Counting Doodles

Saclay, 10 June 2015

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An hommage to Claude Itzykson

Saclay, 10 June 2015

in memory of more than twenty years of collaboration
and friendship

First encounter in Cargèse 1970

1974-75 Teaching Quantum Field Theory in Orsay
and first paper on Ising model and the sine-Gordon theory

25 joint publications

Claude, the researcher
A career embracing a wide spectrum

From Particle Physics

(Thesis with Maurice Jacob on weak interactions 1967)

to Classical and Quantum Field Theory (translation of Akhiezer-Berestetskii's book from the Russian; *Soft Quanta*, eikonal approximation, pair production in an e-m field, relativistic Balmer formula (with H. Abarbanel, É. Brézin, J. Bros, Y. Frishman, I. Todorov, A. Voros, J. Zinn-Justin, '65-70)

to Group Theory (with M. Nauenberg and with M. Bander, 1966)

to Statistical Mechanics (e.g. Ising model, a subject of many returns. . .)

to Mathematical Physics. . .

In the mid-seventies, with R. Balian and J.-M. Drouffe, Lattice Gauge Theories, immediately after Wilson's seminal paper : 3 important papers and a Physics Reports

Large orders of perturbation theory of QED, after Lipatov's, and Brézin–Zinn-Justin's works, with R. Balian, G.Parisi and JBZ, '77-78

with André Morel

Matrix Integrals and Combinatorics (counting of planar Feynman diagrams aka maps), (“BIPZ”, Bessis-Itzykson-Z, “HC-IZ” formula, . . . 1978-80)

Lattice models (M. Peskin, J-M Luck, C. De Dominicis, H. Orland,. . .), **Random Geometry, random interactions** (with É. Brézin and D. Gross; J.-M. Luck; B. Derrida;). . .

1985-1995 **Conformal Field Theory**: Under his guidance and leadership, Saclay’s group with a dozen of bright young researchers, postdocs, students becomes one of the hot spots of the field (D. Altschuler, M. Bauer, A. Cappelli, P. Di Francesco, H. Saleur,. . .)

Topological Field Theory, from matrix integrals to Combinatorics and Algebraic Geometry (with M. Bauer and P. Di Francesco)

Les Houches 1982, with Michael Peskin

Claude, the lover of mathematics

From Group Theory (representations of SU(N) (w. Nauenberg), symmetries of H atom (w. Bander), non compact groups, . . .)

to Combinatorics (matrix integrals and maps)

to Number Theory (billiards and affine algebras, (E. Aurell, J.-M. Luck, P. Moussa), Les Houches 1989)

to Algebraic Geometry (Kontsevich integral and moduli spaces, Grothendieck dessins d'enfants, . . .)

Unfinished work on permutation group applied to replica symmetry

Claude played an important rôle in bringing together mathematicians and physicists (e.g. Les Houches 1989 Winter School on number theory)

Saclay, 1994, with Louis Michel

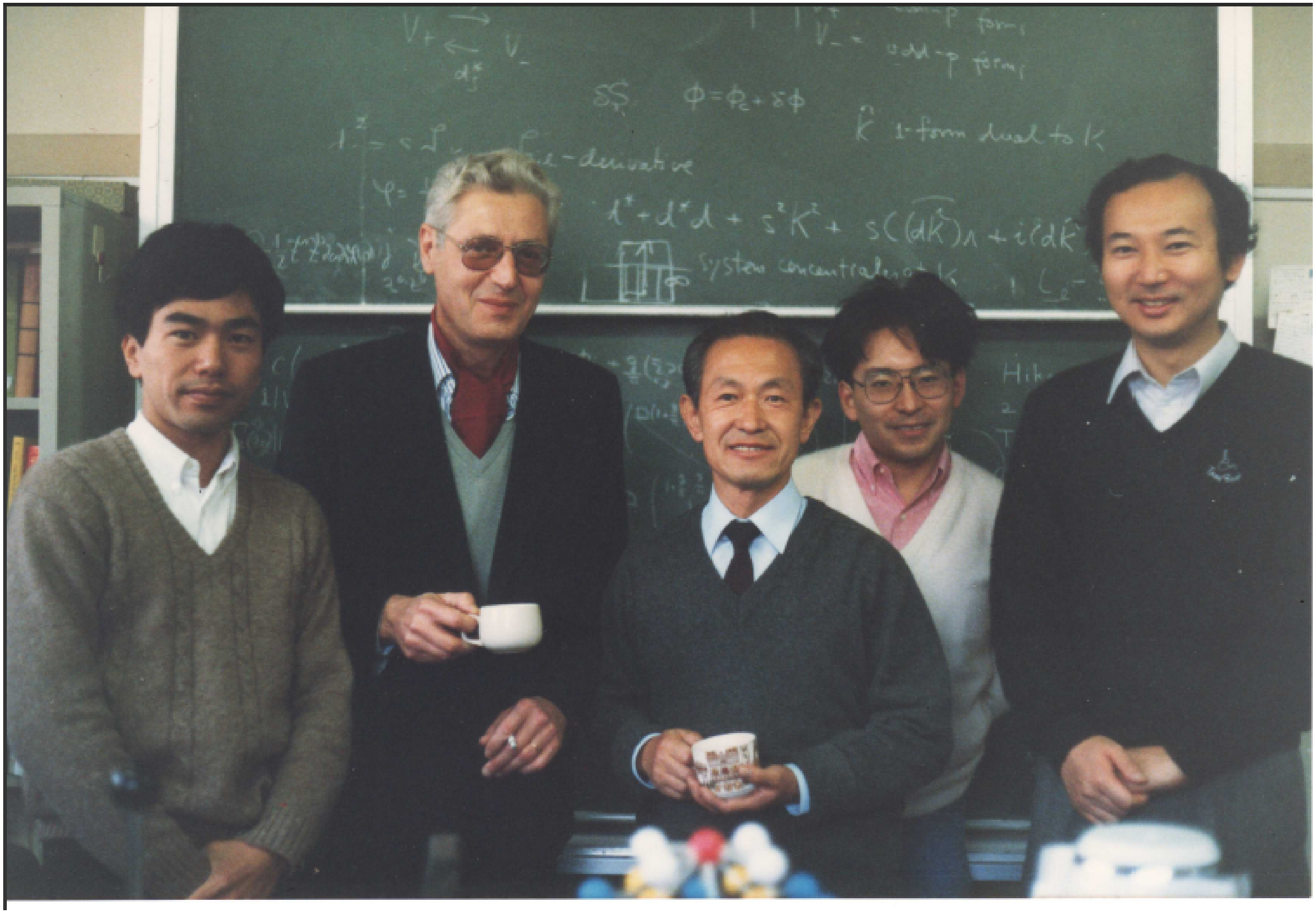
Claude, the teacher and the pedagogue

Claude loved to understand new things and to teach them.

Many series of lectures in various institutions (École Polytechnique) and advanced courses and master classes (DEA Orsay 1974, 1976), EPFL Lausanne, Marseille, CERN, Japan, Cargèse, Les Houches, Trieste, MIT, . . .

from which two influential books grew
"Quantum Field Theory" (1980)
"Statistical Field Theory" (1989) with J.-M. Drouffe

The mentor of many junior physicists . . .



Kyoto ? 1989

Claude, the man of ever open mind and alert curiosity

A unique rôle of go-between in this lab, capable of interacting with everybody, on every subject

A man of culture, not only in science, but also in literature, history . . .

An elegant and witty personality, with a lot of charm and personal charisma, and an influence lasting to this day. . .

with Cirano De Dominicis

Counting doodles

with Robert Coquereaux (CPT Marseille)

Counting doodles

with Robert Coquereaux (CPT Marseille)

ou **Comptage de gribouillis**. . .

By courtesy of Victor Zuber-Doumat

Counting doodles

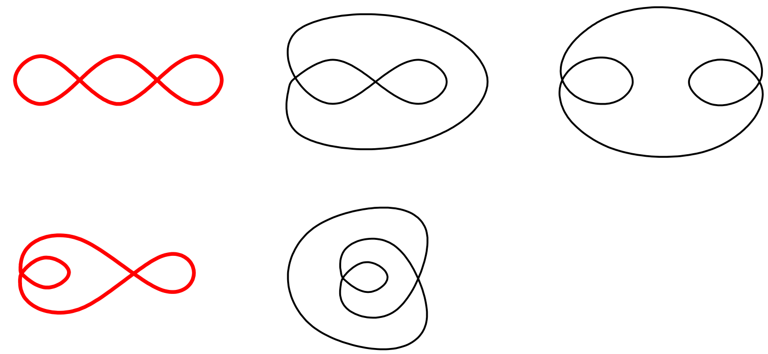
with Robert Coquereaux (CPT Marseille)

doodle: an (open or) closed smooth curve with one component and n self-crossings, drawn in the plane, on the sphere or on a higher genus surface; only double points, with distinct tangents.

In more mathematical terms : image of an *immersion* of an (oriented/unoriented) circle into a 2-dim (oriented/unoriented) surface Σ, defined up to topological equivalence (by the diffeomorphism group $\mathrm{Diff}^+(\Sigma)$, resp. $\mathrm{Diff}(\Sigma)$).

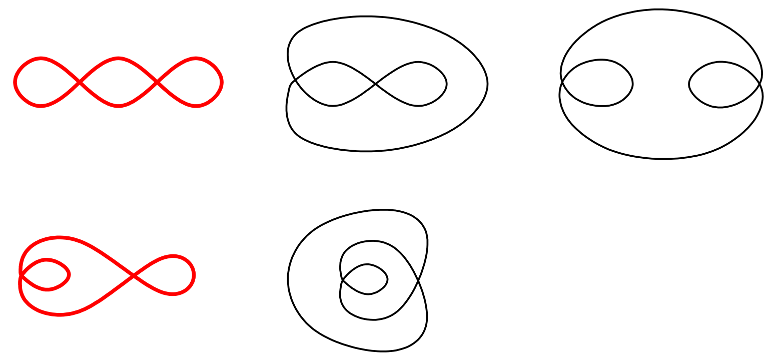
In this talk, mainly immersions into closed surfaces: sphere or higher genus surface.

$n = 2$ Plane vs Sphere



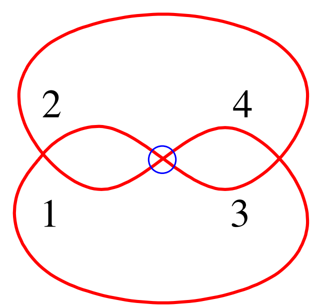
5 immersions of the circle in the plane, 2 in the sphere

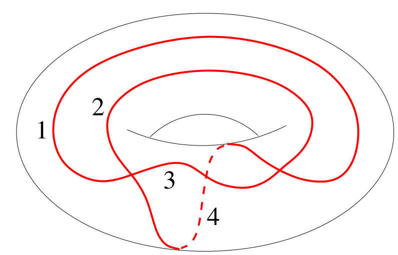
$n = 2$ Plane vs Sphere



5 immersions of the circle in the plane, 2 in the sphere

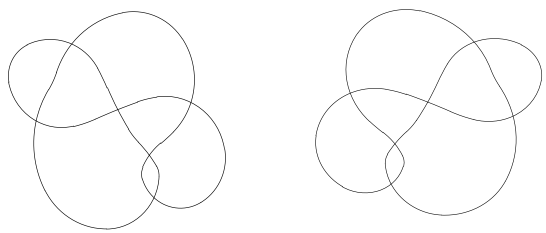
Higher genus ?





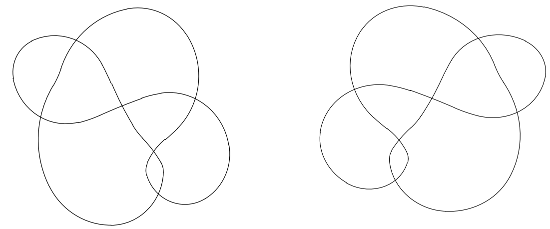
an immersion of a circle into the torus

Oriented surface ?



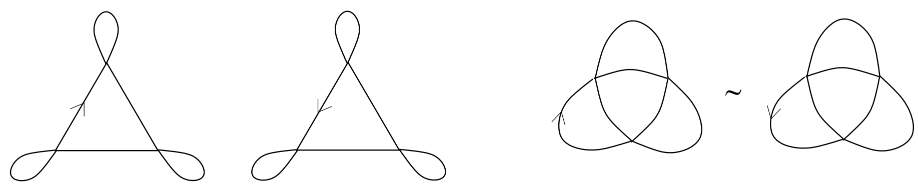
Two immersions of an unoriented circle with $n = 6$ double points. Distinct on an oriented sphere, but equivalent on an unoriented sphere.

Oriented surface ?



Two immersions of an unoriented circle with $n = 6$ double points. Distinct on an oriented sphere, but identical on an unoriented sphere.

Oriented circle ?



Immersions of an oriented circle. Left : an $n = 3$ immersion not equivalent to its reverse; in contrast, the trefoil *is* equivalent to its reverse.

Problem: How to count and how to list such immersions?

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Why is that interesting ?

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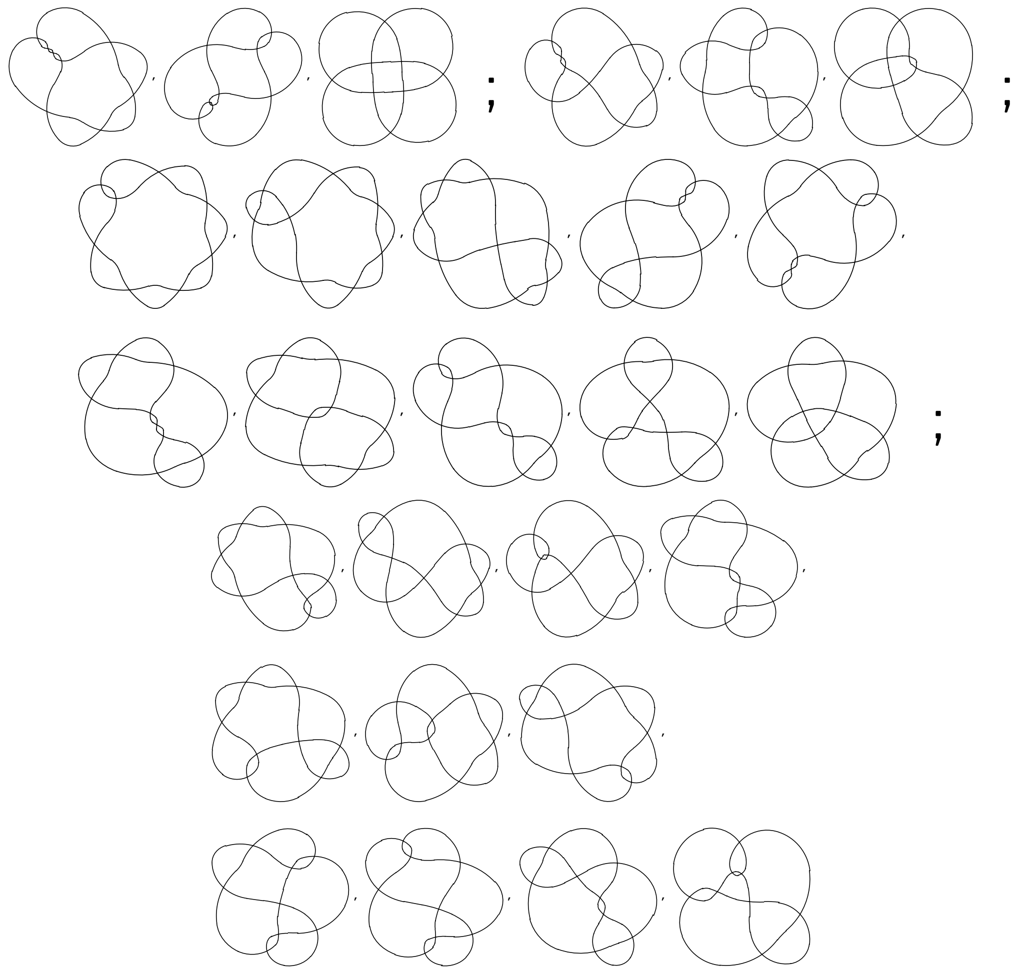
– it's fun !



The 27 indecomposable irreducible immersions of an unoriented circle into an unoriented sphere with $n = 8$ double points.

Problem: How to count and how to list such immersions?

Why is that interesting ?

– it's fun !

– statistics of random curves

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Why is that interesting ?

– it's fun !

– statistics of random curves

– a mathematical challenge [Gauss, Arnold,. . .]

– relevant for knot theory:

counting of (alternating) *links* ✓

(Sundberg & Thistlethwaite; P Zinn-Justin & JBZ) , but *knots* ??

Problem: How to count and how to list such immersions?
Why is that interesting ?

– it's fun !
– statistics of random curves
– a mathematical challenge [Gauss, Arnold,. . .]
– relevant for knot theory
– also a challenge for the theoretical physicist:
  matrix integral techniques fail ! ($n \to 0$ limit of n replicas ?)
  find a substitute ?

Previous works

Arnold; Gusein-Zade–Duzhin; Valette, reps. closed/open/closed curves up to resp. $n = 5, 10, 7$

J. Jacobsen and P. Zinn-Justin: transfer matrix techniques, open curves up to $n = 19$

G. Schaeffer and P. Zinn-Justin: asymptotics by random sampling of "doodles" up to $n = 2^{24}$!!

Problem: How to count and how to list such immersions?

The main idea

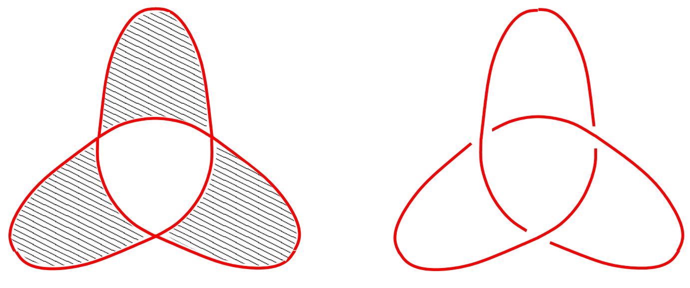
Regard the curve as a 4-valent map, (a graph embedded into a surface, with faces homeomorphic to disks),
make use of permutations to encode the map (an old idea, [Walsh-Lehman 1972, Drouffe 1980, . . .]),
and look at orbits of these permutations under a certain "reparametrization" group.

Diffeomorphism group $\mapsto$ finite group of permutations
Several options, hence several sets of permutations and subgroups of permutations. . .

Colored maps

A simple observation: any planar 4-valent map may be 2-coloured (coloring of faces).

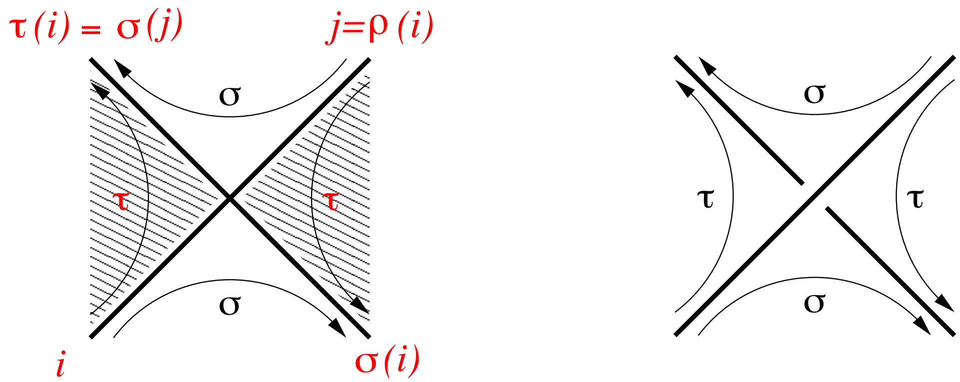
Equivalently, crossings may be drawn as alternatingly over- and under- (“alternating knot”)



(a priori, two distinct colorings, sometimes equivalent.)

Make use of this property to encode a map by a pair (σ, τ) of permutations of the $2n$ edge labels, thus $\sigma, \tau \in S_{2n}$

[? , P Zinn-Justin–JBZ 2003]



Call ρ the involution at over-crossings: $j = \tau^{-1}\sigma(i) =: \rho(i)$

Interest: easy to impose constraints of

– "one-componentness" : $\sigma\rho\sigma^{-1}\rho \in [n^2]$

– genus: $2 - 2g = \#\,\text{faces} - 2n + n$ hence $\#cy(\sigma) + \#cy(\tau) = 2 - 2g + n$

– reduces the diffeomorphism group to a discrete group (subgroup of S_{2n})

Labeled maps $\leftrightarrow$ pairs (σ, τ),
unlabeled ones: *orbits* of (σ, τ) under reparametrization/change of labels:

$$(\sigma, \tau) \mapsto (\gamma\, \sigma\, \gamma^{-1}, \gamma\, \tau\, \gamma^{-1}) \qquad \gamma \in S_{2n}$$

May reduce the reparametrization freedom by imposing some constraint
e.g. $\rho \equiv \rho_0 = (1,2)(3,4) \cdots (2n-1, 2n)$

This leaves σ as the single variable, while $\tau = \sigma \rho$,
and restricts γ to the *centralizer* C_ρ of ρ_0 in S_{2n}: $\gamma\, \rho_0\, \gamma^{-1} = \rho_0$.

A tighter "gauge fixing"

Can impose a further condition on σ, namely that along the circuit, edges are labelled sequentially by $1, 2, 3, \cdots, 2n-1, 2n$ (with again pairs $(2i-1, 2i)$ on over-crossings).

Call $U' := \{\sigma | \sigma\rho\sigma^{-1}\rho = (1, 3, 5, \cdots, 2n-1)(2, 2n, 2n-2, \cdots, 4)\}$

What is the group of reparametrization ? Dihedral group D_n, or if one fixes an orientation, cyclic group $\mathbb{Z}_n$.

Note that U' is the *left coset* $(1, 2, 3, \cdots, 2n)\, C_\rho$: easy to generate! It is nothing else than the set of *open* doodles (or *rooted* 4-valent maps)!

Thus, orbits of U'

– under D_n : bicolored immersions of an unoriented circle

– under $\mathbb{Z}_n$: bicolored immersions of an oriented circle.

A trivial consequence: the "symmetry factor", i.e. the ratio $\frac{|C_\rho|}{\ell_O}$, is a divisor of $2n$ (or n).

How to study orbits ?

– Brute force : construct all conjugates $\gamma\sigma\gamma^{-1}$, $\sigma \in Y' := \{\sigma | \sigma\rho\sigma^{-1}\rho \in [n^2]\}$, but $|C_\rho| = 2^n n!$, $|Y'| = 2^{2n-1} n!(n-1)!$, unpractical for $n \geq 7$;

– Variant: Random sampling of set Y': compute the orbit O of $\sigma \in Y'$ and its *length* ℓ_O, collect all such distinct orbits until $\sum_O \ell_O = |Y'|$.

– Burnside lemma ? $\# C_\rho$–orbits in $Y' = \frac{\sum_k |Y'^k|}{|C_\rho|}$, unpractical

– Y' union of left cosets of C_ρ, U' left coset of C_ρ . . .

– In some cases, orbits $\leftrightarrow$ double cosets $K\backslash G/H$; Frobenius formula on number of double cosets; make use of software `Magma`. . .

– Sort out orbits by genus.

Program carried out up to $n = 8$, 9 or 10.

In that way, we get

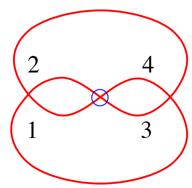
Number and list of *bicolored* immersions of an *(un)oriented* circle in the *oriented* sphere or in a higher genus (oriented) surface Σ.

Can we dispose of the color? Is the immersion described by some σ equivalent to (= in the same D_n- or $\mathbb{Z}_n$-orbit as) its *dual* $\sigma_d = \sigma^{-1}\rho$, in which the two colors have been swapped ?

Can we dispose of the orientation of Σ ? Is the immersion described by some σ *achiral* or not, *i.e.*, equivalent or not to its *mirror image* $\sigma_m = \sigma\rho$?

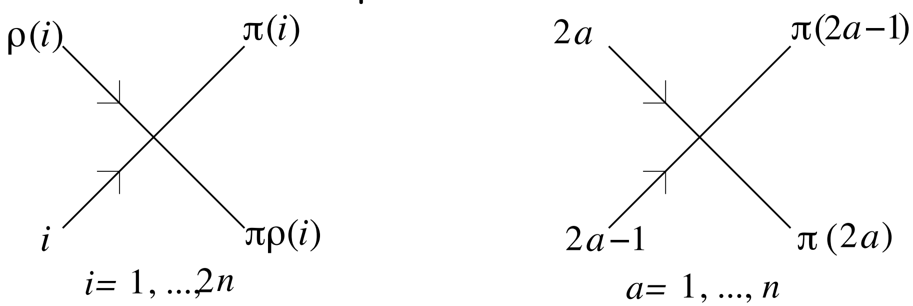
Another family of immersions/curves

Relax bi-colorability assumption. In > 0 genus, it makes a difference !

For example, 

is *not* bi-colorable.

Use a different parametrization of oriented curves by permutations of S_{2n},



$\pi, \rho \in S_{2n}$

Can fix again $\rho = (1,2)(3,4)\cdots(2n-1,2n)$, compute the orbits of π's under the permutation group S_n

(the centralizer of ρ that respects the order $1 < 2, \ldots$) etc.

Summary

We have been able to count and list all curves up to n=9 or 10 crossings for immersions of different types

OOc: bicolourable and bicolored oriented S^1 into oriented Σ

UOc: bicolourable and bicolored unoriented S^1 into oriented Σ

OOb: bicolourable uncolored oriented S^1 into oriented Σ

OUb: bicolourable uncolored oriented S^1 into unoriented Σ

UUc: bicolourable bicolored unoriented S^1 into unoriented Σ

etc

Also, counting when bicolorability is relaxed...

Curious identities, *e.g.* for n even, $\#$ UOc $= \#$ OOb, etc

Counting of spherical immersions

n	1	2	3	4	5	6	7	8	9	10
OO	1	3	9	37	182	1143	7553	54 559	412 306	3 251 240
UO	1	2	6	21	99	588	3829	27 404	206 543	1 626 638
OU	1	2	6	21	97	579	3812	27 328	206 410	1 625 916
UU	1	2	6	19	76	376	2194	14 614	106 421	823 832
UOc	2	3	12	37	198	1143	7658	54 559	413 086	3 251 240

Counting of irreducible indecomposable spherical immersions

Results for $n = 8$ confirmed by independent analysis by Valette 2015

n	1	2	3	4	5	6	7	8	9	10
OO	0	0	1	1	2	6	17	73	290	1274
UO	0	0	1	1	2	4	12	41	161	658
OU	0	0	1	1	2	3	11	38	156	638
UU	0	0	1	1	2	3	10	27	101	364
UOc	0	0	2	1	4	6	24	73	322	1274

Conclusions and Questions

What we have obtained

– computations up to $n = 10$: numbers and lists of curves

– relations between numbers different types of curves,
for ex. for n even, $\#$ UOc $= \#$ OOb

– importance of bicolorability

What remains to do

– extend computations and improve algorithms

– general formulae ?

– asymptotic behavior for large n ?
on the basis of KPZ formula, expected to be (for fixed genus g)

$$\# \sim \kappa \, n^{\gamma(1-g)-3} \, a^n$$

approached very slowly, with $\gamma = \frac{-1-\sqrt{13}}{6}$ (Schaeffer & Zinn-Justin)

– apply this orbit approach to other problems? . . .