BREMSSTRAHLUNG

On the perturbative road to production of hadrons

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Abstract The QCD picture of multiparticle production in hard interactions is reviewed. We illustrate the basic properties of high energy bremsstrahlung with practical examples such as hadroproduction of J/ψ, photon radiation in $W^{\pm}$ decays, hadronic accompanyment in Higgs production. Special emphasis is given to coherence effects which determine the structure of particle multiplication in QCD jets (humpbacked plateau) and the pattern of inter-jet hadron flows (QCD string/drag effects).

Introduction

I planned these lectures as a stress-lifting rather than fact-finding mission. My aim is to convince you that the most simple calculations, backed by a physical intuition, are capable of delivering many amusing verifiable consequences. You will find some scattered formulae in the text. They are all simple. Trust me. They are included not to scare you away from reading but for the benefit of those (ideal) readers who would like to see how things actually come about.

We shall play some selected themes: the concept of QCD partons, physics of bremsstrahlung, conservation of color current, QCD coherence, basics of multiparticle production in hard interactions. If, having worked through these themes you will start hearing a symphony with your inner ear, then the aim of the course will have been reached.

One confession is due before we start: you will find no bibliography attached. This is because all the statements you will meet below fall into two cathegories: they are either well-known or unpublished.

1. SMALL COUPLING, LARGE LOGARITHMS AND EVOLUTION

QCD is a *quantum* theory; have no doubt about it. This very statement seems to make the problem of describing parton systems involving $n > n_0$ gluons and quarks (with the actual value of $n_0 \sim 1$ depending on your computer) look hopeless: solving such a problem would call for sorting out and calculating $\mathcal{O}((n!)^2)$ Feynman diagrams.

Why should we worry about multiparton systems in the first place? Is it not true that the squared matrix element in the $n^{\rm th}$ order of perturbation theory is proportional to $(\alpha_s/\pi)^n \lesssim (0.1)^n$ and, thus, vanishingly small for large n? The answer to this (as to many other questions, according to the celebrated Hegel's dialectic wisdom) is: "Yes and No". Indeed,

Yes, it is very small, if we talk about a "multijet" configuration of 10 energetic quarks and gluons with large angles between them;

No, it is of order *unity*, if we address the *total* probability of having 8 extra gluons (and quarks) in addition to, say, a $q\overline{q}$ pair produced in e^+e^- annihilation at LEP.

Allowing small relative angles between partons in a process with a large hardness Q^2 results in a logarithmic enhancement of the emission probability:

$$\alpha_s \;\Longrightarrow\; \alpha_s \frac{\mathrm{d}\Theta^2}{\Theta^2} \;\to\; \alpha_s \log Q^2\,. \tag{1.1a}$$

As a result, the total probability of one parton (E) turning into two ($E_1 \sim E_2 \sim \frac{1}{2}E$) may become of order 1, in spite of the smallness of the characteristic coupling, $\alpha_s(Q^2) \propto 1/\log Q^2$. A typical example of such a "collinear" enhancement — the splitting process $g \to q\overline{q}$.

Moreover, when we consider the *gluon* offspring, another — "soft" — enhancement enters the game, which is due to the fact that the gluon bremsstrahlung tends to populate the region of *relatively* small energies ($E \simeq E_1 \gg E_2 \equiv \omega$):

$$\alpha_s \;\Longrightarrow\; \alpha_s \frac{\mathrm{d}\omega}{\omega}\frac{\mathrm{d}\Theta^2}{\Theta^2} \;\to\; \alpha_s \log^2 Q^2\,. \tag{1.1b}$$

Thus the true perturbative "expansion parameter" responsible for parton multiplication via $q \to qg$ and $g \to gg$ may actually become *much larger* that 1!

In such circumstances we cannot trust the expansion in $\alpha_s \ll 1$ unless the logarithmically enhanced contributions (1.1) are taken full care of in all orders.

Fortunately, in spite of the complexity of high order Feynman diagrams, such a programme can be carried out. There is a physical reason for that: large contributions (1.1) originate from a specific region of phase space, which can be viewed as a sequence of parton decays strongly ordered in fluctuation times. Given such a separation in time, successive parton splittings become independent, so that the emerging picture is essentially classical. This is how the parton cascades described by the classical equations of parton balance (evolution equations) come about.

1.1. LOGARITHM IS NOT A FUNCTION

The very fact that the all-order logarithmic asymptotes can be written down in a closed form and, more than that, that they *a posteriori* prove to be quite simple, follows from the statement that constitutes a field-theoretical "article of faith":

Logarithm is not a function[1] *but a signal of simple underlying physics.*

In our context this simplicity has to do with the **classical nature** of

- *soft* enhancement of bremsstrahlung amplitudes ("infrared" singularities) and
- *collinear* enhancement of basic $1 \to 2$ parton splitting amplitudes (or "mass" singularities).

As a result, the leading logarithmic asymptotes can be found without performing laborious calculations. It suffices to invoke an intuitively clear picture of parton cascades described in probabilistic fashion in terms of sequential independent elementary parton branchings.

Whether the main enhanced terms are *Double Logarithmic* (DL) as in (1.1b) or *Single Logarithmic* (SL), (1.1a), depends on the nature of the problem under focus. The very distinction between DL and SL asymptotic regimes is often elusive. To illustrate the latter statement one may recall the QCD analysis of structure functions describing deep inelastic scattering (DIS), the subject to be discussed in detail later on in this lecture.

DIS (Q^2 large, x moderate) is a typical SL problem, with the following perturbative (PT) expansion:

$$D^{(n)}(x,Q^2) \quad = \quad C_n(x)\cdot\frac{1}{n!}\left[\frac{\alpha_s}{\pi}\ln\frac{Q^2}{\mu^2}\right]^n + \text{ less sing. terms;} \qquad (1.2)$$

[1] ascribed to L.D.Landau

$$D(x,Q^2) \;=\; \sum_{n=0}^{\infty} D^{(n)}\,.$$

Here x is the Bjorken variable and μ^2 the finite initial virtuality of the target parton A (quark, gluon).

In general, $|C_n(x)| \sim 1$. However, in the quasi-elastic limit of $x \to 1$, when the invariant mass of the produced parton system becomes relatively small, $W^2 = Q^2(1-x)/x \ll Q^2$, the expansion coefficients in (1.2) take the form

$$\begin{aligned} C_n &\propto [C_F \ln(1-x)]^n\,; \\ D(x,Q^2) &\propto (1-x)^{-1} \exp\left\{ \frac{C_F\alpha_s}{\pi} \ln(1-x) \ln\frac{Q^2}{\mu^2} \right\}\,, \end{aligned} \tag{1.3a}$$

and the problem turns out to be DL. Another important example of such a permutation has to do with the opposite limit of numerically small x. In this region the dominant contribution comes from sea-quark pairs copiously produced via gluon cascades, and the answer again exhibits the DL asymptote:

$$\begin{aligned} C_n(x) &\propto \frac{[N_c \ln x^{-1}]^n}{(n-1)!}; \\ D(x,Q^2) &\propto x^{-1}\, I_1\left(2\sqrt{\frac{N_c\alpha_s}{\pi} \ln x^{-1} \ln\frac{Q^2}{\mu^2}} \right)\,, \end{aligned} \tag{1.3b}$$

with I_1 the modified Bessel function.

1.2. PUZZLE OF DIS AND QCD PARTONS

Let us invoke the deep inelastic lepton-hadron scattering — a classical example of a hard process and the standard QCD laboratory for carrying out the perturbative resummation programme.

Here, the momentum q with a large space-like virtuality $Q^2 = |q^2|$ is transferred from an incident electron (muon, neutrino) to the target proton, which then breaks up into the final multiparton $\to$ multihadron system. Introducing an invariant energy $s = 2(Pq)$ between the exchange photon (Z^0, $W^{\pm}$) q and the proton with 4-momentum P ($P^2 = M_p^2$), one writes the invariant mass of the produced hadron system which measures *inelasticity* of the process as

$$W^2 \;\equiv\; (q+P)^2 - M_p^2 = q^2 + 2(Pq) = s(1-x)\,, \qquad x \equiv \frac{Q^2}{2(Pq)} \le 1\,.$$

The cross section of the process depends on two variables: the *hardness* Q^2 and Bjorken x. For the case of *elastic* lepton-proton scattering one has $x \equiv 1$ and it is natural to write the cross section as

$$\frac{d\sigma_{el}}{dQ^2\,[dx]} \;=\; \frac{d\sigma_{\rm Ruth}}{dQ^2}\cdot f_{el}^2(Q^2)\;\cdot\;[\delta(1-x)]\,. \tag{1.4a}$$

Here $\sigma_{\rm Ruth} \propto \alpha^2/Q^4$ is the standard Rutherford cross section for e.m. scattering off a point charge and f_{el} stands for the elastic proton form factor.

For inclusive *inelastic* cross section one can write an analogous expression by introducing an inelastic proton "form factor" which now depends on both the momentum transfer Q^2 and the inelasticity parameter x:

$$\frac{d\sigma_{in}}{dQ^2\,dx} \;=\; \frac{d\sigma_{\rm Ruth}}{dQ^2}\cdot f_{in}^2(x,Q^2)\,. \tag{1.4b}$$

What kind of Q^2-behavior of the form factors (1.4) could one expect in the Bjorken limit $Q^2\to\infty$? Quantum mechanics tells us how the Q^2-behavior of the electromagnetic form factor can be related to the charge distribution inside a proton:

$$f_{el}(Q^2) \;=\; \int d^3r\;\rho(\vec{r})\;\exp\left\{\mathrm{i}\vec{Q}\vec{r}\right\}.$$

For a point charge $\rho(\vec{r}) = \delta^3(\vec{r})$, it is obvious that $f\equiv 1$. On the contrary, for a smooth charge distribution $f(Q^2)$ falls with increasing Q^2, the faster the smoother ρ is. Experimentally, the elastic e-p cross section does decrease with Q^2 *much faster* that the Rutherford one ($f_{el}(Q^2)$ decays as a large power of Q^2). Does this imply that $\rho(\vec{r})$ is indeed regular so that there is no well-localized — point-charge inside a proton? If it were the case, the *inelastic* form factor would decay as well in the Bjorken limit: a tiny photon with the characteristic size $\sim 1/Q\to 0$ would penetrate through a "smooth" proton like a knife through butter, inducing neither elastic nor inelastic interactions.

However, as was first observed at SLAC in the late sixties, for a fixed x, f_{in}^2 stays practically constant with Q^2, that is, the inelastic cross section (with a given inelasticity) is similar to the Rutherford cross section (Bjorken scaling). It looks *as if* there was a point-like scattering in the guts of it, but in a rather strange way: it results in inelastic break-up dominating over the elastic channel. Quite a paradoxical picture emerged; Feynman-Bjorken partons came to the rescue.

Imagine that it is not the proton itself that is a point-charge-bearer, but some other guys (quark-partons) inside it. If those constituents were *tightly* bound to each other, the elastic channel would be bigger than, or comparable with, the inelastic one: an excitation of the parton that takes an impact would be transferred, with the help of rigid links between partons, to the proton as a whole, leading to the elastic scattering or to the formation of a quasi-elastic finite-mass system ($N\pi$, $\Delta\pi$ or so), $1-x \ll 1$.

To match the experimental pattern $f_{el}^2(Q^2) \ll f_{in}^2(Q^2)$ one has instead to view the parton ensemble as a *loosely* bound system of quasi-free particles. Only under these circumstances does knocking off one of the partons inevitably lead to deep inelastic breakup, with a negligible chance of reshuffling the excitation among partons.

The parton model, forged to explain the DIS phenomenon, was intrinsically paradoxical by itself. In sixties and seventies, there was no other way of discussing particle interactions but in the field-theoretical framework, where it remains nowadays. But all reliable (renormalizable, 4-dimensional) quantum field theories (QFTs) known by then had one feature in common: an effective interaction strength (the running coupling $g^2(Q^2)$) *increasing* with the scale of the hard process Q^2. Actually, this feature was widely believed to be a general law of nature, and for a good reason[2]. At the same time, it would be preferable to have it the other way around so as to be in accord with the parton model, which needs parton-parton interaction to *weaken* at small distances (large Q^2).

Only with the advent of non-Abelian QFTs (and QCD among them) exhibiting an anti-intuitive asymptotic-freedom behavior of the coupling, the concept of partons was to become more than a mere phenomenological model.

1.3. QCD DIS MINUTES

Typical QCD graphs for DIS amplitudes are shown in Fig. 1.

For moderate x-values (say, $x > 0.1$) the process is dominated by lepton scattering off a valence quark in the proton. The scattering cross section has a standard energy behavior $\sigma \propto x^{-2(J_{\rm ex}-1)}$, where $J_{\rm ex}$ is the spin of the exchanged particle in the t-channel. It is the quark with $J_{\rm ex} = \frac{1}{2}$ in the left picture of Fig. 1, so that the valence contribution to the cross section decreases at small x as $\sigma \propto x$.

For high-energy scattering, $x \ll 1$, the Bethe-Heitler mechanism takes over which corresponds to the t-channel *gluon* exchange: $J_{\rm ex} = 1$, $\sigma \propto x^0$ =const (modulo logarithms).

In the Leading Logarithmic Approximation (LLA) one insists on picking up, for each new parton taken into consideration, a logarithmic en-

[2] relation between screening and unitarity

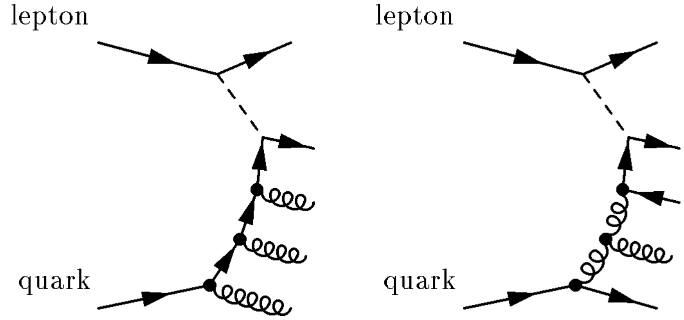


Figure 1 Valence (left) and Bethe-Heitler mechanism (right) of DIS.

hancement factor $\alpha_s \to \alpha_s \log Q^2$. In this approximation the scattering *probability* can be simply obtained by convoluting elementary probabilities of independent $1 \to 2$ parton splittings.

To cut a long story short, the appearance of the log-enhanced contributions in (1.1a) is due to the following structure

$$
\begin{aligned}
&\frac{1}{n!}\left[\frac{\alpha_s}{\pi}\ln\frac{Q^2}{\mu^2}\right]^n \qquad (1.5)\\
&= \left[\frac{\alpha_s}{\pi}\right]^n \int^{Q^2}\frac{dk_{\perp n}^2}{k_{\perp n}^2}\int^{k_{\perp n}^2}\frac{dk_{\perp n-1}^2}{k_{\perp n-1}^2}\cdots\int^{k_{\perp 3}^2}\frac{dk_{\perp 2}^2}{k_{\perp 2}^2}\int_{\mu^2}^{k_{\perp 2}^2}\frac{dk_{\perp 1}^2}{k_{\perp 1}^2}\,,
\end{aligned}
$$

with $k_{\perp i}^2$ the squared transverse momenta of produced partons.

To contribute to the LLA, the transverse momenta of produced partons should be strongly ordered, increasing up the "ladder": $k_{1\perp}^2 \ll \ldots \ll k_{n\perp}^2 \ll Q^2$. (At the level of Feynman *amplitudes* the ladder diagrams dominate, provided a special physical gauge is chosen for gluon fields.)

The expressions (1.2) and (1.3) exhibit an unpleasant (or wonderful, according to taste) feature: they become senseless (diverge) in the zero-quark-mass limit, $\mu \to 0$. Well, when you see a nasty thing happen beyond your reach, you can do no better than make use of it. This "mass singularity", according to (1.5), occurs in the lower limit of the $k_\perp$ integration of the very first (and only!) parton branch. Let us drag this misbehaving integral to the left by rewriting (1.5) as

$$
\begin{aligned}
&(1.5)^{[n]}(Q^2,\, \mu^2) \\
&= \frac{\alpha_s}{\pi}\int_{\mu^2}^{Q^2} \frac{dk_{\perp 1}^2}{k_{\perp 1}^2}\cdot\left[\frac{\alpha_s}{\pi}\right]^{n-1}\int^{Q^2}\frac{dk_{\perp n}^2}{k_{\perp n}^2}\int^{k_{\perp n}^2}\frac{dk_{\perp n-1}^2}{k_{\perp n-1}^2}\cdots\int_{k_{\perp 1}^2}^{k_{\perp 3}^2}\frac{dk_{\perp 2}^2}{k_{\perp 2}^2} \\
&= \frac{\alpha_s}{\pi}\int_{\mu^2}^{Q^2} \frac{dk_{\perp 1}^2}{k_{\perp 1}^2}\;\cdot\;(1.5)^{[n-1]}(Q^2,\, k_{\perp 1}^2)\,,
\end{aligned}
$$

where we have combined the internal $(n-1)$ integrals into the same expression that corresponds to the previous order in α_s-expansion and has a new lower limit $k_{\perp 1}^2$ substituted for the original μ^2. Now, we can localize the μ-dependence by evaluating the logarithmic derivative:

$$
\mu^2\frac{\partial}{\partial\mu^2}\;(1.5)^{[n]} \;=\; -\frac{\alpha_s}{\pi}\cdot\,(1.5)^{[n-1]}\,.
$$

This equation relates the $n^{\rm th}$ order of the PT expansion to the previous one. To put this symbolic relation at work one first has to recall the satellite x-dependence.

By extracting the first step one may look upon the rest as DIS off a new "target" — the parton with transverse momentum $k_{\perp 1}^2$ and a finite fraction z of the initial longitudinal momentum P. As a result, there appears an additional integration with the probability of the first splitting, $\phi(z)$, and the differential equation for the resummed $F^{(\rm LLA)}$ takes the form

$$
\mu^2\frac{\partial}{\partial\mu^2}\;F\left(x,\,Q^2,\mu^2\right) \;=\; -\int_x^1\frac{dz}{z}\,\phi(z)\,\frac{\alpha_s}{\pi}\cdot F\left(\frac{x}{z},\,Q^2,\mu^2\right). \quad (1.6\text{a})
$$

Since a logarithm (like a stick) has two ends, differentiation over the overall hardness scale Q^2 would do the same job, the result being the *evolution equation* in a familiar form:

$$
Q^2\frac{\partial}{\partial Q^2}F(x,Q^2) = \frac{\alpha_s}{\pi}\phi(x)\otimes F(x,Q^2)\,, \quad (1.6\text{b})
$$

where the symbol $\otimes$ stands for convolution in the x-space.

1.4. LLA PARTON EVOLUTION

1.4.1 Space-like parton evolution. The decay phase space for the *space-like* evolution determining the DIS structure functions is

$$
dw^{A\to B+C} = \frac{dk_\perp^2}{k_\perp^2}\,\frac{\alpha_s(k_\perp^2)}{2\pi}\,\frac{dz}{z}\,\Phi_A^{BC}(z) \quad (1.7\text{a})
$$

with z the longitudinal momentum fraction carried by the parton B. The functions Φ play the role of "Hamiltonian" (x-dependent kernels) of the evolution equation (1.6) for the LLA parton distributions.

In the DIS environment the initial parton A with a negative (space-like) virtuality decays into $B[z]$ with the large space-like virtual momentum $|k_B^2| \gg |k_A^2|$ and a positive virtuality (time-like) $C[1-z]$. The parton C generates a subjet of secondary partons ($\to$ hadrons) in the final state. Since no details of the final-state structure are measured in an inclusive process, the invariant mass of the subjet C is integrated over. In the dominant integration region $k_C^2 \ll |k_B^2|$ the parton C looks *quasi-real*, compared with the hard scale of $|k_B^2|$. The same is true for the initial parton A.

Splitting can be viewed as a large momentum-transfer process of scattering (turnover) of a "real" target parton A into a "real" C on the external field mediated by high-virtuality B. At the next step of evolution it is B's turn to play a role of a next target $B \equiv A'$, "real" with respect to yet deeper probe $|k_B'^2| \gg |k_B^2|$, and so on.

Successive parton decays with step-by-step *increasing* space-like virtualities (transverse momenta) constitute the picture of parton wave-function fluctuations inside the proton. The sequence proceeds until the overall hardness scale Q^2 is reached.

1.4.2 Time-like parton cascades. A similar picture emerges for the time-like branching processes determining the internal structure of jets produced, for example, in e^+e^- annihilation. Here the flow of *hardness* is opposite to that in DIS: evolution starts from a highly virtual quark with positive virtuality, originating from the e.m. vertex, while time-like virtualities of its products ("quasi-real" with respect to predecessors; "high-virtuality" with respect to offspring) degrade.

In the time-like case the longitudinal phase space is symmetric in offspring parton energy fractions, and the differential decay probability reads

$$dw^{A\to B+C} = \frac{d\Theta^2}{\Theta^2}\,\frac{\alpha_s(k_\perp^2)}{2\pi}\,dz\,\Phi_A^{BC}(z)\,. \tag{1.7b}$$

It is important to notice that the flow of *energy* (longitudinal momentum) is governed, in the LLA, by the same functions Φ_A^{BC}: it does not matter that now, in contrast to space-like evolution, A is the "virtual" one and B and C are "real". This relation (known as the Gribov-Lipatov reciprocity) is one of many wonderful symmetries that our "parton Hamiltonian" Φ_A^{BC} obeys.

1.4.3 Apparent and hidden in parton dynamics. When studying inclusive characteristics of parton cascades, one traces a single route of successive parton splittings. Having this in mind, we can drop the label that marks partons C whose fate does not concern us, $\Phi_A^{BC}(z) \Longrightarrow \Phi_A^B(z)$ ($\equiv P_{BA}$ in the standard Altarelli-Parisi notation).

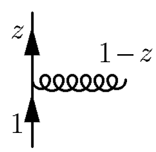
For discussion of the relations between splitting functions it is convenient to strip off color factors and introduce

$$\begin{aligned} \Phi_F^F(z) &= C_F\, V_F^F(z)\,, \qquad \Phi_F^G(z) = C_F\, V_F^G(z)\,, \\ \Phi_G^F(z) &= T_R\, V_G^F(z)\,, \qquad \Phi_G^G(z) = N_c\, V_G^G(z)\,. \end{aligned} \tag{1.8}$$

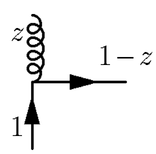
Here C_F and N_c are the familiar quark and gluon "color charges", while T_R is a scientific name for one half:

$$\mathrm{Tr}(t^a t^b) = \sum_{i,k=1}^{N_c} t^a_{ik}\, t^b_{ki} \;\equiv\; T_R\, \delta^{ab} = \tfrac{1}{2}\delta^{ab}\,.$$

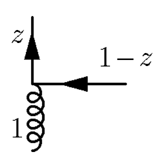
The splitting functions then read



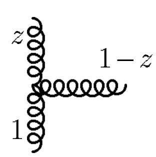
$$V_F^F(z) = \frac{1+z^2}{1-z}\,, \tag{1.9a}$$



$$V_F^G(z) = \frac{1+(1-z)^2}{z}\,, \tag{1.9b}$$



$$V_G^F(z) = z^2 + (1-z)^2\,, \tag{1.9c}$$



$$V_G^G(z) = 2\left[z(1-z) + \frac{1-z}{z} + \frac{z}{1-z} \right]. \tag{1.9d}$$

They have the following remarkable symmetry properties.

Parton Exchange results in an obvious relation between probabilities to find decay products with complementary momentum fractions:

$$V_A^{B(C)}(z) \;=\; V_A^{C(B)}(1-z)\,. \tag{1.10a}$$

Drell-Levy-Yan Crossing Relation emerges when one links together two splitting processes corresponding to opposite evolution "time" sequences:

$$V_B^A(z) \;=\; (-1)^{2s_A+2s_B-1}\, z\, V_A^B\left(\frac{1}{z}\right), \tag{1.10b}$$

with s_A the spin of particle A. Strictly speaking, the crossing $z \to 1/z$ relates the space-like and time-like evolution kernels, and vice versa, $V \leftrightarrow \overline{V}$. Representing (1.10b) in terms of V relies upon an identity of the LLA "Hamiltonians" mentioned above,

Gribov-Lipatov Reciprocity Relation:

$$\overline{V}(z)_{\text{time-like}} \;=\; V(z)_{\text{space-like}}\,. \tag{1.10c}$$

As we see, these relations do not leave much freedom for splitting functions. In fact, one could borrow V_F^F from QED textbooks, reconstruct V_F^G by exchanging the decay products (1.10a), and then obtain V_G^F using the crossing (1.10b). This is the way to generate all three splitting functions relevant for QED (1.9a)–(1.9c). The last gluon-gluon splitting function (1.9d) transforms into itself under both (1.10a) and (1.10b). The more surprising is the fact that the gluon self-interaction kernel actually could have been obtained "from QED" using

the Super-Symmetry Relation:

$$V_F^F(z) + V_F^G(z) = V_G^F(z) + V_G^G(z)\,. \tag{1.10d}$$

This relation exploits the existence of the supersymmetric QFT closely related to real QCD. In the SUSY-QCD, "quark" and "gluon" belong to the same (adjoint) representation of the color group, so that all color factors become identical $C_F = C_A = T_R$ (cf. (1.8)). Bearing this in mind one can spell out (1.10d) as an equality between the *total* probabilities of "quark" and "gluon" decays. The fact that it holds identically in z means that there is an infinite number of non-trivial conservation laws in this theory!

Even this is not the end of the story.

Conformal Invariance leads to a number of relations (involving derivatives) between splitting functions, the simplest of which reads

$$\left(z\frac{d}{dz} - 2\right) V_G^F(z) = \left(z\frac{d}{dz} + 1\right) V_F^G(z)\,. \tag{1.10e}$$

The general character of the symmetry properties makes them practically useful when studying subleading effects in parton distributions where one faces technically difficult calculations. For example, the supersymmetric QCD analogue had been used to choose between two contradictory calculations in the next-to-LLA (the two-loop anomalous dimensions). We illustrate the idea by another example of the most advanced next-to-next-to-leading result obtained by Gaffney and Mueller for the ratio of mean parton multiplicities in gluon and quark jets, which reads

$$\begin{aligned}\frac{C_F}{N_c}\frac{\mathcal{N}_g}{\mathcal{N}_q} \;&\simeq\; 1-\left(1+\frac{T_f}{N_c}-2\frac{T_fC_F}{N_c^2}\right)\\ &\cdot\left[\sqrt{\frac{\alpha_sN_c}{18\pi}}+\frac{\alpha_sN_c}{18\pi}\left(\frac{25}{8}-\frac{3T_f}{4N_c}-\frac{T_fC_F}{N_c^2}\right)\right].\end{aligned} \tag{1.11}$$

Here $T_f \equiv 2n_fT_R$, with $2n_f$ the number of fermions (quarks and antiquarks of n_f flavours).

The symmetry between quarks (fermions) and gluons (bosons) is hidden in QCD. It becomes manifest in another QFT which is a *supersymmetric partner* of QCD. In that theory fermions and vector bosons have the same color properties (both correspond to the adjoint representation). Equating the color factors, $N_c = C_F = T_R$, and bearing in mind another subtlety, $n_f = \frac{1}{2}$ (since the "quark" is a Majorana fermion there), it is straightforward to verify that the ratio of multiplicities in "quark" and "gluon" jets (1.11) indeed turns to unity, in all known (as well as in all unknown) orders.

1.4.4 Fluctuation Time and Evolution Times: Coherence.

Once the Hamiltonian is known, it suffices to propagate it in time to find all we want to know about the system. But was is the *time* in our context?

An attentive reader has noticed that back in (1.7) we wrote the parton splitting phase space differently: for the space-like case (1.7a) in terms of transverse momentum $k_\perp$ and for the time-like evolution (1.7b) via the decay angle Θ. Logarithmic differentials by themselves are identical, since $k_\perp^2$ and Θ^2 are proportional for fixed z. We have made this distinction to stress an important difference between a probabilistic interpretation of DIS and the e^+e^- evolution: the different *evolution times*.

To be honest, within the LLA framework it does not make much sense to argue which of evolution parameters $\ln k_{\perp i}^2$, $\ln \Theta_i^2$ or $\ln |k_i^2|$ (with k^2 the total parton virtuality) does a better job: these choices differ by *subleading* terms. A mismatch is of the order of

$$\frac{\alpha_s}{\pi}\ln^2 z\,. \tag{1.12}$$

Strictly speaking, such contributions should be treated as $\mathcal{O}(\alpha_s)$ within the LLA logic and neglected as compared to $\alpha_s \ln Q \sim 1$. These *next-to-LLA* terms become significant, however, and should be taken care of and "resummed" in all orders when numerically small values of the Bjorken x are concerned[3] such that $\frac{\alpha_s}{\pi}\ln^2 x \sim 1$.

In the DIS environment, the *transverse momentum* ordering proves to be the one that takes care of potentially disturbing corrections (1.12) in all orders, and in this sense becomes a preferable choice for constructing the probabilistic scheme for space-like parton cascades (DIS structure functions).

On the other hand, in the case of the time-like cascades — yes, you've guessed it right from (1.7b)! — it is the *relative angle* between the offspring partons which has to be kept ordered, decreasing along the evolutionary decay chain.

What is the difference between the two prescriptions, and how do they relate with the fluctuation time ordering which was claimed in the beginning of the lecture to ensure the probabilistic picture?

The time ordering is there. Simply by examining the Feynman denominators it is easy to see that the maximally enhanced contribution — collinear logarithm per each splitting, as in the DIS "ladder" (1.5) — implies the ordering in

$$t_{\rm fluct} \simeq \frac{k_{||}}{k_\perp^2}\,.$$

In the DIS kinematics the evolution goes from the proton side and, on the way towards the virtual probe Q^2, parton fluctuations are successively shorter-lived (the "probe" is faster than the lifetime of the "target"):

$$\text{time ordering} — \qquad t_i = \frac{k_{||i}}{k_{\perp i}^2} \;>\; t_{i+1} = \frac{k_{||i+1}}{k_{\perp i+1}^2} \qquad (1.13a)$$

$$k_\perp \text{ ordering} — \qquad k_{\perp i} \;<\; k_{\perp i+1}, \qquad (1.13b)$$

$$\text{mismatch} \Longrightarrow \qquad z\cdot k_{\perp i}^2 < k_{\perp i+1}^2 < k_{\perp i}^2. \qquad (1.13c)$$

In the case of a time-like jet the order of events is opposite. The process starts from a large scale $s = Q^2$, and the partons of the generation $(i+1)$ live *longer* than their parent (i):

$$\text{time ordering} — \qquad t_i = \frac{k_{||i}}{k_{\perp i}^2} \;<\; t_{i+1} = \frac{k_{||i+1}}{k_{\perp i+1}^2} \qquad (1.14a)$$

[3]The word "numerically" stands here as a warning for not confusing this kinematical region with a "parametrically" small x, such that $\alpha_s \ln 1/x \sim 1$ — the Regge region — where essentially different physics comes onto stage.

$$\text{angular ordering} — \qquad \theta_i = \frac{k_{\perp i}}{k_{||i}} \; > \; \theta_{i+1} = \frac{k_{\perp i+1}}{k_{||i+1}}, \qquad (1.14b)$$

$$\text{mismatch} \Longrightarrow \qquad \theta_i^2 < \theta_{i+1}^2 < \frac{\theta_i^2}{z}. \qquad (1.14c)$$

We see that in both situations the mismatch, (1.13c) and (1.14c), may become significant in the case of a relatively *soft* decay,

$$z \equiv \frac{k_{||i+1}}{k_{||i}} \ll 1,$$

that is, when soft gluon emission comes onto stage. Here we better be careful: the catch is, to be emitted *later* does not guarantee being emitted *independently*. Quantum mechanics, you know. The time-ordering (1.13a), (1.14a) proves to be too liberal in both cases. In fact, the parton multiplication in the regions (1.13c) and (1.14c) is suppressed. The sum of the contributing Feynman diagrams vanish.

Let us sort out here the DIS puzzle (1.13), and leave the time-like *angular ordering* phenomenon (1.14) to be slowly enjoyed in the next lecture.

1.4.5 Vanishing of the forward inelastic diffraction. Consider a bit of the DIS ladder — a two-step process shown by the first graph in Fig. 2. Let the second decay be *soft*, $z_2 \ll 1$. (The first one can be either soft or hard, $z_1 \lesssim 1$.) In the kinematical region (1.13c),

$$z_2 \cdot k_{1\perp}^2 \ll k_{2\perp}^2 \ll k_{1\perp}^2 , \qquad (1.15)$$

the *time-ordering* is still intact, which means that the momentum k_2 is transferred fast as compared with the lifetime of the first fluctuation $P \to P' + k_1$.

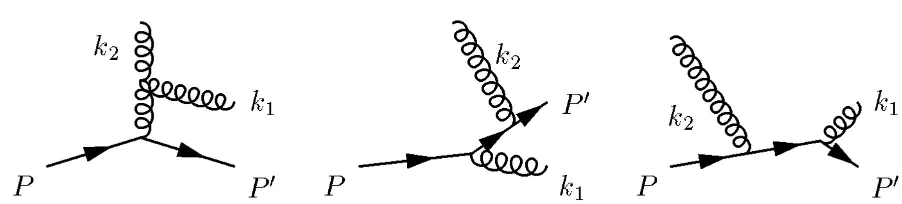


Figure 2 In the "wrong" kinematics $k_{2\perp} < k_{1\perp}$, the sum of the two space-like evolution amplitudes cancels against the final state time-like decay

Since k_2 is the softest, energy-wise, the process can be viewed as inelastic relativistic scattering $P \to P' + k_1$ in the external gluon field (k_2). The

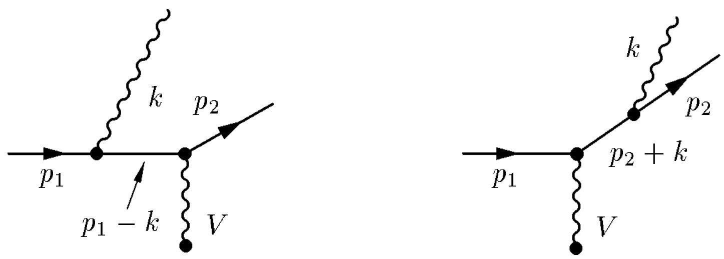


Figure 3 Photon Bremsstrahlung diagrams for scattering off an external field.

transverse size of the field is $\rho_\perp \sim k_{2\perp}^{-1}$. The characteristic size of the fluctuation $P' + k_1$, according to (1.15), is smaller: $\Delta r_{01\perp} \sim k_{1\perp}^{-1} \ll \rho_\perp$. We thus have a *compact* state propagating through the field that is smooth at distances of the order of the size of the system. In such circumstances the field cannot resolve the internal structure of the fluctuation. Components of the fluctuation, partons P' and k_1 in the first two graphs of Fig. 2, scatter *coherently*, and the total amplitude turns out to be identical and opposite in sign to that for the scattering of the initial state P (the last graph): inelastic breakup does not occur.

The cancellation between the amplitudes of Fig. 2 in the region (1.15), and thus the $k_\perp$ ordering, is a direct consequence of the conservation of current.

2. BREMSSTRAHLUNG, COHERENCE, CONSERVATION OF CURRENT

The purpose of the second lecture is to recall the basic properties of photon radiation. Once the QED bremsstrahlung is understood as an essentially coherent and, at the same time, intrinsically classical phenomenon, the physics of gluon radiation will readily follow suit. So, we shall start from the electromagnetic radiation and turn to gluons only once in this lecture, when there is something interesting to say in the specific QCD context (J/ψ production and the Low theorem).

2.1. PHOTON BREMSSTRAHLUNG

Let us consider photon bremsstrahlung induced by a charged particle (electron) which scatters off an external field (e.g., a static electromagnetic field). The derivation is included in every textbook on QED, so we confine ourselves to the essential aspects.

The lowest order Feynman diagrams for photon radiation are depicted in Fig. 3, where p_1, p_2 are the momenta of the incoming and outgoing electron respectively and k represents the momentum of the emitted photon. The corresponding amplitudes, according to the Feynman rules, are given in momentum space by

$$M_{\rm i}^{\mu} = e\,\bar{u}(p_2,s_2)\,V(p_2+k-p_1)\,\frac{m+\not{p}_1-\not{k}}{m^2-(p_1-k)^2}\,\gamma^{\mu}\,u(p_1,s_1), \quad (2.16a)$$

$$M_{\rm f}^{\mu} = e\,\bar{u}(p_2,s_2)\,\gamma^{\mu}\,\frac{m+\not{p}_2+\not{k}}{m^2-(p_2+k)^2}\,V(p_2+k-p_1)\;u(p_1,s_1). \quad (2.16b)$$

Here V stands for the basic interaction amplitude which may depend in general on the momentum transfer (for the case of scattering off the static e.m. field, $V=\gamma^0$).

First we apply the soft-photon approximation, $\omega \ll p_1^0, p_2^0$, to neglect $\not{k}$ terms in the numerators. To deal with the remaining matrix structure in the numerators of (2.16) we use the identity $\not{p}\,\gamma^{\mu} = -\gamma^{\mu}\,\not{p} + 2\,p^{\mu}$ and the Dirac equation for the on-mass-shell electrons,

$$(m+\not{p}_1)\,\gamma^{\mu}\,u(p_1) = (2p_1^{\mu} + [\,(m-\not{p}_1\,])\,u(p_1) = 2p_1^{\mu}\,u(p_1)\,,$$
$$\bar{u}(p_2)\,\gamma^{\nu}\,(m+\not{p}_2) = \bar{u}(p_2)\,([\,(m-\not{p}_2\,] + 2p_1^{\nu}) = 2p_2^{\nu}\,\bar{u}(p_2)\,.$$

Denominators for real electrons ($p_i^2 = m^2$) and the photon ($k^2 = 0$) become $m^2-(p_1-k)^2 = 2(p_1k)$ and $m^2-(p_2+k)^2 = -2(p_2k)$, so that for the total amplitude we obtain the factorized expression

$$M^{\mu} = e\,j^{\mu}\;\times\;M_{\rm el}\,. \quad (2.17a)$$

Here $M_{\rm el}$ is the Born matrix element for non-radiative (elastic) scattering,

$$M_{\rm el} = \bar{u}(p_2,s_2)\,V(p_2-p_1)\,u(p_1,s_1) \quad (2.17b)$$

(in which the photon recoil effect has been neglected, $q = p_2+k-p_1 \simeq p_2-p_1$), and j^{μ} is the *soft accompanying radiation current*

$$j^{\mu}(k) \;=\; \frac{p_1^{\mu}}{(p_1k)} - \frac{p_2^{\mu}}{(p_2k)}\,. \quad (2.17c)$$

Factorization (2.17a) is of the most general nature. The form of j^{μ} does not depend on the details of the underlying process, neither on the nature of participating charges (electron spin, in particular). The only thing which matters is the momenta and charges of incoming and outgoing particles. Generalization to an arbitrary process is straightforward and results in assembling the contributions due to all initial and final particles, weighted with their respective charges.

The soft current (2.17c) has a classical nature. It can be derived from the classical electrodynamics by considering the potential induced by change of the e.m. current due to scattering.

2.2. CLASSICAL CONSIDERATION

From classical field theory we know that it is the acceleration of a charge that causes electromagnetic radiation. Electromagnetic current participating in field formation in the course of scattering consists of two terms (we suppress the charge e for simplicity)

$$\vec{C} = \vec{C}_1 + \vec{C}_2 \,, \quad \begin{cases} \vec{C}_1 \;=\; \vec{v}_1\, \delta^3(\vec{r} - \vec{v}_1\, t) \cdot \vartheta(t_0 - t) \,, \\ \vec{C}_2 \;=\; \vec{v}_2\, \delta^3(\vec{r} - \vec{v}_2\, t) \cdot \vartheta(t - t_0) \,, \end{cases} \tag{2.18a}$$

with $\vec{v}_{1(2)}$ the velocity of the initial (final) charge moving along the classical trajectory $\vec{r} = \vec{v}_i t$. By t_0 we denote the moment in time when the scattering occurs and the velocity abruptly changes. To achieve a Lorentz-covariant description one adds to (2.18a) an equation for the propagation of the charge-density D to be treated as the zero-component of the 4-vector current C^μ,

$$C_i^\mu(t, \vec{r}) \;=\; \Big(D_i(t, \vec{r}),\, \vec{C}_i(t, \vec{r})\Big) \;\equiv\; v_i^\mu\, D_i \,, \tag{2.18b}$$

with v_i^μ the 4-velocity vectors

$$v_i^\mu \;=\; (1,\, \vec{v}_i) \,.$$

The emission amplitude for a field component with 4-momentum $(\omega, \vec{k}\,)$ is proportional to the Fourier transform of the total current. For the two terms of the current we have

$$\begin{aligned} C_1^\mu(k) \;&=\; \int\limits_{-\infty}^{\infty} dt \int d^3r\; e^{i\,x^\nu k_\nu}\; C_1^\mu(t, \vec{r}) = v_1^\mu \int_{-\infty}^{0} d\tau\; e^{ik^0(t_0+\tau) - i(\vec{k}\cdot\vec{v}_i)\tau} \\ &=\; \frac{-i\;\, v_1^\mu\, e^{ik^0 t_0}}{k^0 - (\vec{k} \cdot \vec{v}_1)} \,, \end{aligned} \tag{2.19a}$$

$$\begin{aligned} C_2^\mu(k) \;&=\; \int\limits_{-\infty}^{\infty} dt \int d^3r\; e^{i\,x^\nu k_\nu}\; C_2^\mu(t, \vec{r}) = v_2^\mu \int_{0}^{+\infty} d\tau\; e^{ik^0(t_0+\tau) - i(\vec{k}\cdot\vec{v}_i)\tau} \\ &=\; \frac{i\;\, v_2^\mu\, e^{ik^0 t_0}}{k^0 - (\vec{k} \cdot \vec{v}_2)} \,. \end{aligned} \tag{2.19b}$$

The solution of the Maxwell equation for the field potential induced by the current (2.19) reads

$$\begin{aligned} A^\mu(x) &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ix^\mu k_\mu}\, [-2\pi i \delta(k^2)] \cdot C^\mu(k) \\ &= \int \frac{d^3k}{2\omega(2\pi)^3}\, e^{-i\omega x^0 + i(\vec{k}\cdot\vec{x})} \cdot A^\mu(k) \,, \end{aligned} \tag{2.20}$$

where

$$A^\mu(k) = A_2^\mu(k) - A_1^\mu(k)\,;$$
$$A_i^\mu(k) = \frac{v_i^\mu \cdot e^{i\omega t_0}}{\omega(1 - v_i \cos\Theta_i)}; \quad \omega = |\vec{k}|\,,\ (\vec{k}\cdot\vec{v}_i) \equiv \omega v_i \cos\Theta_i\,. \tag{2.21}$$

Here Θ_i are the angles between the direction of the photon momentum and that of the corresponding (initial/final) charge.

Rewriting (2.21) in the covariant form

$$\frac{v_i^\mu}{\omega - (\vec{k}\cdot\vec{v}_i)} = \frac{E_i\, v_i^\mu}{E_i\,(\omega - (\vec{k}\cdot\vec{v}_i))} \;=\; \frac{p_i^\mu}{(p_i\,k)}\,,$$

we observe that the classical 4-vector "potential" (2.21), as expected, is identical to the quantum amplitude j^μ (2.17c), apart from an overall phase factor $\exp(i\omega t_0)$. The latter is irrelevant for calculating the observable *cross section* (see, however, section 2.4.3).

We conclude that the classical consideration gives the correct accompanying radiation pattern in the *soft-photon* limit. This is natural because in such circumstances (negligible recoil) it is legitimate to keep charges moving along their classical trajectories, which remain unperturbed in the course of sending away radiation.

2.3. SOFT RADIATION CROSS SECTION

To calculate the radiation probability we square the amplitude projected onto a photon polarization state ε_μ^λ, sum over λ and supply the photon phase space factor to write down

$$\mathrm{d}W = e^2 \sum_{\lambda=1,2} \left|\varepsilon_\mu^\lambda\, j^\mu\right|^2 \frac{\omega^2\,\mathrm{d}\omega\,\mathrm{d}\Omega_\gamma}{2\,\omega\,(2\pi)^3}\,\mathrm{d}W_{\mathrm{el}}\ . \tag{2.22}$$

The sum runs over two physical polarization states of the real photon, described by normalized polarization vectors orthogonal to its momentum:

$$\epsilon_\lambda^\mu(k)\cdot\epsilon_{\mu,\lambda'}^*(k) = -\delta_{\lambda\lambda'}\,,\quad \epsilon_\lambda^\mu(k)\cdot k_\mu = 0\,;\qquad \lambda\,,\lambda' = 1,2\,.$$

Within these conditions the polarization vectors may be chosen differently. Due to the gauge invariance such an uncertainty does not affect physical observables. Indeed, the polarization tensor may be represented as

$$\sum_{\lambda=1,2} \epsilon_\lambda^\mu \epsilon_\lambda^{*\,\nu} \;=\; -g^{\mu\nu} \;+\quad \text{tensor proportional to } k^\mu \text{ and/or } k^\nu\,. \tag{2.23}$$

The latter, however, can be dropped since the classical current (2.17c) is explicitly conserving, $(j^\mu k_\mu) = 0$. Therefore one may enjoy the gauge invariance and employ an arbitrary gauge, instead of using the physical polarizations, to calculate accompanying photon production.

The Feynman gauge being the simplest choice,

$$\sum_{\lambda=1,2} \epsilon_\lambda^\mu \epsilon_\lambda^{*\,\nu} \implies -g^{\mu\nu} ,$$

we arrive at

$$\begin{aligned} \mathrm{d}N \;\equiv\; \frac{\mathrm{d}W}{\mathrm{d}W_{\mathrm{el}}} &= -\frac{\alpha}{4\pi^2}\,(j^\mu)^2\,\omega\,\mathrm{d}\omega\,\mathrm{d}\Omega_\gamma \\ &\simeq \frac{\alpha}{\pi}\,\frac{\mathrm{d}\omega}{\omega}\,\frac{\mathrm{d}\Omega_\gamma}{2\pi}\,\frac{1-\cos\Theta_s}{(1-\cos\Theta_1)(1-\cos\Theta_2)}\,. \end{aligned} \tag{2.24}$$

The latter expression corresponds to the relativistic approximation $1 - v_1, 1 - v_2 \ll 1$:

$$-(j^\mu)^2 = \frac{2(p_1p_2)}{(p_1k)(p_2k)} + \mathcal{O}\left(\frac{m^2}{p_0^2}\right) \simeq \frac{2}{\omega^2}\,\frac{(1-\vec{n}_1\cdot\vec{n}_2)}{(1-\vec{n}_1\cdot\vec{n})(1-\vec{n}_2\cdot\vec{n})}\,;$$

it disregards the contribution of *very* small emission angles $\Theta_i^2 \lesssim (1 - v_i^2) = m^2/p_{0i}^2 \ll 1$, where the soft radiation vanishes (the so-called "Dead Cone" region).

If the photon is emitted at a small angle with respect to, say, the incoming particle, i.e. $\Theta_1 \ll \Theta_2 \simeq \Theta_\mathrm{s}$, the radiation spectrum (2.24) simplifies to

$$\mathrm{d}N \;\simeq\; \frac{\alpha}{\pi}\,\frac{\sin\Theta_1\,\mathrm{d}\Theta_1}{(1-\cos\Theta_1)}\,\frac{\mathrm{d}\omega}{\omega} \simeq \frac{\alpha}{\pi}\,\frac{\mathrm{d}\Theta_1^2}{\Theta_1^2}\,\frac{\mathrm{d}\omega}{\omega}\;.$$

Two bremsstrahlung cones appear, centered around incoming and outgoing electron momenta. Inside these cones the radiation has a *double-logarithmic* structure, exhibiting both the *soft* $(\mathrm{d}\omega/\omega)$ and *collinear* $(\mathrm{d}\Theta^2/\Theta^2)$ enhancements.

2.3.1 Low-Barnett-Kroll wisdom. Soft factorization (2.17a) is an essence of the celebrated soft bremsstrahlung theorem, formulated by Low in 1956 for the case of scalar charged particles and later generalized by Barnett and Kroll to charged fermions. The very classical nature of soft radiation makes it universal with respect to intrinsic quantum properties of participating objects and the nature of the underlying scattering process: it is only the classical movement of electromagnetic charges that matters.

It is interesting that according to the LBK theorem both the leading $d\omega/\omega$ and the first subleading, $\propto d\omega$, pieces of the soft photon spectrum prove to be "classical".

For the sake of simplicity we shall leave aside the angular structure of the accompanying photon emission and concentrate on the energy dependence. Then, the relation between the basic cross section $\sigma^{(0)}$ and that with one additional photon with energy ω can be represented symbolically as

$$d\sigma^{(1)}(p_i,\omega) \propto \frac{\alpha}{\pi}\frac{d\omega}{\omega}\left[\left(1-\frac{\omega}{E}\right)\cdot\sigma^{(0)}(p_i)+\left(\frac{\omega}{E}\right)^2\cdot\tilde{\sigma}(p_i,\omega)\right]. \qquad (2.25)$$

The first term in the right-hand side is proportional to the non-radiative cross section $\sigma^{(0)}$. The second term involves the new ω-dependent cross section $\tilde{\sigma}$ which is finite at $\omega = 0$, so that this contribution is suppressed for small photon energies as $(\omega/E)^2$.

This general structure has important consequences, the most serious of which can be formulated, in a dramatic fashion, as

2.3.2 Soft Photons don't carry quantum numbers.

We are inclined to think that the photon has definite quantum numbers (negative C-parity, in particular). Imagine that the basic process is forbidden, say, by C-parity conservation. Why not to take off the veto by adding a photon to the system? Surely enough it can be done. There is, however, a price to pay: the selection rules cannot be overcome by *soft* radiation. Since the classical part of the radiative cross section in (2.25) is explicitly proportional to the non-radiative cross section $\sigma^{(0)} = 0$, only *energetic* photons (described by the $\tilde{\sigma}$ term) could do the job. The energy distribution

$$|M|^2\cdot\frac{d^3k}{\omega} \;\propto\; \omega d\omega$$

is typical for a quantum particle, where the production matrix element M is finite in the $\omega \to 0$ limit, $M = \mathcal{O}(1)$. An enhanced radiation matrix element, $M \propto \omega^{-1}$ characterizes a classical field rather than a quantum object.

So, the price one has to pay to overrule the quantum-number veto by emitting a soft photon with $\omega \ll E$ is the suppression factor

$$\left(\frac{\omega}{E}\right)^2 \ll 1\,.$$

We conclude that the photons that are capable of changing the quantum numbers of the system (be it parity, C-parity or angular momentum) cannot be *soft*. Neither can they be *collinear*, by the way, as it follows from the

2.3.3 Gribov Bremsstrahlung theorem. This powerful generalisation of the Low theorem states that a simple factorization holds at the level of the *matrix element*, provided the photon transverse momentum with respect to the radiating charged particle is small compared to the momentum transfers characterizing the underlying scattering process:

$$M^{(1)} \propto \frac{(\vec{k}_\perp \cdot \vec{e})}{k_\perp^2} \cdot M^{(0)} + \tilde{M}. \tag{2.26}$$

Here again $\tilde{M} = \mathrm{const}$ in the $k_\perp \to 0$ limit. This factorization holds for *hard* photons ($\omega \sim E$) as well as for soft ones.

Both the Low-Barnett-Kroll and the Gribov theorems hold in QCD as well. In particular, it is the Gribov collinear factorization that leads to the probabilistic evolution picture describing collinear QCD parton multiplication we have discussed in the first lecture.

In the QCD context, our statement that "soft photons don't carry quantum numbers" should be strengthened to even more provocative (but true)

2.3.4 Soft Gluons don't carry away no color. Don't rush to protest. Just think it over. In more respectable terms this title could be abbreviated as the NSFL (no-soft-free-lunch) theorem.

Imagine we want to produce a heavy quark $Q\overline{Q}$ bound state ("onium") in a hadron-hadron collision. The C-even (χ_Q) mesons can be produced by fusing two quasi-real gluons (with opposite colors) from the QCD parton clouds of the colliding hadrons:

$$(g+g)_{(1)} \;\to\; Q+\overline{Q} \;\to\; \chi_Q\,. \tag{2.27}$$

In particular, radiative decays of such χ_c mesons are responsible for about 40% of the J/ψ yield. How about the remaining 60% ? To directly create a J/ψ (or ψ' — ${}^3S_1$ C-odd $c\bar{c}$ states) two gluons isn't enough. A C-odd meson can decay into, or couple to, *three* photons (like para-positronium does), a photon plus two gluons, or *three* gluons (in a color-symmetric d_{abc} state).

So, we need one more gluon to attach, for example, in the final state:

$$(g+g)_{(8)} \;\to\; \left(Q+\overline{Q}\right)_{(8)} \;\to\; J/\psi + g\,. \tag{2.28}$$

To pick up an initial gg pair in a color octet state is easier than in the singlet as in (2.27). This, however, does not help to avoid the trouble:

the perturbative cross section turns out to be too small to meet the need. It underestimates the Tevatron $p\bar{p}$ data on direct J/ψ and ψ' production by a large factor (up to 50, at large $p_\perp$).

That very same effect that makes the J/ψ so narrow a meson with the small hadronic decay width $\Gamma_{J/\psi}/M \propto \alpha_s^3(M)$, suppresses its perturbative production cross section (2.28) as well.

Since the PT approach apparently fails, it seemed natural to blame the non-perturbative (NP) physics. Why not to perturbatively form a color-octet "J/ψ" and then to get rid of color in a smooth (free of charge) non-perturbative way? To *evaporate* color does not look problematic: on the one hand, the soft glue distribution is $\mathrm{d}\omega/\omega = \mathcal{O}(1)$, on the other hand, the coupling α_s/π in the NP domain may be of the order of unity as well. So why not?

The LBK theorem tells us that either the radiation is soft-enhanced, $\propto \mathrm{d}\omega/\omega = \mathcal{O}(1)$, and *classical*, or hard, $\propto \omega \mathrm{d}\omega$ and capable of changing the quantum state of the system. Therefore, to rightfully participate in the J/ψ formation as a quantum field, a NP gluon with $\omega \sim \Lambda_{\mathrm{QCD}}$ would have to bring in the suppression factor

$$\left(\frac{\Lambda_{\mathrm{QCD}}}{M_c}\right)^2 \ll 1\,.$$

The *language* of the LBK is perturbative, 'tis true. The question is, and a serious one indeed, whether the NP phenomena respect the basic dynamical features that its PT counterpart does? Or shall we rather forget about quantum mechanics, color conservation, etc. and accept an "anything goes" motto in the NP domain?

To avoid our discussion turning theological, we better address another verifiable issue namely, photoproduction of J/ψ at HERA. Here we have instead of (2.28) the fusion process of a real (photoproduction) or virtual (electroproduction) photon with a quasi-real space-like gluon from the parton cloud of the target proton:

$$\gamma^{(*)} + g \;\to\; \left(Q + \overline{Q}\right)_{(8)} \;\to\; J/\psi + g\,. \tag{2.29}$$

If the final-state gluon were soft NP junk, the J/ψ meson would carry the whole photon momentum and its distribution in Feynman z would peak at $z = 1$ as $(1-z)^{-1}$. The HERA experiments have found instead a flat*ish* (if not vanishing) z-spectrum at large z. The NSFL theorem seems to be up and running.

By the way, the conventional PT treatment of the photoproduction (2.29) is reportedly doing well. So, what is wrong with the hadroproduction then? Strictly speaking, the problem is still open. An alternative to (2.28) would be to look for the third (hard or hard*ish*) gluon in the initial state.[4]

The NSFL QCD discourse has taken us quite far from the mainstream of the introductory lecture. Let us return to the basic properties of QED bremsstrahlung and make a comparative study of

2.4. INDEPENDENT AND COHERENT RADIATION

In the Feynman gauge, the accompanying radiation factor $\mathrm{d}N$ in (2.24) is dominated by the *interference* between the two emitters:

$$\mathrm{d}N \;\propto\; -\left[\frac{p_1^\mu}{(p_1k)}-\frac{p_2^\mu}{(p_2k)}\right]^2 \;\approx\; \frac{2(p_1p_2)}{(p_1k)(p_2k)}\,.$$

Therefore it does not provide a satisfactory answer to the question, which part of radiation is due to the initial charge and which is due to the final one?

There is a way, however, to give a reasonable answer to this question. To do that one has to sacrifice simplicity of the Feynman-gauge calculation and recall the original expression (2.22) for the cross section in terms of physical photon polarizations. It is natural to choose the so-called *radiative* (temporal) gauge based on the 3-vector potential $\vec{A}$, with the scalar component set to zero, $A_0\equiv 0$. Our photon is then described by (real) 3-vectors orthogonal to one another and to its 3-momentum:

$$(\vec{\epsilon}_\lambda\cdot\vec{\epsilon}_{\lambda'}) = \delta_{\lambda\lambda'}\,, \qquad (\vec{\epsilon}_\lambda\cdot\vec{k}\,) = 0\,. \tag{2.30}$$

This explicitly leaves us with *two* physical polarization states. Summing over polarizations obviously results in

$$dN \;\propto\; \sum_{\lambda=1,2}\left|\vec{j}(k)\cdot\vec{e}_\lambda\right|^2 \;=\; \sum_{\alpha,\beta=1\ldots3} \vec{j}^{\,\alpha}(k)\cdot[\,\delta_{\alpha\beta}-\vec{n}_\alpha\vec{n}_\beta\,]\cdot\vec{j}^{\,\beta}(k)\,, \tag{2.31}$$

with α, β the 3-dimensional indices. We now substitute the soft current (2.17c) in the 3-vector form, $p_i^\mu\to\vec{v}_i p_{0i}$, and make use of the relations

$$(\vec{v}_i)_\alpha\left[\delta_{\alpha\beta}-\frac{k_\alpha\,k_\beta}{\vec{k}^2}\right](\vec{v}_i)_\beta \;=\; v_i^2\sin^2\Theta_i\,, \tag{2.32a}$$

$$(\vec{v}_1)_\alpha\left[\delta_{\alpha\beta}-\frac{k_\alpha\,k_\beta}{\vec{k}^2}\right](\vec{v}_2)_\beta \;=\; v_1v_2(\cos\Theta_{12}-\cos\Theta_1\cos\Theta_2)\,, \tag{2.32b}$$

[4] an interesting, reliable and predictive model for production of onia in the gluon field of colliding hadrons is being developed by Paul Hoyer and collaborators, see hep-ph/0004234 and references therein

to finally arrive at

$$dN = \frac{\alpha}{\pi} \left\{ \mathcal{R}_1 + \mathcal{R}_2 - 2\,\mathcal{J} \right\} \cdot \frac{d\omega}{\omega} \frac{d\Omega}{4\pi} . \tag{2.33a}$$

Here

$$\mathcal{R}_i = \frac{v_i^2 \sin^2\Theta_i}{(1 - v_i \cos\Theta_i)^2} , \quad i = 1, 2 , \tag{2.33b}$$

$$\mathcal{J} \equiv \frac{v_1 v_2 (\cos\Theta_{12} - \cos\Theta_1 \cos\Theta_2)}{(1 - v_1\cos\Theta_1)(1 - v_2 \cos\Theta_2)} . \tag{2.33c}$$

The contributions $\mathcal{R}_{1,2}$ can be looked upon as being due to *independent radiation* off initial and final charges, while the $\mathcal{J}$-term accounts for *interference* between them. The independent and interference contribution, taken together, describe the *coherent* emission. It is straightforward to verify that (2.33) is identical to the Feynman-gauge result (2.24):

$$\mathcal{R}_{\rm coher} \equiv \mathcal{R}_{\rm indep} - 2\mathcal{J} = -\omega^2 (j^\mu)^2 , \qquad \mathcal{R}_{\rm indep} \equiv \mathcal{R}_1 + \mathcal{R}_2 . \tag{2.34}$$

2.4.1 The role of interference: strict angular ordering.

In the relativistic limit we have

$$\mathcal{R}_1 \simeq \frac{\sin^2\Theta_1}{(1 - \cos\Theta_1)^2} = \frac{2}{a_1} - 1 , \tag{2.35a}$$

$$\mathcal{J} \simeq \frac{\cos\Theta_{12} - \cos\Theta_1 \cos\Theta_2}{(1 - \cos\Theta_1)(1 - \cos\Theta_2)} = \frac{a_1 + a_2 - a_{12}}{a_1 a_2} - 1 \tag{2.35b}$$

where we introduced a convenient notation

$$a_1 = 1 - \vec{n}\vec{n}_1 = 1 - \cos\Theta_1 , \quad a_2 = 1 - \cos\Theta_2 ,$$
$$a_{12} = 1 - \vec{n}_1\vec{n}_2 = 1 - \cos\Theta_s .$$

The variables a are small when the angles are small: $a \simeq \frac{1}{2}\Theta^2$.

The independent radiation has a typical logarithmic behavior up to large angles $a_1 \lesssim 1$:

$$dN_1 \propto \mathcal{R}_1 \sin\Theta d\Theta \propto \frac{da_1}{a_1} .$$

However, the interference effectively cuts off the radiation at angles exceeding the scattering angle:

$$dN \propto \mathcal{R}_{\rm coher} \sin\Theta d\Theta = 2a_{12}\frac{da}{a_1 a_2} \propto \frac{da}{a^2} \propto \frac{d\Theta^2}{\Theta^4} , \quad a_1 \simeq a_2 \gg a_{12} .$$

To quantify this coherent effect, let us combine an independent contribution with a half of the interference one to define

$$\begin{aligned} V_1 = \mathcal{R}_1 - \mathcal{J} = \frac{2}{a_1} - \frac{a_1+a_2-a_{12}}{a_1 a_2} &= \frac{a_{12}+a_2-a_1}{a_1 a_2}\,, \\ V_2 = \mathcal{R}_2 - \mathcal{J} = \frac{2}{a_2} - \frac{a_1+a_2-a_{12}}{a_1 a_2} &= \frac{a_{12}+a_1-a_2}{a_1 a_2}\,; \end{aligned} \qquad (2.36a)$$

$$\mathcal{R}_{\text{coher}} = V_1 + V_2\,. \qquad (2.36b)$$

The emission probability V_i can be still considered as "belonging" to the charge $\#i$ (V_1 is singular when $a_1 \to 0$, and vice versa). At the same time these are no longer *independent* probabilities, since V_1 explicitly depends on the direction of the partner-charge $\#2$; *conditional* probabilities, so to say.

It is straightforward to verify the following remarkable property of the "conditional" distributions V: after *averaging* over the azimuthal angle of the radiated quantum, $\vec{n}$, with respect to the direction of the parent charge, $\vec{n}_1$, the probability $V_1(\vec{n},\vec{n}_1;\vec{n}_2)$ *vanishes* outside the Θ_s-cone. Namely

$$\langle V_1 \rangle_{\text{azimuth}} \equiv \int_0^{2\pi} \frac{d\phi_{n,n_1}}{2\pi} V_1(\vec{n},\vec{n}_1;\vec{n}_2) = \frac{2}{a_1}\vartheta\left(a_{12}-a_1\right)\,. \qquad (2.37)$$

It is only a_2 that changes under the integral (2.37), while a_1, and obviously a_{12}, stay fixed. The result follows from the angular integral

$$\int_0^{2\pi} \frac{d\phi_{n,n_1}}{2\pi}\frac{1}{a_2} = \frac{1}{|\cos\Theta_1 - \cos\Theta_s|} = \frac{1}{|\,a_{12}-a_1|}\,.$$

Naturally, a similar expression for V_2 emerges after the averaging over the azimuth around $\vec{n}_2$ is performed.

We conclude that as long as the *total* (angular-integrated) emission probability is concerned, the result can be expressed as a sum of two independent bremsstrahlung cones centered around $\vec{n}_1$ and $\vec{n}_2$, both having the finite opening half-angle Θ_s.

This nice property is known as a "strict angular ordering". It is an essential part of the so-called Modified Leading Log Approximation (MLLA), which describes the internal structure of parton jets with a single-logarithmic accuracy.

2.4.2 Angular ordering on the back of envelope. What is the reason for radiation at angles exceeding the scattering angle to be suppressed? Let us try our physical intuition and consider semi-classically how the radiation process really develops.

A physical electron is a charge surrounded by its proper Coulomb field. In quantum language the Lorentz-contracted Coulomb-disk attached to a relativistic particle may be treated as consisting of photons virtually emitted and, in due time, reabsorbed by the core charge. Such virtual emission and absorption processes form a coherent state which we call a physical electron ("dressed" particle).

This coherence is partially destroyed when the charge experiences an impact. As a result, a part of intrinsic field fluctuations gets released in the form of real photon radiation: the bremsstrahlung cone in the direction of the initial momentum develops. On the other hand, the deflected charge now leaves the interaction region as a "half-dressed" object with its proper field-coat lacking some field components (eventually those that were lost at the first stage). In the process of regenerating the new Coulomb-disk adjusted to the final-momentum direction, an extra radiation takes place giving rise to the second bremsstrahlung cone.

Now we need to be more specific to find out which momentum components of the electromagnetic coat do actually take leave.

A typical time interval between emission and reabsorption of the photon k by the initial electron p_1 may be estimated as the Lorentz-dilated lifetime of the virtual intermediate electron state $(p_1 - k)$ (see the left graph in Fig.3),

$$t_{\rm fluct} \sim \frac{E_1}{|m^2 - (p_1 - k)^2|} = \frac{E_1}{2p_1 k} \sim \frac{1}{\omega\Theta^2} \simeq \frac{\omega}{k_\perp^2}\,. \tag{2.38}$$

Here we restricted ourselves, for simplicity, to small radiation angles, $k_\perp \approx \omega\Theta \ll k_{||} \approx \omega$. The fluctuation time (2.38) may become macroscopically large for small photon energies ω and enters as a characteristic parameter in a number of QED processes. As an example, let us mention the so called Landau-Pomeranchuk effect — suppression of soft radiation off a charge that experiences multiple scattering propagating through a medium. Quanta with too large a wavelength get not enough time to be properly formed before successive scattering occurs, so that the resulting bremsstrahlung spectrum behaves as $dN \propto d\omega/\sqrt{\omega}$ instead of the standard logarithmic $d\omega/\omega$ distribution.

The characteristic time scale (2.38) responsible for this and many other radiative phenomena is often referred to as the *formation time*.

Now imagine that within this interval the core charge was kicked by some external interaction and has changed direction by some Θ_s. Whether the photon will be reabsorbed or not depends on the position of the scattered charge with respect to the point where the photon was expecting to meet it "at the end of the day". That is, we need to compare the spatial displacement of the core charge $\Delta\vec{r}$ with the characteristic size of the photon field, $\lambda_{||} \sim \omega^{-1}$, $\lambda_\perp \sim {k_\perp}^{-1}$:

$$\begin{aligned}\Delta r_{||} &\sim \left| v_{2||} - v_{1||} \right| \cdot t_{\text{fluct}} \sim \Theta_s^2 \cdot \frac{1}{\omega\Theta^2} = \left(\frac{\Theta_s}{\Theta}\right)^2 \lambda_{||} \ \Leftrightarrow \ \lambda_{||}\,; \\ \Delta r_\perp &\sim \quad c\Theta_s \cdot t_{\text{fluct}} \sim \Theta_s \cdot \frac{1}{\omega\Theta^2} \quad = \left(\frac{\Theta_s}{\Theta}\right) \lambda_\perp \ \Leftrightarrow \ \lambda_\perp\,.\end{aligned} \tag{2.39}$$

For large scattering angles, $\Theta_s \sim 1$, the charge displacement exceeds the photon wavelength for arbitrary Θ, so that the two full-size bremsstrahlung cones are present. For numerically small $\Theta_s \ll 1$, however, it is only photons with $\Theta \lesssim \Theta_s$ that can notice the charge being displaced and thus the coherence of the state being disturbed. Therefore only the radiation at angles smaller than the scattering angle actually emerges. The other field components have too large a wavelength and are easily reabsorbed *as if* there were no scattering at all.

So what counts is a change in the current, which is sharp enough to be noticed by the "to-be-emitted" quantum within the characteristic formation/field-fluctuation time (2.38) of the latter.

Radiation at large angles has too short a formation time to become aware of the acceleration of the charge. No scattering — no radiation.

The same argument applies to the dual process of production of two opposite charges (decay of a neutral object, vacuum pair production, etc.). The only difference is that now one has to take for $\Delta\vec{r}$ not a displacement between the initial and the final charges, but the actual distance between the produced particles (spatial size of a dipole), to be compared with the radiation wavelength.

2.4.3 Time delay and decoherence effects. Till now we were dealing with particle scattering/production as *instant* processes. Such they usually are (as compared with typical formation times). Nevertheless, let us imagine that our electron in Fig. 3 is delayed by some finite $\Delta t = \tau$ "in the V-vertex". For example, as if some metastable state was formed with characteristic lifetime $\tau = \Gamma^{-1}$.

In such a case one would have to take into consideration an extra *longitudinal* charge displacement due to a finite delay, and (2.39) would be modified as

$$\Delta r_{||} \sim \left(\frac{\Theta_s}{\Theta}\right)^2 \lambda_{||} \ + c\tau \ \Leftrightarrow \ \lambda_{||} \sim \omega^{-1}\,.$$

Now the condition $\Delta r_{||} < \lambda_{||}$ for the radiation at $\Theta > \Theta_s$ to be coherently suppressed implies an additional restriction $\tau < \omega^{-1}$. For large enough values of the delay time, $\tau \gg E^{-1}$, this new condition seriously affects radiation with comparatively large energies $\omega > \tau^{-1}$ (but still *soft* in the overall energy scale, $\omega \ll E_i$). Such photons acquire sufficiently large resolutions for coherence to be completely destroyed by the delay. Therefore they are bound to form two independent bremsstrahlung cones even for $\Theta_s = 0$.

So we would expect the accompanying radiation pattern to be that of the coherent antenna $\mathcal{R}_{\rm coher}$ for softer radiation, $\tau^{-1} > \omega$, and, on the contrary, a sum of two independent sources $\mathcal{R}_1 + \mathcal{R}_2$ for relatively hard photons, $\tau^{-1} < \omega \ll E$. This qualitative expectation has a nice quantitative approval.

The initial and final electron currents in (2.17c) now acquire *different phases* due to the difference between the *Freeze*! and *Move*! times t_{01} and t_{02} (cf. (2.19)):

$$j^\mu \Longrightarrow j^\mu_{\rm del.}(k) = \frac{p_1^\mu}{(p_1 k)}\, e^{{\rm i}\omega t_{01}} - \frac{p_2^\mu}{(p_2 k)}\, e^{{\rm i}\omega t_{02}} \,. \tag{2.40}$$

Now we should be careful when calculating the radiation probability, since the new current (2.40) is no longer conserved: $(j^\mu{}_{\rm del.} k_\mu) \neq 0$. In particular, we cannot use the Feynman-gauge square of this current. The conservation could be formally rescued by adding the term describing our charge being frozen within the time interval $t_{02} - t_{01}$, namely

$$\delta j^\mu_{\rm del.}(k) \;=\; -\frac{\delta^{0\mu}}{\omega} \left\{ e^{{\rm i}\omega t_{01}} - e^{{\rm i}\omega t_{02}} \right\} .$$

However, we can still use the physical polarization method which remains perfectly applicable. The relative phase enters in the interference term, so that the soft radiation pattern gets modified according to

$${\rm d}N \;=\; \frac{\alpha}{\pi} \left\{ \mathcal{R}_1 + \mathcal{R}_2 - 2\,\mathcal{J} \cdot {\rm Re}\left[e^{i\omega(t_{01}-t_{02})} \right] \right\} \frac{{\rm d}\omega}{\omega} \frac{{\rm d}\Omega}{4\pi} \,. \tag{2.41}$$

To make our pedagogical setup more realistic, imagine that it was the formation of a meta-stable (resonant) state that caused the delay. In such a case the delay-time $\tau \equiv t_{01} - t_{02}$ is distributed according to the characteristic decay exponent

$$\left[\Gamma \int_0^\infty d\tau\, e^{-\Gamma\tau} \right] .$$

Averaging (2.41) with this distribution immediately results in a simple Γ-dependent expression, namely,

$${\rm d}N \;=\; \frac{\alpha}{\pi} \left\{ \mathcal{R}_1 + \mathcal{R}_2 - 2\,\mathcal{J} \cdot \chi_\Gamma\left(\omega\right) \right\} \frac{{\rm d}\omega}{\omega} \frac{{\rm d}\Omega}{4\pi}$$

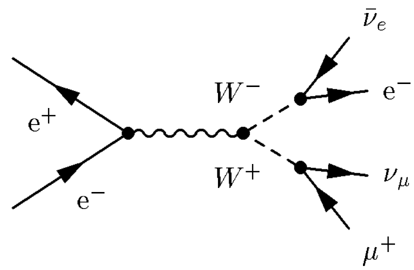


Figure 4 Leptonic decay of a W^+W^- pair as an illustration of time-dependent decoherence effects.

with the *profile factor*

$$\chi_\Gamma \equiv \mathrm{Re}\left[\Gamma\int_0^\infty d\tau\, e^{-\Gamma\tau}\cdot e^{i\omega\tau}\right] = \mathrm{Re}\left[\frac{\Gamma}{\Gamma - i\omega}\right] = \frac{\Gamma^2}{\Gamma^2+\omega^2}\,.$$

The answer can be written as a *mixture* of independent and coherent patterns with the weights depending on the ratio ω/Γ via the profile function χ_Γ,

$$dN \;=\; \frac{\alpha}{\pi}\,\frac{d\omega}{\omega}\,\frac{d\Omega}{4\pi}\left\{\,[\,1-\chi_\Gamma(\omega)\,]\cdot\mathcal{R}_{\mathrm{indep}}(\vec{n}) + \chi_\Gamma(\omega)\cdot\mathcal{R}_{\mathrm{coher}}(\vec{n})\right\}\,. \quad (2.42)$$

$\chi(\omega)$ acts as a "switch": for long-wave radiation $\chi(\omega \ll \Gamma) \to 1$, the standard coherent antenna pattern appears; vice versa, for large frequencies $\chi(\omega \gg \Gamma) \to 0$, the coherence between charges is dashed away, as we expected.

Example: soft photons and the W-width. This simple phenomenon finds an intriguing and important practical implication in the dual channel. Suppose that in the e^+e^- annihilation process a pair of non-relativistic W^+W^- is produced. An intermediate boson has a finite life-time, $\Gamma \simeq 2\,\mathrm{GeV}$, and decays either leptonically or into a quark pair that produces two hadron jets at the end of the day. Thus, the W^+ and W^- decay independently of one another and produce ultra-relativistic electromagnetic currents within a characteristic time interval $|\Delta t_0| \sim \Gamma^{-1}$. The process is displayed in Fig. 4.

Therefore one meets exactly the same "delayed acceleration" scenario as applied to the final-state currents. As a result, (2.42) describes the photon radiation accompanying *leptonic* decays of non-relativistic W^+ and W^-.

Integrating over photon emission angles we derive the total photon multiplicity:

$$\omega\frac{\mathrm{d}N}{\mathrm{d}\omega} \propto \left[\ln\frac{2}{1-v_1} + \ln\frac{2}{1-v_2} - 4\right] + 2\chi_\Gamma(\omega)\left[\ln\frac{1-\cos\Theta_{12}}{2} + 1\right]. \tag{2.43}$$

where v_1 and v_2 are velocities of final charged leptons (e, μ or τ). The main "collinear" contributions $\sim \ln(1-v_1)^{-1} \gg 1$ are naturally ω- and Θ_{12}-independent.

A non-trivial ω-dependence of the profile function χ_Γ comes together with the functional dependence on the angle Θ_{12} between the leptons. The ω-dependent term in (2.43) *enhances* the accompanying photon multiplicity for large values of Θ_{12} and, vice versa, acts *destructively* if the angle between leptons happens to be below the critical value

$$\begin{aligned} \cos\Theta_{\mathrm{crit}} &= 1 - 2\exp(-1) \approx 0.264\,, \\ \Theta_{\mathrm{crit}} &\approx 1.30 \approx \frac{5\pi}{12} = 75^\circ\,. \end{aligned} \tag{2.44}$$

This suggests a programme of measuring the W-width Γ_W by studying the variation of the radiation yield with Θ_{12}.

3. BACK TO QCD

3.1. QCD SCATTERING AND CROSS-CHANNEL RADIATION

Both the qualitative arguments of the previous lecture and the quantitative analysis of the two-particle antenna pattern apply to the QCD process of gluon emission in the course of quark scattering. So two gluon-bremsstrahlung cones with the opening angles restricted by the scattering angle Θ_s would be expected to appear.

There is an important subtlety, however. In the QED case it was deflection of an electron that changed the e.m. current and caused photon radiation. In QCD there is another option, namely to "repaint" the quark. Rotation of the *color state* would affect the color current as well and, therefore, must lead to gluon radiation irrespectively of whether the quark-momentum direction has changed or not.

This is what happens when a quark scatters off a *color* field. To be specific, one may consider as an example two channels of Higgs production in hadron-hadron collisions.

At very high energies two mechanisms of Higgs production become competitive: $W^+W^- \to H$ and the gluon-gluon fusion $gg \to H$ (see Fig. 5).

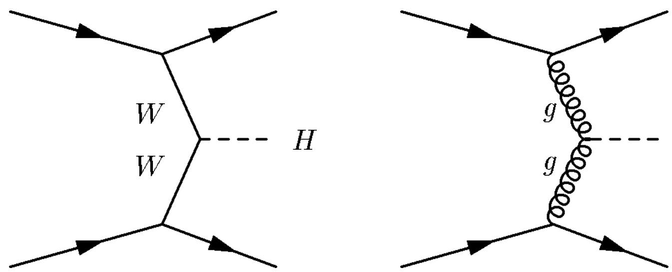


Figure 5 WW and gluon-gluon fusion graphs for Higgs production

Since the typical momentum transfer is large, of the order of the Higgs mass, $(-t) \sim M_H^2$, Higgs production is a *hard* process. Colliding quarks experience hard scattering with characteristic scattering angles $\Theta_s^2 \simeq |t|/s \sim M_H^2/s$. As far as the accompanying gluon radiation is concerned, the two subprocesses differ with respect to the nature of the "external field", which is *colorless* for the W-exchange and *colorful* for the gluon fusion.

The gluon bremsstrahlung amplitudes for the second case are shown in Fig. 6. In principle, a graph with the gluon-gluon interaction vertex should also be considered. However, in the limit $k_\perp \ll q_\perp$, with $\vec{q}_\perp \approx \vec{p}_{2\perp} - \vec{p}_{1\perp}$ the momentum transfer in the scattering process, emission off the external lines dominates (the "soft insertion rules").

The accompanying soft radiation current j^μ factors out from the Feynman amplitudes of Fig. 6, the only difference with the Abelian current (2.17c) being the order of the color generators:

$$j^\mu = \left[t^b \, t^a \left(\frac{p_1^\mu}{(p_1 k)} \right) - t^a \, t^b \left(\frac{p_2^\mu}{(p_2 k)} \right) \right] \; . \tag{3.45}$$

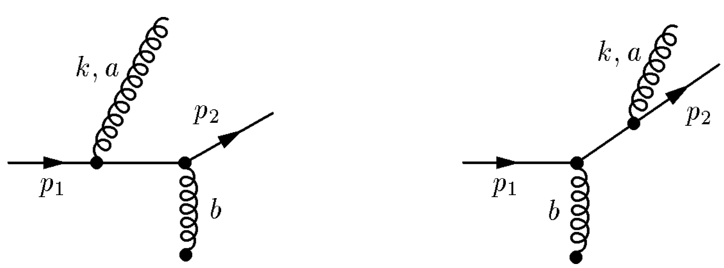


Figure 6 Gluonic Bremsstrahlung diagrams for $k_\perp \ll q_\perp$. The characters a and b denote the colors of the radiated and exchanged gluons.

Introducing the abbreviation $A_i = \frac{p_i^\mu}{(p_i k)}$, we apply the standard decomposition of the product of two triplet color generators,

$$t^a t^b = \frac{1}{2N_c}\delta_{ab} + \tfrac{1}{2}\left(d_{abc} + \mathrm{i} f_{abc}\right)\, t^c \,,$$

to rewrite (3.45) as

$$\begin{aligned} j^\mu &= \tfrac{1}{2}(A_1 - A_2)\,\left\{t^b, t^a\right\} + \tfrac{1}{2}(A_1 + A_2)\,\left[t^b, t^a\right] \\ &= \tfrac{1}{2}(A_1 - A_2)\left(\frac{1}{N}\,\delta^{ab} + d^{abc}\, t^c\right) - \tfrac{1}{2}(A_1 + A_2)\,\mathrm{i} f^{abc}\, t^c \;. \end{aligned}$$

To find the emission *probability* we need to construct the product of the currents and sum over colors. Three color structures do not "interfere", so it suffices to evaluate the squares of the singlet, $\mathbf{8}_s$ and $\mathbf{8}_a$ structures:

$$\begin{aligned} \sum_{a,b}\left(\frac{1}{2N_c}\delta_{ab}\right)^2 &= \left(\frac{1}{2N}\right)^2\,(N_c^2 - 1) = \frac{1}{2N_c}\cdot C_F \,; \\ \sum_{a,b}\left(\tfrac{1}{2}\, d_{abc}\, t^c\right)^2 &= \frac{1}{4}\,\frac{N_c^2 - 4}{N_c}\,(t^c)^2 \;= \frac{N_c^2 - 4}{4N_c}\cdot C_F \,; \\ \sum_{a,b}\left(\tfrac{1}{2}\, i f_{abc}\, t^c\right)^2 &= \quad \frac{1}{4}\, N_c\;(t^c)^2 \quad\;\; = \frac{N_c}{4}\cdot C_F \,. \end{aligned}$$

The common factor $C_F = \left(t^b\right)^2$ belongs to the Born (non-radiative) cross section, so that the radiation spectrum takes the form

$$\begin{aligned} \mathrm{d}N \;\propto\; \frac{1}{C_F}\sum_{\text{color}} j^\mu \cdot (j_\mu)^* &= \left(\frac{1}{2N_c} + \frac{N_c^2 - 4}{4N_c}\right)(A_1 - A_2)\cdot(A_1 - A_2) \\ &+ \frac{N_c}{4}\,(A_1 + A_2)\cdot(A_1 + A_2)\,. \end{aligned}$$

Simple algebra leads to

$$\mathrm{d}N \;\propto\; C_F\,(A_1 - A_2)\cdot(A_1 - A_2) \;+\; N_c\, A_1\cdot A_2 \,. \qquad (3.46)$$

Dots here symbolize the sum over gluon polarization states. Similar to the case of "delayed scattering" discussed above, the current (3.45) *is not conserved* because of non-commuting color matrices. We would need to include gluon radiation from the exchange-gluon line *and* from the source, to be in a position to use an arbitrary gauge (e.g. the Feynman gauge) for the emitted gluon. Once again, the physical polarization technique (2.30) simplifies our task. To obtain the true accompanying radiation pattern (in the $k_\perp \ll q_\perp$ region) it suffices to use the projectors (2.32) for the dots in (3.46). In particular,

$$A_1 \cdot A_2 \equiv \sum_{\lambda=1,2} \left(A_1 e^{(\lambda)}\right) \left(A_2 e^{(\lambda)}\right)^* = \mathcal{J} \qquad \{ \neq -(A_1 A_2) \ sic! \} \ .$$

Accompanying radiation intensity finally takes the form

$$\mathrm{d}N \ \propto \ C_F \, \mathcal{R}_{\text{coher}} + N_c \, \mathcal{J} \, . \tag{3.47}$$

The first term proportional to the squared quark charge is responsible, as we already know, for two narrow bremsstrahlung cones around the incoming and outgoing quarks, $\Theta_1, \Theta_2 \le \Theta_s$. On top of that an additional, purely non-Abelian, contribution shows up, which is proportional to the *gluon* charge. It is given by the interference distribution (2.33c), (2.35b),

$$\mathcal{J} = \frac{a_1 + a_2 - a_{12}}{a_1 a_2} - 1 \, ,$$

which remains *non-singular* in the forward regions $\Theta_1 \ll \Theta_s$ and $\Theta_2 \ll \Theta_s$. At the same time, it populates large emission angles $\Theta = \Theta_1 \approx \Theta_2 \gg \Theta_s$ where

$$\mathrm{d}N \propto \mathrm{d}\Omega \, \mathcal{J} \propto \sin\Theta \, \mathrm{d}\Theta \left(\frac{2}{a} - 1\right) \propto \frac{\mathrm{d}\Theta^2}{\Theta^2} \, . \tag{3.48}$$

Indeed, evaluating the azimuthal average, say, around the *incoming* quark direction we obtain

$$\int \frac{d\phi_1}{2\pi} \mathcal{J} = \frac{1}{a_1}\left(1 + \frac{a_1 - a_{12}}{|a_1 - a_{12}|}\right) - 1 = \frac{2}{a_1} \, \vartheta(\Theta_1 - \Theta_s) - 1 \, .$$

Thus we conclude that the third complementary bremsstrahlung cone emerges. It basically corresponds to radiation at angles *larger* than the scattering angle and its intensity is proportional to the color charge of the t-channel exchange.

We could have guessed without actually performing the calculation that at large angles the gluon radiation is related to the *gluon* color charge. As far as large emission angles $\Theta \gg \Theta_s$ are concerned, one may identify the directions of initial and final particles to simplify the total radiation amplitude as

$$j^\mu = T^b T^a \cdot \frac{p_1^\mu}{p_1 k} - T^a T^b \cdot \frac{p_2^\mu}{p_2 k} \approx \left(T^b T^a - T^a T^b\right) \cdot \frac{p^\mu}{pk} \, .$$

Recalling the general commutation relation for the $SU(N_c)$ generators,

$$\left[T^a(R) \, , \, T^b(R)\right] = \mathrm{i} \sum_c f_{abc} \, T^c(R) \, , \tag{3.49}$$

we immediately obtain the factor $N_c \propto (\mathrm{i} f_{abc})^2$ as the proper color charge. Since (3.49) holds for arbitrary color representation R, we see that the accompanying gluon radiation at large angles $\Theta > \Theta_s$ does not depend on the nature of the projectile.

The bremsstrahlung gluons we are discussing transform, in the end of the day, into observable final hadrons. We are ready now to derive an interesting physical prediction from our QCD soft radiation exercise.

Translating the emission angle into (pseudo)rapidity $\eta = \ln \Theta^{-1}$, the logarithmic angular distribution (3.48) converts into the rapidity plateau. We conclude that in the case of the gluon fusion mechanism, the second in Fig. 5, the hadronic accompaniment should form a practically uniform rapidity plateau. Indeed, the hadron density in the center (small η, large c.m.s. angles) is proportional to the gluon color charge N_c, while in the "fragmentation regions" ($\eta_{\max} > |\eta| > \ln \Theta_s^{-1}$, or $\Theta < \Theta_s$) the two quark-generated bremsstrahlung cones give, roughly speaking, the density $\sim 2 \times C_F \approx N_c$.

At the same time, the WW-fusion events (the first graph in Fig. 5) should have an essentially different final state structure. Here we have a colorless exchange, and the QED-type angular ordering, $\Theta < \Theta_s$, restricts the hadronic accompaniment to the two projectile fragmentation humps as broad as $\Delta\eta = \eta_{\max} - \ln \Theta_s \simeq \ln M_H$, while the central rapidity region should be devoid of hadrons. The "rapidity gap" is expected which spans over $|\eta| < \ln(\sqrt{s}/M_H)$.

3.2. CONSERVATION OF COLOR AND QCD ANGULAR ORDERING

In physical terms universality of the generator algebra is intimately related with *conservation of color*. To illustrate this point let us consider production of a quark-gluon pair in some hard process and address the question of how this system radiates. Let p and k be the momenta of the quark and the gluon, with b the octet color index of the latter. For the sake of simplicity we concentrate on *soft* accompanying radiation, which determines the bulk of particle multiplicity inside jets, the structure of the hadronic plateau, etc. As far as emission of a soft gluon with momentum $\ell \ll k, p$ is concerned, the so-called "soft insertion rules" apply, which tell us that the Feynman diagrams dominate where ℓ is radiated off the external (real) partons — the final quark line p and the gluon k. The corresponding Feynman amplitudes are shown in Fig. 7.

Do two emission amplitudes interfere with each other? It depends on the direction of the radiated gluon $\vec{\ell}$.

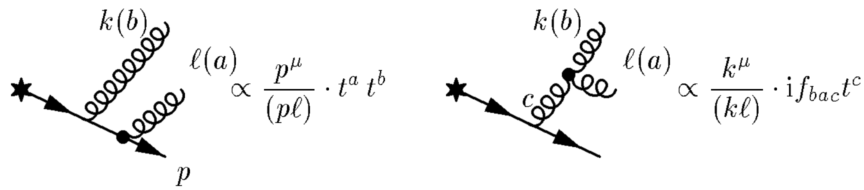


Figure 7 Feynman diagrams for radiation of the soft gluon with momentum ℓ and color a off the qg system.

In the first place, there are two bremsstrahlung cones centered around the directions of $\vec{p}$ and $\vec{k}$:

$$\text{quark cone:}\quad \Theta_{\vec{\ell}} \equiv \Theta_{\vec{\ell},\vec{p}} \;\ll\; \Theta \approx \Theta_{\vec{\ell},\vec{k}}\,,$$
$$\text{gluon cone:}\quad \Theta_{\vec{\ell}} \equiv \Theta_{\vec{\ell},\vec{k}} \;\ll\; \Theta \approx \Theta_{\vec{\ell},\vec{p}}\,,$$

with Θ the angle between $\vec{p}$ and $\vec{k}$ — the aperture of the qg fork. In these regions one of the two amplitudes of Fig. 7 is much larger than the other, and the interference is negligible: the gluon ℓ is radiated independently and participates in the formation of the quark and gluon sub-jets.

If Θ is sufficiently large and the gluon k sufficiently energetic (relatively hard, $k \sim p$), these two sub-jets can be distinguished in the final state. The particle density in q and g jets should be remarkably different. It should be proportional (at least asymptotically) to the probability of soft gluon radiation which, in turn, is proportional to the "squared color charge" of the jet-generating parton, quark or gluon:

$$\left(\ell\frac{\mathrm{d}n}{\mathrm{d}\ell}\right)^{g}_{\Theta_{\vec{\ell}}<\Theta} : \left(\ell\frac{\mathrm{d}n}{\mathrm{d}\ell}\right)^{q}_{\Theta_{\vec{\ell}}<\Theta} \;=\; N_c : \frac{N_c^2-1}{2N_c} = 3 : \frac{4}{3} = \frac{9}{4}\,.$$

Multijet configurations are comparatively rare: emission of an additional hard gluon $k \sim p$ at large angles $\Theta \sim 1$ constitutes a fraction $\alpha_s/\pi \lesssim 10\%$ of all events. Typically k would prefer to belong to the quark bremsstrahlung cone itself, that is to have $\Theta \ll 1$. In such circumstances the question arises about the structure of the accompanying radiation at comparatively *large* angles

$$\Theta_{\vec{\ell}} \;\equiv\; \Theta_{\vec{\ell},\vec{p}} \;\simeq\; \Theta_{\vec{\ell},\vec{k}} \;\gg\; \Theta\,. \tag{3.50}$$

If the quark and the gluon were acting as independent emitters, we would expect the particle density to increase correspondingly and to overshoot the standard quark jet density by the factor

$$\left(\ell\frac{\mathrm{d}n}{\mathrm{d}\ell}\right)^{g+q}_{\Theta_{\vec{\ell}}>\Theta} : \left(\ell\frac{\mathrm{d}n}{\mathrm{d}\ell}\right)^{q}_{\Theta_{\vec{\ell}}>\Theta} \;=\; N_c : \frac{N_c^2-1}{2N_c} + 1 = \frac{13}{4}\,. \tag{3.51}$$

However, in this angular region our amplitudes start to interfere significantly, so that the radiation off the qg pair is no longer given by the *sum of probabilities* $q \to g(\ell)$ plus $g \to g(\ell)$. We have to square the *sum of amplitudes* instead.

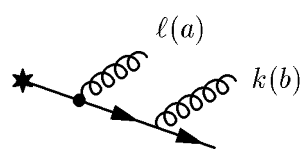
This can be easily done by observing that in the large-angle kinematics (3.50) the angle Θ between $\vec{p}$ and $\vec{k}$ can be neglected, so that the accompanying soft radiation factors in Fig. 7 become indistinguishable,

$$\frac{p^\mu}{(p\ell)} \simeq \frac{k^\mu}{(k\ell)}\,.$$

Thus the Lorentz structure of the amplitudes becomes the same and it suffices to sum the color factors:

$$t^a\, t^b \,+\, \mathrm{i} f_{bac} t^c \;=\; t^a\, t^b + \left[\, t^b\,,\, t^a \,\right] \;\equiv\; t^b\, t^a\,. \tag{3.52}$$

We conclude that the coherent sum of two amplitudes of Fig. 7 results in radiation at large angles *as if* off the initial quark, as shown in Fig. 8.



$$\propto \frac{p^\mu}{(p\ell)} \cdot t^b\, t^a$$

Figure 8 Soft radiation at large angles is determined by the total color charge

This means that the naive probabilistic expectation of enhanced density (3.51) fails and the particle yield is equal to that for the quark-initiated jet instead:

$$\text{(gluon+quark)} \quad \frac{9}{4} + 1 = \frac{13}{4} \qquad \Longrightarrow \qquad 1 \quad \text{(quark)}\,.$$

It actually does not matter whether the gluon k was present at all, or whether there was instead a whole bunch of partons with small relative angles between them. Soft gluon radiation at large angles is sensitive only to the *total* color charge of the final parton system, which equals the color charge of the initial parton. This physically transparent statement holds not only for the quark as in Figs. 7, 8 but for an arbitrary object R (gluon, diquark, ..., you name it) as an initial object. In this case the matrices $t = T(3)$ should be replaced by the generators $T(R)$ corresponding to the color representation R, and (3.52) holds due to the universality of the generator algebra (3.49).

3.3. HUMPBACKED PLATEAU AND LPHD

QCD coherence is crucial for treating particle multiplication **inside** jets, as well as for hadron flows **in-between** jets.

For dessert, we are going to derive together the QCD "prediction" of the inclusive energy spectrum of relatively soft particles from QCD jets. I put the word *prediction* in quotation marks on purpose. This is a good example to illustrate the problem of filling the gap between the QCD formulae, talking quarks and gluons, and phenomena dealing, obviously, with hadrons.

Let me first make a statement:

> It is QCD coherence that allows the prediction of the inclusive soft particle yield in jets practically from the "first principles".

3.3.1 Solving the DIS evolution. You have all the reasons to feel suspicious about this. Indeed, in the first lecture we stressed the similarity between the dynamics of the evolution of space-like (DIS structure functions) and time-like systems (jets). On the other hand, you are definitely aware of the fact that the DIS structure functions cannot be calculated perturbatively. There are input parton distributions for the target proton, which have to be plugged in as an initial condition for the evolution at some finite hardness scale $Q_0 = \mathcal{O}(1\,\mathrm{GeV})$. These initial distributions cannot be calculated "from first principles" nowadays but are subject to fitting. What PT QCD controls then, is the scaling violation pattern. Namely, it tells us how the parton densities change with the changing scale of the transverse-momentum probe:

$$\frac{\partial}{\partial \ln k_\perp} D(x, k_\perp) = \frac{\alpha_s(k_\perp)}{\pi} \int_x^1 \frac{\mathrm{d}z}{z}\, P(z)\, D\left(\frac{x}{z}, k_\perp\right). \tag{3.53}$$

It is convenient to present our "wavefunction" D and "Hamiltonian" P in terms of the complex moment ω, which is Mellin conjugate to the momentum fraction x:

$$D_\omega = \int_0^1 \mathrm{d}x\, x^\omega \cdot D(x)\,, \qquad D(x) = x^{-1} \int_{(\Gamma)} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\, x^{-\omega} \cdot D_\omega\,; \tag{3.54a}$$

$$P_\omega = \int_0^1 \mathrm{d}z\, z^\omega \cdot P(z)\,, \qquad P(z) = z^{-1} \int_{(\Gamma)} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\, z^{-\omega} \cdot P_\omega\,, \tag{3.54b}$$

where the contour Γ runs parallel to the imaginary axis, to the right from singularities of D_ω (P_ω). It is like trading the coordinate ($\ln x$) for the momentum (ω) in a Schrödinger equation.

Substituting (3.54) into (3.53) we see that the evolution equation becomes algebraic and describes propagation in "time" $\mathrm{d}t = \frac{\alpha_s}{\pi}\mathrm{d}\ln k_\perp$ of a free quantum mechanical "particle" with momentum ω and the dispersion law $E(\omega) = P_\omega$:

$$\hat{d}\, D_\omega(k_\perp) \;=\; \frac{\alpha_s(k_\perp)}{\pi} \cdot P_\omega\, D_\omega(k_\perp)\,; \qquad \hat{d} \equiv \frac{\partial}{\partial \ln k_\perp}\,. \tag{3.55}$$

To continue the analogy, our wavefunction D is in fact a multi-component object. It embodies the distributions of valence quarks, gluons and secondary sea quarks which evolve and mix according the 2×2 matrix LLA Hamiltonian (1.9).

At small x, however, the picture simplifies. Here the valence distribution is negligible, $\mathcal{O}(x)$, while the gluon and sea quark components form a system of two coupled oscillators which is easy to diagonalise. What matters is one of the two energy eigenvalues (one of the two branches of the dispersion rule) that is *singular* at $\omega = 0$. The problem becomes essentially one-dimensional. Sea quarks are driven by the gluon distribution while the latter is dominated by gluon cascades. Correspondingly, the leading energy branch is determined by gluon-gluon splitting (1.9d), with a subleading correction coming from the $g \to q(\bar{q}) \to g$ transitions,

$$P_\omega \;=\; \frac{2N_c}{\omega} - a \,+\, \mathcal{O}(\omega)\;, \quad a = \frac{11N_c}{6} + \frac{n_f}{3N_c^2}\,. \tag{3.56}$$

The solution of (3.55) is straightforward:

$$D_\omega(k_\perp) \;=\; D_\omega(Q_0)\cdot \exp\left\{\int_{Q_0}^{k_\perp} \frac{dk}{k}\,\gamma_\omega(\alpha_s(k))\right\}, \tag{3.57a}$$

$$\gamma_\omega(\alpha_s) \;=\; \frac{\alpha_s}{\pi}\,P_\omega\,. \tag{3.57b}$$

The structure (3.57a) is of the most general nature. It follows from *renormalizability* of the theory, and does not rely on the LLA which we used to derive it. The function $\gamma(\alpha_s)$ is known as the "anomalous dimension".[5] It can be perfected by including higher orders of the PT expansion. Actually, modern analyses of scaling violation are based on the improved next-to-LLA (two-loop) anomalous dimension, which includes α_s^2 corrections to the LLA expression (3.57b).

The structure (3.57a) of the x-moments of parton distributions (DIS structure functions) gives an example of a clever separation of PT and

[5] This nice name is a relict of those good old days when particle and solid state physicists used to have common theory seminars. If the coupling α_s were constant (had a "fixed point"), then (3.57a) would produce the function with a non-integer (non-canonical) dimension $D(Q) \propto Q^\gamma$ (analogy — critical indices of thermodynamical functions near the phase transition point).

NP effects; in this particular case — in the form of two factors. It is the ω-dependence of the input function $D_\omega(Q_0)$ ("initial parton distributions") that limits predictability of the Bjorken-x dependence of DIS cross sections.

So, how comes then that in the time-like channel the PT answer turns out to be more robust?

3.3.2 Coherent hump. We are ready to discuss the time-like case, with $D_j^h(x,Q)$ now the inclusive distribution of particles h with the energy fraction (Feynman-x) $x \ll 1$ from a jet (parton j) produced at a large hardness scale Q.

Here the general structure (3.57a) still holds. We need, however, to revisit the expression (3.57b) for the anomalous dimension because, as we have learned, the proper evolution time is now different from the case of DIS.

In the time-like jet evolution, due to Angular Ordering, the evolution equation becomes non-local in $k_\perp$ space:

$$\frac{\partial}{\partial \ln k_\perp} D(x,k_\perp) = \frac{\alpha_s(k_\perp)}{\pi} \int_x^1 \frac{\mathrm{d}z}{z}\, P(z)\, D\left(\frac{x}{z},\, z\cdot k_\perp\right). \tag{3.58}$$

Indeed, successive parton splittings are ordered according to

$$\theta = \frac{k_\perp}{k_{||}} \;>\; \theta' = \frac{k'_\perp}{k'_{||}}\,.$$

Differentiating $D(k_\perp)$ over the scale of the "probe", $k_\perp$, results then in the substitution

$$k'_\perp = \frac{k'_{||}}{k_{||}}\cdot k_\perp \;\equiv\; z\cdot k_\perp$$

in the argument of the distribution of the next generation $D(k'_\perp)$.

The evolution equation (3.58) can be elegantly cracked using the Taylor-expansion trick,

$$D(z\cdot k_\perp) = \exp\left\{\ln z \frac{\partial}{\partial \ln k_\perp}\right\} D(k_\perp) \;=\; z^{\frac{\partial}{\partial \ln k_\perp}}\cdot D(k_\perp). \tag{3.59}$$

Turning as before to moment space (3.54), we observe that the solution comes out similar to that for DIS, (3.57), but for one detail. The exponent $\hat{d}$ of the additional z-factor in (3.59) combines with the Mellin moment ω to make the argument of the splitting function P a *differential operator* rather than a complex number:

$$\hat{d}\cdot D_\omega = \frac{\alpha_s}{\pi}\, P_{\omega+\hat{d}}\cdot D_\omega\,. \tag{3.60}$$

This leads to the differential equation

$$\left(P^{-1}_{\omega+\hat{d}}\,\hat{d}\;-\;\frac{\alpha_s}{\pi}-\left[P^{-1}_{\omega+\hat{d}},\frac{\alpha_s}{\pi}\right]P_{\omega+\hat{d}}\right)\cdot D=0\,. \tag{3.61}$$

Recall that, since we are interested in the small-x region, the essential moments are small, $\omega \ll 1$.

For the sake of illustration, let us keep only the most singular piece in the "dispersion law" (3.56) and neglect the commutator term in (3.61) generating a subleading correction $\propto \hat{d}\alpha_s \sim \alpha_s^2$. In this approximation (DLA),

$$P_\omega \simeq \frac{2N_c}{\omega}\,, \tag{3.62}$$

(3.61) immediately gives a quadratic equation for the anomalous dimension,[6]

$$(\omega+\gamma_\omega)\gamma_\omega\;-\;\frac{2N_c\alpha_s}{\pi}+\mathcal{O}\left(\frac{\alpha_s^2}{\omega}\right)\;=\;0\,. \tag{3.63}$$

The leading anomalous dimension following from (3.63) is

$$\gamma_\omega=\frac{\omega}{2}\left(-1+\sqrt{1+\frac{8N_c\alpha_s}{\pi\,\omega^2}}\right). \tag{3.64}$$

When expanded to first order in α_s, it coincides with that for the space-like evolution, $\gamma_\omega \simeq \alpha_s/\pi\cdot P_\omega$, with P given in (3.62). Such an expansion, however, fails when characteristic $\omega \sim 1/|\ln x|$ becomes as small as $\sqrt{\alpha_s}$, that is when

$$\frac{8N_c\alpha_s}{\pi}\ln^2 x\;\gtrsim\;1\,.$$

This inequality is an elaboration of the estimate (1.12), which we obtained from heuristic arguments in the first lecture.

Now what remains to be done is to substitute our new weird anomalous dimension into (3.57a) and perform the inverse Mellin transform to find $D(x)$. If there were no QCD parton cascading, we would expect the particle *density* $xD(x)$ to be constant (Feynman plateau). It is straightforward to derive that plugging in the DLA anomalous dimension (3.64) results in the plateau density increasing with Q and with a maximum

[6] It suffices to use the next-to-leading approximation to the splitting function (3.56) and to keep the subleading correction coming from differentiation of the running coupling in (3.61) to get the more accurate MLLA anomalous dimension γ_ω.

(hump) "midway" between the smallest and the highest parton energies, namely, at $x_{\max} \simeq \sqrt{Q_0/Q}$. The subleading MLLA effects shift the hump to smaller parton energies,

$$\ln \frac{1}{x_{\max}} = \ln \frac{Q}{Q_0}\left(\frac{1}{2} + c \cdot \sqrt{\alpha_s} + \dots\right) \simeq 0.65 \ln \frac{Q}{Q_0}\,,$$

with c a known analytically calculated number. Moreover, defying naive probabilistic intuition, the softest particles do not multiply at all. The density of particles (partons) with $x \sim Q_0/Q$ stays constant while that of their more energetic companions increases with the hardness of the process Q.

This is a powerful legitimate consequence of PT QCD coherence. We turn now to another, no less powerful though less legitimate, consequence.

3.3.3 Coherent damping of the Landau singularity. The time-like DLA anomalous dimension (3.64), as well as its MLLA improved version, has a curious property. Namely, in sharp contrast with DIS, it allows the momentum integral in (3.57) to be extended to very small scales. Even integrating down to $Q_0 = \Lambda$, the position of the "Landau pole" in the coupling, one gets a finite answer for the distribution (the so-called *limiting spectrum*), simply because the $\sqrt{\alpha_s(k)}$ singularity happens to be integrable!

It would have been poor taste to trust this formal integrability, since the very PT approach to the problem (selection of dominant contributions, parton evolution picture, etc.) relied on α_s being a numerically small parameter. However, the important thing is that, due to time-like coherence effects, the (still perturbative but "smallish") scales, where $\alpha_s(k) \gg \omega^2$, contribute to γ basically in a ω-*independent* way, $\gamma + \omega/2 \propto \sqrt{\alpha_s(k)} \neq f(\omega)$. This means that "smallish" momentum scales k affect only an overall *normalization* without affecting the *shape* of the x-distribution.

Since this is the role of the "smallish" scales, it is natural to expect the same for the truly small — non-perturbative — scales where the partons transform into the final hadrons. This idea has been formulated as a hypothesis of local parton-hadron duality (LPHD). Mathematically, this hypothesis reduces to the statement (guess) that the NP factor in (3.57a) has a finite $\omega \to 0$ limit:

$$D_\omega^{(h)}(Q_0) \;\to\; K^h = \text{const}, \quad \omega \to 0\,.$$

Thus, according to LPHD, the x-shape of the so-called "limiting" parton spectrum which is obtained by formally setting $Q_0 = \Lambda$ in the evolution equations, should be mathematically similar to that of the inclusive distribution of **hadrons** (h). Another essential property is that the "conversion coefficient" K^h should be a true constant independent of the hardness of the process producing the jet under consideration.

3.3.4 Brave gluon counting. The comparison of the limiting spectrum with the inclusive spectrum of all charged hadrons (dominated by $\pi^{\pm}$) was pioneered by Glen Cowan (ALEPH) and by the OPAL collaboration, and has become a standard test of analytic QCD predictions.

In Fig. 9 (DELPHI), the comparison is made of the all-charged hadron spectra at various annihilation energies Q with the so-called "distorted Gaussian" fit which employs the first four moments (the mean, width, skewness and kurtosis) of the MLLA distribution around its maximum.

Is it nothing but one more test of QCQ? Not quite. Such close similarity is deeply puzzling, even worrysome, rather than a successful test.

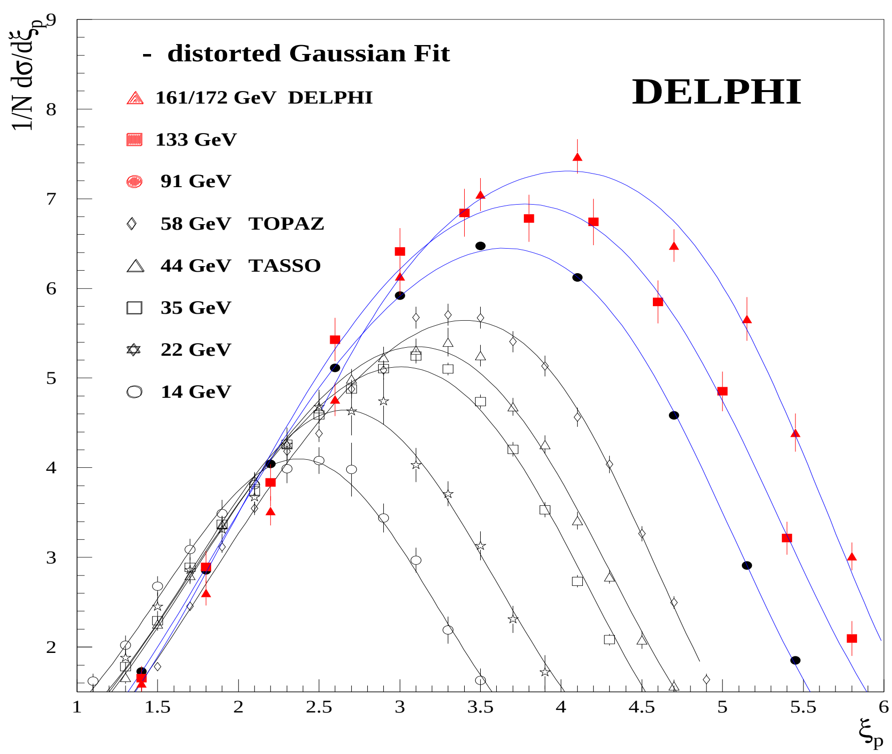


Figure 9 Inclusive distribution of charged hadrons produced in e^+e^- annihilation

Indeed, after a little exercise in translating the values of the logarithmic variable $\xi = \ln(E_{\rm jet}/p)$ in Fig. 9 into GeV you will see that the hadron momenta at the maxima are, for example, $p = \frac{1}{2}Q \cdot e^{-\xi_{\max}} \simeq$ 0.42, 0.85 and 1.0 GeV for Q=14, 35 and 91 GeV, respectively.

Is it not surprising that the PT QCD spectrum is mirrored by that of the pions (which constitute 90% of all charged hadrons produced in jets) with momenta well below 1 GeV?!

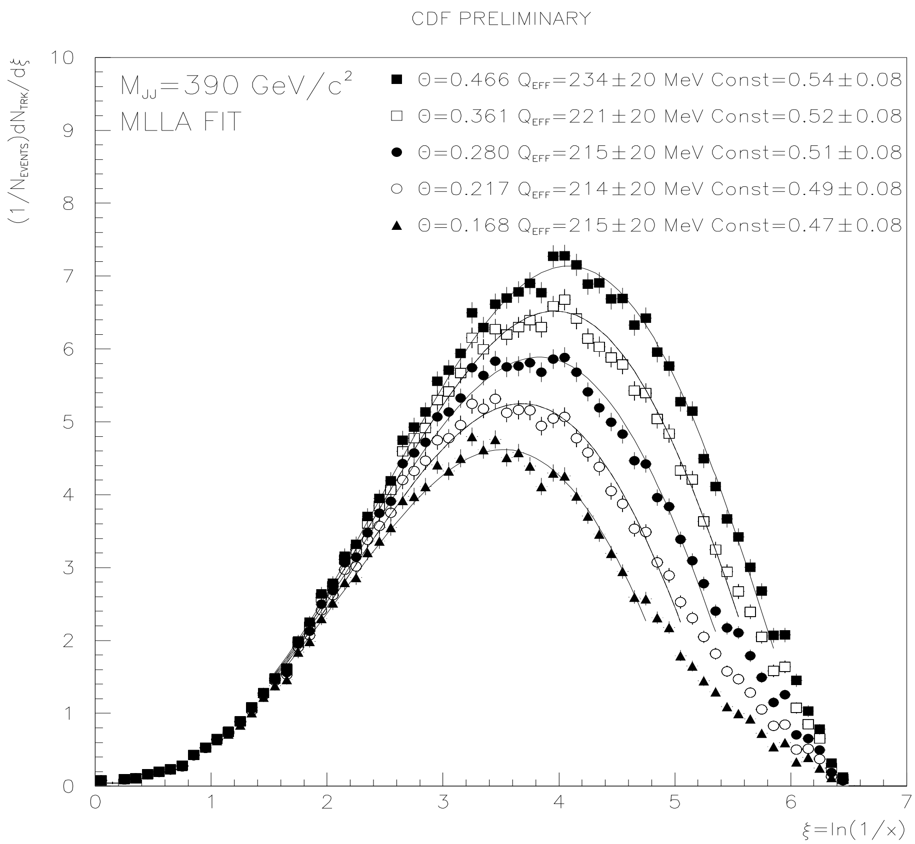


Figure 10 Inclusive energy distribution of charged hadrons in large–$p_\perp$ jets. D.Goulianos, *Proceedings 32nd Recontres de Moriond*, Les Arcs, France, March 1997.

For this very reason, the observation of the parton-hadron similarity was initially met with serious and well-grounded scepticism: it looked more natural (and was more comfortable) to blame the finite hadron mass effects for the falloff of the spectrum at large ξ (small momenta) rather than seriously believe in the applicability of the PT QCD consideration down to such disturbingly small momentum scales.

This worry has recently been answered by CDF.

Theoretically, it is not the energy of the jet but the maximal parton transverse momentum inside it, $k_{\perp\max} \simeq E_{\rm jet} \sin\frac{\Theta}{2}$, that determines the hardness scale and thus the yield and the distribution of the accompanying radiation. This means that by choosing a small opening angle Θ around the jet axis one can study relatively small hardness scales but in a cleaner environment: due to the Lorentz boost effect, eventually all particles that form a short small-Q^2 QCD hump become relativistic and concentrate at the tip of the jet.

For example, by selecting hadrons inside a cone $\Theta \simeq 0.14$ around an energetic quark jet with $E_{\rm jet} \simeq 100$ GeV (LEP-II), one should see the very same curve that corresponds to $Q = 14$ GeV in Fig. 9. Its maximum, however, will now be boosted from dubious 450 MeV into a comfortable 6 GeV range.

A close similarity between the hadron yield and the full MLLA parton spectra (Fig. 10, CDF) can no longer be considered accidental or attributed to non-relativistic kinematical effects.

The fact that even a legitimate finite smearing due to hadronization effects does not look mandatory, suggests a deep duality between the hadron and quark-gluon languages, as applied to such a global characteristic of multihadron production as an inclusive energy spectrum.

3.4. QCD RADIOPHYSICS

Another class of multihadron production phenomena that speak in favor of LPHD is the so-called inter-jet physics. It deals with particle flows in the angular regions between jets in various multi-jet configurations. These particles do not belong to any particular jet, and their production, at the PT level, is governed by coherent soft gluon radiation off the multi-jet system as a whole. Due to QCD coherence, these particle flows are insensitive to internal structure of underlying jets. The only thing that matters is the color topology of the primary system of hard partons and their kinematics.

The ratios of particle flows in different inter-jet valleys are given by parameter-free PT predictions and reveal the so-called "string" or "drag" effects. For a given kinematical jet configuration, such ratios depend only on the number of colors (N_c).

For example, the ratio of the multiplicity flow between a quark (antiquark) and a gluon to that in the $q\bar{q}$ valley in symmetric ("Mercedes") three-jet $q\bar{q}g$ e^+e^- annihilation events is predicted to be

$$\frac{dN_{qg}^{(q\bar{q}g)}}{dN_{q\bar{q}}^{(q\bar{q}g)}} \simeq \frac{5N_c^2-1}{2N_c^2-4} = \frac{22}{7}. \tag{3.65}$$

Comparison of the denominator with the density of radiation in the $q\bar{q}$ valley in $q\bar{q}\gamma$ events with a gluon jet replaced by an energetic photon results in

$$\frac{dN_{q\bar{q}}^{(q\bar{q}\gamma)}}{dN_{q\bar{q}}^{(q\bar{q}g)}} \simeq \frac{2(N_c^2-1)}{N_c^2-2} = \frac{16}{7}. \tag{3.66}$$

Emitting an energetic gluon off the initial quark pair depletes accompanying radiation in the backward direction: color is *dragged* out of the $q\bar{q}$ valley. This destructive interference effect is so strong that the resulting multiplicity flow falls below that in the least favorable direction transverse to the three-jet event plane:

$$\frac{dN_{\perp}^{(q\bar{q}\gamma)}}{dN_{q\bar{q}}^{(q\bar{q}g)}} \simeq \frac{N_C+2C_F}{2(4C_F-N_c)} = \frac{17}{14}. \tag{3.67}$$

At the level of the PT accompanying gluon radiation (QCD radiophysics), such predictions are quite simple and straightforward to derive. The strange thing is, that these and many similar numbers are observed in experiments. The inter-jet particle flow we are discussing is dominated, at present energies, by pions with typical momenta in the 100–300 MeV range! The fact that even such soft junk follows the PT QCD rules is truly amazing.

Since, starting from the LEP-I epoch, the "predictions" of the hump-backed plateau and of the coherent string/drag effects stood up to scrutiny in e^+e^-, DIS and Tevatron experiments, we gained an important piece of knowledge. And this is not that theorists are capable of calculating things, even in the presence of quantum-mechanical effects (see below, in the Acknowledgements). More importantly, we have learned something interesting about the physics of hadronization, about confinement and thus about QCD.

3.5. SOFT CONFINEMENT

Honestly speaking, it makes little sense to treat few-hundred-MeV gluons as PT quanta. What hadron energy spectra and string/drag phenomena are trying to tell us is that the production of hadrons is driven by the strength of the underlying color fields generated by the system of energetic partons produced in a hard interaction. Pushing PT description down into the soft gluon domain is a mere tool for quantifying the strengths of the color field. We conclude:

- The *color field* that is developed by an ensemble of hard primary **partons** determines the structure of the final flow of **hadrons**.
- The Poynting vector of the color field translates into the hadron Poynting vector without visible reshuffling of particle momenta.

Mathematical similarity between the parton and hadron energy and angular distributions means that confinement is very soft and gentle. As far as the global characteristics of final states are concerned, there is no sign of strong forces at the hadronization stage.

When viewed globally, confinement is about *renaming* a flying-away quark into a flying-away pion rather than about forces *pulling* quarks together. To preserve this delicate correspondence is a challenge for the future quantitative theory of hadrons — the "whole QCD".

First steps have been made in this direction in recent years. But this is another story (see below, in the Complaints).

Acknowledgements and Complaints

Particle physics community consists of smart and clever people.

Theorists have to be clever to do the job. They should be clever enough to understand what their colleagues have measured, and doubly-clever to predict what they *will be measuring*. Triply-clever to suggest what experimenters *should measure*.

Experimenters not only need to be clever to do the job, they ought to be smart. They have to be smart to measure, doubly-smart not to listen to what they *will be measuring*, and triply-smart — to what they *should measure*.

Still, I dare to come up with a suggestion.

Organization of the ASI–2000 was perfect, thanks to Harrison Prosper, Misha Danilov and the ASI godfather Tom Ferbel. There is one serious complaint, however. It would be better if the Institute ran (at least) *annually*. The ASI would run out of popular lecturers faster, which would offer a chance to tell more interesting stories. Particularly on such a facinating subject as the physics of hadrons.