

Chapter 3

Simple Lie algebras. Classification and representations. Roots and weights

3.1 Cartan subalgebra. Roots. Canonical form of the algebra

We consider a semi-simple (*i.e.* with no abelian ideal) Lie algebra. We want to construct a canonical form of commutation relations modeled on the case of $SU(2)$

$$[J_z, J_{\pm}] = \pm J_{\pm} \quad [J_+, J_-] = 2J_z . \quad (3-1)$$

It will be important to consider the algebra over $\mathbb{C}$, at the price of “complexifying” it if it was originally real. The adjoint representation will be used. As it is a faithful representation for a semi-simple algebra, (*i.e.* $\text{ad } X = 0 \Rightarrow X = 0$, see exercise B of Chap. 2), no information is lost.

It may also be useful to remember that the complex algebra has a real compact version, in which the real structure constants lead to a negative definite Killing form, and, as the representations can be taken unitary, the elements of the Lie algebra (the infinitesimal generators) may be taken as Hermitian (or antiHermitian, depending on our conventions).

3.1.1 Cartan subalgebra

We define first the notion of *Cartan subalgebra*. This is a *maximal abelian subalgebra* of $\mathfrak{g}$ such that all its elements are diagonalisable (hence simultaneously diagonalisable) in the adjoint representation. That such an algebra exists is non trivial and must be established, but we shall admit it.

If we choose to work with the unitary form of the adjoint representation, the elements of $\mathfrak{g}$ are Hermitian matrices, and assuming that the elements of $\mathfrak{h}$ are commuting among themselves ensures that they are simultaneously diagonalizable.

This Cartan subalgebra is non unique, but one may prove that two distinct choices are related by an automorphism of the algebra $\mathfrak{g}$.

For instance if $\mathfrak{g}$ is the Lie algebra of a Lie group G and if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, any conjugate ghg^{-1} of $\mathfrak{h}$ by an arbitrary element of G is another Cartan subalgebra.

Let $\mathfrak{h}$ be a Cartan subalgebra, call ℓ its dimension, it is independent of the choice of $\mathfrak{h}$ and it is called the *rank* of $\mathfrak{g}$. For $\mathfrak{su}(2)$, this rank is 1, (the choice of J_z for example); for $\mathfrak{su}(n)$, the rank is $n - 1$. Indeed for $\mathfrak{su}(n)$, a Cartan algebra is generated ¹ by diagonal traceless matrices, a basis of which is given by the $n - 1$ matrices

$$H_1 = \text{diag}(1, -1, 0, \dots, 0), H_2 = \text{diag}(0, 1, -1, 0, \dots, 0), \dots, H_{n-1} = \text{diag}(0, \dots, 0, 1, -1) . \quad (3-2)$$

An arbitrary matrix of the Lie algebra, (in that representation), (anti-)Hermitian and traceless, is diagonalisable by a unitary transformation; its diagonal form is traceless and is thus expressed as a linear combination of the h_j ; the original matrix is thus conjugate by a unitary transformation of a linear combination of the h_j . This is a general property, and one proves (Cartan, see [Bu], chap. 16) that

If $\mathfrak{g}$ is the Lie algebra of a group G , any element of $\mathfrak{g}$ is conjugate by G of an element of $\mathfrak{h}$.

Application. Canonical form of antisymmetric matrices. Using the previous statement, prove the

Proposition *If $A = A^* = -A^T$ is a real skew-symmetric matrix of dimension N , one may find a real orthogonal matrix O such that $A = ODO^T$ where $D = \text{diag}\left(\begin{pmatrix} 0 & \mu_j \\ -\mu_j & 0 \end{pmatrix}\right)_{j=1, \dots, n}$ if $N = 2n$ and $D = \text{diag}\left(0, \begin{pmatrix} 0 & \mu_j \\ -\mu_j & 0 \end{pmatrix}\right)_{j=1, \dots, n}$ if $N = 2n + 1$, with real μ_j .*

If one allows the complexification of orthogonal matrices, one may fully diagonalise the matrix A in the form $D = \text{diag}\left(\begin{pmatrix} i\mu_j & 0 \\ 0 & -i\mu_j \end{pmatrix}\right)_{j=1, \dots, n}$ or $D = \text{diag}\left(0, \begin{pmatrix} i\mu_j & 0 \\ 0 & -i\mu_j \end{pmatrix}\right)_{j=1, \dots, n}$. For a proof making only use of matrix theory, see for example [M.L. Mehta, *Elements of Matrix Theory*, p 41].

3.1.2 Canonical basis of the Lie algebra

Let $H_i, i = 1, \dots, \ell$ be a basis of $\mathfrak{h}$. It is convenient to choose the H_i Hermitian. By definition $[H_i, H_j] = 0$, (abelian subalgebra) or more precisely, since we are in the adjoint representation,

$$[\text{ad } H_i, \text{ad } H_j] = 0 . \quad (3-3)$$

We may thus diagonalise simultaneously these $\text{ad } H_i$. We already know (some?) eigenvectors of vanishing eigenvalue since $\forall i, j, \text{ad } H_i H_j = 0$, and we may complete them to make a basis by finding a set of eigenvectors E_α linearly independent of the H_j

$$\text{ad } H_i E_\alpha = \alpha_{(i)} E_\alpha \quad (3-4)$$

i.e. a set of elements of $\mathfrak{g}$ such that

$$[H_i, E_\alpha] = \alpha_{(i)} E_\alpha , \quad (3-5)$$

¹We use momentarily the “representation of definition” (made of $n \times n$ matrices) rather than the adjoint representation.

with the $\alpha_{(i)}$ not all vanishing (otherwise the subalgebra $\mathfrak{h}$ would not be maximal).

The space $\mathfrak{h}^*$. In these expressions, the $\alpha_{(i)}$ are eigenvalues of the operators $\text{ad } H_i$. Since we chose Hermitian $\text{ad } H_i$, their eigenvalues $\alpha_{(i)}$ are real. By linearity, for an arbitrary element of $\mathfrak{h}$ written as $H = \sum_i h^i H_i$,

$$\text{ad}(H)E_\alpha = \alpha(H)E_\alpha, \quad (3-6)$$

and the eigenvalue of $\text{ad}(H)$ on E_α is $\alpha(H) := \sum_i h^i \alpha_{(i)}$, which is a *linear form* on $\mathfrak{h}$. In general linear forms on a vector space E form a vector space E^* , called the dual space of E . One may thus consider the *root* α , of components $\alpha_{(i)}$, as a vector of the dual space of $\mathfrak{h}$, hence $\alpha \in \mathfrak{h}^*$, the *root space*. Note that $\alpha(H_i) = \alpha_{(i)}$.

Roots enjoy the following properties (*)

1. if α is a root, $-\alpha$ in another root;
2. the eigenspace of the eigenvalue α is of dimension 1 (no multiplicity);
3. if α is a root, the only roots of the form $\lambda\alpha$ are $\pm\alpha$;
4. roots α generate all the dual space $\mathfrak{h}^*$.

For proofs of 1., 2., 3., see below, for 4. see exercise A.

Number of roots. Since the H_j are diagonalisable, the total number of their eigenvectors E_α and H_i must be equal to the dimension of the space, here the dimension d of the adjoint representation, *i.e.* of the Lie algebra $\mathfrak{g}$. As any (non vanishing by definition) root comes along with its opposite, the number of roots α is even and equal to $d - \ell$ (with $\ell = \text{rank}(\mathfrak{g})$). We denote Δ the set of roots.

In the basis $\{H_i, E_\alpha\}$ of $\mathfrak{g}$, the Killing form takes a simple form

$$(H_i, E_\alpha) = 0 \quad (E_\alpha, E_\beta) = 0 \quad \text{unless } \alpha + \beta = 0. \quad (3-7)$$

To show that, we write $(H, [H', E_\alpha]) = \alpha(H')(H, E_\alpha)$, and also, using the definition of the Killing form and the cyclicity of the trace

$$(H, [H', E_\alpha]) = \text{tr}(\text{ad } H[\text{ad } H', \text{ad } E_\alpha]) = \text{tr}([\text{ad } H, \text{ad } H'] \text{ad } E_\alpha) = 0 \quad (3-8)$$

since $[\text{ad } H, \text{ad } H'] = 0$. It follows that $\forall H, H' \in \mathfrak{h}$, $\alpha(H')(H, E_\alpha) = 0$, hence that $(H, E_\alpha) = 0$. Likewise

$$([H, E_\alpha], E_\beta) = \alpha(H)(E_\alpha, E_\beta) = -(E_\alpha, [H, E_\beta]) = -\beta(H)(E_\alpha, E_\beta) \quad (3-9)$$

again by the cyclicity of the trace, and thus $(E_\alpha, E_\beta) = 0$ if $\exists H : (\alpha + \beta)(H) \neq 0$, *i.e.* if $\alpha + \beta \neq 0$. Note that the point 1. in (*) above follows simply from (3-7): if $-\alpha$ were not a root, E_α would be orthogonal to all elements of the basis hence to any element of $\mathfrak{g}$, and the form would be degenerate, contrary to the hypothesis of semi-simplicity (and Cartan's criterion). For an elegant proof of points 2. et 3. of (*), see [OR, p. 29].

The restriction of this form to the Cartan subalgebra is non-degenerate, since otherwise one would have $\exists H \in \mathfrak{h}$, $\forall H' \in \mathfrak{h} : (H, H') = 0$, but $(H, E_\alpha) = 0$, thus $\forall X \in \mathfrak{g}$, $(H, X) = 0$ and the form would be degenerate, contrary to the hypothesis of semi-simplicity (and Cartan's criterion, Chap. 1, §4.4). The Killing form being non-degenerate on $\mathfrak{h}$, it induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$: to $\alpha \in \mathfrak{h}^*$ one associates the unique $H_\alpha \in \mathfrak{h}$ such that

$$\forall H \in \mathfrak{h} \quad \alpha(H) = (H_\alpha, H). \quad (3-10)$$

(Or said differently, one solves the linear system $g_{ij}h_\alpha^j = \alpha_{(i)}$ which is of Cramer type since $g_{ij} = (H_i, H_j)$ is invertible.) One has also a bilinear form on $\mathfrak{h}^*$ inherited from the Killing form

$$\langle \alpha, \beta \rangle = (H_\alpha, H_\beta) , \quad (3-11)$$

which we are going to use in § 2 to study the geometry of the root system.

It remains to find the commutation relations of the E_α among themselves. Using the Jacobi identity, one finds that

$$\text{ad } H_i[E_\alpha, E_\beta] = [H_i, [E_\alpha, E_\beta]] = [E_\alpha, [H_i, E_\beta]] - [E_\beta, [H_i, E_\alpha]] = (\alpha + \beta)_{(i)}[E_\alpha, E_\beta] . \quad (3-12)$$

Invoking the trivial multiplicity (=1) of roots, one sees that three cases may occur. If $\alpha + \beta$ is a root, $[E_\alpha, E_\beta]$ is proportional to $E_{\alpha+\beta}$, with a proportionality coefficient $N_{\alpha\beta}$ which will be shown below to be non zero (see § 2.1 and exercise B). If $\alpha + \beta \neq 0$ is not a root, $[E_\alpha, E_\beta]$ must vanish. Finally if $\alpha + \beta = 0$, $[E_\alpha, E_{-\alpha}]$ is an eigenvector of all $\text{ad } H_i$ with a vanishing eigenvalue, thus $[E_\alpha, E_{-\alpha}] = H \in \mathfrak{h}$. To determine that H , let us proceed like in (3-9)

$$\begin{aligned} (H_i, [E_\alpha, E_{-\alpha}]) &= \text{tr}(\text{ad } H_i [\text{ad } E_\alpha, \text{ad } E_{-\alpha}]) = \text{tr}([\text{ad } H_i, \text{ad } E_\alpha] \text{ad } E_{-\alpha}) \\ &= \alpha_{(i)}(E_\alpha, E_{-\alpha}) = (H_i, H_\alpha)(E_\alpha, E_{-\alpha}) \end{aligned} \quad (3-13)$$

hence

$$[E_\alpha, E_{-\alpha}] = (E_\alpha, E_{-\alpha})H_\alpha . \quad (3-14)$$

To recapitulate, we have constructed a canonical basis of the algebra $\mathfrak{g}$

$$\boxed{\begin{aligned} [H_i, H_j] &= 0 \\ [H_i, E_\alpha] &= \alpha_{(i)}E_\alpha \\ [E_\alpha, E_\beta] &= \begin{cases} N_{\alpha\beta}E_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ (E_\alpha, E_{-\alpha})H_\alpha & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}} \quad (3-15)$$

Up to that point, the normalisation of the vectors H_i and E_α has not been fixed. It is common to choose, in accord with (3-7)

$$\boxed{(H_i, H_j) = \delta_{ij} \quad (E_\alpha, E_\beta) = \delta_{\alpha+\beta, 0} .} \quad (3-15)$$

(Indeed, the restriction of the Killing form to $\mathfrak{h}$, after multiplication by i to make the $\text{ad } H_i$ Hermitian, is *positive* definite.) With that normalisation, H_α defined above by (3-10) satisfies also

$$H_\alpha = \alpha.H := \alpha_{(i)}H_i . \quad (3-16)$$

Note that E_α , $E_{-\alpha}$ and H_α form an $\mathfrak{su}(2)$ subalgebra

$$[H_\alpha, E_{\pm\alpha}] = \pm \langle \alpha, \alpha \rangle E_{\pm\alpha} \quad [E_\alpha, E_{-\alpha}] = H_\alpha . \quad (3-17)$$

(This is in fact $H_\alpha / \langle \alpha, \alpha \rangle$ that we identify with J_z , and that observation will be used soon.) Any semi-simple algebra thus contains an $\mathfrak{su}(2)$ algebra associated to each of its roots.

Note that with the normalisations of (3-15), the Killing metric reads in the basis $\{H_i, E_\alpha, E_{-\alpha}\}$

$$g_{ab} = \begin{pmatrix} \mathbb{I}_\ell & & & 0 \\ & 0 & 1 \\ & 1 & 0 \\ 0 & & \ddots & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \quad (3-18)$$

where the first block is an identity matrix of dimension $\ell \times \ell$.

3.2 Geometry of root systems

3.2.1 Scalar products of roots. The Cartan matrix

As noticed in (3-11), the space of roots, *i.e.* the space (of dimension ℓ , see point 4. in (*) above) generated by the $d - \ell$ roots α inherits the Euclidean metric of $\mathfrak{h}$

$$\langle \alpha, \beta \rangle := (H_\alpha, H_\beta) = \alpha(H_\beta) = \beta(H_\alpha) = (\alpha.H, \beta.H) = \sum_i \alpha_{(i)} \beta_{(i)} , \quad (3-19)$$

where the various expressions aim at making the reader familiar with the notations introduced above. (Only the last two expressions depend on the choice of normalisation (3-15).) We shall now show that the geometry –lengths and angles– of roots is strongly constrained. First it is good to remember the lessons of the $\mathfrak{su}(2)$ algebra: in a representation of finite dimension, J_z has integer or half-integer eigenvalues. Thus here, where each $\frac{H_\alpha}{\langle \alpha, \alpha \rangle}$ plays the role of a J_z and has E_β as eigenvectors, $\text{ad } H_\alpha E_\beta = \langle \alpha, \beta \rangle E_\beta$, *i.e.*

$$[H_\alpha, E_\beta] = \langle \alpha, \beta \rangle E_\beta \quad (3-20)$$

we may conclude that

$$2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = m \in \mathbb{Z} . \quad (3-21)$$

Root chains

It is in fact useful to refine the previous discussion. Like in the case of $\mathfrak{su}(2)$, the idea is to repeatedly apply the “raising” E_α and “lowering” $E_{-\alpha}$ operators (aka *ladder operators*) on a given eigenvector E_β . We saw that if α and β are two distinct roots, with $\alpha + \beta \neq 0$, it may happen that $\beta \pm \alpha$ are also roots. Let $p \leq 0$ be the smallest integer such that $(\text{ad } E_{-\alpha})^{|p|} E_\beta$ is non

zero, *i.e.* that $\beta + p\alpha$ is a root, and let $q \geq 0$ be the largest integer such that $(\text{ad } E_\alpha)^q E_\beta$ is non zero, *i.e.* that $\beta + q\alpha$ is a root. We call the subset of roots $\{\beta + p\alpha, \beta + (p+1)\alpha, \dots, \beta, \dots, \beta + q\alpha\}$ the α -chain through β . Note that the $E_{\beta'}$, when β' runs along that chain, form a basis of a finite dimensional representation of the $\text{su}(2)$ algebra generated by H_α and $E_{\pm\alpha}$. According to what we know about these representations of $\text{su}(2)$, the lowest and highest eigenvalues of H_α are opposite

$$\langle \alpha, \beta + p\alpha \rangle = -\langle \alpha, \beta + q\alpha \rangle$$

or $2\langle \beta, \alpha \rangle = -(q+p)\langle \alpha, \alpha \rangle$, thus with the notation (3-21)

$$m = -p - q. \quad (3-22)$$

This construction also shows that $\beta - m\alpha = \beta + (p+q)\alpha$ is in the α -chain through β , (since $p \leq -m \leq q$), hence that this is a root.

Remark. The discussion of § 3.1 left the coefficients $N_{\alpha\beta}$ undetermined. One shows (see Exercise B), using the commutation relations of the E 's along a chain that the coefficients $N_{\alpha\beta}$ satisfy non linear relations and that they are determined up to signs by the geometry of the root system according to

$$|N_{\alpha\beta}| = \sqrt{\frac{1}{2}(1-p)q\langle \alpha, \alpha \rangle}. \quad (3-23)$$

Note that, as stated before, $N_{\alpha\beta}$ vanishes only if $q = 0$, *i.e.* if $\alpha + \beta$ is not a root.

Weyl group

For any vector x in the root space $\mathfrak{h}^*$, define the linear transformation

$$w_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha. \quad (3-24)$$

This is a reflection in the hyperplane orthogonal to α : $(w_\alpha)^2 = I$, $w_\alpha(\alpha) = -\alpha$, and $w_\alpha(x) = x$ if x is orthogonal to α . This is of course an isometry, since it preserves the scalar product: $\langle w_\alpha(x), w_\alpha(y) \rangle = \langle x, y \rangle$. Such a w_α is called a Weyl reflection. By definition the *Weyl group* W is the group generated by the w_α , *i.e.* the set of all possible products of w_α over roots α . Thanks to the remark following (3-22), if α and β are two roots, $w_\alpha(\beta) = \beta - m\alpha$ is also a root. The set of roots is thus globally invariant under the action of the Weyl group. The group W is completely determined by its action on roots, which is a permutation. W is thus a subgroup of the permutation group of the finite set Δ , hence a finite group².

Example : for the algebra $\text{su}(n)$, one finds that $W = \mathcal{S}_n$, the permutation group of n objects, see below in § 3.3.2.

Note that if $\beta_+ = \beta + q\alpha$ is the highest root in the α -chain through β , and $\beta_- = \beta + p\alpha$ the lowest one, $w_\alpha(\beta_\pm) = \beta_\mp$ and more generally, the roots of the chain are swapped by pairs under the action of w_α . The chain is thus invariant by w_α . (This is a generalisation of the $m \leftrightarrow -m$ symmetry of the $\text{su}(2)$ “multiplets” $(-j, -j+1, \dots, j-1, j)$.)

²This property is far from trivial: generically, when m vectors are given in the Euclidean space $\mathbb{R}^m$, the group generated by reflections in the hyperplanes orthogonal to these vectors is infinite. You need very peculiar configurations of vectors to make the group finite. Finite reflection groups have been classified by Coxeter. Weyl groups of simple algebras form a subset of Coxeter groups.

Positive roots, simple roots. Cartan matrix

Roots are not linearly independent in $\mathfrak{h}^*$. One may show that one can partition their set Δ into “positive” and “negative” roots, the opposite of a positive root being negative, and find a basis $\alpha_i, i = 1, \dots, \ell$ of ℓ *simple* roots, such that any positive (resp. negative) root is a linear combination with non negative (resp non positive) integer coefficients of these simple roots. As a consequence, a simple root cannot be written as the sum of two positive roots (check !).

Neither the choice of a set of positive roots, nor that of a basis of simple roots is unique. One goes from a basis of simple roots to another one by some operation of the Weyl group.

If α and β are simple roots, $\alpha - \beta$ cannot be a root (why?). The integer p in the previous discussion thus vanishes and $m = -q \leq 0$. It follows that $\langle \alpha, \beta \rangle \leq 0$.

The scalar product of two simple roots is non positive. (P)

We now define the Cartan matrix

$$C_{ij} = 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}. \quad (3-25)$$

Beware, that matrix is *a priori* non symmetric.³ Its diagonal elements are 2, its off-diagonal elements are ≤ 0 integers.

One must remember that the scalar product appearing in the numerator of (3-25) is positive definite. According to the Schwartz inequality, $\langle \alpha, \beta \rangle^2 \leq \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle$ with equality only if α and β are colinear. This property, together with the integrity properties of their elements, suffices to classify all possible Cartan matrices, as we shall now see.

Write $\langle \alpha_i, \alpha_j \rangle = \|\alpha_i\| \|\alpha_j\| \cos \widehat{\alpha_i, \alpha_j}$. Then by multiplying or dividing the two equations (3-21) for the pair $\{\alpha_i, \alpha_j\}, i \neq j$, namely $C_{ij} = m_i \leq 0$ and $C_{ji} = m_j \leq 0$, where the property (P) above has been taken into account, one finds that if $i \neq j$,

$$\left. \begin{aligned} \cos \widehat{\alpha_i, \alpha_j} &= -\frac{1}{2} \sqrt{m_i m_j} \\ \frac{\|\alpha_i\|}{\|\alpha_j\|} &= \sqrt{\frac{m_i}{m_j}} \end{aligned} \right\} \quad \text{with } m_i, m_j \in \mathbb{N}, \quad (3-26)$$

and the value -1 of the cosinus is impossible, since $\alpha_i \neq -\alpha_j$ by assumption, so that the only possible values of that cosinus are $0, -\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}$, *i.e.* the only possible angles between simple roots are $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$ or $\frac{5\pi}{6}$, with ratios of lengths of roots respectively equal to ?(undetermined), $1, \sqrt{2}, \sqrt{3}$.

There exists of course only one algebra of rank 1, viz the (complexified) $\mathfrak{su}(2)$ algebra, (3-1) or (3-17). It will be called A_1 below. It is then easy to classify the possible algebras of rank 2. The four cases are depicted on Fig. 3.1, with their Cartan matrices reading

$$A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad B_2 : \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad D_2 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \quad (3-27)$$

The nomenclature, A_2, B_2, G_2 and D_2 , is conventional, and so is the numbering of roots. The latter case, D_2 , which has $\langle \alpha_1, \alpha_2 \rangle = 0$, is mentioned here for completeness: it corresponds to

³Also, beware that some authors call Cartan matrix the transpose of (3-25)!

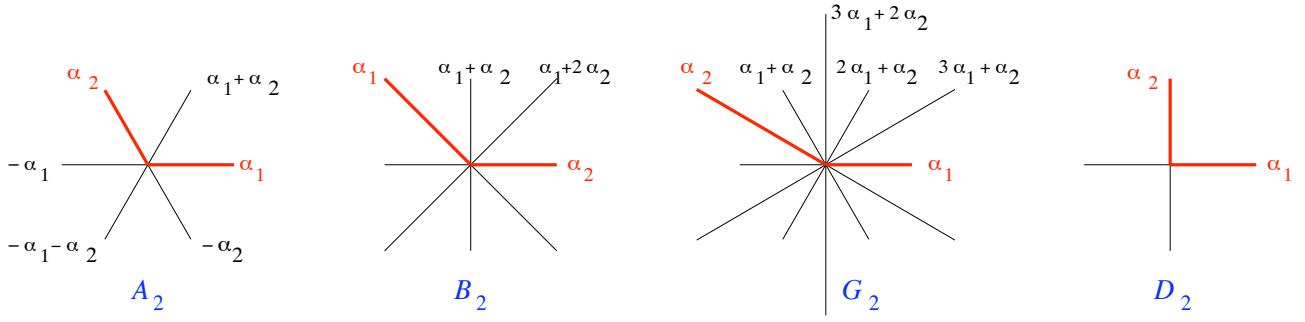


Figure 3.1: Root systems of rank 2. The two simple roots are drawn in thick lines. For the algebras B_2 , G_2 et D_2 , only positive roots have been labelled.

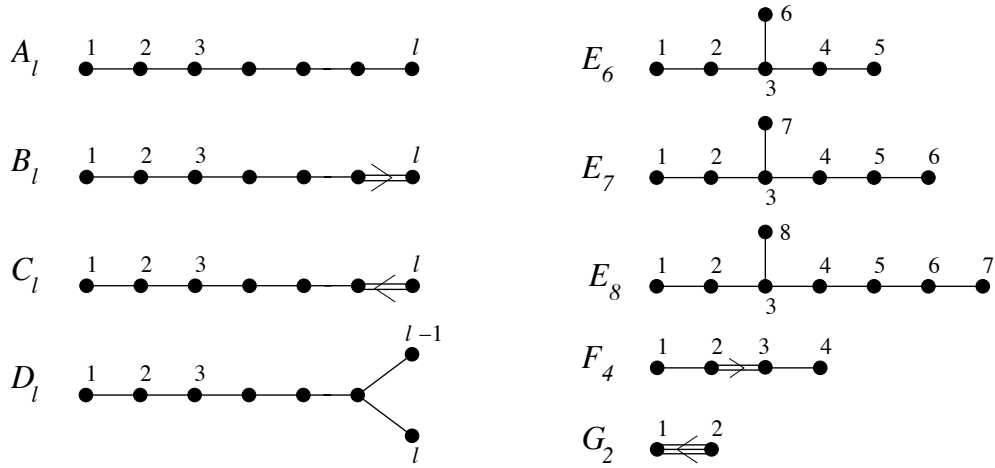


Figure 3.2: Dynkin diagrams

a semi-simple algebra, the direct sum of two A_1 algebras. (Nothing forces its two roots to be of equal length.)

In general, if the set of roots may be split into two mutually orthogonal subsets, one sees that the Lie algebra decomposes into a direct sum of two algebras, and vice versa. Recalling that any semi-simple algebra may be decomposed into the direct sum of simple subalgebras (see end of Chap. 1), in the following we consider only simple algebras.

Dynkin diagram

For higher rank, *i.e.* for higher dimension of the root space, it becomes difficult to visualise the root system. Another representation is adopted, by encoding the Cartan matrix into a diagram in the following way: with each simple root is associated a vertex of the diagram; two vertices are linked by an edge iff $\langle \alpha_i, \alpha_j \rangle \neq 0$; the edge is simple if $C_{ij} = C_{ji} = -1$ (angle of $2\pi/3$, equal lengths); it is double (resp. triple) if $C_{ij} = -2$ (resp. -3) and $C_{ji} = -1$ (angle of $\frac{3\pi}{4}$ resp. $\frac{5\pi}{6}$, with a length ratio of $\sqrt{2}$, resp. $\sqrt{3}$) and then carries an arrow (or rather a sign $>$) from i to j indicating which root is the longest. (Beware that some authors use the opposite convention for arrows !).

3.2.2 Root systems of simple algebras. Cartan classification

The analysis of all possible cases led Cartan⁴ to a classification of simple complex Lie algebras, in terms of four infinite families and five exceptional cases. The traditional notation is the following

$$A_\ell, \quad B_\ell, \quad C_\ell, \quad D_\ell, \quad E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2. \quad (3-28)$$

In each case, the lower index gives the rank of the algebra. The geometry of the root system is encoded in the Dynkin diagrams of Fig. 3.2.

The four infinite families are identified with the (complexified) Lie algebras of classical groups

$$A_\ell = \mathfrak{sl}(\ell + 1, \mathbb{C}), \quad B_\ell = \mathfrak{so}(2\ell + 1, \mathbb{C}), \quad C_\ell = \mathfrak{sp}(2\ell, \mathbb{C}), \quad D_\ell = \mathfrak{so}(2\ell, \mathbb{C}). \quad (3-29)$$

or with their unique compact real form, respectively $A_\ell = \mathfrak{su}(\ell + 1)$, $B_\ell = \mathfrak{so}(2\ell + 1)$, $C_\ell = \mathfrak{usp}(\ell)$, $D_\ell = \mathfrak{so}(2\ell)$.

The “exceptional algebras” $E_6, \dots, G_2$ have respective dimensions 78, 133, 248, 52 and 14. Those are algebras of ... exceptional Lie groups! The *group* G_2 is the group of automorphisms of octonions, F_4 is itself an automorphism group of octonion matrices, etc.

Among these algebras, the algebras A , D , E , whose roots have the same length, are called *simply laced*. A curious observation is that many problems, finite subgroups of $\mathfrak{su}(2)$, “simple” singularities, “minimal conformal field theories”, etc, are classified by the same *ADE* scheme... but this is another story!

The real forms of these simple complex algebras have also been classified by Cartan. One finds 12 infinite series and 23 exceptional cases!

3.2.3 Chevalley basis

There exists another basis of the Lie algebra $\mathfrak{g}$, called Chevalley basis, with brackets depending only on the Cartan matrix. Let h_i, e_i et f_i , $i = 1, \dots, \ell$, be generators attached to simple roots α_i according to

$$e_i = \left(\frac{2}{\langle \alpha_i, \alpha_i \rangle} \right)^{\frac{1}{2}} E_{\alpha_i}, \quad f_i = \left(\frac{2}{\langle \alpha_i, \alpha_i \rangle} \right)^{\frac{1}{2}} E_{-\alpha_i}, \quad h_i = \frac{2\alpha_i \cdot H}{\langle \alpha_i, \alpha_i \rangle}. \quad (3-30)$$

Their commutation relations read

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= C_{ji} e_j \\ [h_i, f_j] &= -C_{ji} f_j \\ [e_i, f_j] &= \delta_{ij} h_j \end{aligned} \quad (3-31)$$

(check!). The algebra is generated by the e_i, f_i, h_i and all their commutators, constrained by (3-31) and by the “Serre relations”

$$\mathrm{ad}(e_i)^{1-C_{ji}} e_j = 0 \quad (3-32)$$

$$\mathrm{ad}(f_i)^{1-C_{ji}} f_j = 0. \quad (3-33)$$

This proves that the whole algebra is indeed encoded in the data of the simple roots and of their geometry (Cartan matrix or Dynkin diagram).

Note also the remarkable and *a priori* not obvious property that in that basis, all the structure constants (coefficients of the commutation relations) are *integers*.

⁴This classification work, undertaken by Killing, was corrected and completed by É. Cartan, and later simplified by van der Waerden, Dynkin, ...

3.2.4 Coroots. Highest root. Coxeter number and exponents

We give here some complements on notations and concepts that are encountered in the study of simple Lie algebras and of their root systems.

As the combination

$$\alpha_i^\vee := \frac{2}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \quad (3-34)$$

for α_i a simple root, appears frequently, it is given the name of *coroot*. The Cartan matrix may be rewritten as

$$C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle. \quad (3-35)$$

The *highest root* θ is the positive root with the property that the sum of its components in a basis of simple roots is maximal: one proves that this characterizes it uniquely. Its components in the basis of simple roots and in that of coroots

$$\theta = \sum_i a_i \alpha_i, \quad \frac{2}{\langle \theta, \theta \rangle} \theta = \sum_i a_i^\vee \alpha_i^\vee, \quad (3-36)$$

called *Kac labels*, *resp dual Kac labels*, play also a role, in particular through their sums,

$$h = 1 + \sum_i a_i, \quad h^\vee = 1 + \sum_i a_i^\vee. \quad (3-37)$$

The numbers h and $h^\vee$ are respectively the *Coxeter number* and the *dual Coxeter number*. When a normalisation of roots has to be picked, which we have not done yet, one usually imposes that $\langle \theta, \theta \rangle = 2$.

Lastly the diagonalisation of the symmetrized Cartan matrix

$$\hat{C}_{ij} := 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\sqrt{\langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle}} \quad (3-38)$$

yields a spectrum of eigenvalues

$$\text{eigenvalues of } \hat{C} = \left\{ 4 \sin^2 \left(\frac{\pi}{2h} m_i \right) \right\}, \quad i = 1, \dots, \ell, \quad (3-39)$$

in which a new set of integers m_i appears, the *Coxeter exponents*, satisfying $1 \leq m_i \leq h - 1$ with possible multiplicities. These numbers are relevant for various reasons. They contain useful information on the Weyl group. After addition of 1, (making them ≥ 2), one gets the degrees of algebraically independent Casimir operators, or the degrees where the Lie group has a non trivial cohomology, etc etc.

Examples: for A_{n-1} alias $\mathfrak{su}(n)$, roots and coroots coincide. The highest root is $\theta = \sum_i \alpha_i$, thus $h = h^\vee = n$, the Coxeter exponents are $1, 2, \dots, n-1$. For D_n alias $\mathfrak{so}(2n)$, roots and coroots are again identical, $\theta = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, $h = 2n - 2$, and the exponents are $1, 3, \dots, 2n-3, n-1$, with $n-1$ double if n is even.

See Appendix F for Tables of data on the classical simple algebras.

3.3 Representations of semi-simple algebras

3.3.1 Weights. Weight lattice

We now turn our attention to representations of semi-simple algebras, with an approach parallel to that of previous sections. In what follows, “representation” means finite dimensional irreducible representation. We also assume these representations to be unitary: this is the case of interest for representations of compact groups. The elements of the Cartan subalgebra commute among themselves, they also commute in any representation. Denoting with “bras” and

“kets” the vectors of that representation, and writing simply X (instead of $d(X)$) for the representative of the element $X \in \mathfrak{g}$, one may find a basis $|\lambda_a\rangle$ which diagonalises simultaneously the elements of the Cartan algebra

$$H|\lambda_a\rangle = \lambda(H)|\lambda_a\rangle \quad (3-40)$$

or equivalently

$$H_i|\lambda_a\rangle = \lambda_{(i)}|\lambda_a\rangle, \quad (3-41)$$

with an eigenvalue λ which is again a linear form on the space $\mathfrak{h}$, hence an element of $\mathfrak{h}^*$, the root space. Such a vector $\lambda = (\lambda_{(i)})$ of $\mathfrak{h}^*$ is called a *weight*. Note that for a unitary representation, the H are Hermitian, hence λ is real-valued: the weights are real vectors of $\mathfrak{h}^*$. As the eigenvalue λ may occur with some multiplicity, we have appended the eigenvectors with a multiplicity index a . The set of weights of a given representation forms in the space $\mathfrak{h}^*$ the *weight diagram* of the representation, see Fig. 3.5 below for examples in the case of $\mathfrak{su}(3)$.

The adjoint representation is a particular representation of the algebra whose weights are the roots. The roots studied in the previous sections thus belong to the set of weights in $\mathfrak{h}^*$.

The vectors $|\lambda_a\rangle$ forming a basis of the representation, their total number, including the multiplicity, equals the dimension of the representation space E . This space E contains representation subspaces for each of the $\mathfrak{su}(2)$ algebras that we identified in § 3.2, generated by $\{H_\alpha, E_\alpha, E_{-\alpha}\}$. By the same argument as in § 2, one now show that any weight λ satisfies

$$\forall \alpha, \quad 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = m' \in \mathbb{Z}, \quad (3-42)$$

and conversely, it may be shown that any $\lambda \in \mathfrak{h}^*$ satisfying (3-42) is the weight of some finite dimensional representation. One may thus use (3-42) as an alternative definition of weights. To convince oneself that the weights of any representation satisfy (3-42), one may, like in § 3.2, define the maximal chain of weights through λ

$$\lambda + p'\alpha, \dots, \lambda, \dots, \lambda + q'\alpha \quad p' \leq 0, \quad q' \geq 0,$$

which form a representation of the $\mathfrak{su}(2)$ subalgebra, and then show that $m' = -p' - q'$.

Let p' be the smallest ≤ 0 integer such that $(E_{-\alpha})^{|p'|}|\lambda_a\rangle \neq 0$, and q' the largest ≥ 0 integer such that $(E_\alpha)^{q'}|\lambda_a\rangle \neq 0$, H_α has respective eigenvalues $\langle \lambda, \alpha \rangle + p'\langle \alpha, \alpha \rangle$, and $\langle \lambda, \alpha \rangle + q'\langle \alpha, \alpha \rangle$ on these vectors. Expressing that the eigenvalues of $2H_\alpha/\langle \alpha, \alpha \rangle$ are opposite integers, one finds

$$2q' + 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2j \quad 2p' + 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = -2j.$$

Subtracting these equations gives $q' - p' = 2j$, and the length of the chain is $2j + 1$ (dimension of the spin j representation of $\mathfrak{su}(2)$), while adding them to get rid of $2j$, one has

$$2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = -(q' + p') =: m', \quad \text{as announced in (3-42).}$$

This chain is invariant under the action of the Weyl reflection w_α . (This is a generalisation of the $\mathbb{Z}_2$ symmetry of $\mathfrak{su}(2)$ “multiplets” $(-j, -j+1, \dots, j-1, j)$.) More generally the set of weights is invariant under the Weyl group: if λ is a weight of a representation, so is $w_\alpha(\lambda)$,

and one shows that they have the same multiplicity. The weight diagram of a representation is thus invariant under the action of W .

The set of weights is split by the Weyl group W into “chambers”, whose number equals the order of W . The chamber associated with the element w of W is the cone

$$\mathcal{C}_w = \{\lambda | \langle w\lambda, \alpha_i \rangle \geq 0, \forall i = 1, \dots, \ell\}, \quad (3-43)$$

where the α_i are the simple roots. (This is not quite a partition, as some weights belong to the “walls” between chambers.) The *fundamental chamber* is $\mathcal{C}_1$, corresponding to the identity in W . The weights belonging to that fundamental chamber are called *dominant weights*. Any weight may be brought into $\mathcal{C}_1$ by some operation of W : it is on the “orbit” (for the Weyl group) of a unique dominant weight. Among the weights of a representation, at least one belongs to $\mathcal{C}_1$.

On the other hand, from $[H_i, E_\alpha] = \alpha_{(i)} E_\alpha$ follows that

$$H_i E_\alpha |\lambda_a\rangle = ([H_i, E_\alpha] + E_\alpha H_i) |\lambda_a\rangle = (\alpha_{(i)} + \lambda_{(i)}) E_\alpha |\lambda_a\rangle$$

hence that $E_\alpha |\lambda_a\rangle$, if non vanishing, is an eigenvector of weight $\lambda + \alpha$. Now, in an irreducible representation, all vectors are obtained from one another by such actions of E_α , and we conclude that

▷ *Two weights of the same (irreducible) representation differ by a integer-coefficient combination of roots,*

(but this combination is in general not a root).

One then introduces a partial order on weights of the same representation: $\lambda' > \lambda$ if $\lambda' - \lambda = \sum_i n_i \alpha_i$, with non negative (integer) coefficients n_i . Among the weights of that representation, one proves there exists a unique highest weight Λ , which is shown to be of multiplicity 1. The highest weight vector will be denoted $|\Lambda\rangle$ (with no index a). It is such that for any positive root $E_\alpha |\Lambda\rangle = 0$, (otherwise, it would not be the highest), hence $q' = 0$ in the discussion above, and $\langle \Lambda, \alpha \rangle = \frac{1}{2} \langle \alpha, \alpha \rangle j > 0$, Λ is thus a dominant weight.

▷ *The highest weight of a representation is a dominant weight, $\Lambda \in \mathcal{C}_1$.*

This highest weight vector characterises the irreducible representation. (In the case of $\mathfrak{su}(2)$, this would be a vector $|j, m = j\rangle$.) In other words, two representations are equivalent iff they have the same highest weight.

One then introduces the *Dynkin labels* of the weight λ by

$$\lambda_i = 2 \frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \in \mathbb{Z} \quad (3-44)$$

with α_i the simple roots. For a dominant weight, thus for any highest weight of a representation, these indices are non negative, *i.e.* in $\mathbb{N}$.

The *fundamental weights* Λ_i satisfy by definition

$$2 \frac{\langle \Lambda_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}. \quad (3-45)$$

Their number equals the rank ℓ of algebra, and they make a basis of $\mathfrak{h}^*$. Each one is the highest weight of an irreducible representation called *fundamental*; hence there are ℓ fundamental representations. We have thus obtained

▷ Any irreducible representation is characterised by its highest weight, and with a little abuse of notation, we denote (Λ) the irreducible representation of highest weight Λ .

▷ Any highest weight decomposes on fundamental weights, and its components are its Dynkin labels (3-44),

$$\Lambda = \sum_{j=1}^{\ell} \lambda_j \Lambda_j \quad , \quad \lambda_i \in \mathbb{N} . \quad (3-46)$$

and any Λ of the form (3-46) is the highest weight of an irreducible representation.

Stated differently, the knowledge of the fundamental weight suffices to construct all irreducible representations of the algebra.

Using the properties just stated, show that the highest weight of the adjoint representation is necessarily θ , defined in § 2.4.

Weight and root lattices

Generally speaking, given a basis of vectors $e_1, \dots, e_p$ in a p dimensional space, the *lattice* generated by these vectors is the set of vectors $\sum_{i=1}^p z^i e_i$ with coefficients $z^i \in \mathbb{Z}$. This lattice is also denoted $\mathbb{Z}e_1 + \dots + \mathbb{Z}e_p$.

The *weight lattice* P is the lattice generated by the fundamental ℓ weights Λ_i . The root lattice Q is the one generated by the ℓ simple roots α_i . This is a sublattice of P . Any weight of an irreducible representation belongs to P .

One may consider the *congruence classes* of the additive group P wrt its subgroup Q , that are the classes for the equivalence relation $\lambda \sim \lambda'$ iff $\lambda - \lambda' \in Q$. The number $|P/Q|$ of these classes turns out to be equal to the determinant of the Cartan matrix(!). In the case of $\mathfrak{su}(n)$, there are n classes, we shall return to that point later.

One may also introduce the lattice $Q^\vee$ generated by the ℓ coroots $\alpha_i^\vee$ (cf §2.4). It is the “dual” of P , in the sense that $\langle \alpha_i^\vee, \Lambda_j \rangle \in \mathbb{Z}$.

One also shows that the subgroups of the finite group P/Q are isomorphic to homotopy groups of groups G having $\mathfrak{g}$ as a Lie algebra! For example for $\mathfrak{su}(n)$, we find below that $P/Q = \mathbb{Z}_n$, and these subgroups are characterised by a divisor d of n . For each of them, $\mathrm{SU}(n)/\mathbb{Z}_d$ has the $\mathfrak{su}(n)$ Lie algebra. The case $n = 2$, with $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, is quite familiar.

Dimension and Casimir operator

It may be useful to know the dimension of a representation with a given highest weight and the value of the quadratic Casimir operator in that representation. These expressions are given in terms of the *Weyl vector* ρ , defined by any of the two (non trivially!) equivalent formulas

$$\begin{aligned} \rho &= \frac{1}{2} \sum_{\alpha > 0} \alpha \\ &= \sum_j \Lambda_j . \end{aligned} \quad (3-47)$$

A remarkable formula, due to Weyl, gives the dimension of the representation of highest weight Λ as a product over positive roots

$$\dim(\Lambda) = \prod_{\alpha > 0} \frac{\langle \Lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \quad (3-48)$$

while the eigenvalue of the quadratic Casimir reads

$$C_2(\Lambda) = \frac{1}{2} \langle \Lambda, \Lambda + 2\rho \rangle . \quad (3-49)$$

A related question is that of the trace of generators of $\mathfrak{g}$ in the representation (Λ) . Let t_a be the basis of $\mathfrak{g}$ such that $\text{tr } t_a t_b = T_A \delta_{ab}$, with a coefficient T_A (“ A ” like adjoint) whose sign depends on conventions (t Hermitian or antiHermitian, see Chap. 1). In the representation of highest weight Λ , one has

$$\text{tr } d_\Lambda(t_a) d_\Lambda(t_b) = T_\Lambda \delta_{ab} . \quad (3-50)$$

But in that basis, the quadratic Casimir reads $C_2 = \sum_a (d_\Lambda(t_a))^2$ hence, taking the trace,

$$\begin{aligned} \text{tr } C_2 &= \sum_a \text{tr } (d_\Lambda(t_a))^2 = T_\Lambda \sum_a 1 = T_\Lambda \dim \mathfrak{g} \\ &= C_2(\Lambda) \text{tr } I_\Lambda = C_2(\Lambda) \dim(\Lambda) \end{aligned} \quad (3-51)$$

whence

$$T_\Lambda = C_2(\Lambda) \frac{\dim(\Lambda)}{\dim \mathfrak{g}} , \quad (3-52)$$

a useful formula in calculations (gauge theories ...). In the adjoint representation, $\dim(A) = \dim \mathfrak{g}$, hence $T_\Lambda = T_A = C_2(A)$.

There is a host of additional, sometimes intriguing, formulas relating various aspects of Lie algebras and representation theory. For example the Freudenthal–de Vries “strange formula”, which connects the norms of the vectors ρ and θ to the dimension of the algebra and the Coxeter number: $\langle \rho, \rho \rangle = \frac{h}{24} \langle \theta, \theta \rangle \dim \mathfrak{g}$.

There is also a formula (Freudenthal) giving the multiplicity of a weight λ within a representation of given highest weight Λ . And last, as a related issue, a formula by Weyl giving the character $\chi_\Lambda(e^H)$ of that representation evaluated on an element of the *Cartan torus*, an abelian subgroup resulting from the exponentiation of the Cartan algebra $\mathfrak{h}$.

Conjugate representation

Given a representation of highest weight Λ , its complex conjugate representation is generally non equivalent. One may characterize what is its highest weight $\bar{\Lambda}$ thanks to the Weyl group. The non-equivalence of representations (Λ) and $(\bar{\Lambda})$ has to do with the symmetries of the Dynkin diagram. For the algebras of type B, C, E_7, E_8, F_4, G_2 for which there is no non trivial symmetry, the representations are self-conjugate. This is also the case of D_{2r} . For the others, conjugation corresponds to the following symmetry on Dynkin labels

$$\begin{array}{lll} A_\ell = \text{su}(\ell + 1) & \lambda_i \leftrightarrow \lambda_{\ell+1-i} & \ell > 1 \\ D_{2r+1} = \text{so}(4r + 2) & \lambda_\ell \leftrightarrow \lambda_{\ell-1} , & \ell = 2r + 1 \\ E_6 & \lambda_i \leftrightarrow \lambda_{6-i}, & i = 1, 2 . \end{array} \quad (3-53)$$

3.3.2 Roots and weights of $\text{su}(n)$

Let us construct explicitly the weights and thus the irreducible representations of $\text{su}(n)$.

We first pick a convenient parametrization of the space $\mathfrak{h}^*$, which is of dimension $n - 1$. Let e_i , $i = 1, \dots, n$, be n vectors of $\mathfrak{h}^* = \mathbb{R}^{n-1}$ (hence necessarily dependent), satisfying $\sum_1^n e_i = 0$.

They are obtained starting from an orthonormal basis $\hat{e}_i$ of $\mathbb{R}^n$ by projecting the $\hat{e}_i$ on the hyperplane orthogonal to $\hat{\rho} := \sum_{i=1}^n \hat{e}_i$, thus $e_i = \hat{e}_i - \frac{1}{n}\hat{\rho}$. (In other words, one is considering the hyperplane $\sum_i x^i = 1$ in the space $\mathbb{R}^n$). These vectors have scalar products given by

$$\langle e_i, e_j \rangle = \delta_{ij} - \frac{1}{n}. \quad (3-54)$$

In terms of these vectors, the positive roots of $\mathfrak{su}(n) = A_{n-1}$, whose number equals $|\Delta_+| = n(n-1)/2$, are

$$\alpha_{ij} = e_i - e_j, \quad 1 \leq i < j \leq n, \quad (3-55)$$

and the $\ell = n - 1$ simple roots are

$$\alpha_i = \alpha_{i i+1} = e_i - e_{i+1}, \quad 1 \leq i \leq n - 1. \quad (3-56)$$

These roots have been normalized by $\langle \alpha, \alpha \rangle = 2$. The sum of positive roots is easily computed to be

$$\begin{aligned} 2\rho &= (n-1)e_1 + (n-3)e_2 + \cdots + (n-2i+1)e_i + \cdots + (n-1)e_n \\ &= (n-1)\alpha_1 + 2(n-2)\alpha_2 + \cdots + i(n-i)\alpha_i + \cdots + (n-1)\alpha_{n-1}. \end{aligned} \quad (3-57)$$

One checks that the Cartan matrix is

$$C_{ij} = \langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \end{cases}$$

in accord with the Dynkin diagram of type A_{n-1} , thus justifying (3-56). The fundamental weights Λ_i , $i = 1, \dots, n-1$ are then readily written

$$\Lambda_i = \sum_{j=1}^i e_j, \quad (3-58)$$

$$e_1 = \Lambda_1, \quad e_i = \Lambda_i - \Lambda_{i-1} \text{ for } i = 2, \dots, n-1, \quad e_n = -\Lambda_{n-1} \quad (3-59)$$

with scalar products

$$\langle \Lambda_i, \Lambda_j \rangle = \frac{i(n-j)}{n}, \quad i \leq j. \quad (3-60)$$

The Weyl group $W \equiv \mathcal{S}_N$ acts on roots and on weights by permuting the e_i :

$$w \in W \leftrightarrow \bar{w} \in \mathcal{S}_N : w(e_i) = e_{\bar{w}(i)}.$$

Dimension of the representation of weight Λ

Combining formulas (3-48) and (3-55), prove the following expression

$$\dim(\Lambda) = \prod_{1 \leq i < j \leq n} \frac{f_i - f_j + j - i}{j - i} \quad \text{où} \quad f_i := \sum_{k=i}^{n-1} \lambda_k, \quad f_n = 0. \quad (3-61)$$

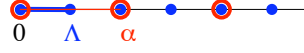


Figure 3.3: Weights of $\mathfrak{su}(2)$. The positive parts of the weight (small dots) and root (big dots) lattices.

Conjugate representations

If $\Lambda = (\lambda_1, \dots, \lambda_{n-1})$ is the highest weight of an irreducible representation of $\mathfrak{su}(n)$, $\bar{\Lambda} = (\lambda_{n-1}, \dots, \lambda_1)$ is that of the complex conjugate, generally inequivalent, representation (see above in § 3.3.1). Note that neither the dimension, nor the value of the Casimir operator distinguish the representations Λ and $\bar{\Lambda}$.

“ n -ality”.

There are n congruence classes of P with respect to Q . They are distinguished by the value of

$$\nu(\lambda) := \lambda_1 + 2\lambda_2 + \dots + (n-1)\lambda_{n-1} \pmod{n}, \quad (3-62)$$

to which one may give the unimaginative name of “ n -ality”, by extension of the “triality” of $\mathfrak{su}(3)$, see below. The elements of the root lattice thus have $\nu(\lambda) = 0$.

Examples of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$.

In the case of $\mathfrak{su}(2)$, there is one fundamental weight $\Lambda = \Lambda_1$ and one positive root α , normalised by $\langle \alpha, \alpha \rangle = 2$, hence $\langle \Lambda, \alpha \rangle = 1$, $\langle \Lambda, \Lambda \rangle = \frac{1}{2}$. Thus $\alpha = 2\Lambda$, Λ corresponds to the spin $\frac{1}{2}$ representation, α to spin 1. The weight lattice and the root lattice are easy to draw, see Fig. 3.3. The Dynkin label λ_1 is identical to the integer $2j$, the two congruence classes of P wrt Q correspond to representations of integer and half-integer spin, the dimension $\dim(\Lambda) = \lambda_1 + 1 = 2j + 1$ and the Casimir operator $C_2(\Lambda) = \frac{1}{4}\lambda_1(\lambda_1 + 2) = j(j + 1)$, in accord with well known expressions.

For $\mathfrak{su}(3)$, the weight lattice is triangular, see Fig. (3.4), on which the triality $\tau(\lambda) := \lambda_1 + 2\lambda_2 \pmod{3}$ has been shown and the fundamental weights and the highest weights of the “low lying” representations have been displayed. Following the common use, representations are referred to by their dimension⁵

$$\dim(\Lambda) = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2), \quad (3-63)$$

supplemented by a bar to distinguish a representation from its conjugate, whenever necessary. The conjugate of representation of highest weight $\Lambda = (\lambda_1, \lambda_2)$ has highest weight $\bar{\Lambda} = (\lambda_2, \lambda_1)$. Only the representations lying on the bisector of the Weyl chamber are thus self-conjugate.

Exercise. Compute the eigenvalue of the quadratic Casimir operator in terms of Dynkin labels λ_1, λ_2 using the formulas (3-49) and (3-57).

⁵which may be ambiguous; for example, identify on Fig. (3.4) the weight of another representation of dimension 15.

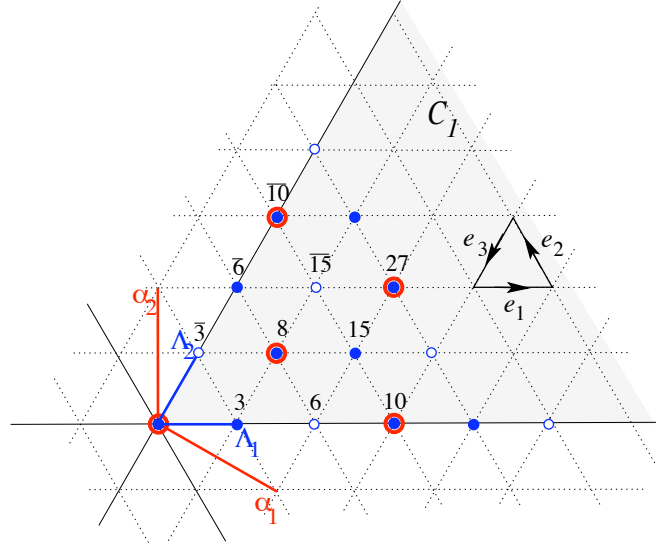


Figure 3.4: Weights of $\mathfrak{su}(3)$. Only the first Weyl chamber $\mathcal{C}_1$ has been detailed, with some highest weights. The weights of triality 0 (forming the root lattice) are represented by a wide disk, those of triality 1, resp. 2, by a full, resp. open disk.

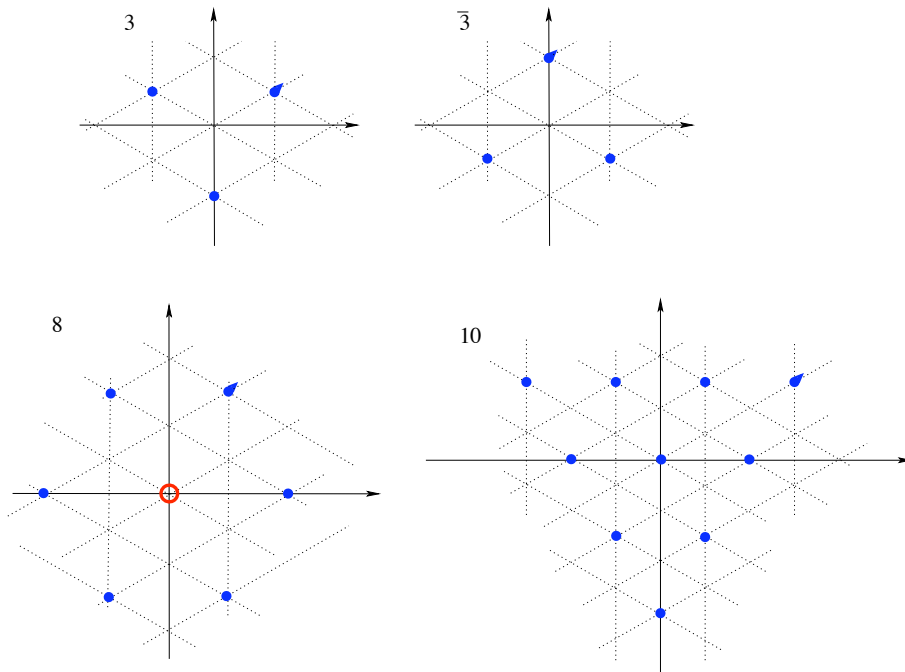


Figure 3.5: The weight diagrams of low lying representations of $\mathfrak{su}(3)$, denoted by their dimension. Note that a rotation of 30° of the weight lattice has been performed with respect to the previous figure. In each representation, the highest weight is marked by a small indentation. The small dots are weights of multiplicity 1, the wider and open dots have multiplicity 2.

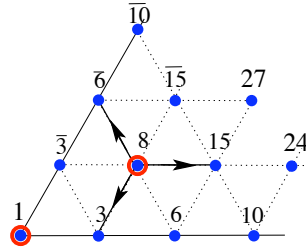


Figure 3.6: Tensor product of the 8 representation by the 3 representation, depicted on the weight diagram of $\mathfrak{su}(3)$.

The set of weights of low lying representations is displayed on Fig. (3.5), after a rotation of the axes of the previous figures. The horizontal axis, colinear to α_1 , and the vertical axis, colinear to Λ_2 , will indeed acquire a physical meaning: that of axes of isospin and “hypercharge” coordinates, see next chapter.

Remark. The case of $\mathfrak{su}(n)$ has been detailed. Analogous formulas for roots, fundamental weights, etc of other simple algebras are of course known explicitly and tabulated in the literature. See for example Appendix F for the identity card of “classical algebras”, of type A, B, C, D , and Bourbaki, chap.6, for more details on the other algebras.

3.4 Tensor products of representations of $\mathfrak{su}(n)$

3.4.1 Littlewood-Richardson rules

Given two irreducible representations of $\mathfrak{su}(n)$ (or of any other Lie algebra), a frequently encountered problem is to decompose their tensor product into a direct sum of irreducible representations. If one is only interested in multiplicities and if one has a character table of the corresponding compact *group*, one may use the formulae proved in Chap. 2, § 2.3.2.

There exists also fairly complex rules giving that decomposition into irreducible representations of a product of two irreducible representations (Λ) and (Λ') of $\mathfrak{su}(n)$. Those are the Littlewood-Richardson rules, which appeal to the expression in terms of Young tableaux (see next §). But it is often simpler to proceed step by step, noticing that the irreducible representation (Λ') is found in an adequate product of fundamental representations, and examining the successive products of representation Λ by these fundamental representations. By the associativity of the tensor product, one brings the original problem back to that of the tensor product of (Λ) by the various fundamental representations.

The latter operation is easy to describe on the weight lattice. Given the highest weight Λ in the first Weyl chamber $\mathcal{C}_1$, the tensor product of (Λ) by the fundamental representation of highest weight Λ_i is decomposed into irreducible representations in the following way: one adds in all possible ways the $\dim(\Lambda_i)$ weights of the fundamental to the vector Λ and one keeps as highest weights in the decomposition only the weights resulting from this addition that belong to $\mathcal{C}_1$.