



M2/International Centre for Fundamental Physics  
Parcours of Physique Théorique

# Invariances in Physics and Group Theory

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Niels Henrik Abel  
1802 - 1829



Elie Cartan  
1869 - 1951



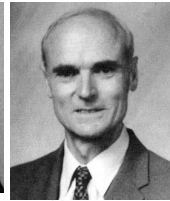
Hendrik Casimir  
1909-2000



Claude Chevalley  
1909 - 1984



Rudolf F. A. Clebsch  
1833 - 1872



Harold S. M. Coxeter  
1907 - 2003



Eugene B. Dynkin  
1924 -



Hans Freudenthal  
1905 - 1990



Ferdinand Frobenius  
1849 - 1917



Paul Albert Gordan  
1837 - 1912



Alfred Haar  
1885 - 1933



Sir William R. Hamilton  
1805 - 1865



Wilhelm K. J. Killing  
1847 - 1923



Sophus Lie  
1842 - 1899



Dudley E. Littlewood  
1903 - 1979



Hendrik A. Lorentz  
1853 - 1928



Hermann Minkowski  
1864 - 1909



Emmy A. Noether  
1882 - 1935



Henri Poincaré  
1854 - 1912



Archibald R. Richardson  
1881 - 1954



Olinde Rodrigues  
1795-1851



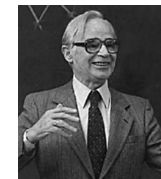
Issai Schur  
1875 - 1941



Jean-Pierre Serre  
1926 -



Miguel Virasoro  
1940-



Bartel van der Waerden  
1903 - 1996



André Weil  
1906 - 1998



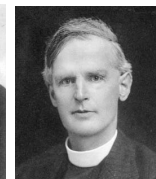
Hermann Weyl  
1885 - 1955



Eugene P. Wigner  
1902 - 1995



Ernst Witt  
1911 - 1991



Alfred Young  
1873 - 1940

Some of the major actors of group theory mentioned in the first part of these notes

## Foreword

The following notes cover the content of the course “Invariances in Physique and Group Theory” given in the fall 2013. Additional lectures were given during the week of “prérentrée” on the  $SO(3)$ ,  $SU(2)$ ,  $SL(2, \mathbb{C})$  groups, see below Chap. 0.

Chapters 1 to 5 also contain, in sections in smaller characters and Appendices, additional details that are not treated in the oral course.

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# Chapter 0

## Some basic elements on the groups $\text{SO}(3)$ , $\text{SU}(2)$ and $\text{SL}(2, \mathbb{C})$

### 0.1 Rotations of $\mathbb{R}^3$ , the groups $\text{SO}(3)$ and $\text{SU}(2)$

#### 0.1.1 The group $\text{SO}(3)$ , a 3-parameter group

Let us consider the rotation group in three-dimensional Euclidean space. These rotations leave invariant the squared norm of any vector  $\mathbf{OM}$ ,  $\mathbf{OM}^2 = x_1^2 + x_2^2 + x_3^2 = x^2 + y^2 + z^2$ <sup>1</sup> and preserve orientation. They are represented in an orthonormal bases by  $3 \times 3$  orthogonal real matrices, of determinant 1 : they form the “special orthogonal” group  $\text{SO}(3)$ .

*Olinde Rodrigues formula*

Any rotation of  $\text{SO}(3)$  is a rotation by some angle  $\psi$  around an axis colinear to a unit vector  $\mathbf{n}$ , and the rotations associated with  $(\mathbf{n}, \psi)$  and  $(-\mathbf{n}, -\psi)$  are identical. We denote  $R_{\mathbf{n}}(\psi)$  this rotation. In a very explicit way, one writes  $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp} = (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + (\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n})$  and  $\mathbf{x}' = \mathbf{x}_{\parallel} + \cos \psi \mathbf{x}_{\perp} + \sin \psi \mathbf{n} \times \mathbf{x}_{\perp}$ , whence Rodrigues formula

$$\mathbf{x}' = R_{\mathbf{n}}(\psi)\mathbf{x} = \cos \psi \mathbf{x} + (1 - \cos \psi)(\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \sin \psi (\mathbf{n} \times \mathbf{x}) . \quad (0.1)$$

As any unit vector  $\mathbf{n}$  in  $\mathbb{R}^3$  depends on two parameters, for example the angle  $\theta$  it makes with the  $Oz$  axis and the angle  $\phi$  of its projection in the  $Ox, Oy$  plane with the  $Ox$  axis (see Fig. 1) an element of  $\text{SO}(3)$  is parametrized by 3 continuous variables. One takes

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \psi \leq \pi . \quad (0.2)$$

But there remains an innocent-looking redundancy,  $R_{\mathbf{n}}(\pi) = R_{-\mathbf{n}}(\pi)$ , the consequences of which we see later ...

---

<sup>1</sup>In this chapter, we use alternately the notations  $(x, y, z)$  or  $(x_1, x_2, x_3)$  to denote coordinates in an orthonormal frame.

$SO(3)$  is thus a dimension 3 manifold. For the rotation of axis  $\mathbf{n}$  colinear to the  $Oz$  axis, we have the matrix

$$\mathcal{R}_z(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0.3)$$

whereas around the  $Ox$  and  $Oy$  axes

$$\mathcal{R}_x(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{pmatrix} \quad \mathcal{R}_y(\psi) = \begin{pmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{pmatrix}. \quad (0.4)$$

*Conjugation of  $R_{\mathbf{n}}(\psi)$  by another rotation*

A relation that we are going to use frequently reads

$$\boxed{RR_{\mathbf{n}}(\psi)R^{-1} = R_{\mathbf{n}'}(\psi)} \quad (0.5)$$

where  $\mathbf{n}'$  is the transform of  $\mathbf{n}$  by rotation  $R$ ,  $\mathbf{n}' = R\mathbf{n}$  (check it!). Conversely any rotation of angle  $\psi$  around a vector  $\mathbf{n}'$  can be cast under the form (0.5) : we'll say later that the "conjugation classes" of the group  $SO(3)$  are characterized by the angle  $\psi$ .

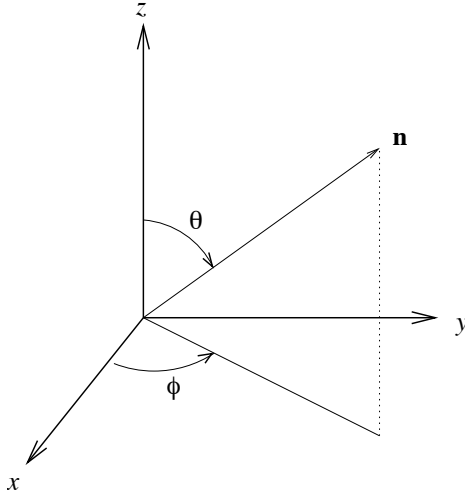


**Fig. 1**

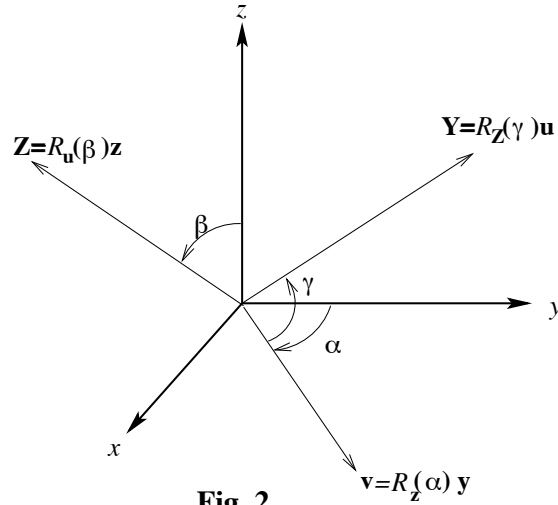


**Fig. 2**

*Euler angles*

Another description makes use of *Euler angles* : given an orthonormal frame  $(Ox, Oy, Oz)$ , any rotation around  $O$  that maps it onto another frame  $(OX, OY, OZ)$  may be regarded as resulting from the composition of a rotation of angle  $\alpha$  around  $Oz$ , which brings the frame onto  $(Ou, Ov, Oz)$ , followed by a rotation of angle  $\beta$  around  $Ov$  bringing it on  $(Ou', Ov, OZ)$ , and lastly, by a rotation of angle  $\gamma$  around  $OZ$  bringing the frame onto  $(OX, OY, OZ)$ , (see Fig. 2). One thus takes  $0 \leq \alpha < 2\pi$ ,  $0 \leq \beta \leq \pi$ ,  $0 \leq \gamma < 2\pi$  and one writes

$$R(\alpha, \beta, \gamma) = R_Z(\gamma)R_v(\beta)R_z(\alpha) \quad (0.6)$$

but according to (0.5)

$$R_Z(\gamma) = R_v(\beta)R_z(\gamma)R_v^{-1}(\beta) \quad R_v(\beta) = R_z(\alpha)R_y(\beta)R_z^{-1}(\alpha)$$

thus, by inserting into (0.6)

$$R(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma) . \quad (0.7)$$

where one used the fact that  $R_z(\alpha)R_z(\gamma)R_z^{-1}(\alpha) = R_z(\gamma)$  since rotations around a given axis commute (they form an abelian subgroup, isomorphic to  $SO(2)$ ).

Exercise : using (0.5), write the expression of a matrix  $R$  which maps the unit vector  $\mathbf{z}$  colinear to  $Oz$  to the unit vector  $\mathbf{n}$ , in terms of  $R_z(\phi)$  and  $R_y(\theta)$  ; then write the expression of  $R_{\mathbf{n}}(\psi)$  in terms of  $R_y$  and  $R_z$ . Write the explicit expression of that matrix and of (0.7) and deduce the relations between  $\theta, \phi, \psi$  and Euler angles. (See also below, equ. (0.66).)

### 0.1.2 From $SO(3)$ to $SU(2)$

Consider another parametrization of rotations. To the rotation  $R_{\mathbf{n}}(\psi)$ , we associate the unitary 4-vector  $u : (u_0 = \cos \frac{\psi}{2}, \mathbf{u} = \mathbf{n} \sin \frac{\psi}{2})$ ; we have  $u^2 = u_0^2 + \mathbf{u}^2 = 1$ , and  $u$  belongs to the unit sphere  $S^3$  in the space  $\mathbb{R}^4$ . Changing the determination of  $\psi$  by an odd multiple of  $2\pi$  changes  $u$  into  $-u$ . There is thus a bijection between  $R_{\mathbf{n}}(\psi)$  and the pair  $(u, -u)$ , *i.e.* between  $SO(3)$  and  $S^3/\mathbb{Z}_2$ , the sphere in which diametrically opposed points are identified. We shall say that the sphere  $S^3$  is a “covering group” of  $SO(3)$ . In which sense is this sphere a group? To answer that question, introduce Pauli matrices  $\sigma_i$ ,  $i = 1, 2, 3$ .

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (0.8)$$

Together with the identity matrix  $\mathbf{I}$ , they form a basis of the vector space of  $2 \times 2$  Hermitian matrices. They satisfy the identity

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{I} + i \epsilon_{ijk} \sigma_k , \quad (0.9)$$

with  $\epsilon_{ijk}$  the completely antisymmetric tensor,  $\epsilon_{123} = +1$ ,  $\epsilon_{ijk}$  = the signature of permutation  $(ijk)$ .

From  $u$  a real unit 4-vector unitary (*i.e.* a point of  $S^3$ ), we form the matrix

$$U = u_0 \mathbf{I} - i \mathbf{u} \cdot \boldsymbol{\sigma} \quad (0.10)$$

which is unitary and of determinant 1 (check it and also show the converse: any unimodular (= of determinant 1) unitary  $2 \times 2$  matrix is of the form (0.10), with  $u^2 = 1$ ). These matrices form the *special unitary* group  $SU(2)$  which is thus isomorphic to  $S^3$ . By a power expansion of the exponential and making use of  $(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = \mathbf{I}$ , a consequence of (0.9), one may verify that

$$e^{-i \frac{\psi}{2} \mathbf{n} \cdot \boldsymbol{\sigma}} = \cos \frac{\psi}{2} - i \sin \frac{\psi}{2} \mathbf{n} \cdot \boldsymbol{\sigma} . \quad (0.11)$$

It is then suggested that the multiplication of matrices

$$U_n(\psi) = e^{-i \frac{\psi}{2} \mathbf{n} \cdot \boldsymbol{\sigma}} = \cos \frac{\psi}{2} - i \sin \frac{\psi}{2} \mathbf{n} \cdot \boldsymbol{\sigma}, \quad 0 \leq \psi \leq 2\pi, \quad \mathbf{n} \in S^2 \quad (0.12)$$

gives the desired group law in  $S^3$ . Let us show indeed that to a matrix of  $SU(2)$  one may associate a rotation of  $SO(3)$  and that to the product of two matrices of  $SU(2)$  corresponds the product of the  $SO(3)$  rotations (this is the *homomorphism* property). To the point  $x$  of  $\mathbb{R}^3$  of coordinates  $x_1, x_2, x_3$ , we associate the Hermitian matrix

$$X = \mathbf{x} \cdot \boldsymbol{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \quad (0.13)$$

with conversely  $x_i = \frac{1}{2} \text{tr}(X \sigma_i)$ , and we let  $SU(2)$  act on that matrix according to

$$X \mapsto X' = UXU^\dagger, \quad (0.14)$$

which defines a linear transform  $x \mapsto x' = \mathcal{T}x$ . One readily computes that

$$\det X = -(x_1^2 + x_2^2 + x_3^2) \quad (0.15)$$

and as  $\det X = \det X'$ , the linear transform  $x \mapsto x' = \mathcal{T}x$  is an isometry, hence  $\det \mathcal{T} = 1$  or  $-1$ . To convince oneself that this is indeed a rotation, *i.e.* that the transformation has a determinant 1, it suffices to compute that determinant for  $U = \mathbf{I}$  where  $\mathcal{T}$  = the identity, hence  $\det \mathcal{T} = 1$ , and then to invoke the connexity of the manifold  $SU(2) (\cong S^3)$  to conclude that the continuous function  $\det \mathcal{T}(U)$  cannot jump to the value  $-1$ . In fact, using identity (0.9), the explicit calculation of  $X'$  leads, after some algebra, to

$$\begin{aligned} X' &= (\cos \frac{\psi}{2} - i \mathbf{n} \cdot \boldsymbol{\sigma} \sin \frac{\psi}{2}) X (\cos \frac{\psi}{2} + i \mathbf{n} \cdot \boldsymbol{\sigma} \sin \frac{\psi}{2}) \\ &= (\cos \psi \mathbf{x} + (1 - \cos \psi)(\mathbf{x} \cdot \mathbf{n}) \mathbf{n} + \sin \psi (\mathbf{n} \times \mathbf{x})) \cdot \boldsymbol{\sigma} \end{aligned} \quad (0.16)$$

which is nothing else than the Rodrigues formula (0.1). We thus conclude that the transformation  $x \mapsto x'$  performed by the matrices of  $SU(2)$  in (0.14) is indeed the rotation of angle  $\psi$  around  $\mathbf{n}$ . To the product  $U_{n'}(\psi')U_n(\psi)$  in  $SU(2)$  corresponds in  $SO(3)$  the composition of the two rotations  $R_{\mathbf{n}'}(\psi')R_{\mathbf{n}}(\psi)$  of  $SO(3)$ . There is thus a “homomorphism” of the group  $SU(2)$  into  $SO(3)$ . This homomorphism maps the two matrices  $U$  and  $-U$  onto one and the same rotation of  $SO(3)$ .

Let us summarize what we have learnt in this section. The group  $SU(2)$  is a covering group (of order 2) of the group  $SO(3)$  (the precise topological meaning of which will be given in Chap. 1), and the 2-to-1 homomorphism from  $SU(2)$  to  $SO(3)$  is given by equations (0.12)-(0.14).

Exercise : prove that any matrix of  $SU(2)$  may be written as  $\begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix}$  with  $|a|^2 + |b|^2 = 1$ . What is the connection with (0.10) ?

## 0.2 Infinitesimal generators. The $su(2)$ Lie algebra

### 0.2.1 Infinitesimal generators of $SO(3)$

Rotations  $R_{\mathbf{n}}(\psi)$  around a given axis  $\mathbf{n}$  form a *one-parameter subgroup*, isomorphic to  $SO(2)$ . In this chapter, we follow the common use (among physicists) and write the infinitesimal generators

of rotations as *Hermitian* operators  $J = J^\dagger$ . Thus

$$R_{\mathbf{n}}(d\psi) = (I - id\psi J_{\mathbf{n}}) \quad (0.17)$$

where  $J_{\mathbf{n}}$  is the “generator” of these rotations, a Hermitian  $3 \times 3$  matrix. Let us first show that we may reconstruct the finite rotations from these infinitesimal generators. By the group property,

$$R_{\mathbf{n}}(\psi + d\psi) = R_{\mathbf{n}}(d\psi)R_{\mathbf{n}}(\psi) = (I - id\psi J_{\mathbf{n}})R_{\mathbf{n}}(\psi) , \quad (0.18)$$

or equivalently

$$\frac{\partial R_{\mathbf{n}}(\psi)}{\partial \psi} = -iJ_{\mathbf{n}}R_{\mathbf{n}}(\psi) \quad (0.19)$$

which, on account of  $R_{\mathbf{n}}(0) = I$ , may be integrated into

$$\boxed{R_{\mathbf{n}}(\psi) = e^{-i\psi J_{\mathbf{n}}} .} \quad (0.20)$$

To be more explicit, introduce the three basic  $J_1$ ,  $J_2$  and  $J_3$  describing the infinitesimal rotations around the corresponding axes<sup>2</sup>. From the infinitesimal version of (0.3) it follows that

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (0.21)$$

which may be expressed by a unique formula

$$\boxed{(J_k)_{ij} = -i\epsilon_{ijk}} \quad (0.22)$$

with the completely antisymmetric tensor  $\epsilon_{ijk}$ .

We now show that matrices (0.21) form a basis of infinitesimal generators and that  $J_{\mathbf{n}}$  is simply expressed as

$$J_{\mathbf{n}} = \sum_k J_k n_k \quad (0.23)$$

which allows us to rewrite (0.20) in the form

$$\boxed{R_{\mathbf{n}}(\psi) = e^{-i\psi \sum_k n_k J_k} .} \quad (0.24)$$

The expression (0.23) follows simply from the infinitesimal form of Rodrigues formula,  $R_{\mathbf{n}}(d\psi) = (I + d\psi \mathbf{n} \times)$  hence  $-iJ_{\mathbf{n}} = \mathbf{n} \times$  or alternatively  $-i(J_{\mathbf{n}})_{ij} = \epsilon_{ikj}n_k = n_k(-iJ_k)_{ij}$ , q.e.d. (Here and in the following, we make use of the convention of summation over repeated indices:  $\epsilon_{ikj}n_k \equiv \sum_k \epsilon_{ikj}n_k$ , etc.)

A comment about (0.24): it is obviously wrong to write in general  $R_{\mathbf{n}}(\psi) = e^{-i\psi \sum_k n_k J_k} \stackrel{?}{=} \prod_{k=1}^3 e^{-i\psi n_k J_k}$  because of the non commutativity of the  $J$ 's. On the other hand, formula (0.7) shows that any rotation of  $SO(3)$  may be written under the form

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3} . \quad (0.25)$$

---

<sup>2</sup>Do not confuse  $J_{\mathbf{n}}$  labelled the unit vector  $\mathbf{n}$  with  $J_k$ ,  $k$ -th component of  $\mathbf{J}$ . The relation between the two will be explained shortly.

The three matrices  $J_i$ ,  $i = 1, 2, 3$  satisfy the very important commutation relations

$$\boxed{[J_i, J_j] = i\epsilon_{ijk}J_k} \quad (0.26)$$

which follow from the identity (Jacobi) verified by the tensor  $\epsilon$

$$\epsilon_{iab}\epsilon_{bjc} + \epsilon_{icb}\epsilon_{baj} + \epsilon_{ijb}\epsilon_{bca} = 0 . \quad (0.27)$$

Exercise: note the structure of this identity ( $i$  is fixed,  $b$  summed over, cyclic permutation over the three others) and check that it implies (0.26).

In view of the importance of relations (0.23–0.26), it may be useful to recover them by another route. Note first that equation (0.5) implies that for any  $R$

$$Re^{-i\psi J_{\mathbf{n}}}R^{-1} = e^{-i\psi RJ_{\mathbf{n}}R^{-1}} = e^{-i\psi J_{\mathbf{n}'}} \quad (0.28)$$

with  $\mathbf{n}' = R\mathbf{n}$ , whence

$$RJ_{\mathbf{n}}R^{-1} = J_{\mathbf{n}'} . \quad (0.29)$$

The tensor  $\epsilon_{ijk}$  is invariant under rotations

$$\epsilon_{lmn}\mathcal{R}_{il}\mathcal{R}_{jm}\mathcal{R}_{kn} = \epsilon_{ijk} \det \mathcal{R} = \epsilon_{ijk} \quad (0.30)$$

since the matrix  $\mathcal{R}$  is of determinant 1. That matrix being also orthogonal, one may push one  $\mathcal{R}$  to the right-hand side

$$\epsilon_{lmn}\mathcal{R}_{jm}\mathcal{R}_{kn} = \epsilon_{ijk}\mathcal{R}_{il} \quad (0.31)$$

which thanks to (0.22) expresses that

$$\mathcal{R}_{jm}(J_l)_{mn}\mathcal{R}_{nk}^{-1} = (J_l)_{jk}\mathcal{R}_{il} \quad (0.32)$$

*i.e.* for any  $R$  and its matrix  $\mathcal{R}$ ,

$$RJ_lR^{-1} = J_i\mathcal{R}_{il} . \quad (0.33)$$

Let  $R$  be a rotation which maps the unit vector  $\mathbf{z}$  colinear to  $Oz$  on the vector  $\mathbf{n}$ , thus  $n_k = \mathcal{R}_{k3}$  and

$$J_{\mathbf{n}} \stackrel{(0.29)}{=} RJ_3R^{-1} \stackrel{(0.33)}{=} J_k\mathcal{R}_{k3} = J_k n_k , \quad (0.34)$$

which is just (0.23). Note that equations (0.33) and (0.34) are compatible with (0.29)

$$J_{\mathbf{n}'} \stackrel{(0.29)}{=} RJ_{\mathbf{n}}R^{-1} \stackrel{(0.34)}{=} RJ_k n_k R^{-1} \stackrel{(0.33)}{=} J_l \mathcal{R}_{lk} n_k = J_l n'_l .$$

As we shall see later in a more systematic way, the commutation relation (0.26) of infinitesimal generators  $J$  encodes an infinitesimal version of the group law. Consider for example a rotation of infinitesimal angle  $d\psi$  around  $Oy$  acting on  $J_1$

$$R_2(d\psi)J_1R_2^{-1}(d\psi) \stackrel{(0.33)}{=} J_k[\mathcal{R}_2(d\psi)]_{k1} \quad (0.35)$$

but to first order,  $R_2(d\psi) = \mathbf{I} - id\psi J_2$ , and thus the left hand side of (0.35) equals  $J_1 - id\psi[J_2, J_1]$  while on the right hand side,  $[\mathcal{R}_2(d\psi)]_{k1} = \delta_{k1} - id\psi(J_2)_{k1} = \delta_{k1} - d\psi\delta_{k3}$  by (0.22), whence  $i[J_1, J_2] = -J_3$ , which is one of the relations (0.26).

### 0.2.2 Infinitesimal generators of $SU(2)$

Let us examine now things from the point of view of  $SU(2)$ . Any unitary matrix  $U$  (here  $2 \times 2$ ) may be diagonalized by a unitary change of basis  $U = V \exp\{i \operatorname{diag}(\lambda_k)\} V^\dagger$ ,  $V$  unitary, and hence written as

$$U = \exp iH = \sum_0^\infty \frac{(iH)^n}{n!} \quad (0.36)$$

with  $H$  Hermitian,  $H = V \operatorname{diag}(\lambda_k) V^\dagger$ . The sum converges (for the norm  $\|M\|^2 = \operatorname{tr} MM^\dagger$ ). The unimodularity condition  $1 = \det U = \exp i \operatorname{tr} H$  is ensured if  $\operatorname{tr} H = 0$ . The set of such Hermitian traceless matrices forms a vector space  $\mathcal{V}$  of dimension 3 over  $\mathbb{R}$ , with a basis given by the three Pauli matrices

$$H = \sum_{k=1}^3 \eta_k \frac{\sigma_k}{2}, \quad (0.37)$$

which may be inserted back into (0.36). (In fact we already observed that any unitary  $2 \times 2$  matrix may be written in the form (0.11)). Comparing that form with (0.24), or else comparing its infinitesimal version  $U_n(d\psi) = (I - i d\psi \mathbf{n} \cdot \frac{\boldsymbol{\sigma}}{2})$  with (0.17), we see that matrices  $\frac{1}{2}\sigma_j$  play in  $SU(2)$  the role played by infinitesimal generators  $J_j$  in  $SO(3)$ . But these matrices  $\frac{1}{2}\sigma$  verify the same commutation relations

$$\left[ \frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i \epsilon_{ijk} \frac{\sigma_k}{2} \quad (0.38)$$

with the same *structure constants*  $\epsilon_{ijk}$  as in (0.26). In other words, we have just discovered that infinitesimal generators  $J_i$  (eq. (0.21) of  $SO(3)$  and  $\frac{1}{2}\sigma_i$  of  $SU(2)$ ) satisfy the same commutation relations (we shall say later that they are the bases of two different representations of the same Lie algebra  $su(2) = so(3)$ ). This has the consequence that calculations carried out with the  $\frac{1}{2}\vec{\sigma}$  and *making only use of commutation relations* are also valid with the  $\vec{J}$ , and vice versa. For instance, from (0.33), for example  $R_2(\beta) J_k R_2^{-1}(\beta) = J_l \mathcal{R}_y(\beta)_{lk}$ , it follows immediately, with no further calculation, that for Pauli matrices, we have

$$e^{-i\frac{\beta}{2}\sigma_2} \sigma_k e^{i\frac{\beta}{2}\sigma_2} = \sigma_l \mathcal{R}_y(\beta)_{lk} \quad (0.39)$$

where the matrix elements  $\mathcal{R}_y$  are read off (0.4). Indeed there is a general identity stating that  $e^A B e^{-A} = B + \sum_{n=1}^\infty \frac{1}{n!} \underbrace{[A, [\dots, [A, B] \dots]]}_{n \text{ commutators}}$ , see Chap. 1, eq. (1.29), and that computation thus involves only commutators. On the other hand, the relation

$$\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k$$

(which does *not* involve only commutators) is specific to the dimension 2 representation of the  $su(2)$  algebra.

### 0.2.3 Lie algebra $su(2)$

Let us recapitulate: we have just introduced the commutation algebra (or Lie algebra) of infinitesimal generators of the group  $SU(2)$  (or  $SO(3)$ ), denoted  $su(2)$  or  $so(3)$ . It is defined by

relations (0.26), that we write once again

$$[J_i, J_j] = i\epsilon_{ijk}J_k . \quad (0.26)$$

We shall also make frequent use of the three combinations

$$J_z \equiv J_3, \quad J_+ = J_1 + iJ_2, \quad J_- = J_1 - iJ_2 . \quad (0.40)$$

It is then immediate to compute

$$\begin{aligned} [J_3, J_+] &= J_+ \\ [J_3, J_-] &= -J_- \\ [J_+, J_-] &= 2J_3 . \end{aligned} \quad (0.41)$$

One also verifies that the *Casimir operator* defined as

$$\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 = J_3^2 + J_3 + J_-J_+ \quad (0.42)$$

commutes with all the  $J$ 's

$$[\mathbf{J}^2, J_i] = 0 , \quad (0.43)$$

which means that it is invariant under rotations.

Anticipating a little on the following, we shall be mostly interested in “unitary representations”, where the generators  $J_i$ ,  $i = 1, 2, 3$  are Hermitian, hence

$$J_i^\dagger = J_i, \quad i = 1, 2, 3 \quad J_\pm^\dagger = J_\mp . \quad (0.44)$$

Let us finally mention an interpretation of the  $J_i$  as differential operators acting on differentiable functions of coordinates in the space  $\mathbb{R}^3$ . In that space  $\mathbb{R}^3$ , an infinitesimal rotation acting on the vector  $\mathbf{x}$  changes it into

$$\mathbf{x}' = R\mathbf{x} = \mathbf{x} + \delta\psi\mathbf{n} \times \mathbf{x}$$

hence a *scalar* function of  $\mathbf{x}$ ,  $f(\mathbf{x})$ , is changed into  $f'(\mathbf{x}') = f(\mathbf{x})$  or

$$\begin{aligned} f'(\mathbf{x}) &= f(R^{-1}\mathbf{x}) = f(\mathbf{x} - \delta\psi\mathbf{n} \times \mathbf{x}) \\ &= (1 - \delta\psi\mathbf{n} \cdot \nabla) f(\mathbf{x}) \\ &= (1 - i\delta\psi\mathbf{n} \cdot \mathbf{J})f(\mathbf{x}) . \end{aligned} \quad (0.45)$$

We thus identify

$$\mathbf{J} = -i\mathbf{x} \times \nabla, \quad J_i = -i\epsilon_{ijk}x_j \frac{\partial}{\partial x_k} \quad (0.46)$$

which allows us to compute it in arbitrary coordinates, for example spherical, see Appendix 0. (Compare also (0.46) with the expression of (orbital) angular momentum in Quantum Mechanics  $L_i = \frac{\hbar}{i}\epsilon_{ijk}x_j \frac{\partial}{\partial x_k}$ ). Exercise: check that these differential operators do satisfy the commutation relations (0.26).

Among the combinations of  $J$  that one may construct, there is one that must play a particular role, namely the Laplacian on the sphere  $S^2$ , a second order differential operator which is invariant under changes of coordinates (see Appendix 0). It is in particular rotation invariant, of degree 2 in the  $J$ ., this may only be the Casimir operator  $\mathbf{J}^2$  (up to a factor). In fact the Laplacian in  $\mathbb{R}^3$  reads in spherical coordinates

$$\begin{aligned} \Delta_3 &= \frac{1}{r} \frac{\partial^2}{\partial r^2} r - \frac{\mathbf{J}^2}{r^2} \\ &= \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\Delta_{\text{sphere } S^2}}{r^2} . \end{aligned} \quad (0.47)$$

For the sake of simplicity we have restricted this discussion to scalar functions, but one might more generally consider the transformation of a collection of functions “forming a representation” of SO(3), *i.e.* transforming linearly among themselves under the action of that group

$$A'(\mathbf{x}') = D(R)A(\mathbf{x})$$

or else

$$A'(\mathbf{x}) = D(R)A(R^{-1}\mathbf{x}) ,$$

for example a vector field transforming as

$$\mathbf{A}'(\mathbf{x}) = R\mathbf{A}(R^{-1}\mathbf{x}) .$$

What are now the infinitesimal generators for such objects ? Show that they now have two contributions, one given by (0.46) and the other coming from the infinitesimal form of  $R$ ; in physical terms, these two contributions correspond to the orbital and to the intrinsic (spin) angular momenta.

## 0.3 Representations of SU(2)

### 0.3.1 Representations of the groups SO(3) and SU(2)

We are familiar with the notions of vectors or tensors in the geometry of the space  $\mathbb{R}^3$ . They are objects that transform linearly under rotations

$$V_i \mapsto \mathcal{R}_{ii'} V_{i'} \quad (V \otimes W)_{ij} = V_i W_j \mapsto \mathcal{R}_{ii'} \mathcal{R}_{jj'} (V \otimes W)_{i'j'} = \mathcal{R}_{ii'} \mathcal{R}_{jj'} V_{i'} W_{j'} \quad \text{etc.}$$

More generally we call *representation* of a group  $G$  in a vector space  $E$  a homomorphism of  $G$  into the group  $\text{GL}(E)$  of linear transformations of  $E$  (see Chap. 2). Thus, as we just saw, the group SO(3) admits a representation in the space  $\mathbb{R}^3$  (the vectors  $V$  of the above example), another representation in the space of rank 2 tensors, etc. We now want to build the general representations of SO(3) and SU(2). For the needs of physics, in particular of quantum mechanics, we are mostly interested in unitary representations, in which the representation matrices are unitary. In fact, as we'll see, it is enough to study the representations of SU(2) to also get those of SO(3), and even better, it is enough to study the way the group elements close to the identity are represented, *i.e.* to find the representations of the infinitesimal generators of SU(2) (and SO(3)).

To summarize : to find all the unitary representations of the group SU(2), it is thus sufficient to find the representations by Hermitian matrices of its Lie algebra  $su(2)$ , that is, Hermitian operators satisfying the commutation relations (0.26).

### 0.3.2 Representations of the algebra $su(2)$

We now proceed to the classical construction of representations of the algebra  $su(2)$ . As above,  $J_{\pm}$  and  $J_z$  denote the representatives of infinitesimal generators in a certain representation. They thus satisfy the commutation relations (0.41) and hermiticity (0.44). Commutation of operators  $J_z$  and  $\mathbf{J}^2$  ensures that one may find common eigenvectors. The eigenvalues of these

Hermitian operators are real, and moreover,  $\mathbf{J}^2$  being semi-definite positive, one may always write its eigenvalues in the form  $j(j+1)$ ,  $j$  real non negative (*i.e.*  $j \geq 0$ ), and one thus considers a common eigenvector  $|j m\rangle$

$$\begin{aligned}\mathbf{J}^2|j m\rangle &= j(j+1)|j m\rangle \\ J_z|j m\rangle &= m|j m\rangle ,\end{aligned}\tag{0.48}$$

with  $m$  a real number, a priori arbitrary at this stage. By a small abuse of language, we call  $|j m\rangle$  an “eigenvector of eigenvalues  $(j, m)$ ”.

(i) Act with  $J_+$  and  $J_- = J_+^\dagger$  on  $|j m\rangle$ . Using the relation  $J_\pm J_\mp = \mathbf{J}^2 - J_z^2 \pm J_z$  (a consequence of (0.41)), the squared norm of  $J_\pm|j m\rangle$  is computed to be:

$$\begin{aligned}\langle j m|J_-J_+|j m\rangle &= (j(j+1) - m(m+1)) \langle j m|j m\rangle \\ &= (j-m)(j+m+1) \langle j m|j m\rangle \\ \langle j m|J_+J_-|j m\rangle &= (j(j+1) - m(m-1)) \langle j m|j m\rangle \\ &= (j+m)(j-m+1) \langle j m|j m\rangle .\end{aligned}\tag{0.49}$$

These squared norms cannot be negative and thus

$$\begin{aligned}(j-m)(j+m+1) &\geq 0 & : & \quad -j-1 \leq m \leq j \\ (j+m)(j-m+1) &\geq 0 & : & \quad -j \leq m \leq j+1\end{aligned}\tag{0.50}$$

which implies

$$-j \leq m \leq j .\tag{0.51}$$

Moreover  $J_+|j m\rangle = 0$  iff  $m = j$  and  $J_-|j m\rangle = 0$  iff  $m = -j$

$$J_+|j j\rangle = 0 \quad J_-|j -j\rangle = 0 .\tag{0.52}$$

(ii) If  $m \neq j$ ,  $J_+|j m\rangle$  is non vanishing, hence is an eigenvector of eigenvalues  $(j, m+1)$ . Indeed

$$\begin{aligned}\mathbf{J}^2 J_+|j m\rangle &= J_+ \mathbf{J}^2|j m\rangle = j(j+1) J_+|j m\rangle \\ J_z J_+|j m\rangle &= J_+(J_z + 1)|j m\rangle = (m+1) J_+|j m\rangle .\end{aligned}\tag{0.53}$$

Likewise if  $m \neq -j$ ,  $J_-|j m\rangle$  is a (non vanishing) eigenvector of eigenvalues  $(j, m-1)$ .

(iii) Consider now the sequence of vectors

$$|j m\rangle, J_-|j m\rangle, J_-^2|j m\rangle, \dots, J_-^p|j m\rangle \dots$$

If non vanishing, they are eigenvectors of  $J_z$  of eigenvalues  $m, m-1, m-2, \dots, m-p \dots$ . As the allowed eigenvalues of  $J_z$  are bound by (0.51), this sequence must stop after a finite number of steps. Let  $p$  be the integer such that  $J_-^p|j m\rangle \neq 0$ ,  $J_-^{p+1}|j m\rangle = 0$ . By (0.52),  $J_-^p|j m\rangle$  is an eigenvector of eigenvalues  $(j, -j)$  hence  $m-p = -j$ , *i.e.*

$$(j+m) \text{ is a non negative integer} .\tag{0.54}$$

Acting likewise with  $J_+, J_+^2, \dots$  sur  $|j m\rangle$ , we are led to the conclusion that

$$(j - m) \text{ is a non negative integer .} \quad (0.55)$$

and thus  $j$  and  $m$  are simultaneously integers or half-integers. For each value of  $j$

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

$m$  may take the  $2j + 1$  values <sup>3</sup>

$$m = -j, -j + 1, \dots, j - 1, j . \quad (0.56)$$

Starting from the vector  $|j m = j\rangle$ , (“highest weight vector”), now chosen of norm 1, we construct the orthonormal basis  $|j m\rangle$  by iterated application of  $J_-$  and we have

$$\begin{aligned} J_+ |j m\rangle &= \sqrt{j(j+1) - m(m+1)} |j m+1\rangle \\ J_- |j m\rangle &= \sqrt{j(j+1) - m(m-1)} |j m-1\rangle \\ J_z |j m\rangle &= m |j m\rangle . \end{aligned} \quad (0.57)$$

These  $2j + 1$  vectors form a basis of the “spin  $j$  representation” of the  $su(2)$  algebra.

In fact this representation of the algebra  $su(2)$  extends to a representation of the group  $SU(2)$ , as we now show.

Remark. The previous discussion has given a central role to the unitarity of the representation and hence to the hermiticity of infinitesimal generators, hence to positivity:  $\|J_\pm |j m\rangle\|^2 \geq 0 \implies -j \leq m \leq j$ , etc, which allowed us to conclude that the representation is necessarily of finite dimension. Conversely one may insist on the latter condition, and show that it suffices to ensure the previous conditions on  $j$  and  $m$ . Starting from an eigenvector  $|\psi\rangle$  of  $J_z$ , the sequence  $J_+^p |\psi\rangle$  yields eigenvectors of  $J_z$  of increasing eigenvalue, hence linearly independent, as long as they do not vanish. If by hypothesis the representation is of finite dimension, this sequence is finite, and there exists a vector denoted  $|j\rangle$  such that  $J_+ |j\rangle = 0$ ,  $J_z |j\rangle = j |j\rangle$ . By the relation  $\mathbf{J}^2 = J_- J_+ + J_z(J_z + 1)$ , it is also an eigenvector of eigenvalue  $j(j+1)$  of  $\mathbf{J}^2$ . It thus identifies with the highest weight vector denoted previously  $|j j\rangle$ , a notation that we thus adopt in the rest of this discussion. Starting from this vector, the  $J_-^p |j j\rangle$  form a sequence that must also be finite

$$\exists q \quad J_-^{q-1} |j j\rangle \neq 0 \quad J_-^q |j j\rangle = 0 . \quad (0.58)$$

One easily shows by induction that

$$J_+ J_-^q |j j\rangle = [J_+, J_-^q] |j j\rangle = q(2j + 1 - q) J_-^{q-1} |j j\rangle = 0 \quad (0.59)$$

hence  $q = 2j + 1$ . The number  $j$  is thus integer or half-integer, the vectors of the representation built in that way are eigenvectors of  $\mathbf{J}^2$  of eigenvalue  $j(j+1)$  and of  $J_z$  of eigenvalue  $m$  satisfying (0.56). We have recovered all the previous results. In this form, the construction of these “highest weight representations” generalizes to other Lie algebras, (even of infinite dimension, such as the Virasoro algebra, see Chap. 1, § 1.3.6).

The matrices  $\mathcal{D}^j$  of the spin  $j$  representation are such that under the action of the rotation  $U \in SU(2)$

$$|j m\rangle \mapsto D^j(U) |j m\rangle = |j m'\rangle \mathcal{D}_{m'm}^j(U) . \quad (0.60)$$

---

<sup>3</sup>In fact, we have just found a necessary condition on the  $j, m$ . That all these  $j$  give indeed rise to representations will be verified in the next subsection.

Depending on the parametrization ((**n**,  $\psi$ ), angles d'Euler, ...), we write  $\mathcal{D}_{m'm}^j(\mathbf{n}, \psi)$ ,  $\mathcal{D}_{m'm}^j(\alpha, \beta, \gamma)$ , etc. By (0.7), we thus have

$$\begin{aligned}\mathcal{D}_{m'm}^j(\alpha, \beta, \gamma) &= \langle j m' | D(\alpha, \beta, \gamma) | j m \rangle \\ &= \langle j m' | e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} | j m \rangle \\ &= e^{-i\alpha m'} d_{m'm}^j(\beta) e^{-i\gamma m}\end{aligned}\quad (0.61)$$

where the Wigner matrix  $d^j$  is defined by

$$d_{m'm}^j(\beta) = \langle j m' | e^{-i\beta J_y} | j m \rangle. \quad (0.62)$$

An explicit formula for  $d^j$  will be given in the next subsection. We also have

$$\begin{aligned}\mathcal{D}_{m'm}^j(\mathbf{z}, \psi) &= e^{-i\psi m} \delta_{mm'} \\ \mathcal{D}_{m'm}^j(\mathbf{y}, \psi) &= d_{m'm}^j(\psi).\end{aligned}\quad (0.63)$$

Exercise : Compute  $\mathcal{D}^j(\mathbf{x}, \psi)$ . (Hint : use (0.5).)

One notices that  $\mathcal{D}^j(\mathbf{z}, 2\pi) = (-1)^{2j} I$ , since  $(-1)^{2m} = (-1)^{2j}$  using (0.55), and this holds true for any axis **n** by the conjugation (0.5)

$$\mathcal{D}^j(\mathbf{n}, 2\pi) = (-1)^{2j} I. \quad (0.64)$$

This shows that a  $2\pi$  rotation in SO(3) is represented by  $-I$  in a half-integer-spin representation of SU(2). Half-integer-spin representations of SU(2) are said to be “projective”, (*i.e.* here, up to a sign), representations of SO(3); we return in Chap. 2 to this notion of projective representation.

We also verify the *unimodularity* of matrices  $\mathcal{D}^j$  (or equivalently, the fact that representatives of infinitesimal generators are traceless). If  $\mathbf{n} = R\mathbf{z}$ ,  $\mathcal{D}(\mathbf{n}, \psi) = \mathcal{D}(R)\mathcal{D}(\mathbf{z}, \psi)\mathcal{D}^{-1}(R)$ , hence

$$\det \mathcal{D}(\mathbf{n}, \psi) = \det \mathcal{D}(\mathbf{z}, \psi) = \det e^{-i\psi J_z} = \prod_{m=-j}^j e^{-im\psi} = 1. \quad (0.65)$$

It may be useful to write explicitly these matrices in the cases  $j = \frac{1}{2}$  and  $j = 1$ . The case of  $j = \frac{1}{2}$  is very simple, since

$$\begin{aligned}\mathcal{D}^{\frac{1}{2}}(U) &= U = e^{-i\frac{1}{2}\psi \mathbf{n} \cdot \boldsymbol{\sigma}} = \begin{pmatrix} \cos \frac{\psi}{2} - i \cos \theta \sin \frac{\psi}{2} & -i \sin \frac{\psi}{2} \sin \theta e^{-i\phi} \\ -i \sin \frac{\psi}{2} \sin \theta e^{i\phi} & \cos \frac{\psi}{2} + i \cos \theta \sin \frac{\psi}{2} \end{pmatrix} \\ &= e^{-i\frac{\alpha}{2}\sigma_3} e^{-i\frac{\beta}{2}\sigma_2} e^{-i\frac{\gamma}{2}\sigma_3} = \begin{pmatrix} \cos \frac{\beta}{2} e^{-i\frac{\alpha+\gamma}{2}} & -\sin \frac{\beta}{2} e^{-i\frac{\alpha-\gamma}{2}} \\ \sin \frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}} & \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}} \end{pmatrix}\end{aligned}\quad (0.66)$$

an expected result since the matrices  $U$  of the group form obviously a representation. (As a by-product, we have derived relations between the two parametrizations, (**n**,  $\psi$ ) = ( $\theta$ ,  $\phi$ ,  $\psi$ ) and Euler angles ( $\alpha$ ,  $\beta$ ,  $\gamma$ ).) For  $j = 1$ , in the basis  $|1, 1\rangle$ ,  $|1, 0\rangle$  and  $|1, -1\rangle$  where  $J_z$  is diagonal (which *is not* the basis (0.21) !)

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad J_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad J_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (0.67)$$

whence

$$d^1(\beta) = e^{-i\beta J_y} = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \quad (0.68)$$

as the reader may check.

In the following subsection, we write more explicitly these representation matrices of the group SU(2), and in Appendix E of Chap. 2, give more details on the differential equations they satisfy and on their relations with “special functions”, orthogonal polynomials and spherical harmonics. . .

### *Irreducibility*

A central notion in the study of representations is that of irreducibility. A representation is irreducible if it has no invariant subspace. Let us show that the spin  $j$  representation of SU(2) that we have just built is irreducible. We show below in Chap. 2 that, as the representation is unitary, it is either irreducible or “completely reducible” (there exists an invariant subspace and its supplementary space is also invariant) ; in the latter case, there would exist block-diagonal operators, different from the identity and commuting the matrices of the representation, in particular with the generators  $J_i$ . But in the basis (0.5) any matrix  $M$  that commutes with  $J_z$  is diagonal,  $M_{mm'} = \mu_m \delta_{mm'}$ , (check it !), and commutation with  $J_+$  forces all  $\mu_m$  to be equal: the matrix  $M$  is a multiple of the identity and the representation is indeed irreducible.

One may also wonder why the study of finite dimensional representations that we just carried out suffices to the physicist’s needs, for instance in quantum mechanics, where the scene usually takes place in an infinite dimensional Hilbert space. We show below (Chap. 2) that *Any representation of SU(2) or SO(3) in a Hilbert space is equivalent to a unitary representation, and is completely reducible to a (finite or infinite) sum of finite dimensional irreducible representations.*

### 0.3.3 Explicit construction

Let  $\xi$  and  $\eta$  be two complex variables on which matrices  $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of SU(2) act according to  $\xi' = a\xi + c\eta$ ,  $\eta' = b\xi + d\eta$ . In other terms,  $\xi$  and  $\eta$  are the basis vectors of the representation of dimension 2 (representation of spin  $\frac{1}{2}$ ) of SU(2). An explicit construction of the previous representations is then obtained by considering homogenous polynomials of degree  $2j$  in the two variables  $\xi$  and  $\eta$ , a basis of which is given by the  $2j + 1$  polynomials

$$P_{jm} = \frac{\xi^{j+m}\eta^{j-m}}{\sqrt{(j+m)!(j-m)!}} \quad m = -j, \dots, j. \quad (0.69)$$

(In fact, the following considerations also apply if  $U$  is an arbitrary matrix of the group GL(2,  $\mathbb{C}$ ) and provide a representation of that group.) Under the action of  $U$  on  $\xi$  and  $\eta$ , the  $P_{jm}(\xi, \eta)$  transform into  $P_{jm}(\xi', \eta')$ , also homogenous of degree  $2j$  in  $\xi$  and  $\eta$ , which may thus be expanded on the  $P_{jm}(\xi, \eta)$ . The latter thus span a dimension  $2j + 1$  representation of SU(2) (or of

$\text{GL}(2, \mathbb{C})$ ), which is nothing else than the previous spin  $j$  representation. This enables us to write quite explicit formulae for the  $\mathcal{D}^j$

$$P_{jm}(\xi', \eta') = \sum_{m'} P_{jm'}(\xi, \eta) \mathcal{D}_{m'm}^j(U) . \quad (0.70)$$

We find explicitly

$$\mathcal{D}_{m'm}^j(U) = ((j+m)!(j-m)!(j+m')!(j-m')!)^{\frac{1}{2}} \sum_{\substack{n_1, n_2, n_3, n_4 \geq 0 \\ n_1+n_2=j+m'; \quad n_3+n_4=j-m' \\ n_1+n_3=j+m; \quad n_2+n_4=j-m}} \frac{a^{n_1} b^{n_2} c^{n_3} d^{n_4}}{n_1! n_2! n_3! n_4!} . \quad (0.71)$$

For  $U = -\mathbf{I}$ , one may check once again that  $\mathcal{D}^j(-\mathbf{I}) = (-1)^{2j} I$ . In the particular case of  $U = e^{-i\psi \frac{\sigma_2}{2}} = \cos \frac{\psi}{2} \mathbf{I} - i \sin \frac{\psi}{2} \sigma_2$ , we thus have

$$d_{m'm}^j(\psi) = ((j+m)!(j-m)!(j+m')!(j-m')!)^{\frac{1}{2}} \sum_{k \geq 0} \frac{(-1)^{k+j-m} \cos \frac{\psi}{2}^{2k+m+m'} \sin \frac{\psi}{2}^{2j-2k-m-m'}}{(m+m'+k)!(j-m-k)!(j-m'-k)!k!} \quad (0.72)$$

where the sum runs over  $k \in [\inf(0, -m-m'), \sup(j-m, j-m')]$ . The expression of the infinitesimal generators acting on polynomials  $P_{jm}$  is obtained by considering  $U$  close to the identity. One finds

$$J_+ = \xi \frac{\partial}{\partial \eta} \quad J_- = \eta \frac{\partial}{\partial \xi} \quad J_z = \frac{1}{2} \left( \xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right) \quad (0.73)$$

on which it is easy to check commutation relations as well as the action on the  $P_{jm}$  in accordance with (0.57). This completes the identification of (0.69) with the spin  $j$  representation.

#### Remarks

1. Repeat the proof of irreducibility of the spin  $j$  representation in that new form.
2. Notice that the space of the homogenous polynomials of degree  $2j$  in the variables  $\xi$  and  $\eta$  is nothing else than the symmetrized  $2j$ -th tensor power of the representation of dimension 2 (see the definition below).

## 0.4 Direct product of representations of $\text{SU}(2)$

### 0.4.1 Direct product of representations and the “addition of angular momenta”

Consider the direct (or tensor) product of two representations of spin  $j_1$  and  $j_2$  and their decomposition on vectors of given total spin (“decomposition into irreducible representations”). We start with the product representation spanned by the vectors

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle \equiv |j_1 m_1; j_2 m_2\rangle \quad \text{written in short as} \quad |m_1 m_2\rangle \quad (0.74)$$

on which the infinitesimal generators act as

$$\mathbf{J} = \mathbf{J}^{(1)} \otimes \mathbf{I}^{(2)} + \mathbf{I}^{(1)} \otimes \mathbf{J}^{(2)} . \quad (0.75)$$

The upper index indicates on which space the operators act. By an abuse of notation, one frequently writes, instead of (0.75)

$$\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)} \quad (0.75')$$

and (in Quantum Mechanics) one refers to the “addition of angular momenta”  $\mathbf{J}^{(1)}$  and  $\mathbf{J}^{(2)}$ . The problem is thus to decompose the vectors (0.74) onto a basis of eigenvectors of  $\mathbf{J}$  and  $J_z$ . As  $\mathbf{J}^{(1)2}$  and  $\mathbf{J}^{(2)2}$  commute with one another and with  $\mathbf{J}^2$  and  $J_z$ , one may seek common eigenvectors that we denote

$$|(j_1 j_2) J M\rangle \quad \text{or more simply} \quad |J M\rangle \quad (0.76)$$

where it is understood that the value of  $j_1$  and  $j_2$  is fixed. The question is thus twofold: which values can  $J$  and  $M$  take; and what is the matrix of the change of basis  $|m_1 m_2\rangle \rightarrow |J M\rangle$ ? In other words, what is the (Clebsch-Gordan) decomposition and what are the Clebsch-Gordan coefficients?

The possible values of  $M$ , eigenvalue of  $J_z = J_z^{(1)} + J_z^{(2)}$ , are readily found

$$\begin{aligned} \langle m_1 m_2 | J_z | J M \rangle &= (m_1 + m_2) \langle m_1 m_2 | J M \rangle \\ &= M \langle m_1 m_2 | J M \rangle \end{aligned} \quad (0.77)$$

and the only value of  $M$  such that  $\langle m_1 m_2 | J M \rangle \neq 0$  is thus

$$M = m_1 + m_2. \quad (0.78)$$

For  $j_1$ ,  $j_2$  and  $M$  fixed, there are as many independent vectors with that eigenvalue of  $M$  as there are couples  $(m_1, m_2)$  satisfying (0.78), thus

$$n(M) = \begin{cases} 0 & \text{if } |M| > j_1 + j_2 \\ j_1 + j_2 + 1 - |M| & \text{if } |j_1 - j_2| \leq |M| \leq j_1 + j_2 \\ 2\inf(j_1, j_2) + 1 & \text{if } 0 \leq |M| \leq |j_1 - j_2| \end{cases} \quad (0.79)$$

(see the left Fig. 3 in which  $j_1 = 5/2$  and  $j_2 = 1$ ). Let  $N_J$  be the number of times the representation of spin  $J$  appears in the decomposition of the representations of spin  $j_1$  et  $j_2$ . The  $n(M)$  vectors of eigenvalue  $M$  for  $J_z$  may also be regarded as coming from the  $N_J$  vectors  $|J M\rangle$  for the different values of  $J$  compatible with that value of  $M$

$$n(M) = \sum_{J \geq |M|} N_J \quad (0.80)$$

hence, by subtracting two such relations

$$\begin{aligned} N_J &= n(J) - n(J+1) \\ &= 1 \quad \text{iff si } |j_1 - j_2| \leq J \leq j_1 + j_2 \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (0.81)$$

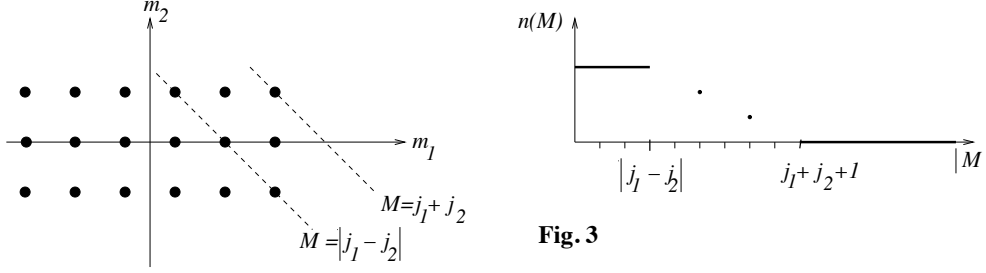


Fig. 3

To summarize, we have just shown that the  $(2j_1 + 1)(2j_2 + 1)$  vectors (0.74) (with  $j_1$  and  $j_2$  fixed) may be reexpressed in terms of vectors  $|J M\rangle$  with

$$\begin{aligned} J &= |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 \\ M &= -J, -J + 1, \dots, J. \end{aligned} \quad (0.82)$$

Note that multiplicities  $N_J$  take the value 0 or 1 ; it is a peculiarity of SU(2) that multiplicities larger than 1 do not occur in the decomposition of the tensor product of irreducible representations, *i.e.* here of fixed spin.

#### 0.4.2 Clebsch-Gordan coefficients, 3- $j$ and 6- $j$ symbols...

The change of orthonormal basis  $|j_1 m_1; j_2 m_2\rangle \rightarrow |(j_1 j_2) J M\rangle$  is carried out by the Clebsch-Gordan coefficients (C.G.)  $\langle (j_1 j_2); J M | j_1 m_1; j_2 m_2 \rangle$  which form a unitary matrix

$$|j_1 m_1; j_2 m_2\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^J \langle (j_1 j_2) J M | j_1 m_1; j_2 m_2 \rangle |(j_1 j_2) J M\rangle \quad (0.83)$$

$$|(j_1 j_2) J M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle (j_1 j_2) J M | j_1 m_1; j_2 m_2 \rangle^* |j_1 m_1; j_2 m_2\rangle. \quad (0.84)$$

Note that in the first line,  $M$  is fixed in terms of  $m_1$  and  $m_2$ ; and that in the second one,  $m_2$  is fixed in terms of  $m_1$ , for given  $M$ . Each relation thus implies only one summation. The value of these C.G. depends in fact on a choice of a relative phase between vectors (0.74) and (0.76); the usual convention is that for each value of  $J$ , one chooses

$$\langle J M = J | j_1 m_1 = j_1; j_2 m_2 = J - j_1 \rangle \quad \text{real}. \quad (0.85)$$

The other vectors are then unambiguously defined by (0.57) and we shall now show that all C.G. are real. C.G. satisfy recursion relations that are consequences of (0.57). Applying indeed  $J_{\pm}$  to the two sides of (0.83), one gets

$$\begin{aligned} \sqrt{J(J+1) - M(M \pm 1)} \quad & \langle (j_1 j_2) J M | j_1 m_1; j_2 m_2 \rangle \\ &= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)} \langle (j_1 j_2) J M \pm 1 | j_1 m_1 \pm 1; j_2 m_2 \rangle \\ &+ \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)} \langle (j_1 j_2) J M \pm 1 | j_1 m_1; j_2 m_2 \pm 1 \rangle \end{aligned} \quad (0.86)$$

which, together with the normalization  $\sum_{m_1, m_2} |\langle j_1 m_1; j_2 m_2 | (j_1 j_2) J M \rangle|^2 = 1$  and the convention (0.85), allows one to determine all the C.G. As stated before, they are clearly all real.

The C.G. of the group SU(2), which describe a change of orthonormal basis, form a unitary matrix and thus satisfy orthogonality and completeness properties

$$\sum_{m_1=-j_1}^{j_1} \langle j_1 m_1; j_2 m_2 | (j_1 j_2) J M \rangle \langle j_1 m_1; j_2 m_2 | (j_1 j_2) J' M' \rangle = \delta_{JJ'} \delta_{MM'} \quad \text{if } |j_1 - j_2| \leq J \leq j_1 + j_2 \quad (0.87)$$

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} \langle j_1 m_1; j_2 m_2 | (j_1 j_2) J M \rangle \langle j_1 m'_1; j_2 m'_2 | (j_1 j_2) J M \rangle = \delta_{m_1 m'_1} \delta_{m_2 m'_2} \quad \text{if } |m_1| \leq j_1, |m_2| \leq j_2.$$

Once again, each relation implies only one non trivial summation.

Rather than the C.G. coefficients, one may consider another set of equivalent coefficients, called *3-j symbols*. They are defined through

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} = \frac{(-1)^{j_1-j_2+M}}{\sqrt{2J+1}} \langle j_1 m_1; j_2 m_2 | (j_1 j_2) J M \rangle \quad (0.88)$$

and they enjoy simple symmetry properties:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

is invariant under cyclic permutation of its three columns, and changes by the sign  $(-1)^{j_1+j_2+j_3}$  when two columns are interchanged or when the signs of  $m_1$ ,  $m_2$  and  $m_3$  are reversed. The reader will find a multitude of tables and explicit formulas of the C.G. and 3j coefficients in the literature.

Let us just give some values of C.G. for low spins

$$\begin{aligned} \frac{1}{2} \otimes \frac{1}{2} : \quad & \begin{aligned} |(\frac{1}{2}, \frac{1}{2})1, 1\rangle &= | \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \rangle \\ |(\frac{1}{2}, \frac{1}{2})1, 0\rangle &= \frac{1}{\sqrt{2}} (| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \rangle + | \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \rangle) \\ |(\frac{1}{2}, \frac{1}{2})0, 0\rangle &= \frac{1}{\sqrt{2}} (| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \rangle - | \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2} \rangle) \\ |(\frac{1}{2}, \frac{1}{2})1, -1\rangle &= | \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \rangle \end{aligned} \end{aligned} \quad (0.89)$$

and

$$\begin{aligned} \frac{1}{2} \otimes 1 : \quad & \begin{aligned} |(\frac{1}{2}, 1)\frac{3}{2}, \frac{3}{2}\rangle &= | \frac{1}{2}, \frac{1}{2}; 1, 1 \rangle \\ |(\frac{1}{2}, 1)\frac{3}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} (\sqrt{2} | \frac{1}{2}, \frac{1}{2}; 1, 0 \rangle + | \frac{1}{2}, -\frac{1}{2}; 1, 1 \rangle) \\ |(\frac{1}{2}, 1)\frac{3}{2}, -\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} (| \frac{1}{2}, \frac{1}{2}; 1, -1 \rangle + \sqrt{2} | \frac{1}{2}, -\frac{1}{2}; 1, 0 \rangle) \\ |(\frac{1}{2}, 1)\frac{3}{2}, -\frac{3}{2}\rangle &= | \frac{1}{2}, -\frac{1}{2}; 1, -1 \rangle \\ |(\frac{1}{2}, 1)\frac{1}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} (-| \frac{1}{2}, \frac{1}{2}; 1, 0 \rangle + \sqrt{2} | \frac{1}{2}, -\frac{1}{2}; 1, 1 \rangle) \\ |(\frac{1}{2}, 1)\frac{1}{2}, -\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} (-\sqrt{2} | \frac{1}{2}, \frac{1}{2}; 1, -1 \rangle + | \frac{1}{2}, -\frac{1}{2}; 1, 0 \rangle) \end{aligned} \end{aligned} \quad (0.90)$$

One notices on the case  $\frac{1}{2} \otimes \frac{1}{2}$  the property that vectors of total spin  $j = 1$  are symmetric under the exchange of the two spins, while those of spin 0 are antisymmetric. This is a general property: in the decomposition of the tensor product of two representations of spin  $j_1 = j_2$ , vectors of spin  $j = 2j_1, 2j_1 - 2, \dots$  are symmetric, those of spin  $2j_1 - 1, 2j_1 - 3, \dots$  are antisymmetric.

This is apparent on the expression (0.88) above, given the announced properties of the 3-j symbols.

In the same circle of ideas, consider the completely antisymmetric product of  $2j+1$  copies of a spin  $j$  representation. One may show that this representation is of spin 0 (following exercise).

(This has consequences in atomic physics, in the filling of electronic orbitals: a complete shell has a total orbital momentum and a total spin that are both vanishing, hence also a vanishing total angular momentum.)

Exercise. Consider the completely antisymmetric tensor product of  $N = 2j + 1$  representations of spin  $j$ . Show that this representation is spanned by the vector  $\epsilon_{m_1 m_2 \dots m_N} |j m_1, j m_2, \dots, j m_N\rangle$ , that it is invariant under the action of SU(2) and thus that the corresponding representation has spin  $J = 0$ .

One also introduces the *6-j symbols* that describe the two possible recombinations of 3 representations of spins  $j_1, j_2$  and  $j_3$

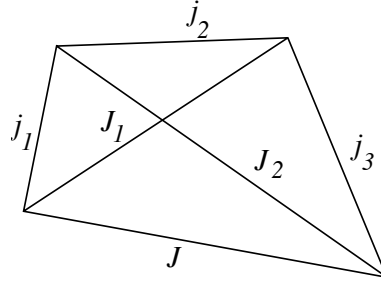


Fig. 4

$$\begin{aligned} |j_1 m_1; j_2 m_2; j_3 m_3\rangle &= \sum \langle (j_1 j_2) J_1 M_1 | j_1 m_1; j_2 m_2 \rangle \langle (J_1 j_3) J M | J_1 M_1; j_3 m_3 \rangle | (j_1 j_2) j_3; J M \rangle \\ &= \sum \langle (j_2 j_3) J_2 M_2 | j_2 m_2; j_3 m_3 \rangle \langle (j_1 J_2) J' M' | j_1 m_1; J_2 M_2 \rangle | j_1 (j_2 j_3); J' M' \rangle \end{aligned} \quad (0.91)$$

depending on whether one composes first  $j_1$  and  $j_2$  into  $J_1$  and then  $J_1$  and  $j_3$  into  $J$ , or first  $j_2$  and  $j_3$  into  $J_2$  and then  $j_1$  and  $J_2$  into  $J'$ . The matrix of the change of basis is denoted

$$\langle j_1 (j_2 j_3); J M | (j_1 j_2) j_3; J' M' \rangle = \delta_{JJ'} \delta_{MM'} \sqrt{(2J_1 + 1)(2J_2 + 1)} (-1)^{j_1 + j_2 + j_3 + J} \begin{Bmatrix} j_1 & j_2 & J_1 \\ j_3 & J & J_2 \end{Bmatrix}. \quad (0.92)$$

and the  $\begin{Bmatrix} \end{Bmatrix}$  are the 6- $j$  symbols. One may visualise this operation of “addition” of the three spins by a tetrahedron (see Fig. 4) the edges of which carry  $j_1, j_2, j_3, J_1, J_2$  and  $J$  and the symbol is such that two spins carried by a pair of opposed edges lie in the same column. These symbols are tabulated in the literature.

## 0.5 A physical application: isospin

The group SU(2) appears in physics in several contexts, not only as related to the rotation group of the 3-dimensional Euclidian space. We shall now illustrate another of its avatars by the isospin symmetry.

There exists in nature elementary particles subject to nuclear forces, or more precisely to “strong interactions”, and thus called *hadrons*. Some of those particles present similar properties but have different electric charges. This is the case with the two “nucleons”, *i.e.* the proton and the neutron, of respective masses  $M_p = 938,28 \text{ MeV}/c^2$  and  $M_n = 939,57 \text{ MeV}/c^2$ , and also with the “triplet” of pi mesons,  $\pi^0$  (mass  $134,96 \text{ MeV}/c^2$ ) and  $\pi^\pm$  ( $139,57 \text{ MeV}/c^2$ ), with  $K$  mesons etc. According to a great idea of Heisenberg these similarities are the manifestation of a symmetry broken by electromagnetic interactions. In the absence of electromagnetic interactions proton and neutron on the one hand, the three  $\pi$  mesons on the other, etc, would have the same mass, differing only by an “internal” quantum number, in

the same way as the two spin states of an electron in the absence of a magnetic field. In fact the group behind that symmetry is also  $SU(2)$ , but a  $SU(2)$  group acting in an abstract space differing from the usual space. One gave the name *isotopic spin* or in short, *isospin*, to the corresponding quantum number. To summarize, the idea is that there exists a  $SU(2)$  group of symmetry of strong interactions, and that different particles subject to these strong interactions (hadrons) form representations of  $SU(2)$ : representation of isospin  $I = \frac{1}{2}$  for the nucleon (proton  $I_z = +\frac{1}{2}$ , neutron  $I_z = -\frac{1}{2}$ ), isospin  $I = 1$  pour the pions ( $\pi^\pm : I_z = \pm 1$ ,  $\pi^0 : I_z = 0$ ) etc. The isospin is thus a “good quantum number”, conserved in these interactions. Thus the “off-shell” process  $N \rightarrow N + \pi$ , ( $N$  for nucleon) important in nuclear physics, is consistent with addition rules of isospins ( $\frac{1}{2} \otimes 1$  “contains”  $\frac{1}{2}$ ). The different scattering reactions  $N + \pi \rightarrow N + \pi$  allowed by conservation of electric charge

$$\begin{array}{lll}
 p + \pi^+ & \rightarrow & p + \pi^+ & I_z = \frac{3}{2} \\
 p + \pi^0 & \rightarrow & p + \pi^0 & I_z = \frac{1}{2} \\
 & & \rightarrow & n + \pi^+ & \text{''} \\
 p + \pi^- & \rightarrow & p + \pi^- & I_z = -\frac{1}{2} \\
 & & \rightarrow & n + \pi^0 & \text{''} \\
 n + \pi^- & \rightarrow & n + \pi^- & I_z = -\frac{3}{2}
 \end{array}$$

also conserve total isospin  $I$  and its  $I_z$  component but the hypothesis of  $SU(2)$  isospin invariance tells us more. The matrix elements of the transition operator responsible for the two reactions in the channel  $I_z = \frac{1}{2}$ , for example, must be related by addition rules of isospin. Inverting the relations (0.90), one gets

$$\begin{aligned}
 |p, \pi^- \rangle &= \sqrt{\frac{1}{3}} |I = \frac{3}{2}, I_z = -\frac{1}{2} \rangle - \sqrt{\frac{2}{3}} |I = \frac{1}{2}, I_z = -\frac{1}{2} \rangle \\
 |n, \pi^0 \rangle &= \sqrt{\frac{2}{3}} |I = \frac{3}{2}, I_z = -\frac{1}{2} \rangle + \sqrt{\frac{1}{3}} |I = \frac{1}{2}, I_z = -\frac{1}{2} \rangle
 \end{aligned}$$

whereas for  $I_z = 3/2$

$$|p, \pi^+ \rangle = |I = \frac{3}{2}, I_z = \frac{3}{2} \rangle .$$

Isospin invariance implies that  $\langle I I_z | \mathcal{T} | I' I'_z \rangle = \mathcal{T}_I \delta_{II'} \delta_{I_z I'_z}$ , as we shall see later (Schur lemma or Wigner–Eckart theorem, Chap. 2): not only are  $I$  and  $I_z$  conserved, but the resulting amplitude depends only on  $I$ , not  $I_z$ . Calculating then the matrix elements of the transition operator  $\mathcal{T}$  between the different states,

$$\begin{aligned}
 \langle p\pi^+ | \mathcal{T} | p\pi^+ \rangle &= \mathcal{T}_{3/2} \\
 \langle p\pi^- | \mathcal{T} | p\pi^- \rangle &= \frac{1}{3} (\mathcal{T}_{3/2} + 2\mathcal{T}_{1/2}) \\
 \langle n\pi^0 | \mathcal{T} | p\pi^- \rangle &= \frac{\sqrt{2}}{3} (\mathcal{T}_{3/2} - \mathcal{T}_{1/2})
 \end{aligned} \tag{0.93}$$

one finds that amplitudes satisfy a relation

$$\sqrt{2} \langle n, \pi^0 | \mathcal{T} | p, \pi^- \rangle + \langle p, \pi^- | \mathcal{T} | p, \pi^- \rangle = \langle p, \pi^+ | \mathcal{T} | p, \pi^+ \rangle = \mathcal{T}_{3/2}$$

a non trivial consequence of isospin invariance, which implies triangular inequalities between squared modules of these amplitudes and hence between cross-sections of the reactions

$$\begin{aligned} [\sqrt{\sigma(\pi^- p \rightarrow \pi^- p)} - \sqrt{2\sigma(\pi^- p \rightarrow \pi^0 n)}]^2 &\leq \sigma(\pi^+ p \rightarrow \pi^+ p) \leq \\ &\leq [\sqrt{\sigma(\pi^- p \rightarrow \pi^- p)} + \sqrt{2\sigma(\pi^- p \rightarrow \pi^0 n)}]^2 \end{aligned}$$

which are experimentally well verified. Even better, one finds experimentally that at a certain energy of about 180 MeV, cross sections (proportional to squares of amplitudes) are in the ratios

$$\sigma(\pi^+ p \rightarrow \pi^+ p) : \sigma(\pi^- p \rightarrow \pi^0 n) : \sigma(\pi^- p \rightarrow \pi^- p) = 9 : 2 : 1$$

which is what one would get from (0.93) if  $\mathcal{T}_{\frac{1}{2}}$  were vanishing. This indicates that at that energy, scattering in the channel  $I = 3/2$  is dominant. In fact, this signals the existence of an intermediate  $\pi N$  state, a very unstable particle called “resonance”, denoted  $\Delta$ , of isospin  $3/2$  and hence with four states of charge

$$\Delta^{++}, \Delta^+, \Delta^0, \Delta^-,$$

the contribution of which dominates the scattering amplitude. This particle has a spin  $3/2$  and a mass  $M(\Delta) \approx 1230 \text{ MeV}/c^2$ .

In some cases one may obtain more precise predictions. This is for instance the case with the reactions

$${}^2\text{H}p \rightarrow {}^3\text{He}\pi^0 \quad \text{and} \quad {}^2\text{H}p \rightarrow {}^3\text{H}\pi^+$$

which involve nuclei of deuterium  ${}^2\text{H}$ , of tritium  ${}^3\text{H}$  and of helium  ${}^3\text{He}$ . To these nuclei too, one may assign an isospin, 0 to the deuteron which is made of a proton and a neutron in an antisymmetric state of their isospins (so that the wave function of these two *fermions*, symmetric in space and in spin, be antisymmetric),  $I_z = -\frac{1}{2}$  to  ${}^3\text{H}$  and  $I_z = \frac{1}{2}$  to  ${}^3\text{He}$  which form an isospin  $\frac{1}{2}$  representation. Notice that in all cases, the electric charge is related to the  $I_z$  component of isospin by the relation  $Q = \frac{1}{2}\mathcal{B} + I_z$ , with  $\mathcal{B}$  the baryonic charge, equal here to the number of nucleons (protons or neutrons).

Exercise: show that the ratio of cross-sections  $\sigma({}^2\text{H}p \rightarrow {}^3\text{He}\pi^0)/\sigma({}^2\text{H}p \rightarrow {}^3\text{H}\pi^+)$  is  $\frac{1}{2}$ .

Remark : invariance under isospin  $\text{SU}(2)$  that we just discussed is a symmetry of *strong interactions*. There exists also in the framework of the Standard Model a notion of “weak isospin”, a symmetry of *electroweak interactions*, to which we return in Chap. 5.

## 0.6 Representations of $\text{SO}(3,1)$ and $\text{SL}(2, \mathbb{C})$

### 0.6.1 A short reminder on the Lorentz group

Minkowski space is a  $\mathbb{R}^4$  space endowed with a pseudo-Euclidean metric of signature  $(+, -, -, -)$ . In an orthonormal basis with coordinates  $(x^0 = ct, x^1, x^2, x^3)$ , the metric is diagonal

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

and thus the squared norm of a 4-vector reads

$$x.x = x^\mu g_{\mu\nu} x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

The isometry group of that quadratic form, called  $O(1,3)$  or the Lorentz group  $\mathcal{L}$ , is such that

$$\Lambda \in O(1,3) \quad x' = \Lambda x \quad : \quad x' \cdot x' = \Lambda^\mu_\rho x^\rho g_{\mu\nu} \Lambda^\nu_\sigma x^\sigma = x^\rho g_{\rho\sigma} x^\sigma$$

i.e.

$$\Lambda^\mu_\rho g_{\mu\nu} \Lambda^\nu_\sigma = g_{\rho\sigma} \quad \text{or} \quad \Lambda^T g \Lambda = g. \quad (0.94)$$

These pseudo-orthogonal matrices satisfy  $(\det \Lambda)^2 = 1$  and (by taking the 00 matrix element of (0.94))  $(\Lambda^0_0)^2 = 1 + \sum_{i=1}^3 (\Lambda^0_i)^2 \geq 1$  and thus  $\mathcal{L} \equiv O(1,3)$  has four connected components (or “sheets”) depending on whether  $\det \Lambda = \pm 1$  and  $\Lambda^0_0 \geq 1$  or  $\leq -1$ . The subgroup of *proper orthochronous* transformations satisfying  $\det \Lambda = 1$  and  $\Lambda^0_0 \geq 1$  is denoted  $\mathcal{L}^\dagger_+$ . Any transformation of  $\mathcal{L}^\dagger_+$  may be written as the product of an “ordinary” rotation of  $SO(3)$  and a “special Lorentz transformation” or “boost”.

A major difference between the  $SO(3)$  and the  $\mathcal{L}^\dagger_+$  groups is that the former is *compact* (the range of parameters is bounded and closed, see (0.2)), whereas the latter is not : in a boost along the 1 direction, say,  $x'_1 = \gamma(x_1 - vx_0/c)$ ,  $x'_0 = \gamma(x_0 - vx_1/c)$ , with  $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ , the velocity  $|v| < c$  does not belong to a compact domain (or alternatively, the “rapidity” variable  $\beta$ , defined by  $\cosh \beta = \gamma$  can run to infinity). This compactness/non-compactness has very important implications on the nature and properties of representations, as we shall see.

The Poincaré group, or inhomogeneous Lorentz group, is generated by Lorentz transformations  $\Lambda \in \mathcal{L}$  and space-time translations; generic elements denoted  $(a, \Lambda)$  have an action on a vector  $x$  and a composition law given by

$$\begin{aligned} (a, \Lambda) \quad : \quad x &\mapsto x' = \Lambda x + a \\ (a', \Lambda')(a, \Lambda) &= (a' + \Lambda' a, \Lambda' \Lambda) ; \end{aligned} \quad (0.95)$$

the inverse of  $(a, \Lambda)$  is  $(-\Lambda^{-1}a, \Lambda^{-1})$  (check it !).

## 0.6.2 Lie algebra of the Lorentz and Poincaré groups

An infinitesimal Poincaré transformation reads  $(\alpha^\mu, \Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu)$ . By taking the infinitesimal form of (0.94), one easily sees that the tensor  $\omega_{\rho\nu} = \omega^\mu_\nu g_{\rho\mu}$  has to be antisymmetric:  $\omega_{\nu\rho} + \omega_{\rho\nu} = 0$ . This leaves 6 real parameters: the Lorentz group is a 6-dimensional group, and the Poincaré group is 10-dimensional.

To find the Lie algebra of the generators, let us proceed like in § 0.2.3: look at the Lie algebra generated by differential operators acting on functions of space-time coordinates; if  $x'^\lambda = x^\lambda + \delta x^\lambda = x^\lambda + \alpha^\lambda + \omega^{\lambda\nu} x_\nu$ ,  $\delta f(x) = f(x^\mu - \alpha^\mu - \omega^{\lambda\nu} x_\nu) - f(x) = (I - i\alpha^\mu P_\mu - \frac{i}{2}\omega^{\mu\nu} J_{\mu\nu} - I)f(x)$ , (see (0.45)), thus

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \quad P_\mu = -i\partial_\mu \quad (0.96)$$

the commutators of which are then easily computed

$$\begin{aligned} [J_{\mu\nu}, P_\rho] &= i(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu) \\ [J_{\mu\nu}, J_{\rho\sigma}] &= i(g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho} - g_{\nu\sigma} J_{\mu\rho}) \\ [P_\mu, P_\nu] &= 0. \end{aligned} \quad (0.97)$$

Note the structure of these relations: antisymmetry in  $\mu \leftrightarrow \nu$  of the first one, in  $\mu \leftrightarrow \nu$ , in  $\rho \leftrightarrow \sigma$  and in  $(\mu, \nu) \leftrightarrow (\rho, \sigma)$  of the second one; the first one shows how a vector (here  $P_\rho$ ) transforms under the infinitesimal transformation by  $J_{\mu\nu}$ , and the second then has the same pattern in the indices  $\rho$  and  $\sigma$ , expressing that  $J_{\rho\sigma}$  is a 2-tensor.

Generators that commute with  $P_0$  (which is the generator of time translations, hence the Hamiltonian) are the  $P_\mu$  and the  $J_{ij}$  but not the  $J_{0j}$  :  $i[P_0, J_{0j}] = P_j$ .

Set

$$J_{ij} = \epsilon_{ijk} J^k \quad K^i = J_{0i} . \quad (0.98)$$

Then

$$\begin{aligned} [J^i, J^j] &= i\epsilon_{ijk} J^k \\ [J^i, K^j] &= i\epsilon_{ijk} K^k \\ [K^i, K^j] &= -i\epsilon_{ijk} J^k \end{aligned} \quad (0.99)$$

and also

$$\begin{aligned} [J^i, P^j] &= i\epsilon_{ijk} P^k & [K^i, P^j] &= iP^0 \delta_{ij} \\ [J^i, P^0] &= 0 & [K^i, P^0] &= iP^i . \end{aligned} \quad (0.100)$$

Remark. The first two relations (0.99) and the first one of (0.100) express that, as expected,  $\mathbf{J} = \{J^j\}$ ,  $\mathbf{K} = \{K^j\}$  and  $\mathbf{P} = \{P^j\}$  transform like vectors under rotations of  $\mathbb{R}^3$ . Now form the combinations

$$M^j = \frac{1}{2}(J^j + iK^j) \quad N^j = \frac{1}{2}(J^j - iK^j) \quad (0.101)$$

which have the following commutation relations

$$\begin{aligned} [M^i, M^j] &= i\epsilon_{ijk} M^k \\ [N^i, N^j] &= i\epsilon_{ijk} N^k \\ [M^i, N^j] &= 0 . \end{aligned} \quad (0.102)$$

By considering the complex combinations  $M$  and  $N$  of its generators, one thus sees that the Lie algebra of  $\mathcal{L} = O(1, 3)$  is isomorphic to  $su(2) \oplus su(2)$ . The introduction of  $\pm i$ , however, implies that *unitary* representations of  $\mathcal{L}$  do not follow in a simple way from those of  $\text{SU}(2) \times \text{SU}(2)$ . On the other hand, representations of finite dimension of  $\mathcal{L}$ , which are *non unitary*, are labelled by a pair  $(j_1, j_2)$  of integers or half-integers.

Exercise. Show that this algebra admits *two* independent quadratic Casimir operators, and express them in terms of  $\mathbf{M}$  and  $\mathbf{N}$  first, and then in terms of  $\mathbf{J}$  and  $\mathbf{K}$ .

### 0.6.3 Covering groups of $\mathcal{L}_+^\uparrow$ and $\mathcal{P}_+^\uparrow$

We have seen that the study of  $\text{SO}(3)$  led us to  $\text{SU}(2)$ , its “covering group” (the deep reasons of which will be explained in Chap. 1 and 2). Likewise in the case of the Lorentz group its “covering group” turns out to be  $\text{SL}(2, \mathbb{C})$ .

There is a simple way to see how  $\text{SL}(2, \mathbb{C})$  and  $\mathcal{L}_+^\dagger$  are related, which is a 4-dimensional extension of the method followed in § 0.1.2. One considers matrices  $\sigma_\mu$  made of  $\sigma_0 = \mathbf{I}$  and of the three familiar Pauli matrices. Note that

$$\text{tr } \sigma_\mu \sigma_\nu = 2\delta_{\mu\nu} \quad \sigma_\mu^2 = \mathbf{I} \quad \text{with no summation over the index } \mu .$$

With any real vector  $x \in \mathbb{R}^4$ , associate the Hermitian matrix

$$X = x^\mu \sigma_\mu \quad x^\mu = \frac{1}{2} \text{tr } (X \sigma_\mu) \quad \det X = x^2 = (x^0)^2 - \mathbf{x}^2 . \quad (0.103)$$

A matrix  $A \in \text{SL}(2, \mathbb{C})$  acts on  $X$  according to

$$X \mapsto X' = AXA^\dagger \quad (0.104)$$

which is indeed Hermitian and thus defines a real  $x'^\mu = \frac{1}{2} \text{tr } (X' \sigma_\mu)$ , with  $\det X' = \det X$ , hence  $x^2 = x'^2$ . This is a linear transformation of  $\mathbb{R}^4$  that preserves the Minkowski norm  $x^2$ , and thus a Lorentz transformation, and one checks by an argument of continuity that it belongs to  $\mathcal{L}_+^\dagger$  and that  $A \rightarrow \Lambda$  is a homomorphism of  $\text{SL}(2, \mathbb{C})$  into  $\mathcal{L}_+^\dagger$ . In the following we denote  $x' = Ax$  if  $X' = AXA^\dagger$ .

As is familiar from the case of  $\text{SU}(2)$ , the transformations  $A$  and  $-A \in \text{SL}(2, \mathbb{C})$  give the same transformation of  $\mathcal{L}_+^\dagger$ :  $\text{SL}(2, \mathbb{C})$  is a covering of order 2 of  $\mathcal{L}_+^\dagger$ . For the Poincaré group, likewise, its covering is the (“semi-direct”) product of the translation group by  $\text{SL}(2, \mathbb{C})$ . If one denotes  $\underline{a} := a^\mu \sigma_\mu$ , then

$$(\underline{a}, A)(\underline{a}', A') = (\underline{a} + A\underline{a}'A^\dagger, AA') \quad (0.105)$$

and one sometimes refers to it as the “inhomogeneous  $\text{SL}(2, \mathbb{C})$  group” or  $\text{ISL}(2, \mathbb{C})$ .

### 0.6.4 Irreducible finite-dimensional representations of $\text{SL}(2, \mathbb{C})$

The construction of § 0.3.3 yields an explicit representation of  $\text{GL}(2, \mathbb{C})$  and hence of  $\text{SL}(2, \mathbb{C})$ .

For  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ , (0.71) gives the following expression for  $\mathcal{D}_{mm'}^j(A)$ :

$$\mathcal{D}_{mm'}^j(A) = [(j+m)!(j-m)!(j+m')!(j-m')!]^{\frac{1}{2}} \sum_{\substack{n_1, n_2, n_3, n_4 \geq 0 \\ n_1 + n_2 = j+m; \quad n_3 + n_4 = j-m' \\ n_1 + n_3 = j+m; \quad n_2 + n_4 = j-m}} \frac{a^{n_1} b^{n_2} c^{n_3} d^{n_4}}{n_1! n_2! n_3! n_4!} \quad (0.71)$$

Note that  $\mathcal{D}^T(A) = \mathcal{D}(A^T)$  (since exchanging  $m \leftrightarrow m'$  amounts to  $n_2 \leftrightarrow n_3$ , hence to  $b \leftrightarrow c$ ) and  $(\mathcal{D}(A))^* = \mathcal{D}(A^*)$  (since the numerical coefficients in (0.71) are real) thus  $\mathcal{D}^\dagger(A) = \mathcal{D}(A^\dagger)$ .

This representation is called  $(j, 0)$ , it is of dimension  $2j+1$ . There exists another one of dimension  $2j+1$ , which is non equivalent, denoted  $(0, j)$ , this is the “contragredient conjugate” representation (in the sense of Chap 2. § 2.1.3.b)  $\mathcal{D}^j(A^{\dagger-1})$ . Replacing  $A$  by  $A^{\dagger-1}$  may be interpreted in the construction of § 0.6.3 if instead of associating  $X = x^\mu \sigma_\mu$  with  $x$ , one associates  $\tilde{X} = x^0 \sigma_0 - \mathbf{x} \cdot \boldsymbol{\sigma}$ . Notice that  $\sigma_2(\sigma_i)^T \sigma_2 = -\sigma_i$  for  $i = 1, 2, 3$  hence  $\tilde{X} = \sigma_2 X^T \sigma_2$ . For the transformation  $A : X \mapsto X' = AXA^\dagger$ , we have

$$\tilde{X}' = \sigma_2 (X')^T \sigma_2 = \sigma_2 (AXA^\dagger)^T \sigma_2 = (\sigma_2 A^T \sigma_2)^\dagger \tilde{X} (\sigma_2 A^T \sigma_2) .$$

Any matrix  $A$  of  $\text{SL}(2, \mathbb{C})$  may itself be written as  $A = a^\mu \sigma_\mu$ , with  $(a^\mu) \in \mathbb{C}^4$ , and as  $\det A = (a^0)^2 - \mathbf{a}^2 = 1$  (the “S” of  $\text{SL}(2, \mathbb{C})$ ), one verifies immediately that  $A^{-1} = a^0 \sigma_0 - \mathbf{a} \cdot \boldsymbol{\sigma}$ , donc

$$\sigma_2 A^T \sigma_2 = A^{-1} . \quad (0.106)$$

Finally

$$X' = A X A^\dagger \iff \tilde{X}' = (A^{-1})^\dagger \tilde{X} A^{-1} . \quad (0.107)$$

Remark. The two representations  $(j, 0)$  and  $(0, j)$  are inequivalent on  $\text{SL}(2, \mathbb{C})$ , but equivalent on  $\text{SU}(2)$ . Indeed in  $\text{SU}(2)$ ,  $A = U = (U^\dagger)^{-1}$ .

Finally, one proves that any finite-dimensional representation of  $\text{SL}(2, \mathbb{C})$  is completely reducible and may be written as a direct sum of irreducible representations. The most general finite-dimensional irreducible representation of  $\text{SL}(2, \mathbb{C})$  is denoted  $(j_1, j_2)$ , with  $j_1$  and  $j_2 \geq 0$  integers or half-integers; it is defined by

$$(j_1, j_2) = (j_1, 0) \otimes (0, j_2) . \quad (0.108)$$

All these representations may be obtained from the representations  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  by tensoring:  $(j_1, 0)$  and  $(0, j_2)$  are obtained by symmetrized tensor product of representations  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$ , respectively, as was done for  $\text{SU}(2)$ . Only representations  $(j_1, j_2)$  with  $j_1$  and  $j_2$  simultaneously integers or half-integers are true representations of  $\mathcal{L}_1^+$ . The others are representations up to a sign.

Exercise : show that the representation  $(0, j)$  is “equivalent” (equal up to a change of basis) to the complex conjugate of representation  $(j, 0)$ . (Hint: show it first for  $j = \frac{1}{2}$  by recalling that  $(A^{-1})^\dagger = \sigma_2 A^* \sigma_2$ , then for representations of arbitrary  $j$  obtained by  $2j$ -th tensor power of  $j = \frac{1}{2}$ .)

## Spinor representations

Return to the “spinor representations”  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$ . Those are representations of dimension 2 (two-component spinors). It is traditional to note the indices of components with “pointed” or “unpointed” indices, for representation  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, 0)$ , respectively. With  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ , we thus have

$$\begin{aligned} (\tfrac{1}{2}, 0) \quad \xi = (\xi^\alpha) &\mapsto \xi' = A \xi = \begin{pmatrix} a \xi^1 + b \xi^2 \\ c \xi^1 + d \xi^2 \end{pmatrix} \\ (0, \tfrac{1}{2}) \quad \xi = (\xi_{\dot{\alpha}}) &\mapsto \xi' = A^* \xi = \begin{pmatrix} a^* \xi^{\dot{1}} + b^* \xi^{\dot{2}} \\ c^* \xi^{\dot{1}} + d^* \xi^{\dot{2}} \end{pmatrix} \end{aligned} \quad (0.109)$$

Note that the alternating (=antisymmetric) form  $(\xi, \eta) = \xi^1 \eta^2 - \xi^2 \eta^1 = \xi^T (i \sigma_2) \eta$  is invariant in  $(\frac{1}{2}, 0)$  (and also in  $(0, \frac{1}{2})$ ), which follows once again from (0.106)

$$(\sigma_2 A^T \sigma_2) A = A^{-1} A = \mathbf{I} \iff A^T (i \sigma_2) A = i \sigma_2 .$$

One may thus use that form to lower indices  $\alpha$  (or  $\dot{\alpha}$ ). Thus

$$\begin{aligned} \text{in } (\tfrac{1}{2}, 0) : (\xi, \eta) &= \xi_\alpha \eta^\alpha & \xi_2 &= \xi^1 & \xi_1 &= -\xi^2 \\ \text{in } (0, \tfrac{1}{2}) : (\xi, \eta) &= \xi_{\dot{\alpha}} \eta^{\dot{\alpha}} & \xi_2 &= \xi^{\dot{1}} & \xi_1 &= -\xi^{\dot{2}} \end{aligned} \quad (0.110)$$

### $(j_1, j_2)$ representation

Tensors  $\{\xi^{\alpha_1 \alpha_2 \dots \alpha_{2j_1}} \dot{\beta}_1 \dot{\beta}_2 \dots \dot{\beta}_{2j_2}\}$  symmetric in  $\alpha_1, \alpha_2, \dots, \alpha_{2j_1}$  and in  $\dot{\beta}_1, \dot{\beta}_2, \dots, \dot{\beta}_{2j_2}$ , form the irreducible representation  $(j_1, j_2)$ . (One cannot lower the rank by taking traces, since the only invariant tensor is the previous *alternating* form). The dimension of that representation is  $(2j_1 + 1)(2j_2 + 1)$ . The most usual representations encountered in field theory are  $(0, 0)$ ,  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$ ,  $(\frac{1}{2}, \frac{1}{2})$ . The reducible representation  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$  describes the (4-component) Dirac fermion; the  $(\frac{1}{2}, \frac{1}{2})$  corresponds to 4-vectors, as seen above:

$$x \mapsto X = x^0 \sigma_0 + \mathbf{x} \cdot \boldsymbol{\sigma} \xrightarrow{A \in SL(2, \mathbb{C})} X' = A X A^\dagger$$

*i.e.*

$$X = X^{\alpha\dot{\beta}} \rightarrow (X')^{\alpha\dot{\beta}} = A^{\alpha\alpha'} (A^{\dot{\beta}\dot{\beta'}})^* X^{\alpha'\dot{\beta'}} ,$$

which shows that  $X$  transforms indeed according to the  $(\frac{1}{2}, \frac{1}{2})$  representation .

Exercise. Show that representations  $(1, 0)$  and  $(0, 1)$ , of dimension 3, describe rank 2 tensors  $F^{\mu\nu}$  that are “self-dual” ou “anti-self-dual”, *i.e.* satisfy

$$F^{\mu\nu} = \pm \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} .$$

## 0.6.5 Irreducible unitary representations of the Poincaré group. One particle states.

According to a theorem of Wigner which will be discussed in Chap. 2, the action of proper orthochronous transformations of the Lorentz or Poincaré groups on state vectors of a quantum theory is described by means of *unitary* representations of these groups, or rather of their “universal covers”  $SL(2, \mathbb{C})$  and  $ISL(2, \mathbb{C})$ . As will be seen below (Chap. 2), unitary representations (of class  $L^2$ ) of the non compact group  $SL(2, \mathbb{C})$  are necessarily of infinite dimension (with the possible exception of the trivial representation  $(0, 0)$ , which describes a state invariant by rotation and by boosts, *i.e.* the vacuum !, ..., and which is in fact not of class  $L^2$ !).

Returning to commutation relations of the Lie algebra (0.97), one seeks a maximal set of commuting operators. The four  $P_\mu$  commute. Let  $(p_\mu)$  be an eigenvalue for a common eigenvector of  $P_\mu$ , describing a “one-particle state”. We assume that the eigenvector denoted  $|p\rangle$  is labelled only by  $p^\mu$  and by discrete indices: (this is indeed the meaning of “one-particle state”, in contrast with a two-particle state that would depend on a relative momentum, a continuous variable)

$$P_\mu |p\rangle = p_\mu |p\rangle . \quad (0.111)$$

One also considers the *Pauli-Lubanski tensor*

$$W^\lambda = \frac{1}{2} \epsilon^{\lambda\mu\nu\rho} J_{\mu\nu} P_\rho \quad (0.112)$$

and one verifies (exercise !) that (0.97) implies

$$[W_\mu, P_\nu] = 0$$

$$\begin{aligned} [W^\mu, W^\nu] &= -i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma \\ [J_{\mu\nu}, W_\lambda] &= i(g_{\nu\lambda} W_\mu - g_{\mu\lambda} W_\nu) . \end{aligned} \quad (0.113)$$

The latter relation means that  $W$  is a Lorentz 4-vector (compare with (0.97)). One also notes that  $W.P = 0$  because of the antisymmetry of tensor  $\epsilon$ . One finally shows (check it!) that  $P^2 = P_\mu P^\mu$  and  $W^2 = W_\mu W^\mu$  commute with all generators  $P$  and  $J$ : those are the Casimir operators of the algebra. According to a lemma by Schur, (see below Chap. 2, § 2.1.4), these Casimir operators are in any irreducible representation proportional to the identity, in other words, their eigenvalues may be used to label the irreducible representations.

In physics, one encounters only two types of representations for these one-particle states<sup>4</sup>: representations with  $P^2 > 0$  and those with  $P^2 = 0$ ,  $W^2 = 0$ . Their detailed study will be done in Adel Bilal's course.

## Bibliography

The historical reference for the physicist is the book by E. Wigner [Wi].

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J.-M. Normand, *A Lie group : Rotations in Quantum Mechanics*, North-Holland.

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## Problem

One considers two spin  $\frac{1}{2}$  representations of the group  $SU(2)$  and their direct (or tensor) product. One denotes  $\mathbf{J}^{(1)}$  and  $\mathbf{J}^{(2)}$  the infinitesimal generators acting in each representation, and  $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$  those acting in their direct product, see (0.75), (0.75').

- What can be said about the operators  $\mathbf{J}^{(1)2}$ ,  $\mathbf{J}^{(2)2}$  and  $\mathbf{J}^2$  and their eigenvalues ?
- Show that  $\mathbf{J}^{(1)}.\mathbf{J}^{(2)}$  may be expressed in terms of these operators and that operators

$$\frac{1}{4}(3\mathbf{I} + 4\mathbf{J}^{(1)}.\mathbf{J}^{(2)}) \quad \text{et} \quad \frac{1}{4}(\mathbf{I} - 4\mathbf{J}^{(1)}.\mathbf{J}^{(2)})$$

are projectors on spaces to be identified.

- Taking into account the symmetries of the vectors under exchange, what can you say about the operator

$$\frac{1}{2}\mathbf{I} + 2\mathbf{J}^{(1)}.\mathbf{J}^{(2)} \quad ?$$

## Appendix 0. Measure and Laplacian on the $S^2$ and $S^3$ spheres

Consider a Riemannian manifold, *i.e.* a manifold endowed with a metric :

$$ds^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta \quad (0.114)$$

---

<sup>4</sup>which does not mean that there are no other irreducible representations; for example “unphysical” representations where  $P^2 = -M^2 < 0$

with a metric tensor  $g$  and (local) coordinates  $\xi^\alpha$ ,  $\alpha = 1, \dots, n$ ;  $n$  is the dimension of the manifold. This  $ds^2$  must be invariant under changes of coordinates,  $\xi \rightarrow \xi'$ , which dictates the change of the tensor  $g$

$$\xi \mapsto \xi' , \quad g \mapsto g' \quad : \quad g'_{\alpha'\beta'} = \frac{\partial \xi^\alpha}{\partial \xi'^{\alpha'}} \frac{\partial \xi^\beta}{\partial \xi'^{\beta'}} g_{\alpha\beta} , \quad (0.115)$$

meaning that  $g$  is a covariant rank-2 tensor. The metric tensor is assumed to be non singular, *i.e.* invertible, and its inverse tensor is denoted with upper indices

$$g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma . \quad (0.116)$$

Also, its determinant is traditionnally denoted  $g$

$$g = \det(g_{\alpha\beta}) . \quad (0.117)$$

There is then a general method for constructing a volume element on the manifold (*i.e.* an integration measure) and a Laplacian, both invariant under changes of coordinates

$$\begin{aligned} d\mu(\xi) &= \sqrt{g} \prod_{\alpha=1}^n d\xi^\alpha \\ \Delta &= \frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g} g^{\alpha\beta} \partial_\beta \end{aligned} \quad (0.118)$$

where  $\partial_\alpha$  is a shorthand notation for the differential operator  $\frac{\partial}{\partial \xi^\alpha}$ .

Exercise: check that  $d\mu(\xi)$  and  $\Delta$  are invariant under a change of coordinates  $\xi \mapsto \xi'$ .

This may be applied in many contexts, and will be used in Chap. 1 to define an integration measure on compact Lie groups.

Let us apply it here to the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . In spherical coordinates, one writes

$$ds^2 = dr^2 + r^2 d\Omega^2$$

where  $d\Omega$  is a generic notation that collects all the angular variables. The metric tensor is thus of the general form  $\begin{pmatrix} 1 & 0 \\ 0 & r^2 A \end{pmatrix}$  with a  $(n-1) \times (n-1)$  matrix  $A$  which is  $r$ -independent and depends only on angular variables. The latter give rise to the Laplacian on the unit sphere  $S^{n-1}$ , denoted  $\Delta_{S^{n-1}}$ ;  $\sqrt{g} = r^{n-1} \sqrt{\det A}$ ; and (0.118) tells us that the Laplacian on  $\mathbb{R}^n$  takes the general form

$$\Delta_{\mathbb{R}^n} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} .$$

Let us write more explicit formulae for the  $S^2$  and  $S^3$  unit spheres. Consider first the unit sphere  $S^2$  with angular coordinates  $0 \leq \theta \leq \pi$ ,  $0 \leq \phi \leq 2\pi$  (Fig. 1). We thus have

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2 \theta d\phi^2 \\ \sqrt{g} &= \sin \theta \end{aligned}$$

$$\begin{aligned} d\mu(x) &= \sin \theta \, d\theta \, d\phi \\ \Delta_{S^2} &= \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} . \end{aligned} \quad (0.119)$$

The generators  $J_i$  read (see (0.46))

$$\begin{aligned} J_3 &= -i \frac{\partial}{\partial \phi} \\ J_1 &= -i \left[ -\cos \phi \cotg \theta \frac{\partial}{\partial \phi} - \sin \phi \frac{\partial}{\partial \theta} \right] \\ J_2 &= -i \left[ -\sin \phi \cotg \theta \frac{\partial}{\partial \phi} + \cos \phi \frac{\partial}{\partial \theta} \right] \end{aligned} \quad (0.120)$$

and one verifies that  $-\Delta_{S^2} = \vec{J}^2 = J_1^2 + J_2^2 + J_3^2$ .

For the unit sphere  $S^3$  one finds similar formulas. In the parametrization (0.12), one takes for example

$$ds^2 = \frac{1}{2} \text{tr} \, dU dU^\dagger = \left( d\frac{\psi}{2} \right)^2 + \sin^2 \frac{\psi}{2} (d\theta^2 + \sin^2 \theta \, d\phi^2) \quad (0.121)$$

invariant under  $U \rightarrow UV$ ,  $U \rightarrow VU$  or  $U \rightarrow U^{-1}$ , whence a measure invariant under the same transformations

$$d\mu(U) = \frac{1}{2} \left( \sin \frac{\psi}{2} \right)^2 \sin \theta \, d\psi \, d\theta \, d\phi . \quad (0.122)$$

In the Euler angles parametrization,

$$U = e^{-i\alpha \frac{\sigma_3}{2}} e^{-i\beta \frac{\sigma_2}{2}} e^{-i\gamma \frac{\sigma_3}{2}} \quad (0.123)$$

thus

$$ds^2 = \frac{1}{2} \text{tr} \, dU dU^\dagger = \frac{1}{4} (d\alpha^2 + 2d\alpha d\gamma \cos \beta + d\gamma^2 + d\beta^2) \quad (0.124)$$

and with  $\sqrt{g} = \sin \beta$  one computes

$$d\mu(U) = \frac{1}{8} \sin \beta \, d\alpha \, d\beta \, d\gamma \quad (0.125)$$

$$\Delta_{S^3} = \frac{4}{\sin^2 \beta} \left[ \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \gamma^2} + \frac{\partial^2}{\partial \alpha \partial \gamma} \right] + \frac{4}{\sin \beta} \frac{\partial}{\partial \beta} \sin \beta \frac{\partial}{\partial \sin \beta} . \quad (0.126)$$

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# Chapter 1

## Groups. Lie groups and Lie algebras

### 1.1 Generalities on groups

#### 1.1.1 Definitions and first examples

Let us consider a group  $G$ , with an operation denoted  $\cdot$ ,  $\times$  or  $+$  depending on the case, a neutral element (or “identity”)  $e$  (or 1 or  $I$  or 0), and an inverse  $g^{-1}$  (or  $-a$ ). If the operation is commutative, the group is called *abelian*. If the group is finite, *i.e.* has a finite number of elements, we call that number the *order* of the group and denote it by  $|G|$ . In these lectures we will be mainly interested in infinite groups, discrete or continuous.

**Examples** (that the physicist may encounter ...)

#### 1. Finite groups

- the cyclic group  $\mathbb{Z}_p$  of order  $p$ , considered geometrically as the invariance rotation group of a circle with  $p$  equidistant marked points, or as the multiplicative group of  $p$ -th roots of the unity,  $\{e^{2i\pi q/p}\}$ ,  $q = 0, 1, \dots, p-1$ , or as the additive group of integers modulo  $p$ ;
- the groups of rotation invariance and the groups of rotation *and* reflexion invariance of regular solids or of regular lattices, of great importance in solid state physics and crystallography;
- the permutation group  $S_n$  of  $n$  objects, called also the *symmetric group*, of order  $n!$ ; etc.

#### 2. Discrete infinite groups.

The simplest example is the additive group  $\mathbb{Z}$ . Let us also mention the translation groups of regular lattices, or the *space groups* in crystallography, which include all isometries (rotations, translations, reflections and their products) leaving a crystal invariant...

Also the groups generated by reflexions in a finite number of hyperplanes of the Euclidean space  $\mathbb{R}^n$ , that are finite or infinite depending on the *arrangement* of these hyperplanes, see Weyl groups in Chap. 4.

Another important example is the modular group  $PSL(2, \mathbb{Z})$  of matrices  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with integer coefficients, of determinant 1,  $ad - bc = 1$ , with matrices  $A$  and  $-A$  identified. Given a 2-dimensional lattice in the complex plane generated by two complex numbers  $\omega_1$  and  $\omega_2$  of non real ratio (why ?), this group describes the changes of basis  $(\omega_1, \omega_2)^T \rightarrow (\omega'_1, \omega'_2)^T = A(\omega_1, \omega_2)^T$  that leave invariant the area of the elementary cell ( $\Im(\omega_2 \omega_1^*) = \Im(\omega'_2 \omega'^*_1)$ ) and that act on  $\tau = \omega_2/\omega_1$  as  $\tau \rightarrow (a\tau + b)/(c\tau + d)$ . This group plays an important role in mathematics in the study of elliptic functions, modular forms, etc, and in physics, in string theory and conformal field theory ...

The homotopy groups, to be encountered soon, are other examples of discrete groups, finite or infinite...

3. Continuous groups. We shall be dealing only with matrix groups of finite dimension, *i.e.* subgroups of the linear groups  $GL(n, \mathbb{R})$  ou  $GL(n, \mathbb{C})$ , for some  $n$ . In particular

- $U(n)$ , the group of complex unitary matrices,  $UU^\dagger = I$ , which is the invariance group of the sesquilinear form  $(x, y) = \sum x^{*i} y^i$  ;
- $SU(n)$  its unimodular subgroup, of unitary matrices of determinant  $\det U = 1$ ;
- $O(n)$  and  $SO(n)$  are orthogonal groups of invariance of the symmetric bilinear form  $\sum_{i=1}^n x_i y_i$ . Matrices of  $SO(n)$  have determinant 1 ;
- $U(p, q)$ ,  $SU(p, q)$ , resp.  $O(p, q)$ ,  $SO(p, q)$ , invariance groups of a sesquilinear, resp. bilinear form, of signature  $((+)^p, (-)^q)$  (*e.g.* the Lorentz group  $O(1, 3)$ ).

Most often one considers groups  $O(n, \mathbb{R})$ ,  $SO(n, \mathbb{R})$  of matrices with real coefficients, but groups  $O(n, \mathbb{C})$ ,  $SO(n, \mathbb{C})$  of invariance of the same bilinear form over the complex numbers may also play a role.

- $Sp(2n, \mathbb{R})$  : Let  $Z$  be the matrix  $2n \times 2n$  made of a diagonal of  $n$  blocks  $i\sigma_2$ :

$$Z = \text{diag} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and consider the bilinear skew-symmetric form}$$

$$(X, Y) = X^T Z Y = \sum_{i=1}^n (x_{2i-1} y_{2i} - y_{2i-1} x_{2i}). \quad (1.1)$$

The symplectic group  $Sp(2n, \mathbb{R})$  is the group of real  $2n \times 2n$  matrices  $B$  that preserve that form:  $B^T Z B = Z$ . That form appears naturally in Hamiltonian mechanics with the symplectic 2-form  $\omega = \sum_{i=1}^n dp_i \wedge dq_i = \frac{1}{2} Z_{ij} d\xi_i \wedge d\xi_j$  in the coordinates  $\xi = (p_1, q_1, p_2, \dots, q_n)$ ;  $\omega$  is invariant by action of  $Sp(2n, \mathbb{R})$  on  $\xi$ . For  $n = 1$ , verify that  $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$ .

One may also consider the complex symplectic group  $Sp(2n, \mathbb{C})$ . A related group, often denoted  $Sp(n)$  but that I shall denote  $USp(n)$  to avoid confusion with the previous ones, the *unitary symplectic group*, is the invariance group of a Hermitian quaternionic form,  $USp(n) = U(2n) \cap Sp(2n, \mathbb{C})$ . See Appendix A.

- the group of “motions” in  $\mathbb{R}^3$  – compositions of  $O(3)$  transformations and translations –, and groups obtained by adjoining dilatations, and then inversions with respect to a point;
- the group of *conformal transformations*, *i.e.* angle preserving, in  $\mathbb{R}^n$  (see Problem at the end of this chapter).
- the Galilean group of transformations  $\mathbf{x}' = O\mathbf{x} + \mathbf{v}t + \mathbf{x}_0$ ,  $t' = t + t_0$ ,  $O \in SO(3)$ ;
- the Poincaré group, in which translations are adjoined to the Lorentz group  $O(1, 3)$ ,
- etc etc.

### 1.1.2 Conjugacy classes of a group

In a group  $G$  we define the following equivalence relation:

$$a \sim b \text{ iff } \exists g \in G : a = g.b.g^{-1} \quad (1.2)$$

and the elements  $a$  et  $b$  are said to be *conjugate*.

The equivalence classes (*conjugacy classes*) that follow provide a partition of  $G$ , since any element belongs to a unique class. For a finite group, the different classes generally have different orders (or cardinalities). For instance, the class of the neutral element  $e$  has a unique element,  $e$  itself.

We have already noted (in Chap. 0) that in the rotation group  $SO(3)$ , a conjugacy class is characterized by the rotation angle  $\psi$  (around some unitary vector  $\mathbf{n}$ ). But this notion is also familiar in the group  $U(n)$ , where a class is characterized by an unordered  $n$ -tuple of eigenvalues  $(e^{i\alpha_1}, \dots, e^{i\alpha_n})$ . This notion of class plays an important role in the discussion of representations of groups and will be abundantly illustrated in the following.

What are the conjugacy classes in the symmetric group  $S_n$ ? One proves easily that any permutation  $\sigma$  of  $S_n$  decomposes into a product of cycles (cyclic permutations) on distinct elements. (To show that, construct the cycle  $(1, \sigma(1), \sigma^2(1), \dots)$ ; then, once back to 1, construct another cycle starting from a number not yet reached, etc.). Finally if  $\sigma$  is made of  $p_1$  cycles of length 1,  $p_2$  of length 2, etc, with  $\sum ip_i = n$ , one writes  $\sigma \in [1^{p_1} 2^{p_2} \dots]$ , and one may prove that this decomposition into cycles characterizes the conjugacy classes: two permutations are conjugate iff they have the same decomposition into cycles.

### 1.1.3 Subgroups

The notion of subgroup  $H$ , subset of a group  $G$  itself endowed with a group structure, is familiar. The subgroup is *proper* if it is not identical to  $G$ . If  $H$  is a subgroup, for any  $a \in G$ , the set  $a^{-1}.H.a$  of elements of the form  $a^{-1}.h.a$ ,  $h \in H$  forms also a subgroup, called *conjugate subgroup* to  $H$ .

Examples of particular subgroups are provided by :

- the center  $Z$  :

In a group  $G$ , the *center* is the subset  $Z$  of elements that commute with all other elements of  $G$ :

$$Z = \{a \mid \forall g \in G, a.g = g.a\} \quad (1.3)$$

$Z$  is a subgroup  $G$ , and is proper if  $G$  is nonabelian. Examples: the center of the group  $GL(2, \mathbb{R})$  of regular  $2 \times 2$  matrices is the set of matrices multiple of  $I$ ; the center of  $SU(2)$  is the group  $\mathbb{Z}_2$  of matrices  $\pm I$  (check by direct calculation).

- the centralizer of an element  $a$  :

The *centralizer* (or *commutant*) of a given element  $a$  of  $G$  is the set of elements of  $G$  that commute with  $a$

$$Z_a = \{g \in G \mid a.g = g.a\}. \quad (1.4)$$

The commutant  $Z_a$  is never empty: it contains at least the subgroup generated by  $a$ . The center  $Z$  is the intersection of all the commutants. Example: in the group  $GL(2, \mathbb{R})$ , the commutant of the Pauli matrix  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is the abelian group of matrices of the form  $a\mathbf{I} + b\sigma_1$ ,  $a^2 - b^2 \neq 0$ .

- More generally, given a subset  $S$  of a group  $G$ , its *centralizer*  $Z(S)$  and its *normalizer*  $N(S)$  are the subgroups commuting respectively individually with every element of  $S$  or globally with  $S$  as a whole

$$Z(S) = \{y : \forall s \in S \quad y.s = s.y\} \quad (1.5)$$

$$N(S) = \{x : x^{-1}.S.x = S\} . \quad (1.6)$$

### 1.1.4 Homomorphism of a group $G$ into a group $G'$

A *homomorphism* of a group  $G$  into a group  $G'$  is a map  $\rho$  of  $G$  into  $G'$  which respects the composition law:

$$\forall g, h \in G, \quad \rho(g.h) = \rho(g).\rho(h) \quad (1.7)$$

In particular,  $\rho$  maps the neutral element of  $G$  onto that of  $G'$ , and the inverse of  $g$  onto that of  $g' = \rho(g)$ :  $\rho(g^{-1}) = (\rho(g))^{-1}$ .

An example of homomorphism that we shall study in great detail is that of a linear representation of a group, whose definition has been given in Chap. 0 and that we return to in Chap. 2.

The *kernel* of the homomorphism, denoted  $\ker \rho$ , is the set of *preimages* (or *antecedents*) of the neutral element in  $G'$ :  $\ker \rho = \{x \in G : \rho(x) = e'\}$ . It is a subgroup of  $G$ .

For example, the parity (or signature) of a permutation of  $S_n$  defines a homomorphism from  $S_n$  into  $\mathbb{Z}_2$ . Its kernel is made of even permutations: this is *alternating group*  $A_n$  of order  $n!/2$ .

### 1.1.5 Cosets with respect to a subgroup

Consider a subgroup  $H$  of a group  $G$ . We define the following relation between elements of  $G$  :

$$g \sim g' \iff g.g'^{-1} \in H, \quad (1.8)$$

which may also be rewritten as

$$g \sim g' \iff \exists h \in H : g = h.g' \text{ or equivalently } g \in H.g' . \quad (1.9)$$

This is an equivalence relation (check !), called the *right equivalence*. One defines in a similar way the *left equivalence* by

$$g \sim_L g' \iff g^{-1}.g' \in H \iff g \in g'.H. \quad (1.10)$$

This relation (say, right) defines equivalence classes that give a partition of  $G$ ; if  $g_j$  is a representative of class  $j$ , that class, called *right-coset*, may be denoted  $H.g_j$ . The elements of  $H$  form by themselves a coset. The set of (say right) cosets is denoted  $G/H$  and called the (right) coset “space”. Its cardinality (the number of cosets) is called the *index* of  $H$  in  $G$  and is denoted

by  $|G : H|$ . For example, the (additive) group of even integers  $H = 2\mathbb{Z}$  is of finite index 2 in  $G = \mathbb{Z}$ . In contrast,  $\mathbb{Z}$  is of infinite index in  $\mathbb{R}$ .

If  $H$  is of finite order  $|H|$ , all cosets have  $|H|$  elements, and if  $G$  is itself of finite order  $|G|$ , it is partitioned into  $|G : H| = |G|/|H|$  classes, and one obtains the Lagrange theorem as a corollary: the order  $|H|$  of any subgroup  $H$  divides that of  $G$ , and the index  $|G : H| = |G|/|H|$  is the order (=cardinality) of the coset space  $G/H$ .

The left equivalence gives rise in general to a different partition, but with the same index. For example, the group  $S_3$  has a  $\mathbb{Z}_2$  subgroup generated by the permutation of the two elements 1 et 2. Exercise: check that the left and right cosets do not coincide.

### 1.1.6 Invariant subgroups

Consider a group  $G$  with a subgroup  $H$ .  $H$  is an *invariant subgroup* (one also says *normal*) if one of the following equivalent properties holds true

- $\forall g \in G, \forall h \in H, ghg^{-1} \in H$ ;
- left and right cosets coincide;
- $H$  is equal to all its conjugates  $\forall g \in G, gHg^{-1} = H$ .

Exercise: check the equivalence of these three definitions.

The important property to remember is the following:

- If  $H$  is an invariant subgroup  $G$ , the coset space  $G/H$  may be given a group structure, and is called the *quotient group*.

Note that in general one cannot consider the *quotient group*  $G/H$  as a subgroup of  $G$ .

Let us sketch the proof. If  $g_1 \sim g'_1$  and  $g_2 \sim g'_2$ ,  $\exists h_1, h_2 \in H : g_1 = h_1.g'_1, g_2 = g'_2.h_2$ , hence  $g_1.g_2 = h_1.(g'_1.g'_2).h_2$  i.e.  $g_1.g_2 \sim g'_1.g'_2$  et  $g_1^{-1} = g'^{-1}_1.h_1^{-1} \sim g'^{-1}_1$ . The equivalence relation is thus compatible with the composition and inverse operations, and if  $[g_1]$  and  $[g_2]$  denote two cosets, one defines  $[g_1].[g_2] = [g_1.g_2]$  where on the right hand side (rhs), one takes any representative  $g_1$  of the coset  $[g_1]$  and  $g_2$  of  $[g_2]$ ; and likewise for the inverse. Thus the group structure passes to the coset space. The coset made by  $H$  is the neutral element in the quotient group.

Example of an invariant subgroup: The *kernel* of a homomorphism  $\rho$  of  $G$  into  $G'$  is an invariant subgroup; show that the quotient group is isomorphic to the image  $\rho(G) \subset G'$  of  $G$  by  $\rho$ .

### 1.1.7 Simple, semi-simple groups

A group is *simple* if it has no non-trivial invariant subgroup (non trivial, i.e. different from  $\{e\}$  and from  $G$  itself). A group is *semi-simple* if it has no non-trivial *abelian* invariant subgroup. Any simple group is obviously semi-simple.

This notion is important in representation theory and in the classification of groups.

Examples : The rotation group in two dimensions is not simple, and not even semi-simple (why?). The group  $SO(3)$  is simple (non trivial proof, see below, section 1.2.2). The group

$SU(2)$  is neither simple nor semi-simple, as it contains the invariant subgroup  $\{I, -I\}$ . The group  $S_n$  is not simple, for  $n > 2$  (why?).

### Direct, semi-direct product

Consider two groups  $G_1$  and  $G_2$  and their *direct product*  $G = G_1 \times G_2$ : it is the set of pairs  $(g_1, g_2)$  endowed with the natural product  $(g'_1, g'_2) \cdot (g_1, g_2) = (g'_1 \cdot g_1, g'_2 \cdot g_2)$ . Obviously its subgroups  $\{(g_1, e)\} \simeq G_1$  and  $\{(e, g_2)\} \simeq G_2$  are invariant subgroups, and  $G$  is not simple.

A more subtle construction appeals to the automorphism group of  $G_1$  denoted  $\text{Aut}(G_1)$ : this is the group of bijections  $\beta$  of  $G_1$  into itself that respect its product (group homomorphism):  $\beta(g'_1 g_1) = \beta(g'_1) \beta(g_1)$ . Suppose there is a group homomorphism  $\varphi$  from another group  $G_2$  into  $\text{Aut}(G_1)$ :  $\forall g_2 \in G_2, \varphi(g_2) \in \text{Aut}(G_1)$ . We now define on pairs  $(g_1, g_2)$  the following product

$$(g'_1, g'_2) \cdot (g_1, g_2) = (g'_1 \cdot \varphi(g'_2)g_1, g'_2 \cdot g_2).$$

Exercise: show that this defines on these pairs a group structure. This is the semi-direct product of  $G_1$  and  $G_2$  (for a given  $\varphi$ ) and is denoted  $G_1 \rtimes_{\varphi} G_2$ . Check that the subgroup  $\{(g_1, e)\} \simeq G_1$  is an invariant subgroup of  $G$ .

Examples : the group of (orientation preserving) motions, generated by translations and rotations in Euclidean  $\mathbb{R}^n$ , is the semi-direct product of  $\mathbb{R}^n \rtimes SO(n)$ , with  $(\vec{a}', R')(\vec{a}, R) = (\vec{a}' + R'\vec{a}, R'R)$ . Likewise the Poincaré group in Minkowski space is the semi-direct product  $\mathbb{R}^4 \rtimes \mathcal{L}$ .

## 1.2 Continuous groups. Topological properties. Lie groups.

A continuous group (one also says a *topological group*) is a topological space (hence endowed with a basis of neighbourhoods that allows us to define notions of continuity etc<sup>1</sup>) with a group structure, such that the composition and inverse operations  $(g, h) \mapsto g \cdot h$  et  $g \mapsto g^{-1}$  are continuous functions. In other words, if  $g'$  is nearby  $g$  (in the sense of the topology of  $G$ ), and  $h'$  nearby  $h$ , then  $g' \cdot h'$  is nearby  $g \cdot h$  and  $g'^{-1}$  is nearby  $g^{-1}$ .

The matrix groups presented at the beginning of this chapter all belong to this class of topological groups, but there are also groups of “infinite dimension” like the group of diffeomorphisms invoked in General Relativity, or of gauge transformations in gauge theories.

Let us first study some topological properties of these continuous groups.

### 1.2.1 Connectivity

A group may be connected or not. If  $G$  is not connected, the *connected component of the identity* (i.e. of the neutral element) is an invariant subgroup.

One may be interested in the connectivity in the general topological sense (a topological space  $E$  is connected if its only subspaces that are both open and closed are  $E$  and  $\emptyset$ ), but we shall be mainly concerned by the *arc connectivity*: for any pair of points, there exists a continuous path in the space (here the group) that joins them. Show that the connected component of the identity is an invariant subgroup for both definitions. Ref. [K-S, Po].

<sup>1</sup>See Appendix B for a reminder of some points of vocabulary...

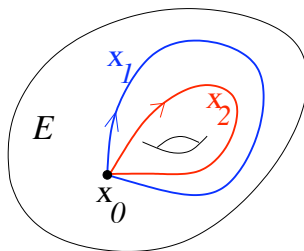


Figure 1.1: The paths  $x_1$  et  $x_2$  are homotopic. But none of them is homotopic to the “trivial” path that stays at  $x_0$ . The space is not simply connected.

Examples.  $O(3)$  is disconnected and the connected component of the identity is  $SO(3)$ ; for the Lorentz group  $\mathcal{L}=O(1,3)$ , the connected component of  $I$  is its proper orthochronous subgroup  $\mathcal{L}_+^\uparrow$ , (see Chap. 0), the other “sheets” then result from the application on it of parity  $P$ , of time reversal  $T$  and their product  $PT$ ...

### 1.2.2 Simple connectivity. Homotopy group. Universal covering

The notion of *simple connectivity* should not be mistaken for the previous one.

As it does not apply only to groups, consider first an arbitrary topological space  $E$ . Let us consider closed paths (or “loops”) drawn in the space, with a fixed end-point  $x_0$ , *i.e.* continuous maps  $x(t)$  from  $[0, 1]$  into  $E$  such that  $x(0) = x(1) = x_0$ . Given two such closed paths  $x_1(\cdot)$  et  $x_2(\cdot)$  from  $x_0$  to  $x_0$ , can one deform them continuously into one another? In other words, is there a continuous function  $f(t, \xi)$  of two variables  $t, \xi \in [0, 1]$ , taking its values in the space  $E$ , such that

$$\begin{aligned} \forall \xi \in [0, 1] \quad f(0, \xi) &= f(1, \xi) = x_0 & : \text{closed paths} \\ \forall t \in [0, 1] \quad f(t, 0) &= x_1(t) \quad f(t, 1) = x_2(t) & : \text{interpolation} . \end{aligned} \quad (1.11)$$

If this is the case, one says that the paths  $x_1$  and  $x_2$  are *homotopic* (this is an equivalence relation between paths), or equivalently that they belong to the same *homotopy class*, see fig. 1.1.

One may also *compose* paths: If  $x_1(\cdot)$  and  $x_2(\cdot)$  are two paths from  $x_0$  to  $x_0$ , their product  $x_2 \circ x_1$  also goes from  $x_0$  to  $x_0$  by following first  $x_1$  and then  $x_2$ . The inverse path of  $x_1(\cdot)$  for that composition is the same path but followed in the reverse direction:  $x_1^{-1}(t) := x_1(1 - t)$ . Both the composition and the inverse are compatible with homotopy: if  $x_1 \sim x'_1$  and  $x_2 \sim x'_2$ , then  $x_2 \circ x_1 \sim x'_2 \circ x'_1$  and  $x_1^{-1} \sim x'^{-1}_1$ . These operations thus pass to classes, giving the set of homotopy classes a group structure: this is *homotopy group*  $\pi_1(E, x_0)$ . Hence, a representative of the identity class is given by the “trivial” path,  $x(t) = x_0, \forall t$ . One finally shows that in a connected space, homotopy groups relative to different end-points  $x_0$  are isomorphic; if  $E$  is a connected group, see below, one may take for example the base point  $x_0$  to be the identity  $x_0 = e$ . One may thus talk of **the** homotopy group (or *fundamental group*)  $\pi_1(E)$ . For more details, see for example [Po], [DNF].

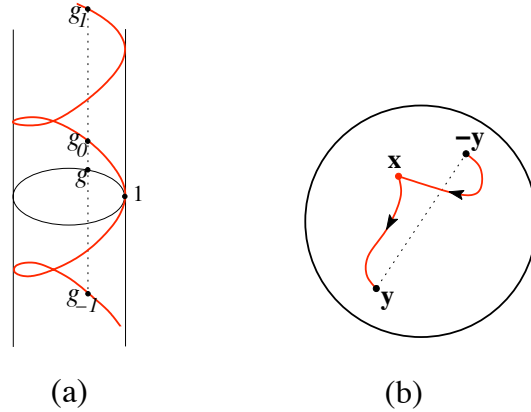


Figure 1.2: (a) The group  $U(1)$ , identified with the circle and its universal covering group  $\mathbb{R}$ , identified with the helix. An element  $g \in U(1)$  is lifted to points  $\cdots, g_{-1}, g_0, g_1, \cdots$  on the helix. (b) In the ball  $B^3$  representing  $SO(3)$ , the points  $\mathbf{y}$  and  $-\mathbf{y}$  of the surface are identified. A path going from  $\mathbf{x}$  to  $\mathbf{x}$  via  $\mathbf{y}$  and  $-\mathbf{y}$  is thus closed but non contractible:  $SO(3)$  is non simply connected.

If all paths from  $x_0$  to  $x_0$  may be continuously contracted into the trivial path  $\{x_0\}$ ,  $\pi_1(E)$  is trivial, and  $E$  is said to be *simply connected*. In the opposite case, one may prove and we shall admit that one may construct a space  $\tilde{E}$ , called *the universal covering space* of  $E$ , such that  $\tilde{E}$  is simply connected and that **locally**,  $E$  and  $\tilde{E}$  are *homeomorphic*. This means that there exists a continuous and surjective mapping  $p$  from  $\tilde{E}$  to  $E$  such that any point  $x$  in  $\tilde{E}$  has a neighborhood  $V_x$  and that  $V_x \mapsto p(V_x)$  is a homeomorphism, *i.e.* a bicontinuous bijection<sup>2</sup>. The universal covering space  $\tilde{E}$  of  $E$  is unique (up to a homeomorphism).

Let us now restrict ourselves to the case where  $E = G$ , a topological group. Then one shows that its covering  $\tilde{G}$  is also a group, the *universal covering group*, and moreover, that the map  $p$  is a homomorphism of  $\tilde{G}$  into  $G$ . Its kernel which is an invariant subgroup of  $\tilde{G}$ , is proved to be isomorphic to the homotopy group  $\pi_1(G)$  ([Po], sect. 51). The quotient group is isomorphic to  $G$

$$\tilde{G}/\pi_1(G) \simeq G, \quad (1.12)$$

(according to a general property of the quotient group by the kernel of a homomorphism, cf. sect. 1.1.6).

One may construct the universal covering group  $\tilde{G}$  by considering paths that join the identity  $e$  to a point  $g$ , and their equivalence classes under continuous deformation **with fixed ends**.  $\tilde{G}$  is the set of these equivalence classes. It is a group for the multiplication of paths defined as follows: if two paths  $g_1(t)$  and  $g_2(t)$  join  $e$  to  $g_1$  and to  $g_2$  respectively, the path  $g_1(t).g_2(t)$  joins  $e$  to  $g_1.g_2$ . This composition law is compatible with equivalence and gives  $\tilde{G}$  a group structure and one shows that  $\tilde{G}$  is simply connected (cf. [Po] sect. 51). The projection  $p$  of  $\tilde{G}$  into  $G$  associates with any class of paths their common end-point. One may verify that this is indeed a local homeomorphism and a group homomorphism, and that its kernel is the homotopy group  $\pi_1(G)$ .

Example : The group  $G = U(1)$  of complex numbers of modulus 1, seen as the unit circle  $S^1$ , is non simply connected: a path from the identity 1 to 1 may wind an arbitrary number

<sup>2</sup>“bicontinuous” means that the map and its inverse are both continuous.

of times around the circle and this (positive or negative) winding number characterizes the different homotopy classes: the homotopy group is  $\pi_1(U(1)) = \mathbb{Z}$ . The group  $\tilde{G}$  is nothing else than the additive group  $\mathbb{R}$  and may be visualised as a helix above  $U(1)$ . The quotient is  $\mathbb{R}/\mathbb{Z} \simeq U(1)$ , which must be interpreted as the fact that a point of  $U(1)$ , *i.e.* an angle, is a real number modulo an integer multiple of  $2\pi$ . One may also say that  $\pi_1(S^1) = \mathbb{Z}$ . More generally one may convince oneself that for spheres,  $\pi_1(S^n)$  is trivial (all loops are contractible) as soon as  $n > 1$ .

Another fundamental example: The rotation group  $SO(3)$  is not simply connected, as foreseen in Chap. 0. To see this fact, represent the rotation  $R_{\mathbf{n}}(\psi)$  by the point  $\mathbf{x} = \tan \frac{\psi}{4} \mathbf{n}$  of an auxiliary space  $\mathbb{R}^3$ ; all these points are in the ball  $B^3$  of radius 1, with the identity rotation at the center and rotations of angle  $\pi$  on the sphere  $S^2 = \partial B^3$ , but because of  $R_{\mathbf{n}}(\pi) = R_{-\mathbf{n}}(\pi)$ , (see Chap 0, sect. 1.1), diametrically opposed points must be identified. It follows that there exists in  $SO(3)$  closed loops that are non contractible: a path from  $\mathbf{x}$  to  $\mathbf{x}$  passing through two diametrically opposed points on the sphere  $S^2$  must be considered as closed but is not contractible (Fig. 1.2.b). There exist two classes of non homotopic closed loops from  $\mathbf{x}$  to  $\mathbf{x}$  and the group  $SO(3)$  is doubly connected, *i.e.* its homotopy group is  $\pi_1(SO(3)) = \mathbb{Z}_2$ . In fact, we already know the universal covering group of  $SO(3)$ : it is the group  $SU(2)$ , which has been shown to be homeomorphic to the sphere  $S^3$ , hence is simply connected, and for which there exists a homomorphism mapping it to  $SO(3)$ , according to  $\pm U_{\mathbf{n}}(\psi) = \pm(\cos \frac{\psi}{2} - i \sin \frac{\psi}{2} \sigma \cdot \mathbf{n}) \mapsto R_{\mathbf{n}}(\psi)$ , see Chap. 0, sect. 1.2.

This property of  $SO(3)$  to be non simply connected may be illustrated by various home experiments, the precise interpretation of which may not be obvious, such as “Dirac’s belt” and “Feynman’s plate”, see <http://gregegan.customer.netSPACE.net.au/APPLETS/21/21.html> and <http://www.math.utah.edu/~palais/links.html> for nice animations, and V. Stojanoska et O. Stoytchev, *Mathematical Magazine*, **81**, 2008, 345-357, for a detailed discussion involving the *braid group*.

The same visualisation of rotations by the interior of the unit ball also permits to understand the above assertion that the group  $SO(3)$  is simple. Suppose it is not, and let  $R = R_{\mathbf{n}}(\psi)$  be an element of an invariant subgroup of  $SO(3)$ , which also contains all the conjugates of  $R$  (by definition of an invariant subgroup). These conjugates are represented by points of the sphere of radius  $\tan \psi/4$ . The invariant subgroup containing  $R_{\mathbf{n}}(\psi)$  and points that are arbitrarily close to its inverse  $R_{-\mathbf{n}}(\psi)$  contains also points that are arbitrarily close to the identity, which by conjugation, fill a small ball in the vicinity of the identity. It remains to show that the products of such elements fill all the bowl, *i.e.* that the invariant subgroup may only be  $SO(3)$  itself; this is in fact true for any connected Lie group, as we shall see below.

Other examples: classical groups. One may prove that

- the groups  $SU(n)$  are all simply connected, for any  $n$ , whereas  $\pi_1(U(n)) = \mathbb{Z}$ ;
- for the group  $SO(2) \cong U(1)$ , we have seen that  $\pi_1(SO(2)) = \mathbb{Z}$ ;
- for any  $n > 2$ ,  $SO(n)$  is doubly connected,  $\pi_1(SO(n)) = \mathbb{Z}_2$ , and its covering group is called  $Spin(n)$ . Hence  $Spin(3) = SU(2)$ .

The notion of homotopy, *i.e.* of continuous deformation, that we have just applied to loops, *i.e.* to maps of  $S^1$  into a manifold  $\mathcal{V}$  (a group  $G$  here), may be extended to maps of a sphere  $S^n$  into  $\mathcal{V}$ . Even though the composition of such maps is less easy to visualise, it may be defined and is again compatible with homotopy, leading to the definition of the homotopy group  $\pi_n(\mathcal{V})$ . For example  $\pi_n(S^n) = \mathbb{Z}$ . See [DNF] for more details and the determination of these groups  $\pi_n$ . This notion is important in physics to describe topological defects,

solitons, instantons, monopoles, etc. See Fig. 1.3 for *vortex* and *anti-vortex* configurations of unit vectors in 2 dimensions, of respective winding number (or *vorticity*)  $\pm 1$ .

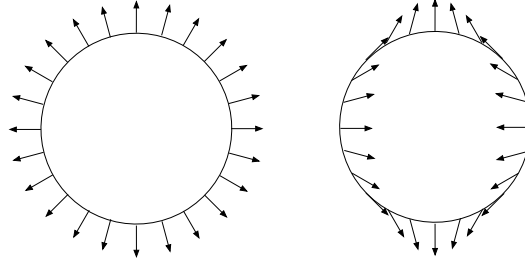


Figure 1.3: Two configurations of unit vectors realizing homotopically non trivial mappings  $S^1 \rightarrow S^1$ . Those are respectively the *vortex* and *anti-vortex* of the XY model of statistical mechanics, see for example <http://www.ibiblio.org/e-notes/Perc/xy.htm> for more details and nice figures.

### 1.2.3 Compact and non compact groups

If the domain  $\mathcal{D}$  in which the parameters of the group  $G$  take their values is compact,  $G$  is said to be a *compact group*.

Recall some of the many equivalent characterizations of a compact space  $E$ . From any infinite sequence one may extract a converging subsequence. Given a covering of  $E$  by a set of open sets  $U_i$ ,  $E$  may be covered by a *finite* number of them. Any continuous function on  $E$  is bounded, etc. For a subset  $\mathcal{D}$  of  $\mathbb{R}^d$ , being compact is equivalent to being closed and bounded.

Examples. The unitary groups  $U(n)$  and their subgroups  $SU(n)$ ,  $O(n)$ ,  $SO(n)$ ,  $USp(n/2)$  ( $n$  even), are compact. The groups  $SL(n, \mathbb{R})$  or  $SL(n, \mathbb{C})$ ,  $Sp(n, \mathbb{R})$  or  $Sp(n, \mathbb{C})$ , the translation group in  $\mathbb{R}^n$ , the Galilean group, the Lorentz and Poincaré groups are not, why ?

### 1.2.4 Invariant measure

When dealing with a *finite* group, one often considers sums over all elements of the group and makes use of the “rearrangement lemma”, in which one writes

$$\sum_{g \in G} f(g'g) = \sum_{h=g'g \in G} f(h) = \sum_{g \in G} f(g) ,$$

(left invariance), the same thing with  $g'g$  changed into  $gg'$  (right invariance), and also

$$\sum_{g \in G} f(g^{-1}) = \sum_{g^{-1} \in G} f(g^{-1}) = \sum_{g \in G} f(g) .$$

Can one do similar operations in continuous groups, the finite sum being replaced by an integral, which converges and enjoys the same invariances ? This requires the existence of an integration measure, with left and right invariance, and invariance under inversion:

$$d\mu(g) = d\mu(g'.g) = d\mu(g.g') = d\mu(g^{-1})$$

such that  $\int d\mu(g)f(g)$  be finite for any continuous function  $f$  on the group.

One may prove (and we admit) that

- if the group is compact, such a measure exists and is unique up to a normalization.

This is the *Haar measure*.

For example, in the unitary group  $U(n)$ , one may construct explicitly the Haar measure, using the method proposed in chap. 0, Appendix 0: one first defines a metric on  $U(n)$  by writing  $ds^2 = \text{tr } dU.dU^\dagger$  in any parametrization; this metric is invariant under  $U \rightarrow UU'$  or  $U \rightarrow U'U$  and by  $U \rightarrow U^{-1} = U^\dagger$ ; the measure  $d\mu(U)$  that follows has the same properties. See Appendix C for the explicit calculation for  $SU(2)$  and  $U(n)$ , and more details in the TD.

Conversely if the group is non compact, left and right measures may still exist, they may even coincide, (for non compact abelian or semi-simple groups) but their integral over the group diverges.

Thus, if  $G$  is *locally compact*, (*i.e.* any point has a basis of compact neighbourhoods), one proves that there exists a left invariant measure, unique up to a multiplicative constant. There exists also a right invariant measure, but they may not coincide. For example, take

$$G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \middle| x, y \in \mathbb{R}, y > 0 \right\}$$

one easily checks that  $d\mu_L(g) = y^{-2}dxdy$ ,  $d\mu_R(g) = y^{-1}dxdy$  are left and right invariant measures, respectively, and that their integrals diverge. See [Bu].

### 1.2.5 Lie groups

Imposing more structure on a continuous group leads us in a natural way to the notion of *Lie groups*.

According to the usual definition, a Lie group is a topological group which is also a differentiable manifold and such that the composition and inverse operations  $G \times G \rightarrow G$  and  $G \rightarrow G$  are infinitely differentiable functions. One sometimes also requests them to be analytic real functions, *i.e.* functions for which the Taylor series converges to the function. That the two definitions coexist in the literature is a hint that the weakest (infinite differentiability) implies the strongest. In fact, according to a remarkable theorem (Montgomery and Zippin, 1955), much weaker hypotheses suffice to ensure the Lie group property.

A topological connected group which is locally homeomorphic to  $\mathbb{R}^d$ , for some *finite*  $d$ , is a Lie group. In other words, the existence of a *finite* number of local coordinates, together with the properties of being a topological group (continuity of the group operations), are sufficient to imply the analyticity properties ! <sup>3</sup> This shows that the structure of Lie group is quite powerful and rigid. There exist, however, infinite dimensional Lie groups.

To avoid a mathematical discussion unnecessary for our purpose, we shall restrict ourselves to continuous groups of finite size matrices. In such a group, the matrix elements of  $g \in G$  depend continuously on real parameters  $(\xi^1, \xi^2, \dots, \xi^d) \in \mathcal{D} \subset \mathbb{R}^d$ , and in the group operations  $g(\xi'') = g(\xi').g(\xi)$ , and  $g(\xi)^{-1} = g(\xi'')$ , the  $\xi''^i$  are continuous (in fact analytic) functions of the  $\xi^j$  (and  $\xi'^j$ ). Such a group is called a *Lie group*, and  $d$  is its *dimension*.

More precisely, in the spirit of differential geometry, one has in general to introduce several domains  $\mathcal{D}_j$ , with continuous (in fact analytic) transition functions between coordinate charts, etc.

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<sup>3</sup>For an elementary example of such a phenomenon, consider a function  $f$  of one real variable, satisfying  $f(x)f(y) = f(x+y)$ . Under the only assumption of continuity, show that  $f(x) = \exp kx$ , hence that it is analytic !

Examples : all the matrix groups presented in §1.1 are Lie groups. Check that the dimension of  $U(n)$  is  $n^2$ , that of  $SU(n)$  is  $n^2 - 1$ , that  $O(n)$  or  $SO(n)$  is  $n(n-1)/2$ . What is the dimension of  $Sp(2n, \mathbb{R})$  ? of the Galilean group in  $\mathbb{R}^3$ ? of the Lorentz and Poincaré groups ?

Show that  $\dim(Sp(2n, \mathbb{R})) = \dim(USp(n)) = \dim(SO(2n+1))$ . We shall see below in Chap. 3 that this is not an accident.

The study of a Lie group and of its representations involves two steps: first a local study of its tangent space in the vicinity of the identity (its Lie algebra), and then a global study of its topology, *i.e.* an information not provided by the local study.

## 1.3 Local study of a Lie group. Lie algebra

### 1.3.1 Algebras and Lie algebras. Definitions

Let us first recall the definition of an algebra.

An *algebra* is a vector space over a field (for physicists,  $\mathbb{R}$  or  $\mathbb{C}$ ), endowed with a product denoted  $X * Y$ , (not necessarily associative), bilinear in  $X$  and  $Y$

$$(\lambda_1 X_1 + \lambda_2 X_2) * Y = \lambda_1 X_1 * Y + \lambda_2 X_2 * Y \quad (1.13)$$

$$X * (\mu_1 Y_1 + \mu_2 Y_2) = \mu_1 X * Y_1 + \mu_2 X * Y_2 . \quad (1.14)$$

Examples: the set  $M(n, \mathbb{R})$  or  $M(n, \mathbb{C})$  of  $n \times n$  matrices with real, resp. complex coefficients, is an associative algebra for the usual matrix product. The set of vectors of  $\mathbb{R}^3$  is a (non associative !) algebra for the vector product (denoted  $\wedge$  in the French literature, and  $\times$  in the anglo-saxon one) .

A *Lie algebra* is an algebra in which the product, denoted  $[X, Y]$  and called *Lie bracket*, has the additional properties of being antisymmetric and of satisfying the Jacobi identity

$$[X, Y] = -[Y, X] \quad (1.15)$$

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0 . \quad (1.16)$$

Examples : Any associative algebra for a product denoted  $*$ , in particular any matrix algebra, is a Lie algebra for a Lie bracket defined by the commutator

$$[X, Y] = X * Y - Y * X .$$

The bilinearity and antisymmetry properties are obvious, and verifying the Jacobi identity takes one line. Another example: the space  $\mathbb{R}^3$  with the above-mentioned vector product is in fact a Lie algebra, with the Jacobi identity following from the “double vector product” formula,  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ . If we write  $(\mathbf{v} \wedge \mathbf{w})_i = \epsilon_{ijk} v_j w_k$  in terms of the completely antisymmetric tensor  $\epsilon$ , the Jacobi identity is indeed what we encountered in (0.27).

(See Chap. 0, (0.27).)

### 1.3.2 Tangent space in a Lie group

Consider a Lie group  $G$  and a *one-parameter subgroup*  $g(t)$ , where  $t$  is a real parameter taking values in a neighborhood of 0, with  $g(0) = e$ ; in other words,  $g(t)$  is a curve in  $G$ , assumed to be differentiable, and passing through the identity, and one assumes that (for  $t$  near 0)

$$g(t_1)g(t_2) = g(t_1 + t_2) \quad g^{-1}(t) = g(-t) . \quad (1.17)$$

The composition law in this subgroup locally amounts to the addition of parameters  $t$ ; thus, locally, this one-parameter subgroup is isomorphic to the abelian group  $\mathbb{R}$ . It is then natural to differentiate

$$g(t + \delta t) = g(t)g(\delta t) \quad \Leftrightarrow \quad g^{-1}(t)g(t + \delta t) = g(\delta t) . \quad (1.18)$$

As we have chosen to restrict ourselves to matrix groups, (with  $e \equiv I$ , the identity matrix), we may write the linear tangent map in the form

$$g(\delta t) = I + \delta t X + \dots$$

which defines a vector  $X$  in the tangent space. One may also write

$$X = \left. \frac{d}{dt} g(t) \right|_{t=0} , \quad (1.19)$$

this is the velocity at  $t = 0$  (or at  $g = e$ ) along the curve. Equation (1.18) thus reads

$$g'(t) = g(t)X . \quad (1.20)$$

As usual in differentiable geometry, (see Appendix B.3), the tangent space  $T_e G$  at  $e$  to the group  $G$ , which we denote  $\mathfrak{g}$  from now on, is the vector space generated by the tangent vectors to all one-parameter subgroups (*i.e.* all velocity vectors at  $t = 0$ ). If coordinates  $\xi^\alpha$  of  $G$  have been chosen in the vicinity of  $e$  ( $\equiv I$ ), a tangent vector is a differential operator  $X = X^\alpha \frac{\partial}{\partial \xi^\alpha}$ . The dimension (as a vector space) of this tangent space is equal to the dimension of the (group) manifold defined above as the number of (real) parameters  $\dim \mathfrak{g} = \dim G$ .

In the case of a group  $G \subset \text{GL}(n, \mathbb{R})$  to which we are restricting ourselves,  $X \in \mathfrak{g} \subset M(n, \mathbb{R})$ , the set of real  $n \times n$  matrices, and one may carry out all calculations in that algebra. In particular, one may integrate (1.20) as

$$g(t) = \exp tX = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n , \quad (1.21)$$

a converging sum. (In fact, the assumption that the group is a matrix group may be relaxed, provided one makes sense out of the map  $\exp$  from  $\mathfrak{g}$  to  $G$ , a map that enjoys some of the usual properties of the exponential, see Appendix B.4.)

### 1.3.3 Relations between the tangent space $\mathfrak{g}$ and the group $G$

1. If  $G$  is the linear group  $GL(n, \mathbb{R})$ ,  $\mathfrak{g}$  is the algebra of real  $n \times n$  matrices, denoted  $M(n, \mathbb{R})$ . If  $G$  is the group of unitary matrices  $U(n)$ ,  $\mathfrak{g}$  is the space of anti-Hermitian  $n \times n$  matrices. Moreover they are traceless if  $G = SU(n)$ . Likewise, for the orthogonal  $O(n)$  group,  $\mathfrak{g}$  is made of skew-symmetric, hence traceless, matrices.

For the symplectic group  $G = USp(n)$ ,  $\mathfrak{g}$  is generated by “anti-selfdual” quaternionic matrices, see Appendix A. For each of these cases, check that the characteristic property (anti-Hermitian, skew-symmetric, tracelessness, ...) is preserved by the commutator, thus making  $\mathfrak{g}$  a Lie algebra.

2. The exponential map plays an important role in the reconstruction of the Lie group  $G$  from its tangent space  $\mathfrak{g}$ . One may prove, and we admit, that

- the map  $X \in \mathfrak{g} \mapsto e^X \in G$  is bijective in the neighborhood of the identity;
- it is surjective (= every element in  $G$  is reached) if  $G$  is connected and compact;
- it is injective (any  $g \in G$  has only one antecedent) only if  $G$  is simply connected. An example of non-injectivity is provided by  $G = U(1)$ , for which  $\mathfrak{g} = i\mathbb{R}$  and all the  $i(x + 2\pi k)$ ,  $k \in \mathbb{Z}$  have the same image by  $\exp$ . The converse is in general wrong: for example in  $SU(2)$  which is simply connected, if  $\mathbf{n}$  is a unit vector,  $e^{i\pi\mathbf{n} \cdot \boldsymbol{\sigma}} = -I$ , hence all elements  $i\pi \mathbf{n} \cdot \boldsymbol{\sigma}$  of  $\mathfrak{g} = \mathfrak{su}(2)$  have the same image!

★ Example of a non-compact group for which the  $\exp$  map is non surjective:  $G = SL(2, \mathbb{R})$ , for which  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ , the set of real traceless matrices. For any  $A \in \mathfrak{g}$ , hence traceless, use its characteristic equation to show that  $\text{tr } A^{2n+1} = 0$ ,  $\text{tr } A^{2n} = 2(-\det A)^n$ , hence  $\text{tr } e^A = 2 \cosh \sqrt{-\det A} \geq -2$ . There exist in  $G$ , however, matrices of trace  $< -2$ , for instance  $\text{diag}(-2, -\frac{1}{2})$ .

★ For a non compact group, the  $\exp$  map may still be useful. One may prove that any element of a matrix group may be written as the product of a finite number of exponentials of elements in its Lie algebra. [Cornwell p 151].

★ Observe that one still has  $\det e^X = e^{\text{tr } X}$ , a property easily established if  $X$  belongs to the set of diagonalizable matrices. As the latter are dense in  $M(d, \mathbb{R})$ , the property holds true in general.

### 1.3.4 The tangent space as a Lie algebra

Let us now show that the tangent space  $\mathfrak{g}$  of  $G$  at  $e \equiv I$  has a Lie algebra structure. Given two one-parameter groups generated by two independent vectors  $X$  et  $Y$  of  $\mathfrak{g}$ , we measure their lack of commutativity by constructing their *commutator* (in a sense different from the usual one!)  $g = e^{tX}e^{uY}e^{-tX}e^{-uY}$ ; for small  $t \sim u$ , this  $g$  is close to the identity, and may be written  $g = \exp Z$ ,  $Z \in \mathfrak{g}$ . Compute  $Z$  to the first non trivial order

$$\begin{aligned} e^{tX}e^{uY}e^{-tX}e^{-uY} &= (I + tX + \frac{1}{2}t^2X^2)(I + uY + \frac{1}{2}u^2Y^2)(I - tX + \frac{1}{2}t^2X^2)(I - uY + \frac{1}{2}u^2Y^2) \\ &= I + (XY - YX)tu + O(t^3) . \end{aligned} \quad (1.22)$$

The computation has been carried out in the associative algebra of matrices, the neutral element being denoted  $I$ . All the neglected terms are of third order since  $t \sim u$ . To order 2, one thus

sees the appearance of the commutator in the usual sense,  $XY - YX$ , *i.e.* the Lie bracket of matrices  $X$  and  $Y$ . In general, for an arbitrary Lie group, the bracket is defined by

$$e^{tX}e^{uY}e^{-tX}e^{-uY} = e^Z \quad , \quad Z = tu[X, Y] + O(t^3) \quad (1.23)$$

and one proves that this bracket has the properties (1.15) of a Lie bracket.

This fundamental result follows from a detailed discussion of the local form of the group operations in a Lie group ("Lie equations", see for example [OR]).

### • Adjoint map in the Lie algebra $\mathfrak{g}$ . Baker-Campbell-Hausdorff formula

Let us introduce a handy notation. For any  $X \in \mathfrak{g}$ , let  $\text{ad } X$  be the linear operator in the Lie algebra defined by

$$Y \mapsto (\text{ad } X)Y := [X, Y] \quad , \quad (1.24)$$

hence

$$(\text{ad}^p X)Y = [X, [X, \dots [X, Y] \dots]]$$

with  $p$  brackets (commutators).

Given two elements  $X$  and  $Y$  in  $\mathfrak{g}$ , and  $e^X$  and  $e^Y$  the elements they generate in  $G$ , does there exist a  $Z \in \mathfrak{g}$  such that  $e^Xe^Y = e^Z$ ? The answer is yes, at least for  $X$  et  $Y$  small enough.

Note first that if  $[X, Y] = 0$ , the ordinary rules of computation apply and  $Z = X + Y$ . In general, the *Baker-Campbell-Hausdorff formula*, that we admit, gives an explicit expression of  $Z$ .

$$\begin{aligned} e^Xe^Y &= e^Z \\ Z &= X + \int_0^1 dt \psi(\exp \text{ad } X \exp t \text{ad } Y)Y \end{aligned} \quad (1.25)$$

where  $\psi(\cdot)$  is the function

$$\psi(u) = \frac{u \ln u}{u - 1} = 1 + \frac{1}{2}(u - 1) - \frac{1}{6}(u - 1)^2 + \dots \quad , \quad (1.26)$$

which is regular at  $u = 1$ . The first terms in the expansion in powers of  $X$  and  $Y$  read explicitly

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \dots \quad (1.27)$$

This complicated formula has some useful particular cases. Hence if  $X$  et  $Y$  commute with  $[X, Y]$ , (1.25) boils down to

$$e^Xe^Y = e^{X+Y+\frac{1}{2}[X,Y]} = e^{X+Y}e^{\frac{1}{2}[X,Y]} \quad , \quad (1.28)$$

a formula that one may prove directly using the general identity

$$e^XYe^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} \text{ad}^n X Y \quad (1.29)$$

(which is nothing else than the Taylor expansion at  $t = 0$  of  $e^{tX}Ye^{-tX}$  evaluated at  $t = 1$ ), and writing and solving the differential equation satisfied by  $f(t) = e^{tX}e^{tY}$ ,  $f(0) = 1$

$$f'(t) = (X + e^{tX}Ye^{-tX})f(t) \quad (1.30)$$

$$= (X + Y + t[X, Y])f(t) \quad . \quad (1.31)$$

On the other hand, to first order in  $Y$ , one may replace the argument of  $\psi$  in (1.25) by  $\exp \text{ad } X$  and then

$$Z = X + \sum_{n=0}^{\infty} \frac{B_n}{n!} (-1)^n (\text{ad } X)^n Y + O(Y^2) \quad (1.32)$$

where the  $B_n$  are the Bernoulli numbers:  $\frac{t}{e^t-1} = \sum_0 B_n \frac{t^n}{n!}$ ,  $B_0 = 1, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}$  and, beside  $B_1 = -\frac{1}{2}$ , all  $B$  of odd index vanish. Still to first order in  $Y$ , one has also

$$e^{X+Y} = e^X + \int_0^1 dt e^{tX} Y e^{(1-t)X} + O(Y^2)$$

which is obtained by writing and solving the differential equation satisfied by  $F(t) = \exp t(X+Y) \cdot \exp -tX$ .

The convergence of these expressions may be proven for  $X$  et  $Y$  small enough. Note that this BCH formula makes only use of the  $\text{ad}$  map in the Lie algebra, and not of the ordinary matrix multiplication in  $\text{GL}(d, \mathbb{R})$ . This is what makes it a canonical and universal formula.

### 1.3.5 An explicit example: the Lie algebra of $\text{SO}(n)$

From the definition of elements of  $\mathfrak{g}$  as tangent vectors in  $G$  at  $e \equiv I$ , or else from the construction of one-parameter subgroups associated with each  $X \in \mathfrak{g}$ , follows the interpretation of  $X$  as “infinitesimal generator” of the Lie group  $G$ . The actual determination of the Lie algebra of a given Lie group  $G$  may be done in several ways, depending on the way the group is defined or represented.

If one has an explicit parametrization of the elements of  $G$  in terms of  $d$  real parameters, infinitesimal generators are obtained by differentiation wrt these parameters. See in Chap. 0, the explicit cases of  $\text{SO}(3)$  and  $\text{SU}(2)$  treated in that way.

If the group has been defined as the invariance group of some quadratic form in variables  $x$ , one may derive an expression of the infinitesimal generators as differential operators in  $x$ . Let us illustrate it on the group  $\text{O}(n)$ , the invariance group of the form  $\sum_{i=1}^n x_i^2$  in  $\mathbb{R}^n$ . The most general linear transformation leaving that form invariant is  $x \rightarrow x' = O x$ , with  $O$  orthogonal. In an infinitesimal form,  $O = I + \omega$ , and  $\omega = -\omega^T$  is an arbitrary skew-symmetric real matrix. An infinitesimal transformation of the form  $\delta x^i = \omega^i_j x^j$  may also be written

$$\delta x^i = \omega^i_j x^j = -\frac{1}{2} \omega^{kl} J_{kl} x^i \quad (1.33)$$

$$J_{kl} = x^k \partial_l - x^l \partial_k \quad : \quad J_{kl} x^i = x^k \delta_{il} - x^l \delta_{ik} \quad (1.34)$$

(note that we allow to raise and lower freely the indices, thanks to the signature  $(+)^n$  of the metric). This yields an explicit representation of infinitesimal generators of the  $\text{so}(n)$  algebra as differential operators. It is then a simple matter to compute the commutation relations<sup>4</sup>

$$[J_{ij}, J_{kl}] = \delta_{il} J_{jk} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{jk} J_{il} . \quad (1.35)$$

(In other words, the only non-vanishing commutators are of the form  $[J_{ij}, J_{ik}] = -J_{jk}$  for any triplet  $i \neq j \neq k \neq i$ , and those that follow by antisymmetry in the indices.)

<sup>4</sup>Note that wrt to the calculation carried out in the  $\text{O}(1,3)$  group in Chap. 0, § 0.6.2, we have changed our conventions and use here anti-Hermitian generators.

One may proceed in a different way, by using a basis of matrices in the Lie algebra, regarded as the space of skew-symmetric  $n \times n$  matrices. Such a basis is provided by matrices  $A_{ij}$  labelled by pairs of indices  $1 \leq i < j \leq n$ , with matrix elements

$$(A_{ij})_k^l = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} .$$

Hence the matrix  $A_{ij}$  has only two non vanishing (and opposite) elements, at the intersection of the  $i$ -th row and  $j$ -th column and vice versa. Check that these matrices  $A_{ij}$  have commutation relations given by (1.35).

Exercise : repeat this discussion and the computation of commutation relations for the group  $SO(p, q)$  of invariance of the form  $\sum_{i=1}^p x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$ . It is useful to introduce the metric tensor  $g = \text{diag}((+1)^p, (-1)^q)$ .

### A physical application: Noether currents for the “O( $n$ ) model”

A field theory frequently considered (see F. David’s course and Chap. 4) is the  $O(n)$  model. Its Lagrangian, written here in the Euclidean version of the theory and for a real bosonic field  $\phi = \{\phi^k\}$  with  $n$  components,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}(\phi^2)^2 \quad (1.36)$$

is invariant under  $O(n)$  rotations. The Noether currents are derived from infinitesimal transformations of the previous type  $\delta\phi = \sum_{i < j} \delta\omega^{ij} A_{ij}\phi$ , or, in components,  $\delta\phi^k = \sum_{i < j} \delta\omega^{ij} (A_{ij})^k_l \phi^l$ , namely (up to a possible factor)  $j_\mu^{(ij)} = \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi^k} (A_{ij})^k_l \phi^l = \partial_\mu\phi^k (A_{ij})^k_l \phi^l$ . Using the antisymmetry of matrices  $A$  and the Euler–Lagrange equations, show that these currents are indeed divergenceless, which implies the conservation of  $\dim \mathfrak{so}(n) = \frac{1}{2}n(n-1)$  “charges”.

### 1.3.6 An example of infinite dimension: the Virasoro algebra

In these notes, we are restricting our attention to Lie groups and algebras of finite dimension. Let us give here an example of infinite dimension. One considers diffeomorphisms  $z \mapsto z' = f(z)$  where  $f$  is an analytic (holomorphic) function of its argument except maybe at 0 and at infinity. (One also speaks of the “diffeomorphisms of the circle”.) This is obviously a group and an infinite dimensional manifold, which manifests itself in the algebra of infinitesimal diffeomorphisms  $z \mapsto z' = z + \epsilon(z)$ , generated by differential operators  $\ell_n$

$$\ell_n = -z^{n+1} \frac{\partial}{\partial z} , \quad n \in \mathbb{Z} \quad (1.37)$$

which satisfy

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} . \quad (1.38)$$

This Lie algebra is the *Witt algebra*. A modified form of this algebra, with a *central extension* (see Chap. 2), i.e. with an additional “central” generator  $c$  commuting with all generators, is called *Virasoro algebra* and appears naturally in physics. Calling  $L_n$  et  $c$  the generators of that algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \quad [c, L_n] = 0 . \quad (1.39)$$

(The  $L_n$  may be thought of as quantum realizations of the operators  $\ell_n$ , with the  $c$  term resulting from quantum effects. . .)

Check that the Jacobi identity is indeed satisfied by this algebra. One proves that this is the most general central extension of (1.38) that respects the Jacobi identity. Show that the subalgebra generated by  $L_{\pm 1}$ ,  $L_0$  is not affected by the central term. What is the geometric interpretation of the corresponding transformations?

The Virasoro algebra plays a central role in the construction of conformal field theories in 2d and in their application to two-dimensional critical phenomena and to string theory. More details in [DFMS].

## 1.4 Relations between properties of $\mathfrak{g}$ and $G$

Let us examine how properties of  $G$  translate in  $\mathfrak{g}$ .

### 1.4.1 Simplicity, semi-simplicity

Let us define the infinitesimal version of the notion of invariant subgroup. An *ideal* (also sometimes called an invariant subalgebra) in a Lie algebra  $\mathfrak{g}$  is a subspace  $\mathfrak{I}$  of  $\mathfrak{g}$  which is stable under multiplication (defined by the Lie bracket) by any element of  $\mathfrak{g}$ , *i.e.* such that  $[\mathfrak{I}, \mathfrak{g}] \subset \mathfrak{I}$ . The ideal is called *abelian* si  $[\mathfrak{I}, \mathfrak{I}] = \{0\}$ .

A Lie algebra  $\mathfrak{g}$  is *simple* if  $\mathfrak{g}$  has no other ideal than  $\{0\}$ . It is *semi-simple* if  $\mathfrak{g}$  has no other abelian ideal than  $\{0\}$ .

Example. Consider the Lie algebra of  $\text{SO}(4)$ , denoted  $\text{so}(4)$ , see the formulae given in (1.35) for  $\text{so}(n)$ . It is easy to check that the combinations

$$A_1 := \frac{1}{2}(J_{12} - J_{34}), \quad A_2 = \frac{1}{2}(J_{13} + J_{24}), \quad A_3 := \frac{1}{2}(J_{14} - J_{23})$$

commute with

$$B_1 := \frac{1}{2}(J_{12} + J_{34}), \quad B_2 = \frac{1}{2}(-J_{13} + J_{24}), \quad B_3 := \frac{1}{2}(J_{14} + J_{23})$$

and that

$$[A_i, A_j] = \epsilon_{ijk} A_k \quad [B_i, B_j] = \epsilon_{ijk} B_k, \quad [A_i, B_j] = 0$$

where one sees two commuting copies of  $\text{so}(3)$ . One writes  $\text{so}(4) = \text{so}(3) \oplus \text{so}(3)$ . Obviously the algebra  $\text{so}(4)$  is not simple, but it is semi-simple.

Notice the difference between this case of  $\text{so}(4)$  and the case of the algebra  $\text{so}(1,3)$  studied in Chap. 0, § 0.6.2. There, the indefinite signature forced us to complexify the algebra to “decouple” the two copies of the algebra  $\text{so}(3)$ .

One has the following relations

$$G \text{ simple} \implies \mathfrak{g} \text{ simple}$$

$$G \text{ semi-simple} \implies \mathfrak{g} \text{ semi-simple}$$

but the converse is not true ! Several different Lie groups may have the same Lie algebra, *e.g.*  $\text{SO}(3)$  which is simple, and  $\text{SU}(2)$  which is not semi-simple, as seen above in §1.1.7. <sup>5</sup>

### 1.4.2 Compacity. Complexification

A semi-simple Lie algebra is said to be compact if it is the Lie algebra of a compact Lie group. At first sight, this definition looks non intrinsic to the algebra and seems to depend on the Lie group from which it derives. We shall see below that a condition (Cartan criterion) allows to remove this dependance.

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<sup>5</sup>Beware! Some authors call “simple” any Lie group whose Lie algebra is simple. This amounts to making a distinction between the concepts of simple group and simple Lie group. The latter is such that it has no non trivial invariant *Lie group*. Thus the Lie group  $\text{SU}(2)$  is a simple Lie group but not a simple group, as it has an invariant subgroup  $\mathbb{Z}_2$  which is not of Lie type...

At this stage one should examine the issue of complexification. Several distinct groups may have different Lie algebras, that become isomorphic when the parameters are complexified. For instance, the groups  $O(3)$  et  $O(2,1)$ , the first compact, the second non compact, have Lie algebras

$$\begin{aligned} \mathfrak{o}(3) & \left\{ \begin{array}{ll} X_1 = z\partial_y - y\partial_z \\ X_2 = x\partial_z - z\partial_x \\ X_3 = y\partial_x - x\partial_y \end{array} \right. & [X_1, X_2] = y\partial_x - x\partial_y = X_3 \text{ etc} \\ \mathfrak{o}(2,1) & \left\{ \begin{array}{ll} \tilde{X}_1 = z\partial_y + y\partial_z \\ \tilde{X}_2 = x\partial_z + z\partial_x \\ \tilde{X}_3 = y\partial_x - x\partial_y \end{array} \right. & \begin{array}{l} [\tilde{X}_1, \tilde{X}_2] = y\partial_x - x\partial_y = \tilde{X}_3 \\ [\tilde{X}_2, \tilde{X}_3] = -z\partial_y - y\partial_z = -\tilde{X}_1 \\ [\tilde{X}_3, \tilde{X}_1] = -x\partial_z - z\partial_x = -\tilde{X}_2 \end{array} \end{aligned} \quad (1.40)$$

that non isomorphic on the real numbers, but  $i\tilde{X}_1$ ,  $i\tilde{X}_2$  et  $-\tilde{X}_3$  verify the  $\mathfrak{o}(3)$  algebra. The algebras  $\mathfrak{o}(3)$  and  $\mathfrak{o}(2,1)$  are said to have the same complexified form  $\mathfrak{g}_\mathbb{C}$ , or else, to be two real forms of  $\mathfrak{g}_\mathbb{C}$ , but only one of them, namely  $\mathfrak{o}(3)$ , (or  $\mathfrak{so}(3)=\mathfrak{su}(2)$ ), is compact. This complexified form is the  $\mathfrak{sl}(2, \mathbb{C})$  algebra, of which  $\mathfrak{sl}(2, \mathbb{R})$  is another non compact real form. (See Exercise B and TD).

The algebras  $\mathfrak{so}(4)$  and  $\mathfrak{so}(1,3)$  studied above and in Chap. 0 provide another example of two algebras, which are two non-isomorphic real forms of the same complexified form.

Another example is provided by  $\mathfrak{sp}(2n, \mathbb{R})$  et  $\mathfrak{usp}(n)$ . (See Appendix A).

More generally, one may prove ([FH] p. 130) that

- any semi-simple complex Lie algebra has a unique real compact form.

To summarize, local topological properties of the Lie group are transcribed in the Lie algebra. The Lie algebra, however, is unable to capture global topological properties of the group, as we discuss now.

### 1.4.3 Connectivity, simple-connectivity

- If  $G$  is non connected and  $G'$  is the subgroup of the connected component of the identity, the Lie algebras of  $G$  and  $G'$  coincide:  $\mathfrak{g} = \mathfrak{g}'$ .
- If  $G$  is non simply connected, let  $\tilde{G}$  be its universal covering group.  $G$  et  $\tilde{G}$  being locally isomorphic, they have the same Lie algebra. Examples:  $U(1)$  and  $\mathbb{R}$ ;  $SO(3)$  and  $SU(2)$ ;  $SO(1,3)$  and  $SL(2, \mathbb{C})$ .

To summarize:

Given a Lie group  $G$ , we have constructed its Lie algebra. Conversely, a theorem by Lie asserts that any (finite-dimensional) Lie algebra is the Lie algebra of some Lie group [Ki-Jr, p.34]. More precisely, to every Lie algebra  $\mathfrak{g}$  corresponds a unique connected and simply connected Lie group  $G$ , whose Lie algebra is  $\mathfrak{g}$ . Any other connected Lie group  $G'$  with the same Lie algebra  $\mathfrak{g}$  has the form  $G' = G/H$  with  $H$  a finite or discrete invariant subgroup of  $G$ . This agrees with what we saw above: if  $G$  is the covering group of  $G'$ ,  $G' = G/\pi_1(G')$ . For example  $U(1)=\mathbb{R}/\mathbb{Z}$ ,  $SO(3)=SU(2)/\mathbb{Z}_2$ . If  $G'$  is non connected, the previous property applies to the connected component of the identity.

### 1.4.4 Structure constants. Killing form. Cartan criteria

Given a basis  $\{t_\alpha\}$  in a  $d$ -dimensional Lie algebra  $\mathfrak{g}$ , any element  $X$  of  $\mathfrak{g}$  reads  $X = \sum_{\alpha=1}^d x^\alpha t_\alpha$ . The *structure constants* of  $\mathfrak{g}$  (in that basis), defined by

$$[t_\alpha, t_\beta] = C_{\alpha\beta}^\gamma t_\gamma, \quad (1.41)$$

are clearly antisymmetric in their two lower indices  $C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma$ . Return to the linear operator  $\text{ad } X$  defined above in (1.24)

$$\text{ad } X \text{ } Z = [X, Z] = \sum x^\alpha z^\beta C_{\alpha\beta}^\gamma t_\gamma,$$

and for  $X, Y \in \mathfrak{g}$  consider the linear operator  $\text{ad } X \text{ ad } Y$  which acts in the Lie algebra according to

$$\text{ad } X \text{ ad } Y \text{ } Z = [X, [Y, Z]] = C_{\alpha\delta}^\epsilon C_{\beta\gamma}^\delta x^\alpha y^\beta z^\gamma t_\epsilon.$$

Exercises (easy !): show that the Jacobi identity is equivalent to the identity

$$\sum_\delta (C_{\alpha\delta}^\epsilon C_{\beta\gamma}^\delta + C_{\beta\delta}^\epsilon C_{\gamma\alpha}^\delta + C_{\gamma\delta}^\epsilon C_{\alpha\beta}^\delta) = 0 \quad (1.42)$$

(note the structure : a cyclic permutation on the three indices  $\alpha, \beta, \gamma$  with  $\epsilon$  fixed and summation over the repeated  $\delta$ ); and show that this identity may also be expressed as

$$[\text{ad } X, \text{ad } Y]Z = \text{ad } [X, Y]Z. \quad (1.43)$$

Taking the trace of this linear operator  $\text{ad } X \text{ ad } Y$  defines the *Killing form*

$$(X, Y) := \text{tr}(\text{ad } X \text{ ad } Y) = \sum_{\gamma, \delta} C_{\alpha\delta}^\gamma C_{\beta\gamma}^\delta x^\alpha y^\beta =: g_{\alpha\beta} x^\alpha y^\beta, \quad (1.44)$$

a symmetric bilinear form (a scalar product) on vectors of the Lie algebra. The symmetric tensor  $g_{\alpha\beta}$  is thus given by

$$g_{\alpha\beta} = \sum_{\gamma, \delta} C_{\alpha\delta}^\gamma C_{\beta\gamma}^\delta = \text{tr}(\text{ad } t_\alpha \text{ ad } t_\beta).$$

(Symmetry in  $\alpha, \beta$  is manifest on the 1st expression, it follows from the cyclicity of the trace in the 2nd.)

Note that this Killing form is invariant under the action of any  $\text{ad } Z$  :

$$\forall X, Y, Z \in \mathfrak{g} \quad ([Z, X], Y) + (X, [Z, Y]) = 0 \quad (1.45)$$

(think of  $\text{ad } Z$  as an infinitesimal generator acting like a derivative, either on the first term, or on the second). Indeed the first term equals  $\text{tr}(\text{ad } Z \text{ ad } X \text{ ad } Y - \text{ad } X \text{ ad } Z \text{ ad } Y)$  while the second is  $\text{tr}(\text{ad } X \text{ ad } Z \text{ ad } Y - \text{ad } X \text{ ad } Y \text{ ad } Z)$ , and they cancel thanks to the cyclicity of the trace. One may prove that in a simple Lie algebra, an invariant symmetric form is necessarily a multiple of the Killing form.

One may then use the tensor  $g_{\alpha\beta}$  to lower the 3d label of  $C_{\alpha\beta}^{\gamma}$ , thus defining

$$C_{\alpha\beta\gamma} = C_{\alpha\beta}^{\delta} g_{\gamma\delta} = C_{\alpha\beta}^{\delta} C_{\gamma\epsilon}^{\kappa} C_{\delta\kappa}^{\epsilon}.$$

Let us then show that this  $C_{\alpha\beta\gamma}$  is completely antisymmetric in  $\alpha, \beta, \gamma$ . Given the already known antisymmetry in  $\alpha, \beta$ , it suffices to show that  $C_{\alpha\beta\gamma}$  is invariant by cyclic permutations. This follows from (1.45) which may be written in a more symmetric form as

$$(X, [Y, Z]) = (Y, [Z, X]) = (Z, [X, Y]) = C_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma} = C_{\beta\gamma\alpha} y^{\beta} z^{\gamma} x^{\alpha} = C_{\gamma\alpha\beta} z^{\gamma} x^{\alpha} y^{\beta}, \quad (1.46)$$

thus proving the announced property.

A quite remarkable theorem of E. Cartan states that:

- (i) *A Lie algebra is semi-simple iff the Killing form is non-degenerate, i.e.  $\det g \neq 0$ .*
- (ii) *A real semi-simple Lie algebra is compact iff the Killing form is negative definite.*

Those are the *Cartan criteria*.

In one way, property (i) is easy to prove. Suppose that  $\mathfrak{g}$  is not semi-simple and let us show that  $\det g = 0$ . Let  $\mathfrak{J}$  be an ideal of  $\mathfrak{g}$ , choose a basis of  $\mathfrak{g}$  made of a basis of  $\mathfrak{J}$ ,  $\{t_i\}$ ,  $i = 1, \dots, r$ , complemented by  $t_a$ ,  $a = r+1, \dots, d$ . For  $1 \leq i, j \leq r$ , compute  $g_{ij} = \sum_{\alpha\beta} C_{i\alpha}^{\beta} C_{j\beta}^{\alpha}$ . By definition of an ideal,  $\alpha$  and  $\beta$  are themselves between 1 and  $r$ ,  $g_{ij} = \sum_{1 \leq k, l \leq r} C_{ik}^l C_{jl}^k$ . Hence the restriction of the Killing form of  $\mathfrak{g}$  to  $\mathfrak{J}$  is the Killing form of  $\mathfrak{J}$ . If moreover the ideal is assumed to be abelian,  $g_{ij} = 0$  and  $g_{ia} = 0$  (Exercise: check that point!). The form is obviously degenerate ( $\det g = 0$ ). The reciprocal,  $\det g = 0 \Rightarrow \mathfrak{g}$  non semi-simple, is more delicate to prove.

Likewise, property (ii) is relatively easy to prove in the sense compactness  $\Rightarrow$  definite negative form. Start from an arbitrary positive definite symmetric bilinear form; for example in a given basis  $\{t_{\alpha}\}$ , consider  $\langle X, Y \rangle = \sum x^{\alpha} y^{\beta}$ . For a compact group  $G$ , one can make this form invariant by averaging over  $G$ :  $\varphi(X, Y) := \int d\mu(g) \langle gXg^{-1}, gYg^{-1} \rangle$ . It is invariant  $\varphi(gXg^{-1}, gYg^{-1}) = \varphi(X, Y)$ , or in infinitesimal form,  $\varphi([Z, X], Y) + \varphi(X, [Z, Y]) = 0$ , (cf (1.45)). It is also positive definite. Let  $e_{\alpha}$  be a basis which diagonalizes it,  $\varphi(e_{\alpha}, e_{\beta}) = \delta_{\alpha\beta}$ . Let us calculate in that basis the matrix of the  $\text{ad } X$  operator and show that it is antisymmetric,  $(\text{ad } X)_{\alpha\beta} = -(\text{ad } X)_{\beta\alpha}$ :

$$(\text{ad } X)_{\alpha\beta} = \varphi(e_{\alpha}, [X, e_{\beta}]) = -\varphi(e_{\beta}, [X, e_{\alpha}]) = -(\text{ad } X)_{\beta\alpha}.$$

Hence the Killing form

$$(X, X) = \text{tr}(\text{ad } X \text{ad } X) = \sum_{\alpha, \beta} (\text{ad } X)_{\alpha\beta} (\text{ad } X)_{\beta\alpha} = - \sum_{\alpha, \beta} ((\text{ad } X)_{\alpha\beta})^2 \leq 0$$

is negative semi-definite, and if the algebra is semi-simple, it is negative definite, q.e.d.

**Example.** The case of  $\text{SO}(3)$  or  $\text{SU}(2)$  is familiar. The structure constants are given by the completely antisymmetric tensor  $C_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$ . The Killing form is  $g_{\alpha\beta} = -2\delta_{\alpha\beta}$ . Exercise: compute the Killing form for the algebra  $\mathfrak{so}(2, 1)$ , (see Exercise B).

A last important theorem (again by Cartan !) states that

- *Any semi-simple Lie algebra  $\mathfrak{g}$  is a direct sum of simple Lie algebras  $\mathfrak{g}_i$*

$$\mathfrak{g} = \oplus_i \mathfrak{g}_i.$$

This is a simple consequence of (1.46). Consider a semi-simple algebra  $\mathfrak{g}$  with an ideal  $\mathfrak{J}$  and call  $\mathfrak{C}$  the complement of  $\mathfrak{J}$  wrt the Killing form, i.e.  $(\mathfrak{J}, \mathfrak{C}) = 0$ . By (1.46),  $([\mathfrak{C}, \mathfrak{J}], \mathfrak{J}) = (\mathfrak{C}, [\mathfrak{J}, \mathfrak{J}]) = (\mathfrak{C}, \mathfrak{J}) = 0$  (since  $\mathfrak{J}$  is a subalgebra), and  $([\mathfrak{C}, \mathfrak{J}], \mathfrak{C}) = (\mathfrak{J}, \mathfrak{C}) = 0$  (since  $\mathfrak{J}$  is an ideal), hence  $[\mathfrak{C}, \mathfrak{J}]$ , orthogonal to any element of  $\mathfrak{g}$  for

the non-degenerate Killing form, vanishes,  $[\mathfrak{C}, \mathfrak{J}] = 0$ , which means that  $\mathfrak{g} = \mathfrak{J} \oplus \mathfrak{C}$ . Iterating the argument on  $\mathfrak{C}$ , one gets the announced property.

Cartan made use of these properties to classify the simple complex and real Lie algebras. We return to this classification in Chap. 3.

### 1.4.5 Casimir operator(s)

With previous notations, given a semi-simple Lie algebra  $\mathfrak{g}$ , hence with an invertible Killing form, and a basis  $\{t_\alpha\}$  of  $\mathfrak{g}$ , we define

$$C_2 = \sum_{\alpha, \beta} g^{\alpha\beta} t_\alpha t_\beta \quad (1.47)$$

where  $g^{\alpha\beta}$  is the inverse of  $g_{\alpha\beta}$ , i.e.  $g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta$ .

Formally, this combinaison of the  $t$ 's, which does not make use of the Lie bracket, does not live in the Lie algebra but in its *universal enveloping algebra*  $U\mathfrak{g}$ , defined as the associative algebra of polynomials in elements of  $\mathfrak{g}$ . Here, since we restricted ourselves to  $\mathfrak{g} \subset M(n, \mathbb{R})$ ,  $U\mathfrak{g}$  may also be considered as a subalgebra of  $M(n, \mathbb{R})$ .

Let us now show that  $C_2$  has a vanishing bracket (commutator) with any  $t_\gamma$  hence with any element of  $\mathfrak{g}$ . This is the quadratic *Casimir operator*.

$$\begin{aligned} [C_2, t_\gamma] &= \sum_{\alpha, \beta} g^{\alpha\beta} [t_\alpha t_\beta, t_\gamma] \\ &= \sum_{\alpha, \beta} g^{\alpha\beta} (t_\alpha [t_\beta, t_\gamma] + [t_\alpha, t_\gamma] t_\beta) \\ &= \sum_{\alpha, \beta, \delta} g^{\alpha\beta} C_{\beta\gamma}^\delta (t_\alpha t_\delta + t_\delta t_\alpha) \\ &= \sum_{\alpha, \beta, \delta, \kappa} g^{\alpha\beta} g^{\delta\kappa} C_{\beta\gamma\kappa} (t_\alpha t_\delta + t_\delta t_\alpha) . \end{aligned} \quad (1.48)$$

The term  $\sum_{\beta\kappa} g^{\alpha\beta} g^{\delta\kappa} C_{\beta\gamma\kappa}$  is antisymmetric in  $\alpha \leftrightarrow \delta$ , while the term in parentheses is symmetric. The sum thus vanishes, q.e.d.

One shows that in a simple Lie algebra, (more precisely in its universal enveloping algebra), a quadratic expression in  $t$  that commutes with all the  $t$ 's is proportional to the Casimir operator  $C_2$ . In other words, the quadratic Casimir operator is unique up to a factor.

Example. In the Lie algebra  $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ , the Casimir operator  $C_2$  is (up to a sign)  $\mathbf{J}^2$ , which, as everybody knows, commutes with the infinitesimal generators  $J^i$  of the algebra. In a non simple algebra, there are as many quadratic operators as there are simple components, see for example the two Casimir operators  $\mathbf{J}^2$  and  $\mathbf{K}^2$  in the (complexified)  $\mathfrak{so}(1,3) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$  algebra of the Lorentz group (see Chap. 0 § 0.6.2); or  $P^2$  and  $W^2$  in the Poincaré algebra, see Chap. 0, § 0.6.5.

There may exist other, higher degree Casimir operators. Check that

$$C_r = g^{\alpha_1 \alpha'_1} g^{\alpha_2 \alpha'_2} \dots g^{\alpha_r \alpha'_r} C_{\alpha_1 \beta_1}^{\beta_2} C_{\alpha_2 \beta_2}^{\beta_3} \dots C_{\alpha_r \beta_r}^{\beta_1} t_{\alpha'_1} t_{\alpha'_2} \dots t_{\alpha'_r} \quad (1.49)$$

has a vanishing bracket with any  $t_\gamma$ . What is that  $C_3$  in  $\mathfrak{su}(2)$ ? See Bourbaki ([Bo], chap. 3.52) for a discussion of these general Casimir operators. See also exercice C below.

If one remembers that infinitesimal generators (vectors of the Lie algebra) may be regarded as differential operators in the group coordinates, one realizes that the Casimir operators yield invariant (since commuting with the generators) differential operators. In particular, the quadratic Casimir operator corresponds to an invariant Laplacian on the group (see Chap. 0, § 0.2.3 for the case of  $\text{SO}(3)$ ).

These Casimir operators will play an important role in the study of group representations.

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## Appendix A. Quaternion field and symplectic groups

### A.1 Quaternions

The set of quaternions is the algebra over  $\mathbb{C}$  generated by 4 elements,  $e_i$ ,  $i = 1, 2, 3$ ,

$$q = q^{(0)}1 + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3 \quad q^{(\cdot)} \in \mathbb{C} \quad (\text{A.1})$$

with multiplication  $e_i^2 = e_1e_2e_3 = -1$ , from which it follows that

$$e_1e_2 = -e_2e_1 = e_3$$

and cyclic permutations. One may represent the  $e_i$  in terms of Pauli matrices :  $e_i \mapsto -i\sigma_i$ .

The conjugate of  $q$  is the quaternion

$$\bar{q} = q^{(0)}1 - q^{(1)}e_1 - q^{(2)}e_2 - q^{(3)}e_3 . \quad (\text{A.2})$$

not to be confused with its complex conjugate

$$q^* = q^{(0)*}1 + q^{(1)*}e_1 + q^{(2)*}e_2 + q^{(3)*}e_3 . \quad (\text{A.3})$$

Note that  $q\bar{q} := |q|^2 = (q^{(0)})^2 + (q^{(1)})^2 + (q^{(2)})^2 + (q^{(3)})^2$ , the square norm of the quaternion, and hence  $q^{-1} = \bar{q}/|q|^2$  if this norm is non-vanishing.

One may also define the Hermitian conjugate of  $q$  as

$$q^\dagger = \bar{q}^* = q^{(0)*}1 - q^{(1)*}e_1 - q^{(2)*}e_2 - q^{(3)*}e_3 \quad (\text{A.4})$$

(in accordance with the fact that Pauli matrices are Hermitian).

Note that conjugation and Hermitian conjugation reverse the order of factors

$$\overline{(q_1q_2)} = \bar{q}_2\bar{q}_1 \quad (q_1q_2)^\dagger = q_2^\dagger q_1^\dagger . \quad (\text{A.5})$$

A *real quaternion* is a quaternion of the form (A.1) with  $q^{(\mu)} \in \mathbb{R}$ , hence identical with its complex conjugate.

The set of real quaternions forms a field, which is also a space of dimension 4 over  $\mathbb{R}$ . It is denoted  $\mathbb{H}$  (from Hamilton).

### A.2 Quaternionic matrices

Let us consider matrices  $Q$  with quaternionic elements  $(Q)_{ij} = q_{ij}$ , or  $Q = (q_{ij})$ . One may apply to  $Q$  the conjugations defined above. One may also transpose  $Q$ . The Hermitian conjugate of  $Q$  is defined by

$$(Q^\dagger)_{ij} = q_{ji}^\dagger . \quad (\text{A.6})$$

The *dual*  $Q^R$  of a quaternionic matrix  $Q$  is the matrix

$$(Q^R)_{ij} = \bar{q}_{ji} . \quad (\text{A.7})$$

(It plays for quaternionic matrices the same role as Hermitian conjugates for complex matrices.) A quaternionic matrix is *self-dual* if

$$Q^R = Q = (q_{ij}) = (\bar{q}_{ji}) , \quad (\text{A.8})$$

it is *real quaternionic* if

$$Q^R = Q^\dagger \quad \text{hence} \quad q_{ij} = q_{ij}^* , \quad (\text{A.9})$$

*i.e.* if its elements are real quaternions.

### A.3 Symplectic groups $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{USp}(n)$ , and the Lie algebras $\mathfrak{sp}(2n)$ et $\mathfrak{usp}(n)$

Consider the  $2n \times 2n$  matrix

$$S = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad (\text{A.10})$$

(with  $I_n$  the  $n \times n$  identity matrix) and the associated “skew-symmetric” bilinear form

$$(X, Y) = X^T S Y = \sum_{i=1}^n (x_i y_{i+n} - y_i x_{i+n}) . \quad (\text{A.11})$$

The symplectic group  $\mathrm{Sp}(2n, \mathbb{R})$  is the group of real  $2n \times 2n$  matrices that preserve that form

$$B^T S B = S . \quad (\text{A.12})$$

In the basis where  $X^T = (x_1, x_{n+1}, x_2, x_{n+2}, \dots)$ , the matrix  $S = \mathrm{diag} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{diag}(-e_2)$  in terms of quaternions, and the symplectic group is then generated by quaternionic  $n \times n$  matrices  $Q$  satisfying  $Q^R \cdot Q = I$ , (check !); the matrix  $B$  being real, however, the elements of  $Q$  are such that  $q_{ij}^{(\alpha)}$  are real for  $\alpha = 0, 2$  and purely imaginary for  $\alpha = 1, 3$ . This group is non compact. Its Lie algebra  $\mathfrak{sp}(2n, \mathbb{R})$  is generated by real matrices  $A$  such that  $A^T S + S A = 0$ . The dimension of that group or of its Lie algebra is  $n(2n + 1)$ . For  $n = 1$ ,  $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$ .

A related group is  $\mathrm{USp}(n)$ , generated by unitary real quaternionic  $n \times n$  matrices  $Q^R = Q^\dagger = Q^{-1}$ . This is the invariance group of the quaternionic Hermitian form  $\sum \bar{x}_i y_i$ ,  $x, y \in \mathbb{H}^n$ . It is compact since it is a subgroup of  $\mathrm{U}(2n)$ . Its Lie algebra  $\mathfrak{usp}(n)$  is generated by antiselfdual real quaternionic matrices  $A = -A^R = -A^\dagger$  (check!). Its dimension is again  $n(2n + 1)$ . For  $n = 1$ ,  $\mathrm{USp}(1) = \mathrm{SU}(2)$ .

Expressing the condition on matrices  $A$  of  $\mathfrak{sp}(n, \mathbb{R})$  in terms of quaternions, one sees that the two algebras  $\mathfrak{sp}(2n, \mathbb{R})$  and  $\mathfrak{usp}(n)$  have the same complexified algebra, namely  $\mathfrak{sp}(2n, \mathbb{C})$ . Only  $\mathfrak{usp}(n)$  is compact.

## Appendix B. A short reminder of topology and differential geometry.

### B.1 A lexicon of some concepts of topology used in these notes

*Topological space* : set  $E$  with a collection of *open subsets*, with the property that the union of open sets and the intersection of a finite number of them is an open subset, and that  $E$  and  $\emptyset$  are open.

*Closed subset* of  $E$  : complement of an open subset of  $E$ .

*Neighborhood of a point  $x$*  : subset  $E$  that contains an open set containing  $x$ . Let  $\mathcal{V}(x)$  be the set of neighborhoods of  $x$ .

A topological space is *separated* (or *Hausdorff*) if two distinct points have distinct neighborhoods. This will always be assumed in these notes.

*Basis of neighborhoods  $\mathcal{B}(x)$  of a point  $x$*  : subset of  $\mathcal{V}(x)$  such that any  $V \in \mathcal{V}(x)$  contains a  $W \in \mathcal{B}(x)$ . (Intuitively, a basis is made of “enough” neighborhoods.)

*Continuous function*: a function  $f$  from topological space  $E$  to topological space  $F$  is called continuous if the inverse image of every open set in  $F$  is open in  $E$ .

*Compact space*  $E$ : topological (separated) space such that from any covering of  $E$  by open sets, one may extract a finite covering.

Consequences: if  $E$  is compact,

- any infinite sequence of points in  $E$  has an accumulation point in  $E$ ;
- if  $f : E \mapsto F$  is continuous,  $f(E)$  is compact;
- any continuous real function on  $E$  is bounded.

If  $E$  is a subspace of  $\mathbb{R}^n$ ,  $E$  compact  $\Leftrightarrow E$  closed and bounded (Heine–Borel).

*Locally compact space* : (separated) space in which any point has at least one compact neighborhood. Examples :  $\mathbb{R}$  is not compact but is locally compact ;  $\mathbb{Q}$  is neither compact nor locally compact.

## B.2 Notion of manifold

A *manifold*  $M$  of dimension  $n$  is a space which locally, in the vicinity of each point, “resembles”  $\mathbb{R}^n$  or  $\mathbb{C}^n$ . Counter-examples are given by two secant lines, or by  $\text{---}\bigcirc$ . More precisely, there exists a collection of neighborhoods  $U_i$  covering  $M$ , with *charts*  $f_i$ , i.e. invertible and bicontinuous (homeomorphisms) functions between  $U_i$  and an open set of  $\mathbb{R}^n$ :  $f_i(U_i) \subset \mathbb{R}^n$ . Let  $m$  be a point of  $M$ ,  $m \in U_i$ , and  $f_i(m) = (x^1, x^2, \dots, x^n)$  its image in  $\mathbb{R}^n$  :  $(x^1, x^2, \dots, x^n)$  are the *local coordinates* of  $m$ , which depend on the map. It is fundamental to know how to change the coordinate chart. The manifold is said to be differentiable of class  $C^k$  if for any pair of open sets  $U_i$  and  $U_j$  with a non-empty intersection,  $f_j \circ f_i^{-1}$  which maps  $f_i(U_i \cap U_j) \subset \mathbb{R}^n$  onto  $f_j(U_i \cap U_j) \subset \mathbb{R}^n$  is of class  $C^k$ .

Example : the sphere  $S^2$  is an analytic manifold of dimension 2. One may choose as two open sets the sphere with its North, resp. South, pole removed, with a map to  $\mathbb{R}^2$  given by the stereographic projection (see Problem below) from that pole.

A *Riemann manifold* is a differentiable real manifold on the tangent vectors of which a positive definite inner product has been defined. If the inner product is only assumed to be a non degenerate form of signature  $(+1)^p, (-1)^{n-p}$ , the manifold is said to be *pseudo-Riemannian*. In local coordinates  $x^i$ , we have  $X = X^i \frac{\partial}{\partial x^i}$ , and the inner product and the squared length element are given by the metric tensor  $g$

$$(X, Y) = g_{ij} X^i Y^j, \quad ds^2 = g_{ij} dx^i dx^j. \quad (\text{B.1})$$

## B.3 Tangent space

In differential geometry, a tangent vector  $X$  to a manifold  $M$  at a point  $x_0$  is a linear differential operator, of first order in the derivatives in  $x_0$ , acting on functions  $f$  on  $M$ . In local coordinates  $x^i$ ,

$$X : f(x) \mapsto \sum_i X^i \frac{\partial}{\partial x^i} \Big|_{x_0} f(x)$$

and under a change of coordinates  $\{x^i\} \rightarrow \{y^j\}$ , these operators transform by the Jacobian matrix  $\frac{\partial}{\partial y^j} = \sum_i \frac{\partial x^i}{\partial y^j} \Big|_{x_0} \frac{\partial}{\partial x^i}$  with the transformation of  $X^i \rightarrow Y^j$  that follows from it.

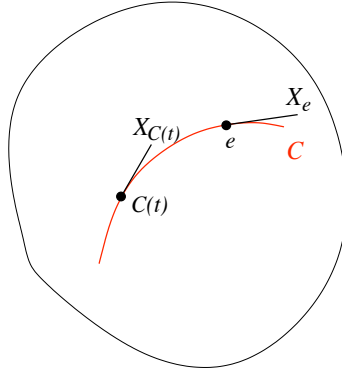


Figure 1.4: The field of tangent vectors to the curve  $C(t)$  is a left-invariant vector field

*Tangent vector to a curve* : if a curve  $C(t)$  passes through the point  $x_0$  at  $t = 0$ , one may differentiate a function  $f$  along that curve

$$f \mapsto \left. \frac{df(C(t))}{dt} \right|_{t=0},$$

which defines the tangent vector to the curve  $C$  at point  $x_0$ , also called velocity vector and denoted  $C'(t)|_{t=0} = C'(0)$ .

The tangent space to  $M$  at  $x_0$ , denoted  $T_{x_0}M$ , is the vector space generated by the velocity vectors of all curves passing through  $x_0$ . The space  $T_{x_0}M$  has a basis made of  $\left. \frac{\partial}{\partial x^i} \right|_{x_0}$  : it has the same dimension as  $M$ .

If a vector  $X_x$  tangent to  $M$  at  $x$  is assigned for any  $x$ , this defines a *vector field* on the manifold  $M$ .

## B.4 Lie group. Exponential map

Take a group  $G$ ,  $e$  its identity. Let  $C(t)$  be a curve passing through  $C(0) = e$ , and let  $X_e = (C'(t))_{t=0}$  be its velocity vector at  $e$ . For  $g \in G$ , one defines the *left translate*  $g.C(t)$  of  $C$  by  $g$ . Its velocity at  $g$ ,  $X_g = (g.C(t))'_{t=0}$ , is called a *left translated vector* of  $X_e$ . The vector field  $g \mapsto X_g$  is said to be left-invariant, it is the set of left translated vectors of  $X_e$ . The tangent space at  $e$  and the space of invariant vector fields are thus isomorphic, and are both denoted  $\mathfrak{g}$ .

Conversely, given a tangent vector  $X_e$  at  $e$ , let

$$C(t) = \exp tX_e \tag{B.2}$$

be the unique solution to the differential equation

$$C'(t) = X_{C(t)} \tag{B.3}$$

which expresses that the curve  $C(t)$  is tangent at any of its points to the left-invariant vector field, that equation being supplemented by the initial condition that  $C(0) = e$ . (This first-order differential equation has a solution, determined up to a constant (in the group), and that constant is fixed uniquely by the initial condition.)

Let us now prove that the function  $\exp$  defined by (B.2) satisfies property (1.17). Note that  $C(t)$  satisfies (B.3), and so does  $C(t+t')$ . Thus  $C(t+t') = k.C(t)$ , (with  $k$  constant in the group), and that constant is fixed by taking  $t = 0$ ,  $C(t') = k$ , hence  $C(t+t') = C(t')C(t)$  and  $C(-t) = C(t)^{-1}$ , qed.

In the case of matrix groups considered in this course, the function  $\exp$  is of course identical to the exponential function defined by its Taylor series (1.21).

## Appendix C. Invariant measure on SU(2) and on U(n)

The group SU(2) being isomorphic to a sphere  $S^3$  is compact and one may thus integrate a function on the group with a wide variety of measures  $d\mu(g)$ . The invariant measure, such that  $d\mu(g.g_1) = d\mu(g_1.g) = d\mu(g^{-1}) = d\mu(g)$ , is, on the other hand, unique up to a factor.

A possible way to determine that measure is to consider the transformation  $U \rightarrow U' = U.V$  where  $U, V$  and hence  $U'$  are unitary of the form (0.10) (*i.e.*  $U = u_0 I - \mathbf{u} \cdot \boldsymbol{\sigma}$ ,  $u \in S^3$  etc) ; if the condition  $u_0^2 + \mathbf{u}^2 = 1$  is momentarily relaxed (but  $v_0^2 + \mathbf{v}^2 = 1$  maintained), this defines a linear transformation  $u \rightarrow u'$  which conserves the norm  $\det U = u_0^2 + \mathbf{u}^2 = u_0'^2 + \mathbf{u}'^2 = \det U'$ . This is thus an isometry of the space  $\mathbb{R}^4$  which preserves the natural measure  $d^4 u \delta(u^2 - 1)$  on the unit sphere  $S^3$  of equation  $\det U = 1$ . In other terms, that measure on the sphere  $S^3$  gives a right invariant measure:  $d\mu(U) = d\mu(U.V)$ . One may prove in a similar way that it is left invariant:  $d\mu(U) = d\mu(V.U)$ . It is also invariant under  $U \rightarrow U^{-1}$ , since inversion in SU(2) amounts to the restriction to  $S^3$  of the orthogonal transformation  $u_0 \rightarrow u_0$ ,  $\mathbf{u} \rightarrow -\mathbf{u}$  in  $\mathbb{R}^4$ , which preserves of course the natural measure on  $S^3$  :

$$d\mu(U) = d\mu(UV) = d\mu(VU) = d\mu(U^{-1}) .$$

The explicit form of the measure depends on the chosen parametrization. If one uses the direction  $\mathbf{n}$  (or its two polar angles  $\theta$  et  $\phi$ ) and the rotation angle  $\psi$ , one finds

$$d\mu(U) = \frac{1}{2} \sin^2 \frac{\psi}{2} \sin \theta d\psi d\theta d\phi \quad (\text{C.1})$$

normalized for SU(2) to

$$v(\text{SU}(2)) = \int_{\text{SU}(2)} d\mu(U) = \frac{1}{2} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} = 2\pi^2 \quad (\text{C.2})$$

which is the “area” of the unit sphere  $S^3$  and the volume of SU(2). For SO(3) where the angle  $\psi$  has a range restricted to  $(0, \pi)$ , one finds instead  $v(\text{SO}(3)) = \int_{\text{SO}(3)} d\mu(g) = \pi^2$ .

The expression in any other coordinate system, like the Euler angles, is then obtained by computing the adequate Jacobian

$$d\mu(U) = \frac{1}{8} \sin \beta d\alpha d\beta d\gamma . \quad (\text{C.3})$$

(Note that  $0 \leq \gamma \leq 4\pi$  for SU(2), whereas  $0 \leq \alpha \leq 2\pi$  and  $0 \leq \beta \leq \pi$ ).

Compare these results with those obtained by a different method in Chap. 0, App. 0.

### • Case of U(n).

Let us discuss rapidly the case of U(n), using the method of Chap. 0, App. 0. Any unitary matrix  $U \in \text{U}(n)$  may be diagonalized in the form

$$U = V \Lambda V^\dagger , \quad (\text{C.4})$$

with  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  and the  $\lambda_i$  are in fact of modulus one,  $\lambda_j = e^{i\alpha_j}$ . These  $\lambda_i$  may be regarded as “radial” variables, while  $V$  represents the “angular” variables. Note that  $V$  has to be restricted **not** to commute with the diagonal matrix  $\Lambda$ . If the latter is generic, with distinct eigenvalues  $\lambda_i$ ,  $V$  lives in  $\text{U}(n)/\text{U}(1)^n$ . The natural metric, invariant under  $U \mapsto U'U$  or  $\mapsto UU'$ , reads  $\text{tr}(dU dU^\dagger)$ . But  $dU = V(d\Lambda + [dX, \Lambda])V^\dagger$ ,

where  $dX := V^\dagger dV$  is anti-Hermitian (and with no diagonal elements, why?). Thus  $\text{tr}(dU dU^\dagger) = \sum_i |d\alpha_i|^2 + 2 \sum_{i < j} |dX_{ij}|^2 |\lambda_i - \lambda_j|^2$  which defines the metric tensor  $g_{\alpha\beta}$  in coordinates  $\xi^\alpha = (\alpha_i, \Re X_{ij}, \Im X_{ij})$  and determines the integration measure

$$d\mu(U) = \sqrt{\det g} \prod d\xi^\alpha = \text{const.} |\Delta(e^{i\alpha})|^2 \prod d\alpha_i d\mu(V) . \quad (\text{C.5})$$

Here  $\Delta(\lambda)$  is the *Vandermonde determinant*

$$\Delta(\lambda) := \prod_{i < j} (\lambda_i - \lambda_j) = \begin{vmatrix} \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} . \quad (\text{C.6})$$

The “radial” part of the integration measure is thus given by  $|\Delta(e^{i\alpha})|^2 \prod d\alpha_i$  up to a constant factor, or equivalently

$$d\mu(U) = \text{const.} \prod_{i < j} \sin^2 \left( \frac{\alpha_i - \alpha_j}{2} \right) \prod d\alpha_i \times \text{angular part} . \quad (\text{C.7})$$

Note that this radial part of the measure suffices if one has to integrate over the group a function of  $U$  which is invariant by  $U \rightarrow VUV^\dagger$ ,  $V \in \text{U}(n)$ . For example  $\int d\mu(U) \text{tr} P(U)$ , with  $P$  a polynomial.

## Exercises and Problem for chapter 1

### A. Action of a group on a set

A group  $G$  is said to act on a set  $E$  if there exists a homomorphism  $\beta$  of  $G$  into the group of bijections of  $E$  into itself.

1. Write explicitly the required conditions.

One then defines the *orbit*  $O(x)$  of a point  $x \in E$  as the set of images  $\beta(g)x$  for all  $g \in G$ .

2. Show that belonging to the same orbit is an equivalence relation.

3. Example : action of  $O(n)$  on  $\mathbb{R}^n$ . What are the orbits?

4. A space is *homogeneous* if it has only one orbit. Show that a trivial example is given by the action of translations on  $\mathbb{R}^n$ . More generally, what can be said of the left action of  $G$  on itself, with  $E = G$ ? Give other examples of homogeneous spaces for  $G = O(3)$  or  $\mathcal{L} = O(1,3)$ .

5. One also defines the *isotropy group*  $S(x)$  of the element  $x \in E$ , (also called *stabilizer*, or, by physicists, *little group*): this is the subgroup of  $G$  leaving  $x$  invariant:

$$S(x) = \{g \in G | \beta(g)x = x\} . \quad (1.49)$$

Show that if  $x$  and  $y$  belong to the same orbit, their isotropy groups are conjugate. What is the isotropy group of a point  $x \in \mathbb{R}^n$  under the action of  $SO(n)$ ? of a time-like vector  $p$  in Minkowski space under the action of the Lorentz group? Is  $S(x)$  an invariant subgroup?

6. Show that there exists a bijection between points of the orbit  $O(x)$  and the coset space  $G/S(x)$ . For a finite group  $G$ , deduce from it a relation between the orders (cardinalities) of  $G$ ,  $O(x)$  and  $S(x)$ . Is this set  $G/S(x)$  homogeneous for the action of  $G$ ?

Chap. 2 will be devoted to the particular case where  $E$  is a vector space, with the linear transformations of  $GL(E)$  acting as bijections: one then speaks of representations of  $G$  in  $E$ .

### B. Lie groups and algebras of dimension 3.

1. Recall the definition of the group  $SU(1,1)$ . What is its dimension?

2. Which equation defines its Lie algebra? What does that imply on the matrix elements of  $X \in \mathfrak{su}(1,1)$ ?

Prove that one may write a basis of  $\mathfrak{su}(1,1)$  in terms of 3 Pauli matrices and compute their commutation relations. Is this algebra isomorphic to the  $\mathfrak{so}(3)$  algebra?

3. One now considers the linear group  $SL(2, \mathbb{R})$ . What is its definition? How is its Lie algebra defined? Give a basis in terms of Pauli matrices.

4. Prove the isomorphism of the two algebras  $\mathfrak{su}(1,1)$  et  $\mathfrak{sl}(2, \mathbb{R})$ .

5. Same questions with the algebra  $\mathfrak{so}(2,1)$ : definition, dimension, commutation relations, isomorphism with one of the previous algebras?

6. Using the Cartan criteria, discuss the semi-simplicity and the compactness of these various algebras. What is their relationship with  $\mathfrak{su}(2)$ ?

(For the geometric relationship between the groups  $SU(1,1)$ ,  $SL(2, \mathbb{R})$  et  $SO(1,2)$ , see §13 and §24, vol. 1 of [DNF].)

### C. Casimir operators in $\mathfrak{u}(n)$ .

1. Prove that the  $n^2$  matrices  $t_{(ij)}$  of size  $n \times n$ ,  $1 \leq i, j \leq n$ , with elements  $(t_{(ij)})_{ab} = \delta_{ia}\delta_{jb}$  form a basis of the algebra  $\mathfrak{u}(n)$ . Compute their commutation relations and the structure constants of the algebra.

2. Compute the Killing form in that basis and check that the properties related to Cartan criteria are satisfied.

3. Show that the elements in the enveloping algebra  $C^{(r)} = \sum_{1 \leq i_1, i_2, \dots, i_r \leq n} t_{(i_1 i_2)} t_{(i_2 i_3)} \cdots t_{(i_r i_1)}$  commute with all  $t_{(ij)}$  and are thus Casimir operators of degree  $r$ .

4. How to modify this discussion for the  $\mathfrak{su}(n)$  algebra? ([Bu], chap 10).

## Problem : Conformal transformations

I-1. We recall that in a (classical) local, translation invariant field theory, one may define a *stress-energy tensor*  $\Theta_{\mu\nu}(x)$  such that

- under an infinitesimal change of coordinates  $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu(x)$ , the action has a variation

$$\delta S = \int d^d x (\partial_\mu a_\nu) \Theta^{\mu\nu}(x) ; \quad (1.50)$$

- $\Theta_{\mu\nu}$  is conserved:  $\partial_\mu \Theta^{\mu\nu}(x) = 0$  ;
- we assume that  $\Theta_{\mu\nu}$  is symmetric in  $\mu, \nu$ .

Prove that if  $\Theta$  is traceless,  $\Theta_\mu^\mu = 0$ , the action is also invariant under dilatations,  $x^\mu \rightarrow x'^\mu = (1 + \delta\lambda)x^\mu$ .

2. In a Riemannian or pseudo-Riemannian manifold of dimension  $d$ , with a metric tensor  $g_{\mu\nu}(x)$  of signature  $\{(+1)^p, (-1)^{d-p}\}$ , a *conformal transformation* is a coordinate transformation  $x^\mu \rightarrow x'^\mu$  which is a *local dilatation* of lengths

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \rightarrow ds'^2 = g_{\mu\nu}(x') dx'^\mu dx'^\nu = \alpha(x) ds^2 \quad (1.51)$$

- a) Write the infinitesimal form of that condition, when  $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu(x)$ . (Hint : One may relate the dilatation parameter  $1 + \delta\alpha$  to  $a^\mu$  by taking some adequate trace.)
- b) Prove that for an Euclidean or pseudo-Euclidean space of metric  $g_{\mu\nu} = \text{diag}\{(+1)^p, (-1)^{d-p}\}$ , that condition may be recast as

$$\partial_\mu a_\nu + \partial_\nu a_\mu = \frac{2}{d} g_{\mu\nu} \partial_\rho a^\rho . \quad (1.52)$$

3. Prove, using (1.50,1.52), that under the conditions of 1. and 2.b, any field theory invariant under translations, rotations and dilatations is also invariant under conformal transformations.

4. We now study consequences of (1.52). We set  $D := \frac{1}{d} \partial_\rho a^\rho$ .

- a) Differentiating (1.52) with respect to  $x^\nu$ , prove that

$$\partial^2 a_\mu = (2 - d) \partial_\mu D. \quad (1.53)$$

- b) Differentiating (1.53) w.r.t.  $x^\mu$ , prove that in dimension  $d > 1$ ,  $D$  is a harmonic function :  $\partial^2 D = 0$ .
- c) We assume in the following that  $d \geq 2$ . Differentiating (1.53) w.r.t.  $x^\nu$ , symmetrizing it in  $\mu$  and  $\nu$  and using (1.52), prove that if  $d > 2$ , then  $\partial_\mu \partial_\nu D = 0$ . Show that it implies the existence of a constant scalar  $h$  and of a constant vector  $k$  such that  $D = k_\mu x^\mu + h$ .
- d) Differentiating (1.52) w.r.t.  $x^\sigma$  and antisymmetrizing it in  $\nu$  and  $\sigma$ , prove that

$$\partial_\mu (\partial_\sigma a_\nu - \partial_\nu a_\sigma) = 2(g_{\mu\nu} k_\sigma - g_{\mu\sigma} k_\nu) = \partial_\mu (2k_\sigma x_\nu - 2k_\nu x_\sigma). \quad (1.54)$$

- e) Show that it implies the existence of a constant skew-symmetric tensor  $l_{\sigma\nu}$  such that

$$\partial_\sigma a_\nu - \partial_\nu a_\sigma = (2k_\sigma x_\nu - 2k_\nu x_\sigma) + 2l_{\sigma\nu} , \quad (1.55)$$

which, together with (1.52), gives

$$\partial_\sigma a_\nu = x_\nu k_\sigma - x_\sigma k_\nu + l_{\sigma\nu} + g_{\nu\sigma} k_\rho x^\rho + h g_{\nu\sigma} .$$

- f) Conclude that the general expression of an infinitesimal conformal transformation in dimension  $d > 2$  reads

$$a_\nu = k_\sigma x^\sigma x_\nu - \frac{1}{2} x_\sigma x^\sigma k_\nu + l_{\sigma\nu} x^\sigma + h x_\nu + c_\nu \quad (1.56)$$

with  $c$  a constant vector<sup>6</sup>. On how many independent real parameters does such a transformation depend in dimension  $d$  ?

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<sup>6</sup>This pretty argument is due to Michel Bauer.

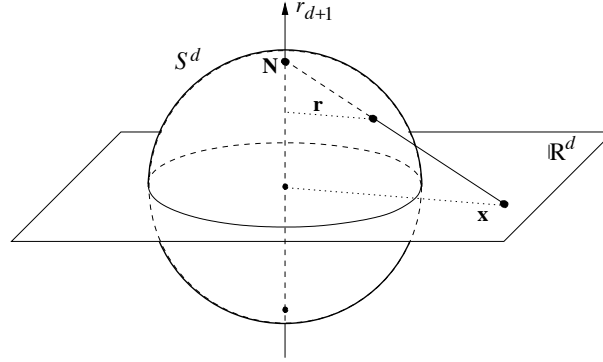


Figure 1.5: Stereographic projection from the North pole

**II-1.** One learns in geometry that in the (pseudo-)Euclidean space of dimension  $d > 2$ , conformal transformations are generated by translations, rotations, dilatations and “special conformal transformations”, obtained by composition of an inversion  $x^\mu \rightarrow x^\mu/x^2$ , a translation and again an inversion. Write the finite and the infinitesimal forms of special conformal transformations, and show that this result is in agreement with (1.56), which justifies the previous assertion.

2. Write the expression of infinitesimal generators  $P_\mu$  of translations,  $J_{\mu\nu}$  of rotations,  $D$  of dilatations and  $K_\mu$  of special transformations, as differential operators in  $x$ .

3. Write with the minimum of calculations the commutation relations of these generators (Hint : use already known results on the generators  $P_\mu$  and  $J_{\mu\nu}$ , and make use of homogeneity and of the definition of special conformal transformations to reduce the only non-trivial computation to that of  $[K_\mu, P_\nu]$ ). Check that these commutation “close” on this set of generators  $P$ ,  $J$ ,  $D$  et  $K$ .

4. What is the dimension of the conformal group in the Euclidean space  $\mathbb{R}^d$  ?

**III-1.** To understand better the nature of the conformal group, one now maps the space  $\mathbb{R}^d$ , completed by the point at infinity and endowed with its metric  $\mathbf{x}^2 = x_1^2 + \dots + x_d^2$ , on the sphere  $S^d$ . This sphere is defined by the equation  $\mathbf{r}^2 + r_{d+1}^2 = 1$  in the space  $\mathbb{R}^{d+1}$ , and the mapping is the stereographic projection from the “North pole”  $\mathbf{r} = 0$ ,  $r_{d+1} = 1$  (see Fig. 1.5). Prove that

$$\mathbf{r} = \frac{2\mathbf{x}}{\mathbf{x}^2 + 1} \quad r_{d+1} = \frac{\mathbf{x}^2 - 1}{\mathbf{x}^2 + 1} .$$

What is the image of the point at infinity ? What is the effect of the inversion in  $\mathbb{R}^d$  on the point  $r = (\mathbf{r}, r_{d+1}) \in S^d$  ?

2. The previous sphere is in turn regarded as the section of the light-cone  $\mathcal{C}$  in Minkowski space  $\mathcal{M}_{1,d+1}$  of equation  $z_0^2 - \mathbf{z}^2 - z_{d+1}^2 = 0$  by the hyperplane  $z_0 = 1$ . Prove that this establishes a one-to-one correspondance between points of  $\mathbb{R}^d \cup \{\infty\}$  and rays of the light-cone (*i.e.* vectors up a dilatation) and that the expression of  $\mathbf{x} \in \mathbb{R}^d$  as a function of  $z = (z_0, \mathbf{z}, z_{d+1}) \in \mathcal{C}$  is

$$\mathbf{x} = \frac{\mathbf{z}}{z_0 - z_{d+1}} .$$

3. We now want to prove that the action of the conformal group in  $\mathbb{R}^d$  follows from *linear* transformations in  $\mathcal{M}_{1,d+1}$  that preserve the light-cone. Without any calculation, show that these transformations must then belong to the Lorentz group in  $\mathcal{M}_{1,d+1}$ , that is  $O(1, d+1)$ .

a) What are the linear transformations of  $z$  corresponding to rotations of  $\mathbf{x}$  in  $\mathbb{R}^d$ ? Show that dilatations of  $x$  correspond to “boosts” of rapidity  $\beta$  in the plane  $(z_0, z_{d+1})$ , by giving the relation between the dilatation parameter and the rapidity.

b) Let us now consider transformations of  $O(1, d+1)$  that preserve  $z_0 - z_{d+1}$ . Write the matrix  $T_a$  of such an infinitesimal transformation acting on coordinates  $(z_0, \mathbf{z}, z_{d+1})$ , and such that  $\delta \mathbf{z} = \mathbf{a}(z_0 - z_{d+1})$  (to first

order in **a**). To which transformation of  $\mathbf{x} \in \mathbb{R}^d$  does it correspond? Compute by exponentiation of  $T_a$  the matrix of a finite transformation (Hint: compute the first powers  $T_a^2, T_a^3 \dots$ ).

c) What is finally the interpretation of the inversion in  $\mathbb{R}^d$  in the Lorentz group of  $\mathcal{M}_{1,d+1}$ ? What can be said about special conformal transformations? What is the dimension of the group  $O(1, d+1)$ ? What can be concluded about the relation between the Lorentz group in Minkowski space  $\mathcal{M}_{1,d+1}$  and the conformal group  $\mathbb{R}^d$ ?

**IV.** Last question: Do you know conformal transformations in the space  $\mathbb{R}^2$  that are *not* of the type discussed in **II.1**?