy of the transformation group. Such a symmetry, called *gauge symmetry*, is familiar in electrodynamics. Its extension by Yang and Mills to non-abelian transformation groups turned out to be one of the most fruitful theoretical ideas of the second half of the XXth century. A full course should be devoted to it. More modestly, the present chapter gives an elementary introduction and overview.

5.1 Gauge invariance. Minimal coupling. Yang–Mills Lagrangian

5.1.1 Gauge invariance

The study of electrodynamics has introduced the notion of local invariance. The Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{5-1}$$

is invariant under infinitesimal gauge transformations

$$\begin{aligned} \delta A_\mu(x) &= -\partial_\mu \delta\alpha(x) \\ \delta\psi(x) &= ie\delta\alpha(x)\psi(x)\ , \end{aligned} \tag{5-2}$$

since the electromagnetic field tensor

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

is invariant, and the combination

$$iD\!\!\!/\psi(x) := (i\partial\!\!\!/ - eA\!\!\!/)\psi(x)$$

also transforms as ψ. The finite form of these transformations is readily written

$$\begin{aligned} A_\mu(x) &\mapsto A_\mu(x) - \partial_\mu \alpha(x) \\ \psi(x) &\mapsto e^{ie\alpha(x)}\psi(x) \ , \end{aligned} \tag{5-3}$$

which shows that the transformations are a local (*i.e.* x dependent) version of those of the group U(1) or $\mathbb{R}$ (depending whether one identifies or not α and $\alpha + 2\pi/e$). The corresponding global transformations are those leading to a conserved Noether current, which implies the conservation of electric charge. The Lagrangian displays the "minimal coupling" of the field ψ to the electromagnetic field[1]. Any other charged field of charge q couples to the electromagnetic field through a term involving the "covariant derivative" $i\partial_\mu - qA_\mu(x)$.

This is for example the case of a charged, hence complex, boson field ϕ, whose contribution to the Lagrangian reads

$$\delta\mathcal{L} = (\partial_\mu - iqA_\mu)\phi^*(\partial^\mu + iqA_\mu)\phi - V(\phi^*\phi) \tag{5-4}$$

which is indeed invariant under $\phi(x) \mapsto e^{iq\alpha(x)}\phi(x)$, $A_\mu(x) \mapsto A_\mu(x) - \partial_\mu\alpha(x)$.

5.1.2 Non abelian Yang–Mills extension

Following the brilliant observation of Yang and Mills (1954), this construction may be transposed to the case of a non-abelian Lie group G, with however a few interesting modifications... Let ψ be a field (which we denote as a fermion field, but this is irrelevant) transforming under G by some representation $\mathcal{D}$. Let T_a be the infinitesimal generators in that representation, which we assume antihermitian: $[T_a, T_b] = C_{ab}{}^c T_c$; the infinitesimal transformation thus reads

$$\delta\psi(x) = T_a \delta\alpha^a \psi(x) \ . \tag{5-5}$$

(In this chapter, we denote t_a the corresponding matrices in the adjoint representation.) To extend the notion of local transformation, we need a gauge field A_μ, which allows to construct a covariant derivative $D_\mu\psi$. It is natural to consider that A_μ lives in the Lie algebra of G, and hence carries indices of the adjoint representation

$$A_\mu(x) = \{A^a_\mu(x)\} \tag{5-6}$$

or equivalently, is represented in any representation by the antihermitian matrix[2]

$$A_\mu(x) = T_a A^a_\mu(x) \ . \tag{5-7}$$

The covariant derivative reads

$$D_\mu\psi(x) := (\partial_\mu - A_\mu(x))\psi(x) \ , \tag{5-8}$$

[1] An additional term in the Lagrangan like $\bar\psi[\gamma_\mu, \gamma_\nu]\psi F^{\mu\nu}$ would be gauge invariant but non minimal.

[2] Caution! this convention implies that some expressions differ by a factor i from the abelian case.

or, componentwise

$$D_\mu \psi_A(x) := \left(\partial_\mu \delta_{AB} - A^a_\mu(x)\,(T_a)_A{}^B\right)\psi_B(x)\ . \tag{5-9}$$

That covariant derivative does transform as ψ, just like in the abelian case, provided one imposes that A_μ transforms according to

$$\delta A^a_\mu(x) = \partial_\mu \delta\alpha^a(x) + C_{bc}{}^a \delta\alpha^b(x) A^c_\mu(x)\ . \tag{5-10}$$

The term $\partial_\mu\delta\alpha^a(x)$ notwithstanding, one sees that $\{A^a_\mu\}$ transforms as the adjoint representation (whose matrices are $(t_a)_b{}^c = -C_{ab}{}^c$). Lastly a field tensor transforming in a covariant way (*i.e.* without any inhomogeneous term in $\partial\delta\alpha^a(x)$) may be constructed

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \tag{5-11}$$

or in components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - C_{bc}{}^a A^b_\mu A^c_\nu\ . \tag{5-12}$$

One proves, after some algebra and using the Jacobi identity, that

$$\delta F^a_{\mu\nu}(x) = C_{bc}{}^a \delta\alpha^b(x) F^c_{\mu\nu}(x)\ , \tag{5-13}$$

which is indeed an infinitesimal transformation in the adjoint representation.

It is in fact profitable, and more enlightening, to look at the effect of a *finite* local transformation $g(x)$ of the group G,

$$\begin{aligned} \psi(x) &\mapsto \mathcal{D}(g(x))\psi(x) \\ A_\mu = A^a_\mu T_a &\mapsto \mathcal{D}(g(x))(-\partial_\mu + A_\mu(x))\mathcal{D}(g^{-1}(x))\ , \end{aligned} \tag{5-14}$$

and for the covariant derivative acting on ψ,

$$D_\mu\psi(x) \mapsto \mathcal{D}(g(x)) D_\mu \psi(x) \tag{5-15}$$

or equivalently[3]

$$D_\mu \mapsto \mathcal{D}(g(x)) D_\mu \mathcal{D}(g^{-1}(x))\ . \tag{5-16}$$

Now one verifies easily that in a given representation

$$[D_\mu, D_\nu] = -F_{\mu\nu} := -F^a_{\mu\nu} T_a \tag{5-17}$$

from which follows that $F_{\mu\nu}(x) \mapsto \mathcal{D}(g(x)) F_{\mu\nu} \mathcal{D}(g^{-1}(x))$, and in particular, in the adjoint representation, the finite transformation of $F_{\mu\nu} = F^a_{\mu\nu} t_a$ is

$$F_{\mu\nu}(x) \mapsto g(x) F_{\mu\nu}(x) g^{-1}(x)\ , \tag{5-18}$$

of which (5-13) is the infinitesimal version.

[3]Beware the notations! In that equation (5-16), which deals with a differential operator, the derivative ∂_μ contained in D_μ acts on everything sitting on its right, whereas in the second equation (5-14), it acts only on $\mathcal{D}(g^{-1}(x))$.

Pure gauge

If the tensor $F_{\mu\nu}$ vanishes in the neighbourhood of a point x_0, one may write locally (*i.e.* in that neighbourhood) $A_\mu(x)$ as a "pure gauge", *i.e.*

$$F_{\mu\nu} = 0 \iff A_\mu(x) = (\partial_\mu g(x))\, g^{-1}(x) \ . \tag{5-19}$$

(The naming "pure gauge" is justified by the fact that such an $A_\mu(x) = (\partial_\mu g(x))\, g^{-1}(x)$ is the gauge transform of a vanishing gauge field! Proving $\Leftarrow$ is a trivial calculation, as for $\Rightarrow$, see a few lines below...) We insist on the local character of that property.

Parallel transport along a curve

Another interesting object is the group element attached to a curve C going from x_0 to x

$$\gamma(C) := P \exp\left(\int_C dx^\mu A_\mu(x)\right) \tag{5-20}$$

where the symbol P means that a parametrization $x(s)$ of the curve being chosen, and terms in the expansion of the exponential are ordered from right to left with increasing s (compare with the T-product in quantum field theory). One shows that under the gauge transformation (5-14)

$$\gamma(C) \mapsto g(x)\gamma(C)g^{-1}(x_0) \ . \tag{5-21}$$

More generally, for any representation $\mathcal{D}$ and with $A = A^a T_a$, (5-20) defines a $\gamma_{\mathcal{D}}(C)$ in the representation $\mathcal{D}$ that transforms as $\gamma_{\mathcal{D}}(C) \mapsto \mathcal{D}(g(x))\gamma_{\mathcal{D}}(C)\mathcal{D}(g^{-1}(x_0))$.
Exercise. Prove that statement by first considering an infinitesimal path from x to $x + dx$, hence $\gamma(C) \approx 1 + A_\mu(x)dx^\mu$, and by performing a finite gauge transformation $A_\mu \to g(x)(-\partial_\mu + A_\mu(x))g^{-1}(x)$, show that $\gamma(C) \to g(x + dx)\gamma(C)g^{-1}(x)$. The result for a finite curve follows by recombining these infinitesimal elements.

Given an objet, like the fieldψ, transforming by some representation $\mathcal{D}$, the role of $\gamma_{\mathcal{D}}(C)$ is to "transport" $\psi(x_0)$ into an object denoted ${}^t\psi(x)$ transforming like $\psi(x)$. Show that for an infinitesimal curve $(x, x + dx)$ the difference ${}^t\psi(x + dx) - \psi(x + dx)$ is expressed in a natural way in terms of the covariant derivative.

Consider then the case where $x = x_0$ in (5-20). From (5-21), it follows that for a closed loop C, $\gamma(C)$ transforms in a covariant way, $\gamma(C) \mapsto g(x_0)\gamma(C)g^{-1}(x_0)$. Let us examine again the case of an infinitesimal closed loop. One finds that then

$$\gamma(C) \approx \exp \frac{1}{2} \int_{\mathcal{S}} dx^\mu \wedge dx^\nu F_{\mu\nu} \ , \tag{5-22}$$

where the integration is carried out on an infinitesimal surface $\mathcal{S}$ of boundary C.
Exercise: Prove that statement by considering an elementary square circuit extending from x along the coordinate axes μ and ν: $(x \to x + dx^\mu \to x + dx^\mu + dx^\nu \to x + dx^\nu \to x)$, and expand to second order in dx to find $\gamma(C) \approx 1 + dx^\mu dx^\nu F_{\mu\nu}$ (with no summation over μ, ν). Hint: the use of the commutator formula of Chap. 1, (1.23), simplifies the computation a great deal!

This has an immediate consequence. If $F = 0$, any $\gamma(C)$ of the form (5-20) is insensitive to small variations of the curve C with fixed end points x_0 and x, hence depends only on these end-points x_0 and x. The element $g(x, x_0) := g(C)$ that follows satisfies $(\partial_\mu - A_\mu)g(x, x_0) = 0$, (check!), thus completing the proof of (5-19).

Wilson loop

Return to the case of a closed loop C with $x = x_0$ in (5-20). As just mentionned, $\gamma(C)$ transforms in a covariant way, $\gamma(C) \mapsto g(x_0)\gamma(C)g^{-1}(x_0)$. Its trace

$$W(C) = \text{tr } \gamma(C) = \text{tr } P \exp \oint dx^\mu A_\mu(x) \tag{5-23}$$

is thus invariant. We postulate that any physical quantity in a gauge theory must be "gauge invariant", *i.e.* invariant under a gauge transformation. This is the case of $\text{tr } F_{\mu\nu}F^{\mu\nu}$, $\bar\psi(i\partial\!\!\!/ - A\!\!\!/)\psi$ etc. The interest of $W(C)$ is that it is a *non local* invariant quantity, which depends on the contour C. Note that it depends on the representation in which $A = A^a T_a$ is evaluated.

This *Wilson loop* was proposed by Wilson and Polyakov as a way to measure the interaction potential between particles propagating along C, and as a good indicator of confinement. See below § 3.1 and the Problem at the end of this Chapter for a discrete version of that quantity.

5.1.3 Geometry of gauge fields

The previous considerations show that the theory of gauge fields has a strong geometric flavor. The appropriate language to discuss these matters is indeed the *theory of fiber bundles*, principal fiber bundle for the gauge group itself, vector bundle for each matter field like ψ, above the base space which is space–time. The gauge field is a connection on the fiber bundle, which permits to define a parallel transport from point to point. The tensor $F_{\mu\nu}$ is its curvature, as expressed by (5-17). All these notions are defined locally, in a system of local coordinates (*a chart*), and changes of chart imply transformations of the form (5-14). This language becomes particularly useful when one looks at topological (instantons etc) or global ("Gribov problem") properties of gauge theories. For a mere introduction to properties of local symmetry and the *perturbative* construction of the standard model, we won't need it.

5.1.4 Yang–Mills Lagrangian

The Lagrangian describing a gauge field coupled to a matter field like ψ via the minimal coupling reads

$$\mathcal{L} = \frac{1}{2\text{g}^2}\text{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \bar\psi\big(i(\partial\!\!\!/ - A\!\!\!/) - m\big)\psi \ , \tag{2-24}$$

with a parameter, the coupling constant g. The value of that coupling depends of course on the normalization of the matrices T_a that appear in $F_{\mu\nu} = F^a_{\mu\nu}T_a$. One proves (see Exercise B at the end of this chapter) that for any simple Lie algebra one may choose a basis such that in any representation R, $\text{tr}\,T_aT_b = -T_R\delta_{ab}$, with T_R a real positive coefficient that depends on the group and on the representation. We will choose in $F_{\mu\nu}$ the fundamental representation of lowest dimension, (the defining representation of dimension N in the case of SU(N)) with a normalization $T_f = \frac{1}{2}$, hence $\text{tr}\,T_aT_b = -\frac{1}{2}\delta_{ab}$. To the Lagrangian $\mathcal{L}$, one may add the

contribution of other fermion fields or of boson fields. Note that the representations "carried" by the fermions and other matter fields, that appear in their covariant derivatives $D_\mu = \partial_\mu - A^a_\mu T_a$, may differ from the fundamental representation.

As such, $\mathcal{L}$ of (2-24) ressembles very much the Lagrangian of the abelian case (5-1), after a change $iA \to \mathrm{g}A$ has been carried out.

Let us review the most salient features of that construction:

- like in the abelian case, the gauge invariance principle implies a coupling of a universal type, namely through the covariant derivative;
- contrary to the abelian case where each charge is independent and unquantized (at least if the gauge group is $\mathbb{R}$ rather than U(1)), the coupling constant g of all fields to the gauge fields is the same, within each simple component of the gauge group; (for example, the standard model, based on the group U(1)×SU(2)×SU(3) possesses three independent couplings, see below.)
- like in the abelian case, the gauge field comes naturally without a mass term: a mass term $\frac{1}{2}M^2 A_\mu A^\mu$ does break gauge invariance. This looks most embarrassing for physical applications, since the massless vector fields (of spin 1) are quite exceptional in Nature (the electromagnetic field and its photonic excitations being the basic counter-example); this will lead us to introduce "soft" mechanisms of (spontaneous) breaking of gauge invariance to remedy it;
- contrary to the abelian case, the gauge field itself carries a charge of the group: we saw that for global (x independent) transformations of the group G, A_μ transforms by the adjoint representation. The property of the gauge field to be charged has important implications in many phenomena, from the infrared effects (confinement), to the ultraviolet ones (sign of the β function), as we shall see below.

5.1.5 Quantization. Renormalizability

The quantization of the Yang–Mills theory requires to overcome serious difficulties that we only briefly evoke. As in electrodynamics, the quadratic form in the gauge field in $\mathcal{L}$, namely

$$(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \quad \text{or in Fourier space} \quad A_\mu(-k)(k^\mu k^\nu - k^2 g^{\mu\nu})A_\nu(k)$$

is degenerate, thus non invertible, which reflects gauge invariance. Consequently the propagator of the field A_μ is *a priori* undefined. One must first "fix the gauge", by imposing a non-invariant "gauge condition" (like the Coulomb gauge in QED), and the Faddeev and Popov procedure, justified by their general study of *constrained systems* leads to the introduction of auxiliary fields and to explicit Feynman rules, (see for example [IZ, chap. 12] and the courses of the second semester).

One then proves, and that was a decisive step in the building of the Standard Model[4], that the theory so quantized is renormalizable: all ultraviolet divergences appearing in Feynman

[4] G. 't Hooft and M. Veltman, Nobel prize 1999

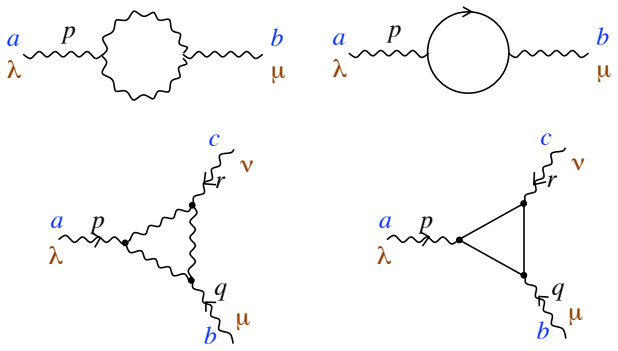


Figure 5.1: Some one-loop diagrams in a gauge theory

diagrams may, at any finite order of perturbation theory, be absorbed into a redefinition of parameters –couplings, field normalization, masses– of the Lagrangian. Thus, to the one loop order, diagrams of Fig. 5.1 have divergences that may be absorbed into a change of normalization of the A field ("wave function renormalization") and a renormalization of the coupling constant g

$$\mathrm{g} \mapsto \mathrm{g}_0 = \left(1 - \frac{\mathrm{g}^2}{(4\pi)^2}\left(\frac{11}{3}C_2 - \frac{4}{3}T_f\right)\log\frac{\Lambda}{\mu}\right)\mathrm{g}\ , \tag{2-25}$$

where Λ is a scale of ultraviolet "cutoff" and μ a mass scale which must be introduced for a definition of the renormalisation procedure.

5.2 Massive gauge fields

5.2.1 Weak interactions and intermediate bosons

We saw in Chap. 4 (equ. (4-30)) that the Fermi Lagrangian

$$\mathcal{L}_{\text{Fermi}} = -\frac{G}{\sqrt{2}}J^{\rho}(x)J^{\dagger}_{\rho}(x) \tag{2-26}$$

gives a good description of the low energy physics of weak interactions: leptonic processes like $\bar{\nu}_e e^- \to \bar{\nu}_e e^-$ or $\bar{\nu}_\mu \mu^-$, semi-leptonic ones like $\pi^+ \to \mu^+\nu_\mu$ or the β decay $n \to pe^-\bar{\nu}_e$, or non-leptonic ones : $\Lambda \to p\pi^-$, $K^0 \to \pi\pi$, etc. But this Lagrangian is theoretically unsatisfactory, since it leads to a non renormalizable theory, making impossible any calculation beyond the "Born term", the first order of perturbation theory, which violates unitarity.

The violation of unitarity appears in the calculation of the total cross section σ of any process, to first order of the perturbation series. A simple dimensional argument gives at high energy

$$\sigma \sim \text{const.}\ G^2 s$$

where s is the square center-of-mass energy. But that behavior contradicts general results based on unitarity that predict that σ must decrease in each partial wave like $1/s$. A violation of unitarity by the Born term is thus expected at an energy of the order of $\sqrt{s} \sim G^{-\frac{1}{2}} \sim 300$ GeV. And the non-renormalisability of the theory precludes an improvement of that Born term.

The idea is thus to regard $\mathcal{L}_{\text{Fermi}}$ as an approximation of a theory where the charged current J^ρ is coupled to a vector field W of mass M, in the large mass limit[5] Consider the new Lagrangian

$$\mathcal{L}_{\text{int.boson}} = gJ^\rho(x)W_\rho^\dagger(x) + \text{h.c.} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + M^2W_\rho^\dagger W^\rho \ . \tag{2-27}$$

In the large mass M limit, one may neglect the kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with respect to the mass term, the field W becomes a simple auxiliary field with no dynamics, that one may integrate out by "completing the square", and one recovers $\mathcal{L}_{\text{Fermi}}$ provided

$$\frac{G}{\sqrt{2}} = \frac{g^2}{M^2} \tag{2-28}$$

which relates the new coupling g to the Fermi constant G. Is the theory (2-27) with its "intermediate boson" W, vector of weak interactions, a good theory of weak interactions? In fact the propagator of the massive W field reads

$$-i\,\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}}{k^2 - M^2} \tag{2-29}$$

which has a bad behavior as $k \gg M$ and makes again the theory non-renormalizable : the problem has just been shifted! The solution stems from a soft and subtle (!) way to introduce the mass of the W field, via a spontaneous breaking of gauge symmetry.

5.2.2 Spontaneous breaking of gauge symmetry. Brout–Englert–Higgs mechanism

Let us return to the abelian case described by (5-1), (5-4) and suppose now that the potential V has a minimum localized at a non–zero value of $\phi^*\phi$. Consequently, the field ϕ acquires a vev $\langle\,\phi\,\rangle = v/\sqrt{2} \neq 0$. Reparametrizing the field ϕ according to

$$\phi(x) = e^{iq\theta(x)/v}\frac{v + \varphi(x)}{\sqrt{2}} \tag{2-30}$$

with v real and φ hermitian, and accompanying it by a U(1) gauge transformation

$$\begin{aligned} \phi(x) &\mapsto \phi'(x) = e^{-iq\theta(x)/v}\phi(x) = \frac{v + \varphi(x)}{\sqrt{2}} \\ A_\mu(x) &\mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{v}\partial_\mu\theta(x) \end{aligned} \tag{2-31}$$

together with the corresponding transformation of other charged fields $(\psi\ \dots)$, one sees that the Lagrangian $\delta\mathcal{L}$ of (5-4) reads

$$\begin{aligned} \delta\mathcal{L} &= (\partial_\mu - iqA'_\mu)\phi'(\partial^\mu + iqA'_\mu)\phi' - V(\phi'^2) \\ &= \frac{1}{2}|(\partial_\mu - iqA'_\mu)\varphi|^2 + \frac{1}{2}q^2v^2A'_\mu A'^\mu - V\left(\frac{1}{2}(v+\varphi)^2\right) \ . \end{aligned} \tag{2-32}$$

[5]The inverse mass M^{-1} represents the range of weak interactions, which is known to be short, and the mass M is thus high (of the order of 100 GeV, as we see below).

Finally, one sees that the spontaneous breaking of the U(1) symmetry by the boson field ϕ leads to the appearance of a mass term for the gauge field A'_μ ! One also notes that the field θ which, in the absence of a gauge field, would be the Goldstone field, has completely disappeared, being "swallowed" by the new massive ("longitudinal") mode of the vector field A_μ ; the total number of degrees of freedom of these fields is thus not modified: we started with 2 transverse modes of the massless electomagnetic field + 2 modes of the charged field (its real and its imaginary parts, say) and we end up with 3 + 1. This is the Brout–Englert–Higgs mechanism, in its abelian version. If the boson ϕ is coupled to a fermion field ψ by a term of the type $\bar{\psi}\phi\psi$, the appearance of its "vev" gives rise to a mass term $\frac{qv}{\sqrt{2}}\bar{\psi}\psi$ for the ψ.

This Brout–Englert–Higgs mechanism extends to a non-abelian group. The details depend on the scheme of breaking and on the choice of representation for the boson field. (See the courses of second semester for a detailed analysis.) In general, if the group G is broken into a subgroup H, the $r = \dim G - \dim H$ would-be Goldstone bosons, that are in correspondence with generators of the "coset" G/H, become longitudinal massive modes of r vectors. It remains $\dim H$ massless vector fields. Example : in the electro-weak standard model of § 5.3.2 below : $G = \mathrm{SU}(2) \times \mathrm{U}(1)$, $H = \mathrm{U}(1)$ (not the U(1) factor of G!), three gauge fields become massive, one remains massless.

A crucial step in the construction of the standard model was to understand that this spontaneous symmetry breaking in a gauge theory, that we just described at the classical level, is compatible with the quantization of the theory. Renormalizability in 4 dimensions of the gauge theory is not affected by that breaking, and the resulting theory is unitary. Only physical states (massive gauge fields, remaining bosons after the symmetry breaking, etc) contribute to the sum over intermediate states in the unitarity relation.

5.3 The standard model

What is presently called the standard model of particle physics is a gauge theory based on a *non simple gauge group*: $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$, in which the different factors play distinct roles. As the group has three simple factors, the theory depends *a priori* on three independent coupling constants, with gauge fields for each, that are coupled to matter fields, quarks and leptons, as well as to boson fields that play an auxiliary but crucial role!

5.3.1 The strong sector

The group $\mathrm{SU}(3) \equiv \mathrm{SU}(3)_c$ is the gauge group of color (see Chap. 4, § 3.2.). The gauge fields A_μ have indices of the adjoint representation (of dimension 8). The associated particles, called *gluons*, with spin 1 and zero mass, have never been directly observed so far. The gluon fields are coupled to color degrees of freedom of fermionic quark fields, ψ_{Ai}, which carry an index A of the representation 3 (or $\bar{3}$ for the $\bar{\psi}$) (and also a flavor index $i = u, d, s, c, b, t$, on which the color group $\mathrm{SU}(3)_c$ does not act). The theory so defined is Quantum Chromodynamics (QCD in short). It describes the physics of all strong interactions. Its Lagrangian is of the type (2-24),

with fermionic mass terms depending on flavor, generated by the electroweak sector.

Asymptotic freedom

Knowing the coupling constant renormalization , (2-25), one may compute the corresponding beta function. One finds[6]

$$\beta(\mathrm{g}) = -\Lambda\frac{\partial}{\partial\Lambda}\mathrm{g}(\Lambda)|_{\mathrm{g}_0} = -\frac{\mathrm{g}^3}{(4\pi)^2}\left(\frac{11}{3}C_2 - \frac{4}{3}T_f\right) + \mathrm{O}(\mathrm{g}^5) \tag{2-33}$$

It thus appears that this beta function is negative in the vicinity of $\mathrm{g} = 0$, as long as the coefficient $\frac{11}{3}C_2 - \frac{4}{3}T_f > 0$ (not too many matter fields!), in other words that $\mathrm{g} = 0$ is an attractive ultraviolet fixed point of the renormalization group: $\frac{d\mathrm{g}^2(\lambda)}{d\log\lambda} < 0 \Rightarrow \mathrm{g}^2(\lambda) \sim (2b\log\lambda)^{-1} \to 0$ as $\lambda \to \infty$, with $b =$ coefficient of the term $-\mathrm{g}^3$ in (2-33). This is *asymptotic freedom*, a fundamental property of strong interactions. Exercise : how many triplets of quarks are compatible with asymptotic freedom of QCD ?

This non-abelian gauge theory is the only local and renormalizable theory in 4 dimensions that possesses that property of asymptotic freedom. As such, it is the only one consistent with results of deep-inelastic scattering experiments of leptons off hadrons, that reveal the internal structure of the latter as made, at very short distances, of quasi-free constituents.

This $\mathrm{SU}(3)_c$ gauge group is not broken, either explicitly, or spontaneously. This is essential for the consistency of the scenario imagined to account for the quark confinement of quarks and gluons (see Chap. 4, § 3.2.) : non singlet particles of the gauge group are supposed inobservable, as bound to one another inside singlet states and being submitted to forces of growing intensity as one attempts to pull them apart.

This "infrared slavery" (infrared = large distance) is the reciprocal of asymptotic freedom. It shows that confinement is a strong coupling phenomenon, which is by essence non perturbative, namely inaccessible to perturbative calculations.

A non-perturbative approach that has provided many qualitative and quantitative results is the discretization of QCD into a *lattice gauge theory*. This opened the possibility to use methods borrowed to Statistical Mechanics of lattice models, either analytical (strong coupling or high temperature calculations, mean field, ...) or numerical (Monte-Carlo). The confinement scenario seems confirmed in that approach by the study of the expectation value of the Wilson loop defined above (§ 5.1.2). Following the idea of Wilson and Polyakov, for a rectangular loop C of dimensions $T \times R$, $T \gg R$, and carrying the representation σ of the gauge group, $W^{(\sigma)}(C)$ describes the evolution during time T of a pair of static particles (of very high mass), belonging to representation σ, and "frozen" at a relative distance R. One wants to compute the potential between these static charges

$$V_\sigma(R) = -\lim_{T\to\infty}\frac{1}{T}\log W^{(\sigma)}(C) \ .$$

If the Wilson loop has an "area law", $\log W(C) \sim -\kappa RT$, the potential between static charges grows linearly at large distance, $V \sim \kappa R$, which is accord with the idea of confinement. This is what happens in general in a lattice gauge theory at strong coupling, see the Problem at the end of this chapter. The Monte-Carlo computations confirm that this behavior persists at weak couplings, which are relevant for making contact with the continuous theory (the coupling of the lattice theory must be thought of as the effective coupling at the scale of the lattice spacing a, thus, according to asymptotic freedom, $\mathrm{g}_0^2 = \mathrm{g}^2(\Lambda = 1/a) \to 0$). These Monte Carlo computations allow to determine numerically the coefficient κ in V, or *string tension*.

[6]David J. Gross, H. David Politzer, Frank Wilczek, Nobel prize 2004

QCD is still a very active research field. Strong interactions are indeed ubiquitous in particle physics and the observation of any other interaction, of any other effect, assumes a knowledge as precise as possible of the strong contribution. At the time LHC has started to yield data, precise calculations of QCD contributions are of fundamental importance.

5.3.2 The electro-weak sector, a sketch

The group U(1) × SU(2) describes the electro-weak interactions (Glashow–Salam–Weinberg model[7]). Generators of these groups SU(2) and U(1) are referred to as weak isopin and weak hypercharge. We present only the main lines of that construction, without explaining the details nor the reasons that led to choices of groups, of representations etc.

Call A^a_μ, W^i_μ and B_μ the gauge fields of SU(3), SU(2) and U(1) respectively. The *left-handed*, $\psi_L := \frac{1}{2}(1-\gamma_5)\psi$, and *right-handed*, $\psi_R := \frac{1}{2}(1+\gamma_5)\psi$, quarks and leptons are coupled to fields W_μ and B_μ in a different way. One writes the covariant derivative of one of these fields as

$$D_\mu\psi = (\partial_\mu - \mathrm{g}_3 A^a_\mu T_a - \mathrm{g}_2 W^j_\mu t_j - i\frac{\mathrm{g}_1}{2} y B_\mu)\psi \tag{2-34}$$

where the representations assigned to each field, either lepton or quark, left or right, are the triplet representation of $\mathrm{SU}(3)_c$ for quarks and the trivial one for leptons, of course, and for the electroweak part, are given in the following Table

Quarks & Leptons	(ν^e_L, e_L)	ν^e_R	e_R	(u_L, d_L)	u_R	d_R
Weak isospin t_z	$(\frac{1}{2}, -\frac{1}{2})$	0	0	$(\frac{1}{2}, -\frac{1}{2})$	0	0
Weak hypercharge y	$(-1, -1)$	0	-2	$(\frac{1}{3}, \frac{1}{3})$	$\frac{4}{3}$	$-\frac{2}{3}$
Electric charge $Q = \frac{1}{2}y + t_z$	$(0, -1)$	0	-1	$(\frac{2}{3}, -\frac{1}{3})$	$\frac{2}{3}$	$-\frac{1}{3}$

Table 1. Weak quantum numbers of leptons ν^e and e and of quarks u, d.
Things repeat themselves identically for the next generations.

A remarkable consequence of the use of SU(2) as a symmetry group of weak interactions is that, beside the two charged currents $J^{1,2}_\mu$ (or $J^\pm_\mu$) of Fermi theory, a third component J^3_μ appears. This neutral current, which is *not* the electromagnetic current and which is coupled to the gauge field W^μ_3, is necessarily present and contributes for example to the $e^-\nu_\mu \to e^-\nu_\mu$ scattering which is forbidden in Fermi theory. The experimental discovery of these neutral currents (1973)[8] was the first confirmation of the validity of the Standard Model.

The group $\mathrm{U}(1)_{em}$ of electromagnetism is identified thanks to the charges of the fields. There is a "mixing" of the initial U(1) factor and of a U(1) subgroup of SU(2). This mixing is characterized by an angle θ_W, called *Weinberg angle*: if the U(1) and SU(2) gauge fields are denoted B_μ and W_μ respectively, the electromagnetic field is $A^{em}_\mu = \cos\theta_W B_\mu + \sin\theta_W W^0_\mu$, with the orthogonal combination corresponding to another neutral vector field called Z^0.

[7] S. Glashow, A. Salam, S. Weinberg, Nobel prize 1979

[8] The history of that discovery may be read in http://cerncourier.com/cws/article/cern/29168

Let us examine the "neutral current" terms that couple for example the electron and its neutrino to neutral boson vectors W^0 and B. They are read off the covariant derivatives (2-34) with quantum numbers of Table 1

$$\frac{1}{2}i\big[\bar{e}_L(-\mathrm{g}_2W^0-\mathrm{g}_1B)e_L+\bar{e}_R(-2\mathrm{g}_1B)e_R+\bar{\nu}_e(\mathrm{g}_2W^0-\mathrm{g}_1B)\nu_e\big]$$

The rotation $W^0=\cos\theta_W Z^0+\sin\theta_W A$, $B=-\sin\theta_W Z^0+\cos\theta_W A$ must be such that the electric charge e (coupling to A) is the same for e_L and e_R and zero for ν_e. One finds

$$2e=\mathrm{g}_2\sin\theta_W+\mathrm{g}_1\cos\theta_W=2\mathrm{g}_1\cos\theta_W \quad \text{et} \quad \mathrm{g}_2\sin\theta_W-\mathrm{g}_1\cos\theta_W=0$$

which are indeed compatible and give

$$\tan\theta_W=\frac{\mathrm{g}_1}{\mathrm{g}_2} \qquad e=\mathrm{g}_1\cos\theta_W=\mathrm{g}_2\sin\theta_W\ . \tag{2-35}$$

The result of this calculation does not of course depend on the representation in which it is carried. At this stage we have just made a change of parameters, $(\mathrm{g}_1,\mathrm{g}_2)\mapsto(e,\theta_W)$ but the latter are physically observable.

The Lagrangian contains also a coupling to a boson field of spin 0, assumed to be a complex doublet of SU(2) $\Phi=\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$ (weak isospin $\frac{1}{2}$, weak hypercharge $y=+1$), $|D\Phi|^2$ où $D_\mu\Phi=(\partial_\mu-\mathrm{g}_2W^i_\mu\frac{\tau_i}{2}-\frac{1}{2}\mathrm{g}_1B_\mu)\Phi$. The field Φ is endowed with a potential $V(\Phi)$ with a "mexican hat" shape, which is responsible of the spontaneous breaking of $\mathrm{U}(1)\times\mathrm{SU}(2)$ into $\mathrm{U}(1)_{em}$, and hence of the generation of the masses of vector fields according to the mechanism described in § 5.2.2, and also of those of fermions. This *Higgs field* (2 complex components, hence 4 Hermitian ones) has three of its components that disappear, traded for longitudinal modes of massive gauge fields. Only one of these four components remains, and this is that component φ (and the "Higgs particle" it creates) that is actively searched in current experiments ATLAS and CMS at LHC. In parallel, three of the four gauge fields, the $W^\pm$ and the Z^0, become massive, whereas the fourth, the electromagnetic field A remains massless.

The symmetry breaking of $\mathrm{U}(1)\times\mathrm{SU}(2)$ by the field Φ occurs in a direction that preserves $\mathrm{U}(1)_{em}$. (Or more exactly the direction of that breaking determines what is called $\mathrm{U}(1)_{em}$.) One writes, by generalizing (2-30) to the SU(2) group of generators $i\frac{\tau^j}{2}$ ($\tau^j=$ Pauli matrices)

$$\Phi(x)=e^{i\xi_j(x)\frac{\tau^j}{2v}}\begin{pmatrix}0\\ \frac{v+\varphi(x)}{\sqrt{2}}\end{pmatrix}\ ,$$

which is accompanied by a gauge transformation, which causes the fields ξ_j to disappear and gives the fields W and B the quadratic form (mass terms)

$$\mathcal{L}_{(2)}=\frac{1}{8}v^2[(\mathrm{g}_1B-\mathrm{g}_2W^3)^2+\mathrm{g}_2^2((W^1)^2+(W^2)^2)]$$

As expected the component $Z^0=(\mathrm{g}_1B-\mathrm{g}_2W^3)/\sqrt{\mathrm{g}_1^2+\mathrm{g}_2^2}$ becomes massive, and so do $W^{1,2}$, whereas the orthogonal combination $A=(\mathrm{g}_2B+\mathrm{g}_1W^3)/\sqrt{\mathrm{g}_1^2+\mathrm{g}_2^2}$ remains massless. One finds

$$M_{W^\pm}=\frac{1}{2}v\mathrm{g}_2 \qquad M_{Z^0}=\frac{1}{2}v\sqrt{\mathrm{g}_1^2+\mathrm{g}_2^2} \tag{2-36}$$

and using (2-35), the relation $G=\frac{\mathrm{g}_2^2}{8M_W^2}$ read off the Lagrangian and the experimental values of e and of $G=10^{-5}m_p^2$ one computes

$$M_{W^\pm}=\frac{38}{\sin\theta_W}\mathrm{GeV} \qquad M_{Z^0}=\frac{M_W}{\cos\theta_W}=\frac{38}{\sin\theta_W\cos\theta_W}\mathrm{GeV}\ .$$

These expressions then undergo small perturbative corrections. Lastly the mass of the famous and so eagerly expected Higgs boson φ is *not* predicted by the theory but should lie in the range 100–200 GeV. The latest results from LHC seem to exclude most of that range, leaving only a "window" between 120 and 140 GeV for its mass in the above simplest scenario.

The "intermediate bosons" associated with the massive vector fields $W^{\pm}$ and Z^0 have been discovered experimentally at the end of the seventies[9]; their masses $M_{W^{\pm}} = 80.4\,\mathrm{GeV}$ and $M_{Z^0} = 91.2\,\mathrm{GeV}$ are compatible with the following value of the Weinberg angle

$$\sin^2\theta_W \approx 0.23\ , \tag{2-37}$$

which is also compatible with all the other experimental results.

To summarize, the Lagrangian that describes all interactions but gravitation has a remarkably simple and compact form

$$\mathcal{L} = -\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + \sum_{\substack{left\ and\ right\\ quarks\ \&\ leptons}} \bar{\psi}\gamma^\mu D_\mu\psi + |D\Phi|^2 - V(\Phi) + \mathrm{Higgs} - \mathrm{fermions\ couplings}\ , \tag{2-38}$$

where $\mathbf{F}_{\mu\nu}$ denote the gauge field tensors of $\mathbf{A}$, $\mathbf{W}$ and B. Note that the $\mathrm{SU}(2)\times\mathrm{U}(1)$ invariance forbids couplings between left and right fermions (which transform under different representations), and thus forbids fermionic mass terms. The only mass scale lies in $V(\Phi)$, and it is the Higgs mechanism and the coupling of Φ to fermions –leptons and quarks– which give rise to the masses of fermions and of (some of) vector bosons. This coupling, called after Yukawa, has the general form (written for quarks),

$$\mathcal{L}_Y = -Y^d_{ij}\bar{\psi}_{Li}.\Phi\, d_{Rj} - Y^u_{ij}\bar{\psi}_{Li}.\widetilde{\Phi}^\dagger u_{Rj} + \mathrm{h.c.}\ , \tag{2-39}$$

with *a priori* arbitrary matrices $Y^d_{ij},\ Y^u_{ij}$: $i,j = 1,2,3$ are generation indices, the dot denotes the scalar product of isospin doublets Φ and $\widetilde{\Phi}^\dagger = \begin{pmatrix} \phi^{0\dagger} \\ -\phi^{+\dagger} \end{pmatrix}$ with quark doublets

$$\psi_{Li} = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L = \left(\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L \right)\ .$$

Couplings of the same type appear between leptons and scalar fields.

The vev $v/\sqrt{2}$ of ϕ^0 then gives rise to a "mass matrix". A complication of the theory described by (2-38) is that the diagonalization of that quark mass matrix involves a unitary rotation of (u_L, c_L, t_L) and of (d_L, s_L, b_L) with respect to the basis coupled to gauge fields in (2-38) : if (u_L, c_L, t_L) and (d_L, s_L, b_L) stand for the mass eigenstates, the charged hadronic current coupled to the field W^+ is

$$J_\mu = (\bar{u}\bar{c}\bar{t})_L\gamma_\mu M \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \tag{2-40}$$

[9]Carlo Rubbia and Simon van der Meer, Nobel prize 1984

with M the *unitary Cabibbo-Kobayashi-Maskawa matrix*[10]. This mechanism generalizes to 3 generations the mixing with the Cabibbo angle encountered in Chap. 4, (equ. (4-33)) in the case of 2 generations. The matrix M is written as

$$M = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with 4 angles δ and θ_{ij}, ($c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$), and $\theta_{12} = \theta_C =$ Cabibbo angle. Experimentally, $0 \ll \theta_{13} \ll \theta_{23} \ll \theta_{12} \ll \pi/2$. The accurate measure of the matrix elements of M is presently the object of an intense experimental activity, in connection with the study of violations of the CP symmetry (due to a large extent to the phase $e^{i\delta}$) and of "flavor oscillations".

For a much more comprehensive discussion of details and achievments of the standard model, see the courses of the 2nd semester.

5.4 Complements

5.4.1 Standard Model and beyond

The Standard Model is both remarkably well verified and not very satisfactory. Beside massive neutrinos, whose existence is now beyond any doubt, and which request little amendments to the Lagrangian (2-38), no significative disagreement has been found to this day between experimental results and predictions of the model. Still, the non satisfactory aspects of the standard model are numerous: the excessively large number (about twenty) of free parameters in the model, the lack of "naturalness" in the way certain terms have to be adjusted in a very fine way; the question of the Higgs boson whose discovery is expected at LHC, but that many physicists regard as an *ad hoc* construction; etc.

Attempts at improving the standard model by fusing the three gauge groups within a larger group, in a "grand-unified" (GUT) theory should be mentionned. The next susbsection is devoted to that issue.

The currently most popular extensions of the standard model are those based on supersymmetry. The "MSSM", ("*Maximally Supersymmetric (extension of the) Standard Model*"), or the "NMSSM" ("*Next-to ...*"), resolve the hierarchy problem, predict a convergence of electroweak and strong couplings and also the existence of supersymmetric partners for all known particles. For confirmation of this scheme, people eagerly await results from LHC...

5.4.2 Grand-unified theories or GUTs

An empirical observation is that the three coupling constants g_1, g_2, g_3, starting from their value experimentally values measured at current energies, seem to converge under the renormalization group flow to a common value at some energy of the order of 10^{15} or ^{16}GeV. This was a very strong incentive towards a grand-unification, see

[10] M. Kobayashi, T. Maskawa, Nobel prize 2008, avec Y. Nambu

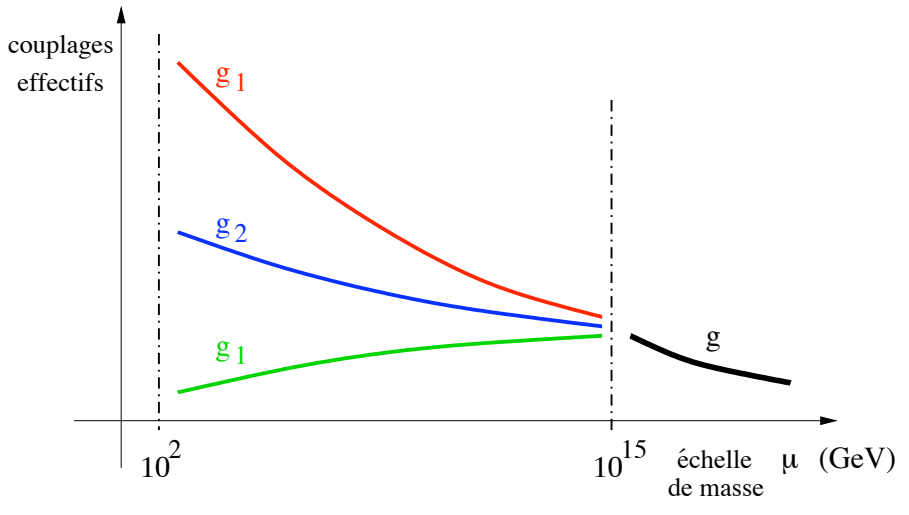


Figure 5.2: Schematic evolutions of the three effective couplings of the standard model and of that of a grand-unified theory

Fig. 5.2. The resulting grand-unified theory should not only be a gauge theory with a single coupling if the unification group G is simple, but also be capable of predicting the matter field and particle content according to the representations of SU(3)× SU(2)× U(1) from some representations of the group G. For various reasons, the group SU(5) turns out to be the best candidate. This GUT possesses dim SU(5)= 24 gauge fields.

The main reason of that choice of SU(5) comes from the number of chiral fermions by generation. Each generation of the standard model contains two quark flavors coming each in 3 colors, plus one lepton, and each of these 6+1 fields may have two chiralities, plus a neutrino assumed to be massless and chiral. In total there are 15 chiral fermions per generation. (Remember that the antiparticle of a right fermion is left-handed: it is thus sufficient to consider left fermions.) One thus seeks a simple group G possessing a representation (reducible or irreducible) of dimension 15, that may accomodate all left-handed fermions of each generation. The only candidate is finally the group SU(5) which has representations of dimension 15 : the symmetric tensor representation, and representations sums of 5 (or $\bar{5}$) and 10 (or $\overline{10}$).

The group SU(5) of unitary 5×5 matrices contains a SU(3) subgroup (3×3 submatrices of the upper left corner, say) and a SU(2) subgroup (2×2 blocks of the lower right corner), which give the corresponding generators of $\mathrm{SU}(3) \times \mathrm{SU}(2)$; the U(1) subgroup is generated by the diagonal traceless matrix diag $(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$. It is clear that these three subgroups commute with one another.

One must then decompose all fields (in representations 5, 10, 15 and 24) into representations of SU(3) × SU(2). This exercise shows that representation 15 must be discarded and that the reducible representation $\bar{5} \oplus 10$ is the appropriate one for fermion fields: the $\bar{5}$ decomposes into representations $(\bar{3}, 1) \oplus (1, 2)$ and contains antiquarks $\bar{d}_L$ and left leptons e_L^- and ν_e ; the 10 decomposes into $(1, 1) \oplus (3, 2) \oplus (\bar{3}, 1)$ containing the left lepton e_L^+, singlet of SU(2) and of SU(3), the two left quarks u_L, d_L which form a doublet of SU(2) and the antiquarks $\bar{u}_L$.

Likewise, the 24 gauge fields include the 8 gluon fields, the 3+1 vectors of the electroweak sector, plus 12 supplementary fields, which acquire a very large mass at the expected breakdown of $\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

The breakdown $\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ should take place at a grand-unification energy of the order of 10^{15} or 10^{16} GeV, an energy where couplings $\mathrm{g}_3, \mathrm{g}_2, \mathrm{g}_1$ of SU(3), SU(2) and U(1) seem to converge (Fig. 5.2). The infinitesimal generators being now rigidly bound within the simple group SU(5), one may related the couplings to the U(1) and SU(2) gauge fields and predict the Weinberg angle: one finds $\sin^2 \theta = \frac{3}{8}, \ldots$ but this calculation applies to the unification energy ! The angle is renormalized between that energy and energies of

current experiments.

A striking consequence of the quarks–leptons unification within SU(5) multiplets is a violation of separate conservations of lepton and baryon numbers. In particular the existence of interaction terms, for example $X^\rho(\bar{d}\gamma_\rho e^+ + \bar{u}^c\gamma_\rho u)$, with one of the new gauge fields (the matrices of the generators have been omitted), allows proton decay $p = duu \to d\bar{d}e^+ = \pi^0 e^+$, and by other channels as well. The decay rate must be carefully computed to see if it is consistent with experimental data on proton lifetime (present bound $10^{32\pm1}$ years), ... which is not the case !

One should also show to which representation the Higgs boson fields belong to permit a two-step breaking $\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{SU}(3) \times \mathrm{U}(1)$ with two very different breakings...

Finally, the SU(5) GUT

- incorporates by construction the structure of fermion generations;
- puts leptons and quarks in the same representation and explains the commensurability of their electric charges and the cancellation of anomalies (see discussion below);
- reduces the number of parameters in the standard model and predicts the value of the Weinberg angle (at the unification scale);
- does not explain the why of the three observed generations;
- does not elucidate the question of "naturalness" (just evoked above), nor the related issue of "hierarchy" (why is the ratio M_{GUT}/M_W so large?) ;
- last but not least, fatal disease, predicts effects such as the proton decay at rates seemingly inconsistent with observation.

This is the latter point that led to abandon this unification scheme and to favor supersymmetric routes to unification.

5.4.3 Anomalies

We mentioned in Chap. 4 the existence of chiral anomalies, that affect the axial current $J^{(5)}_\mu$ of the classical U(1) symmetry. In the gauge theory of the Standard Model, the electroweak gauge fields are coupled differently to left-handed and right-handed fermions, more precisely, they are coupled to axial currents, see the Lagrangian

$$\mathcal{L} = i\bar{\psi}(\partial\!\!\!/ - A\!\!\!/)\frac{(1-\gamma_5)}{2}\psi$$

which contains a term $A^\mu_a J_{\mu a}$ with $J_{\mu a} = \bar{\psi}T_a\frac{(1-\gamma_5)}{2}\psi$. Classically that current $J_{\mu a}$ should have a vanishing covariant derivative (in the adjoint representation) if the fermions are massless. One may again compute the (covariant) divergence of that current to the one-loop order, and one finds that

$$D_\mu J^\mu = \frac{i}{24\pi^2}\partial_\mu \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,T_a(A_\nu\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma) \ .$$

Curiously the right hand side is not gauge invariant, but its forms is not arbitrary and is dictated by geometric considerations ("descent equations") that are beyond the scope of the present discussion. The anomaly of this "non-singlet" current (*i.e.* carrying a non-trivial representation of the gauge group) thus breaks gauge invariance. As such, it jeopardizes all the consistency, renormalisability and unitarity, of the theory. One conceives that controlling this anomaly is crucial for the construction of a physically sensible theory.

Then one observes that the "group theoretical coefficient" of the anomaly is proportional to

$$d_{abc} = \mathrm{tr}\,(T_a\{T_b, T_c\})$$

where $\{T_b, T_c\}$ is the anticommutator of infinitesimal generators, see Exercise B.3.

In practice one ensures the anomaly cancellation in two cases:

- a) Suppose that the fermions all belong to real or pseudoreal representations. One recalls (see Chap 2) that this refers to situations where the representation is (unitarily) equivalent to its complex conjugate representation, $T_a^* = CT_aC^{-1}$. In unitary representations the T_a are antihermitian, $T_a = -T_a^\dagger = -T_a^{T\,*}$. One then verifies (see Exercise B.3) that the group theoretical factor $d_{abc} = -d_{abc} = 0$ vanishes and so does the anomaly. Thus (4-dimensional) theories with gauge group SU(2) (in which all representations are real or pseudoreal) have no anomaly.
- b) Another situation is that there is cancellation of anomalies coming from different fermion representations. This is what takes place in the standard model. According to the argument of a), there is no anomaly associated with the weak isospin currents, coupled to an SU(2) gauge field. But there may *a priori* be some with weak hypercharge currents (U(1) group), as well as mixed anomalies, for example one U(1) current and two SU(2) etc. One must thus check that for all choices of three generators labelled by a, b, c, the constant d_{abc} vanishes when one sums over all fermion representations. Finally one shows that it reduces to the vanishing of $\mathrm{tr}\,(t_3^2 Q)$ for each generation, which is indeed satisfied in the Standard Model. This is also what happens for the SU(5) theory discussed in the previous section: one shows that for each generation, contributions of representations $\bar{5}$ and 10 cancel one another.

$\star$

Further references for Chapter 5

On geometric aspects of gauge theory and an introduction to the theory of fiber bundles, see for example M. Daniel and C. Viallet, *The geometric setting of gauge theories of the Yang-Mills type*, *Rev. Mod. Phys.* **52** (1980) 175-197.

On gauge theories, Yang-Mills, the standard model, etc, one may consult any book of quantum field theory later than 1975, for example [IZ], [PS], [Wf], [Z-J].

On group theoretical aspects of gauge theories, voir L. O'Raifeartaigh, *op. cit.*.

A very good review of grand-unification is given in *Introduction to unified theories of weak, electromagnetic and strong interactions - SU(5)*, A. Billoire and A. Morel, rapport Saclay DPh-T/80/068.

For a historical review of these developments, see *Broken Symmetries*, note of the Nobel Committee, 2008,
http://nobelprize.org/nobel_prizes/physics/laureates/2008/phyadv08.pdf

For a detailed review of the Standard Model and a compilation of all known properties of elementary particles, see *The Review of Particle Physics*, on
http://pdg.lbl.gov/ already cited in Chap. 4.

Exercises for chapter 5

A. Non abelian gauge field

1. Complete the proofs of (5-21) et (5-22).

2. Let A be a non abelian gauge field and F its field tensor. Show that the covariant derivative of F is such that

$$D_{\mu a}{}^{b} F_{\nu\rho b} t^a = [D_\mu, F_{\nu\rho}] = \partial_\mu F_{\nu\rho} - [A_\mu, F_{\nu\rho}] \ .$$

Prove the identity

$$[D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] = 0 \, .$$

Recall what is the abelian version of that identity and its interpretation.

3. Consider the operator $\not{D} = \not{\partial} - \not{A}$ acting on Dirac fermions in a representation R. One wants to compute $\not{D}^2$. Writing $D_\mu D_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} D_\mu D_\nu \{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[D_\mu, D_\nu]\gamma^\mu\gamma^\nu$, show that one may write $\not{D}^2$ as a sum of $D^2 = D_\mu D^\mu$ and of a term of the form $a F_{\mu\nu}\sigma^{\mu\nu}$, where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Compute a.

B. Group theoretical factors...

1. *Casimir operators*

Let G be a simple compact Lie group of dimension d, R one of its representations, that one may assume irreducible and unitary. Let t_a be a basis of the Lie algebra $\mathfrak{g}$ of G, T_a its representatives in representation R. The t_a and T_a are chosen antihermitian. One then considers the bilinear form on the Lie algebra defined by

$$(X, Y)^{(R)} = \mathrm{tr}\,(T_a T_b) x^a y^b$$

if $X = x^a t_a$ and $Y = y^b t_b \in \mathfrak{g}$ (with summation over repeated indices).

- Prove that this form is invariant in the sense that

$$\forall Z \in \mathfrak{g} \quad ([X, Z], Y)^{(R)} + (X, [Y, Z])^{(R)} = 0 \ .$$

  One recalls that any invariant bilinear form on a simple Lie algebra is a multiple of the Killing form.

- b) Prove that one may choose a basis of t_a and hence of T_a such that

$$\mathrm{tr}\,(T_a T_b) = -T_R \delta_{ab}$$

  with T_R a coefficient that depends on the representation.

- c) What is the sign of T_R ?

- d) Consider then the quadratic Casimir operator

$$C_2^{(R)} = -\sum_\alpha (T_a)^2 \ .$$

  On how many values of a does one sum in that expression?

- e) Recall why $C_2^{(R)}$ is a multiple of the identity in the representation space of R

$$C_2^{(R)} = c_2(R)\,\mathbb{I}\ .$$

- f) Why are the assumptions of simplicity of G and of irreductibility of R important for that result?

- g) What is the sign of $c_2(R)$? Justify.

- h) Show that T_R is related to the value $c_2(R)$ of the quadratic Casimir operator. For that purpose, one may compute in two different ways the quantity

$$\mathrm{tr}\sum_a (T_a)^2\ .$$

- i) To what does this relation boil down for the adjoint representation adjoint of G ?

- j) Normalize the (antihermitian) generators of SU(N) in such a way that in the defining representation, $\mathrm{tr}\,T_aT_b = -\frac{1}{2}\delta_{ab}$, thus $T_f = \frac{1}{2}$. Is this verified by infinitesimal generators $i\frac{\sigma_a}{2}$ of SU(2) ? What is then the value of c_2 in that defining representation?

2. *Computation of traces and of Casimir operators in representations of SU(N)*

- a) Show that the expression (3-49) of Chap. 3 $c_2(\Lambda) = \frac{1}{2}\langle\,\Lambda, \Lambda + 2\rho\,\rangle$ may be rewritten as $c_2(\Lambda) = \frac{1}{2}\left(\langle\,\Lambda+\rho, \Lambda+\rho\,\rangle - \langle\,\rho,\rho\,\rangle\right)$, thus for SU($N$), using expressions (3-47) and (3-60) of the same chapter

$$c_2(\Lambda) = \frac{1}{2N}\sum_{i=1}^{N-1}\left([(\lambda_i+1)^2-1]i(N-i) + 2\sum_{j=i+1}^{N-1}[(\lambda_i+1)(\lambda_j+1)-1]i(N-j)\right)\ .$$

- b) Compute that expression for the defining representation. Does the result agree with that found in question 1.j) above?

- c) Recall why the highest weight of the adjoint representation is the highest root (denoted θ in the Appendix F of Chap 3). Is the expression $\theta = \Lambda_1 + \Lambda_{N-1}$ in accord with what is known on the adjoint representation?

- d) Calculate the value of $c_2(\Lambda)$ for the adjoint representation.

- e) Check this value for SU(2) by a direct calculation of $c_2(\mathrm{adj})$.

- f) What is the value of T_{adj} in SU(N), that follows as a consequence of question 1.i) ?

3. *Anomaly coefficients*

We keep the same notations and conventions as above.

- a) In the computation of some Feynman diagrams in a gauge theory of group G, one encounters the coefficient

$$d_{\alpha\beta\gamma} = \text{tr } (T_\alpha(T_\beta T_\gamma + T_\gamma T_\beta)) \ .$$

  Show that $d_{\alpha\beta\gamma}$ is completely symmetric in its three indices.

- b) We recall that a representation is said to be *real* or *pseudoreal* if it is (unitarily) equivalent to its complex conjugate, hence if in a basis where the T_α are antihermitian, one may find a unitary matrix U such that the complex conjugate of each T_α verifies

$$(T_\alpha)^* = U T_\alpha U^{-1} \ .$$

  Show that if that condition is satisfied, $d_{\alpha\beta\gamma}$ vanishes identically. That condition is important to ensure the consistency of gauge theory, this is the condition of *anomaly cancellation.*

- c) Is the spin $\frac{1}{2}$ representation of SU(2) real or pseudoreal ? That of spin j ? Justify your answer.

- d) Give two examples of (non necessarily irreducible) non trivial representations of SU(3) that are real or pseudoreal, and two that are not.

- e) What is the coefficient d for the U(1) groupe and a representation of charge q ?

C. Spontaneous breaking of SU(2) gauge symmetry

Consider an SU(2) gauge theory coupled to a boson field $\vec{\Phi}$ of spin 1, considered as a vector of dimension 3. The potential of that field is denoted $V(\vec{\Phi}^2)$.

1. Write the Lagrangian and the gauge transformations of the fields $\vec{A}_\mu$ and $\vec{\Phi}$.
2. We suppose that the symmetry is spontaneously broken: the field Φ acquires a vev v along some direction, say 3 : $\langle \vec{\Phi} \rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$. What is the residual group of symmetry? What will be the effect of the field A_μ ? Give a description of the fields and physical particles after symmetry breaking.

Problem. Lattice gauge theory

In the following, G denotes a compact Lie group, $\chi^{(\rho)}$ the character of its irreducible unitary representation ρ.

1. Show that the orthogonality relations of $\mathcal{D}^{(\rho)}$ imply the following formulas:

$$\int_G \frac{d\mu(g)}{v(G)} \chi^{(\rho)}(g.g_1.g^{-1}.g_2) = \frac{1}{n_\rho} \chi^{(\rho)}(g_1)\chi^{(\rho)}(g_2) \ , \tag{2-41}$$

and

$$\int_G \frac{d\mu(g)}{v(G)} \chi^{(\rho)}(g.g_1)\chi^{(\sigma)}(g^{-1}.g_2) = \frac{\delta_{\rho,\sigma}}{n_\rho}\chi^{(\rho)}(g_1.g_2) \ . \tag{2-42}$$

Recall why a representation of G may always be regarded as unitary and show that then

$$\chi^{(\rho)}(g^{-1}) = \chi^{(\bar{\rho})}(g) = (\chi^{(\rho)}(g))^* \ , \tag{2-43}$$

where $\bar{\rho}$ is the complex conjugate representation of ρ.
We make a frequent use of these three relations in the following.

2. Let χ be the character of a real representation r (not necessarily irreducible) of G, β a real parameter.
a) Show that one may expand $\exp \beta\chi(g)$ on characters of irreducible representations of G according to

$$e^{\beta\chi(g)} = \sum_\rho n_\rho b_\rho \chi^{(\rho)}(g) \ ,$$

with functions $b_\rho(\beta)$.
Express the function $b_\rho(\beta)$ in terms of a group integral.
Using (2-43), show that the functions $b_\rho(\beta)$ are real, $b_\rho(\beta) = (b_\rho(\beta))^* = b_{\bar{\rho}}(\beta)$.
b) Show that b_ρ is non vanishing provided representation ρ appears in some tensor power $r^{\otimes n}$.
c) For $G = \mathrm{SU}(2)$ and $r = (j = \frac{1}{2})$, the representation of spin $\frac{1}{2}$, is condition **b)** satisfied for any ρ ? Why ?
If $r = (j = 1)$, what are the representations for which b_ρ is a priori zero ?
d) For $G =$SU(3) and $\chi = \chi^{(3)} + \chi^{(\bar{3})}$, show that b_ρ is non zero for all ρ.
For $\beta \to 0$, what is the leading behaviour of $b_a(\beta)$ when $\beta \to 0$ if a denotes the adjoint representation of SU(3) ? More generally what is the leading behaviour of $b_\rho(\beta)$ where ρ is the representation of highest weight $\Lambda = (\lambda_1, \lambda_2)$?

3. One defines a model of statistical mechanics in d dimensions in the following way. On a hypercubic lattice of dimension d and of lattice spacing a, the degrees of freedom are attached to links (edges) between neighbouring sites and take their value in the compact group G. With each oriented link $\ell = \vec{ij}$ one associates the element of G denoted $g_\ell = g_{ij}$, with $-\ell = \vec{ji}$, one associates $g_{ji} = g_\ell^{-1}$. With each elementary square (alias "plaquette") $p = ijkl$, one associates the product of the link elements :

$$g_p = g_{ij}.g_{jk}.g_{kl}.g_{li}$$

and the "energy" of a configuration of these variables is given by

$$E = -\sum_{\text{plaquettes } p} \chi(g_p) \tag{2-44}$$

where χ is, like in question 2, the character of some real representation r of the group. The Boltzmann weight is thus

$$e^{-\beta E} = \prod_p e^{\beta\chi(g_p)} \qquad , \qquad \beta = \frac{1}{kT}$$

and the partition function reads

$$Z = \prod_{\text{links } \ell} \int_G \frac{\mathrm{d}\,\mu(g_\ell)}{v(G)} \prod_{\text{plaquettes}} e^{\beta\chi(g_p)} \ . \tag{2-45}$$

a) Show that the energy E is invariant by redefinition of g_{ij} as $g_{ij} \mapsto g_i.g_{ij}.g_j^{-1}$, where $g_i \in G$, (this is a *local* invariance, the analogue in that discrete formalism of the gauge invariance studied in this chapter), and that E does not depend on the orientation of plaquettes.
b) One wants to understand the relation with the formalism of § 5.1. The degrees of freedom g_{ij} represent the path-variables defined in (5-20), $g_{ij} = g(j,i)$ along the edge from i to site j

$$g_{ij} \equiv P \exp \int_{l=\vec{ij}} A_\mu dx^\mu$$

- For a small lattice spacing a, show, by using for example the BCH formula and by expanding to the first non vanishing order that

  $$g_p = \exp\left(a^2 F_{\mu\nu} + \mathrm{o}(a^2)\right)$$

  where μ and ν denote the directions of the edges of plaquette p. (One is here interested in an Euclidean version of gauge theory, and position of indices μ, ν is irrelevant.) Show then that the energy E_p (2-44) reads

  $$E_p \sim \text{const. } a^4 (F_{\mu\nu})^2 + \text{const.}'$$

  where the first constant will be determined as a function of the representation r chosen for χ.

- Explain why the parameter β identifies (up to a factor) with the inverse of coupling g^2 in the continuous gauge theory. In fact this is rather the "bare (or unrenormalized) coupling constant", why ?

One first restricts oneself for simplicity to $d = 2$ dimensions. For a finite lattice of $\mathcal{N}$ plaquettes, for example a rectangle of size $L_1 \times L_2$ (see Fig.5.3), one wants to calculate Z. One chooses "free boundary conditions", in other words variables g_ℓ on the boundary of the rectangle are independent. One is also interested in the expectation value $W^{(\sigma)}(C)$ of $\chi^{(\sigma)}(g_{_C})$ where $g_{_C}$ is the ordered product of g_ℓ along a closed oriented curve C for some irreducible representation σ of G

$$W^{(\sigma)}(C) := \langle\, \chi^{(\sigma)}(g_{_C}) \,\rangle = \frac{1}{Z} \prod_{\text{liens } \ell} \int_G \frac{\mathrm{d}\,\mu(g_l)}{v(G)} \chi^{(\sigma)}\Big(\prod_{\ell \in C} g_\ell\Big) \prod_p e^{\beta\chi(g_p)} \ . \tag{2-46}$$

d) Using the results of question 2 show that one may expand each $\exp \beta\chi(g_p)$ on characters of irreducible representations of G according to

$$e^{\beta\chi(g_p)} = \sum_\rho n_\rho b_\rho\, \chi^{(\rho)}(g_p) \ . \tag{2-47}$$

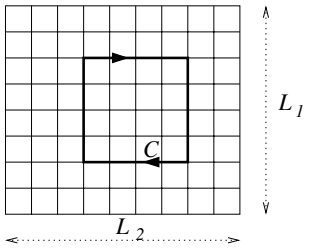


Figure 5.3: Square lattice in 2 d

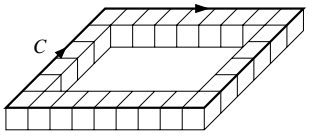


Figure 5.4: A tube configuration contributing to the Wilson loop

e) One inserts in (2-45) or (2-46) the expansion (2-47) for each plaquette. Show that if two plaquettes share one link ℓ, formulas of part 1 permit an integration over the variable g_ℓ of that link and that the two representations carried by the two adjacent plaquettes are then identical. Using repeatedly these formulas of part 1, show that one may integrate over all variables g_ℓ and that

$$Z = b_1^{\mathcal{N}} \qquad\qquad W^{(\sigma)}(C) = n_\sigma \left(\frac{b_\sigma}{b_1}\right)^A \tag{2-48}$$

where A is the area of the curve C, *i.e.* the number of plaquettes it encompasses, and the index 1 refers to the identity representation.

f) One now consider the case of dimension $d = 3$. Variables g_ℓ are attached to links of a cubic lattice. Energy is again given (2-44), where the sum runs over all plaquettes of this 3-dimensional lattice. As before, $W^{(\sigma)}(C) = \langle\, \chi^{(\sigma)}(g_C)\,\rangle$ receives contributions from plaquette configurations that form a surface bounded by C.

Let us show that contributions to the Wilson loop $W^{(\sigma)}(C)$ may also come from plaquette configurations forming a tube resting on the contour C (Fig. 5.4).

– Show that for such a configuration, the repeated application of formulas (2-41) and (2-42) on all variables g_ℓ leads to the following expression

$$W^{(\sigma)}(C)\Big|_{\text{tube}} = \sum_\rho \left(\frac{b_\rho}{b_1}\right)^P \int_G \frac{d\mu(g)}{v(G)} \chi^{(\rho)}(g)\chi^{(\rho)}(g^{-1})\chi^{(\sigma)}(g) \tag{2-49}$$

where P is the number of plaquettes making the tube.

– Under which condition $\mathcal{C}$ on representation σ of the loop C is the contribution of representation ρ to the right hand side of (2-49) non vanishing?

– Give an example for G=SU(2) of representations σ for which this condition $\mathcal{C}$ is *never* satisfied for any ρ, and hence these tubular configurations do not contribute.

– Inversely give an example (again for SU(2)) of a possible choice of σ which satisfies it.

We admit that at high temperature, (small β), the dominant contribution to $W^{(\sigma)}(C)$ is of type (2-49) if condition $\mathcal{C}$ may be satisfied, and of type (2-48) in the opposite case.

4. The evaluation of the expectation value of the Wilson loop $W^{(\sigma)}(C)$ in the limit of a large loop C which is a rectangle $R \times T$ allows to compute the potential $V_\sigma(R)$ between two static "charged" particles separated by distance R, one carrying representation σ of the group and the other one being its antiparticle. More precisely we admit that

$$V_\sigma(R) = -\lim_{T\to\infty} \frac{1}{T} \log\, W^{(\sigma)}(C) \ .$$

Evaluate the dependence of $V_\sigma(R)$ in R which follows either from (2-48), or from the contribution to (2-49) due to representation ρ. What do you conclude on the interaction between the two particles in those two situations?

Physically, this kind of considerations gives a discrete (lattice) and simplified (2 or 3 dimensions) model of QCD. One may repeat these calculations in higher dimension, where the above result appear as the leading term in a small β ("high temperature") expansion. The fact that $W^{(\sigma)}(C)$ decays like x^A ($x = b_\sigma/b_1 < 1$ for β small enough) for large areas is a signal of quark confinement in that theory, that is of the impossibility to separate a pair quark-antiquark at large distance ...

$\star$