Landau-Khalatnikov-Fradkin transformation and even ζ-values in Euclidean massless correlators

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Based on a work with Anatoly V. Kotikov (JINR, Dubna)

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LPTHE
LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

Unexpected relation between

The Landau-Khalatnikov-Fradkin (LKF) transformation

an elegant and powerful transformation allowing one to study the gauge covariance of Green's functions in gauge theories.

&

The multi-loop structure of Euclidean massless correlators

subject to a mysterious cancellation of even zeta-values, ζ_{2n}, *e.g.*, of π^{2n}.

Proof of the "no-π theorem"

even ζ-values can be absorbed in a redefinition of the transcendental basis, *i.e.*, the so-called hatted ζ-values, $\hat{\zeta}_{2n+1}$

Outline

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The LKF transformation (LKFT)

In its original form [Landau and Khalatnikov '55, Fradkin '55]

Relates the QED fermion propagator in two different ξ-gauges

$$S_F(x,\xi) = S_F(x,\eta)\, e^{D(x)-D(0)}, \quad D(x) = \Delta\, e^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-\mathrm{i}px}}{p^4}, \quad \Delta = \xi - \eta$$

Other important works (including generalizations to higher point functions): [Johnson & Zumino '59; Zumino '60; Okubo '60; Bialynicki-Birula '60; Sonoda '01]

Physical quantities should not depend on gauge-fixing parameters ξ and η

Control over gauge dependence & precious information can be obtained by studying the gauge-covariance of correlation functions

Extensively used for decades

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Most important (recent) applications:

- gauge-covariance of Schwinger-Dyson equations [Curtis & Pennington '90; Dong, Munczek & Roberts '94, '96; Bashir, Kizilersu & Pennington '98, '00; Burden & Tjiang '98; Jia & Pennington '16, '17; ...]
- estimation of large orders of perturbation theory (non-perturbative) [Bashir and Raya '02, Jia & Pennington '17; ...]
- generalization brane-worlds [Ahmad et al. '16; James, Kotikov & ST '19]
- generalization to SU(N) gauge theories [De Meerleer et al. '18, '19]

Multi-loop structure of Euclidean massless correlators

We focus on propagator-type (p-type) functions: $\overline{\text{MS}}$-renormalized Euclidean 2-point functions (possible IRR) expressible in terms of massless p-type Feynman integrals (p-integrals or master integrals).

Mysterious cancellations of even ζ-values, ζ_{2n}, *e.g.*, of π^{2n}

Observations (pQCD p-type functions):

- **30 years ago:** all contributions proportional to $\zeta_4 = \pi^4/90$ cancel out in the Adler function at three-loops [Gorishnii, Kataev & Larin '91]
- **10 years ago:** four-loop contribution is also π-free and a similar fact holds for the coefficient function of the Bjorken sum rule [Baikov, Chetyrkin & Kühn '10]
- **recent years:** increasing evidence for other quantities [Jamin et al. '18; Davies & Vogt '18; Ruijl et al. '18; Vogt et al. '18; Baikov et al. '18; Moch et al. '18; Herzog et al. '18, '19; Baikov et al. '19]

Note: first appearance of ζ_4 in some 5-loop correlators (*e.g.*, $\beta_{\rm QCD}$)

"no-π theorem" [Broadhurst '99; Baikov & Chetyrkin '18]

Regularity in terms proportional to π^{2n} explained by *observing* that the ε-dependent ($d = 4 - 2\varepsilon$) transformation of ζ-values

$$\hat{\zeta}_3 \equiv \zeta_3 + \frac{3\varepsilon}{2}\zeta_4 - \frac{5\varepsilon^3}{2}\zeta_6, \qquad \hat{\zeta}_5 \equiv \zeta_5 + \frac{5\varepsilon}{2}\zeta_6, \qquad \hat{\zeta}_7 \equiv \zeta_7, \quad \cdots \tag{1}$$

Eliminates even zetas from the loop expansion of p-integrals/-functions
Defines the existence of a hatted transcendental basis

Recently:

- Eq. (1) has been generalized to 5- and 6-loop p-integrals [Baikov & Chetyrkin '18; Georgoudis et al. '18]
- Eq. (1) has been generalized to 7-loop p-integrals [Baikov & Chetyrkin '19]
- results [Baikov & Chetyrkin '18, '19; Georgoudis et al. '18] also display multi-$\hat{\zeta}$ values

"no-π theorem" [Broadhurst '99; Baikov & Chetyrkin '18]

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Our work: proof of the no-π theorem

- using the LKFT, we generalize (1) to all orders in perturbation theory.
- note: we find the one-fold set of $\hat{\zeta}$-values (no multi-$\hat{\zeta}$ values).
  (the LKFT involves only products of Γ-functions)

View of first terms generated by our exact result

In blue: terms known from [Baikov & Chetyrkin '18] (up to 6 loops).
In red: terms known from [Baikov & Chetyrkin '19] (7 loop).

$$\hat{\zeta}_3 = \zeta_3 + \frac{3\,\varepsilon}{2}\,\zeta_4 - \frac{5\,\varepsilon^3}{2}\,\zeta_6 + \frac{21\,\varepsilon^5}{2}\,\zeta_8 - \frac{153\,\varepsilon^7}{2}\,\zeta_{10} + \frac{1705\,\varepsilon^9}{2}\,\zeta_{12} - \frac{26949\,\varepsilon^{11}}{2}\,\zeta_{14} + \frac{573405\,\varepsilon^{13}}{2}\,\zeta_{16} - \frac{15802673\,\varepsilon^{15}}{2}\,\zeta_{18} + \frac{547591761\,\varepsilon^{17}}{2}\,\zeta_{20} - \frac{23302711005\,\varepsilon^{19}}{2}\,\zeta_{22} + \frac{1194695479813\,\varepsilon^{21}}{2}\,\zeta_{24} + \cdots$$

$$\hat{\zeta}_5 = \zeta_5 + \frac{5\,\varepsilon}{2}\,\zeta_6 - \frac{35\,\varepsilon^3}{4}\,\zeta_8 + 63\,\varepsilon^5\,\zeta_{10} - \frac{2805\,\varepsilon^7}{4}\,\zeta_{12} + \frac{22165\,\varepsilon^9}{2}\,\zeta_{14} - \frac{943215\,\varepsilon^{11}}{4}\,\zeta_{16} + 6498590\,\varepsilon^{13}\,\zeta_{18} - \frac{900752361\,\varepsilon^{15}}{4}\,\zeta_{20} + \frac{19165711635\,\varepsilon^{17}}{2}\,\zeta_{22} - \frac{1965195294755\,\varepsilon^{19}}{4}\,\zeta_{24} + 29867386995325\,\varepsilon^{21}\,\zeta_{26} + \cdots$$

$$\hat{\zeta}_7 = \zeta_7 + \frac{7\,\varepsilon}{2}\,\zeta_8 - 21\,\varepsilon^3\,\zeta_{10} + 231\,\varepsilon^5\,\zeta_{12} - \frac{7293\,\varepsilon^7}{2}\,\zeta_{14} + \frac{155155\,\varepsilon^9}{2}\,\zeta_{16} - 2137954\,\varepsilon^{11}\,\zeta_{18} + 74083926\,\varepsilon^{13}\,\zeta_{20} + \cdots$$

$$\hat{\zeta}_9 = \zeta_9 + \frac{9\,\varepsilon}{2}\,\zeta_{10} - \frac{165\,\varepsilon^3}{4}\,\zeta_{12} + \frac{1287\,\varepsilon^5}{2}\,\zeta_{14} - \frac{109395\,\varepsilon^7}{8}\,\zeta_{16} + 376805\,\varepsilon^9\,\zeta_{18} - \frac{26113581\,\varepsilon^{11}}{2}\,\zeta_{20} + \cdots$$

$$\hat{\zeta}_{11} = \zeta_{11} + \frac{11\,\varepsilon}{2}\,\zeta_{12} - \frac{143\,\varepsilon^3}{2}\,\zeta_{14} + \frac{3003\,\varepsilon^5}{2}\,\zeta_{16} - 41327\,\varepsilon^7\,\zeta_{18} + 1431859\,\varepsilon^9\,\zeta_{20} - 60931689\,\varepsilon^{11}\,\zeta_{22} + \cdots$$

$$\hat{\zeta}_{13} = \zeta_{13} + \frac{13\,\varepsilon}{2}\,\zeta_{14} - \frac{455\,\varepsilon^3}{4}\,\zeta_{16} + 3094\,\varepsilon^5\,\zeta_{18} - \frac{214149\,\varepsilon^7}{2}\,\zeta_{20} + 4555915\,\varepsilon^9\,\zeta_{22} - \frac{467142949\,\varepsilon^{11}}{2}\,\zeta_{24} + \cdots$$

$$\hat{\zeta}_{15} = \zeta_{15} + \frac{15\,\varepsilon}{2}\,\zeta_{16} - 170\,\varepsilon^3\,\zeta_{18} + 5814\,\varepsilon^5\,\zeta_{20} - 247095\,\varepsilon^7\,\zeta_{22} + 12666445\,\varepsilon^9\,\zeta_{24} - 770015850\,\varepsilon^{11}\,\zeta_{26} + \cdots$$

These results provide stringent constraints on multi-loop calculations

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Outline

LKFT in dimensional regularization ($d = 4 - 2\varepsilon$)

$$S_F(x,\xi) = S_F(x,\eta)\, e^{D(x)-D(0)}, \quad D(x) = \Delta\, e^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{-\mathrm{i}px}}{p^4}, \;\; \Delta = \xi - \eta$$

Massless fermion propagator in some gauge ξ:

$$S_F(p,\xi) = \frac{1}{i\hat{p}}\, P(p,\xi), \qquad S_F(x,\xi) = \hat{x}\, X(x,\xi)\,,$$

where $P(p,\xi)$ and $X(x,\xi)$ are scalar functions.

Representations related by d-dimensional Fourier transform:

$$S_F(p,\xi) = \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}}\, e^{\mathrm{i}px}\, S_F(x,\xi), \quad S_F(x,\xi) = \int \frac{\mathrm{d}^d p}{(2\pi)^{d/2}}\, e^{-\mathrm{i}px}\, S_F(p,\xi)\,.$$

Techniques for massless Feynman integral calculations [Kotikov & ST '19]

LKFT in dimensional regularization ($d = 4 - 2\varepsilon$)

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Techniques for massless Feynman integral calculations [Kotikov & ST '19]:

$$D(x) = -\mathrm{i}\,\Delta\, e^2\, (\mu^2 x^2)^{2-d/2} \frac{\Gamma(d/2-2)}{2^4(\pi)^{d/2}}, \qquad D(0) = 0\,,$$

because $D(0)$ is a massless tadpole (no-scale integral).

Position-space LKFT in dimensional regularization ($d = 4 - 2\varepsilon$)

$$S_F(x,\xi) = S_F(x,\eta)\, e^{\mathrm{i}D(x)}$$

$$D(x) = \frac{\mathrm{i}\,\Delta\, A}{\varepsilon}\,\Gamma(1-\varepsilon)\,(\pi\mu^2 x^2)^{\varepsilon}, \qquad A = \frac{\alpha_{\mathrm{em}}}{4\pi} = \frac{e^2}{(4\pi)^2}$$

Let in some gauge η ($\eta = 0$ in the Landau gauge):

$$P(p,\eta) = \sum_{m=0}^{\infty} a_m(\eta)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon}$$

where $a_m(\eta)$ are coefficients of the loop expansion of the propagator and

$$\tilde{\mu}^2 = 4\pi\mu^2 \qquad (\overline{\mu}^2 = \tilde{\mu}^2\, e^{-\gamma}\ \overline{\text{MS}}\text{-scale})$$

the renormalization scale

Momentum-space LKFT in dimensional regularization (I)

For another gauge ξ, the fermion propagator can be expressed as:

$$P(p,\xi) = \sum_{m=0}^{\infty} a_m(\xi)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon}$$

where
$$a_m(\xi) = a_m(\eta)\, \frac{\Gamma(2-(m+1)\varepsilon)}{\Gamma(1+m\varepsilon)} \times$$
$$\times \sum_{l=0}^{\infty} \frac{\Gamma(1+(m+l)\varepsilon)\,\Gamma^l(1-\varepsilon)}{l!\,\Gamma(2-(m+l+1)\varepsilon)}\, \frac{(\Delta\, A)^l}{(-\varepsilon)^l} \left(\frac{\tilde{\mu}^2}{p^2}\right)^{l\varepsilon}$$

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Scale fixing (appropriate choice is crucial)

We work in MS-like schemes

- Popular $\overline{\text{MS}}$-scale: subtracts Euler-γ.
- Popular G-scale [Chetyrkin, Kataev & Tkachov '80]: subtracts Euler-γ and ζ_2

We use (for uniform transcendental weight):

- minimal Vladimirov-scale (MV-scale): new scale based on old calculations of [Vladimirov '79] (it has been used once in [Kataev & Vardiashvili '88])

$$\mu_{\text{MV}}^{2\varepsilon} = \frac{\tilde{\mu}^{2\varepsilon}}{\Gamma(1-\varepsilon)}$$

The MV-scale is the most efficient for our calculations

- g-scale [Broadhurst '99] (small variant of G-scale)

$$\mu_g^{2\varepsilon} = \tilde{\mu}^{2\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

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Momentum-space LKFT in dimensional regularization (II)

In both the MV-scale and g-scales ($p =$MV$, g$):

$$a_m(\xi) = a_m(\eta) \sum_{l=0}^{\infty} \frac{1-(m+1)\varepsilon}{1-(m+l+1)\varepsilon} \Phi_p(m,l,\varepsilon) \frac{(\Delta A)^l}{(-\varepsilon)^l l!} \left(\frac{\mu_p^2}{p^2}\right)^{l\varepsilon}$$

$$\Phi_{MV}(m,l,\varepsilon) = \frac{\Gamma(1-(m+1)\varepsilon)\Gamma(1+(m+l)\varepsilon)\Gamma^{2l}(1-\varepsilon)}{\Gamma(1+m\varepsilon)\Gamma(1-(m+l+1)\varepsilon)}$$

$$\Phi_g(m,l,\varepsilon) = \Phi_{MV}(m,l,\varepsilon) \frac{\Gamma^l(1-2\varepsilon)}{\Gamma^{3l}(1-\varepsilon)\Gamma^l(1+\varepsilon)}$$

The Φ_p-functions can be expressed as expansions in ζ_i ($i \geq 3$) using

$$\Gamma(1+\beta\varepsilon) = \exp\left[-\gamma\beta\varepsilon + \sum_{s=2}^{\infty} (-1)^s \eta_s \beta^s \varepsilon^s\right], \quad \eta_s = \frac{\zeta_s}{s}$$

Momentum-space LKFT in dimensional regularization (III)

In the MV-scale:

$$a_m(\xi) = a_m(\eta) \sum_{l=0}^{\infty} \frac{1-(m+1)\varepsilon}{1-(m+l+1)\varepsilon} \, \Phi_{\mathrm{MV}}(m,l,\varepsilon) \, \frac{(\Delta\, A)^l}{(-\varepsilon)^l \, l!} \left(\frac{\mu^2_{\mathrm{MV}}}{p^2} \right)^{l\varepsilon}$$

$$\Phi_{\mathrm{MV}}(m,l,\varepsilon) = \exp\left[\sum_{s=3}^{\infty} \eta_s \, p_s(m,l) \, \varepsilon^s \right] \qquad \eta_s = \frac{\zeta_s}{s}$$

$$p_s(m,l) = \ (m+1)^s - (m+l+1)^s + 2l + (-1)^s \Big\{ (m+l)^s - m^s \Big\}$$

$$p_1(m,l) = 0, \qquad p_2(m,l) = 0$$

Notice: $\Phi_{\mathrm{MV}}(m,l,\varepsilon)$ contains ζ_s-function values of a given weight (or transcendental level) s in factor of ε^s.

property of uniform transcendentality [Kotikov & Lipatov '00]

For applications see, *e.g.*, [Kotikov & Lipatov '02; Fleischer et al. '98; Kotikov et al. '07; Bajnok et al. '09; Lukowski et al. '10; Marboe et al. '15; Dixon et al. '19; Broedel et al. '19]

Polynomials $p_s(m,l)$

$$p_s(m,l) = (m+1)^s - (m+l+1)^s + 2l + (-1)^s\Big\{(m+l)^s - m^s\Big\}$$

$$p_1(m,l) = 0, \qquad p_2(m,l) = 0$$

Conveniently separated in even and odd s values. Recursion relations:

$$p_{2k} = p_{2k-1} + Lp_{2k-2} + p_3, \qquad L = l(l+1)$$
$$p_{2k-1} = p_{2k-2} + Lp_{2k-3} + p_3$$

simple form holds in the MV-scheme (more complicated otherwise)

p_s takes the form of a polynomial in L in factor of p_3:

$$p_4 = 2p_3\,,$$
$$p_5 = p_4 + Lp_3 + p_3 = (3+L)p_3\,,$$
$$p_6 = p_5 + Lp_4 + p_3 = (4+3L)p_3\,,$$

Possible to eliminate L:

$$Lp_3 = p_5 - 3p_3, \quad p_6 = 3p_5 - 5p_3\,,$$

Key fact: even polynomials are entirely expressible in terms of odd ones

Generalize to arbitrary k:

$$p_{2k} = \sum_{s=2}^{k} p_{2s-1}\, C_{2k,2s-1} = \sum_{m=1}^{k-1} p_{2k-2m+1}\, C_{2k,2k-2m+1}\,,$$

where the coefficients have the following structure

$$C_{2k,2k-2m+1} = b_{2m-1}\, \frac{(2k)!}{(2m-1)!\,(2k-2m+1)!}$$

The first few values read:

$$b_1 = \frac{1}{2}, \quad b_3 = -\frac{1}{4}, \quad b_5 = \frac{1}{2}, \quad b_7 = -\frac{17}{2}, \quad b_9 = \frac{31}{2}\,,$$
$$b_{11} = -\frac{691}{4}, \quad b_{13} = \frac{5461}{2}, \quad b_{15} = -\frac{929569}{16}\,,$$
$$b_{17} = \frac{3202291}{2}, \quad b_{19} = -\frac{221930581}{4}\,,$$
$$b_{21} = \frac{4722116521}{2}, \quad b_{23} = -\frac{968383680827}{8}\,.$$

Looks like they are proportional to the numerators of Bernoulli numbers!

Generalize to arbitrary k:

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$$C_{2k,2k-2m+1} = b_{2m-1}\, \frac{(2k)!}{(2m-1)!\,(2k-2m+1)!}$$

Closer inspection reveals relation with zero values of Euler polynomials:

$$b_{2m-1} \; = -E_{2m-1}(x=0)$$

and therefore to Bernoulli, B_m, and Genocchi, G_m, numbers because

$$E_{2m-1}(x=0) = \frac{G_{2m}}{2m}, \quad G_{2m} = -\frac{(2^{2m}-1)}{m}\, B_{2m}$$

Hence:

$$b_{2m-1} = \frac{(2^{2m}-1)}{m}\, B_{2m}$$

Hatted ζ-values

At this point, we may reconsider:

$$\Phi_{\mathrm{MV}}(m,l,\varepsilon) = \exp\left[\sum_{s=3}^{\infty} \eta_s\, p_s(m,l)\,\varepsilon^s\right] \qquad \eta_s = \frac{\zeta_s}{s}$$

and perform the decomposition:

$$\sum_{s=3}^{\infty} \eta_s\, p_s\,\varepsilon^s = \sum_{k=2}^{\infty} \eta_{2k}\, p_{2k}\,\varepsilon^{2k} + \sum_{k=2}^{\infty} \eta_{2k-1}\, p_{2k-1}\,\varepsilon^{2k-1}\,.$$

Then

$$\sum_{k=2}^{\infty} \eta_{2k}\, p_{2k}\,\varepsilon^{2k} = \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k}\, C_{2k,2s-1}\,\varepsilon^{2k}\,.$$

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Then

$$\sum_{k=2}^{\infty} \eta_{2k}\, p_{2k}\, \varepsilon^{2k} = \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k}\, C_{2k,2s-1}\, \varepsilon^{2k}\,.$$

Hence:

$$\sum_{s=3}^{\infty} \eta_s\, p_s(m,l)\,\varepsilon^s = \sum_{s=2}^{\infty} \hat{\eta}_{2s-1}\, p_{2s-1}\, \varepsilon^{2s-1}$$

$$\hat{\eta}_{2s-1} = \eta_{2s-1} + \sum_{k=s}^{\infty} \eta_{2k}\, C_{2k,2s-1}\, \varepsilon^{2(k-s)+1}$$

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and perform the decomposition:

$$\sum_{s=3}^{\infty} \eta_s\, p_s\, \varepsilon^s = \sum_{k=2}^{\infty} \eta_{2k}\, p_{2k}\, \varepsilon^{2k} + \sum_{k=2}^{\infty} \eta_{2k-1}\, p_{2k-1}\, \varepsilon^{2k-1}\,.$$

Then

$$\sum_{k=2}^{\infty} \eta_{2k}\, p_{2k}\, \varepsilon^{2k} = \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k}\, C_{2k,2s-1}\, \varepsilon^{2k}\,.$$

Hence:

$$\sum_{s=3}^{\infty} \eta_s\, p_s(m,l)\,\varepsilon^s = \sum_{s=2}^{\infty} \hat{\eta}_{2s-1}\, p_{2s-1}\, \varepsilon^{2s-1}$$

$$\hat{\eta}_{2s-1} = \eta_{2s-1} + \sum_{k=s}^{\infty} \eta_{2k}\, C_{2k,2s-1}\, \varepsilon^{2(k-s)+1}$$

Hatted ζ-values

"no-π theorem" [Broadhurst '99; Baikov & Chetyrkin '18]

Final exact analytical expression [Kotikov & ST '19]:

$$\Phi_{\rm MV}(m,l,\varepsilon) = \exp\left[\sum_{s=2}^{\infty} \frac{\hat{\zeta}_{2s-1}}{2s-1}\, p_{2s-1}\, \varepsilon^{2s-1}\right]$$

$$\hat{\zeta}_{2s-1} = \zeta_{2s-1} + \sum_{k=s}^{\infty} \zeta_{2k}\, \hat{C}_{2k,2s-1}\, \varepsilon^{2(k-s)+1}$$

$$\hat{C}_{2k,2s-1} = \frac{2s-1}{2k}\, C_{2k,2s-1} = b_{2k-2s+1}\, \frac{(2k-1)!}{(2s-2)!\,(2k-2s+1)!}$$

$$b_{2m-1} = \frac{(2^{2m}-1)}{m}\, B_{2m}$$

Note 1: identical hatted ζ-values found for the g-scale [Kotikov & ST '19]

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View of first terms generated by our exact result

In blue: terms known from [Baikov & Chetyrkin '18] (up to 6 loops).
In red: terms known from [Baikov & Chetyrkin '19] (7 loop).

$$\hat{\zeta}_3 = \zeta_3 + \frac{3\,\varepsilon}{2}\,\zeta_4 - \frac{5\,\varepsilon^3}{2}\,\zeta_6 + \frac{21\,\varepsilon^5}{2}\,\zeta_8 - \frac{153\,\varepsilon^7}{2}\,\zeta_{10} + \frac{1705\,\varepsilon^9}{2}\,\zeta_{12} - \frac{26949\,\varepsilon^{11}}{2}\,\zeta_{14} + \frac{573405\,\varepsilon^{13}}{2}\,\zeta_{16} - \frac{15802673\,\varepsilon^{15}}{2}\,\zeta_{18} + \frac{547591761\,\varepsilon^{17}}{2}\,\zeta_{20} - \frac{23302711005\,\varepsilon^{19}}{2}\,\zeta_{22} + \frac{1194695479813\,\varepsilon^{21}}{2}\,\zeta_{24} + \cdots$$

$$\hat{\zeta}_5 = \zeta_5 + \frac{5\,\varepsilon}{2}\,\zeta_6 - \frac{35\,\varepsilon^3}{4}\,\zeta_8 + 63\,\varepsilon^5\,\zeta_{10} - \frac{2805\,\varepsilon^7}{4}\,\zeta_{12} + \frac{22165\,\varepsilon^9}{2}\,\zeta_{14} - \frac{943215\,\varepsilon^{11}}{4}\,\zeta_{16} + 6498590\,\varepsilon^{13}\,\zeta_{18} - \frac{900752361\,\varepsilon^{15}}{4}\,\zeta_{20} + \frac{19165711635\,\varepsilon^{17}}{2}\,\zeta_{22} - \frac{1965195294755\,\varepsilon^{19}}{4}\,\zeta_{24} + 29867386995325\,\varepsilon^{21}\,\zeta_{26} + \cdots$$

$$\hat{\zeta}_7 = \zeta_7 + \frac{7\,\varepsilon}{2}\,\zeta_8 - 21\,\varepsilon^3\,\zeta_{10} + 231\,\varepsilon^5\,\zeta_{12} - \frac{7293\,\varepsilon^7}{2}\,\zeta_{14} + \frac{155155\,\varepsilon^9}{2}\,\zeta_{16} - 2137954\,\varepsilon^{11}\,\zeta_{18} + 74083926\,\varepsilon^{13}\,\zeta_{20} + \cdots$$

$$\hat{\zeta}_9 = \zeta_9 + \frac{9\,\varepsilon}{2}\,\zeta_{10} - \frac{165\,\varepsilon^3}{4}\,\zeta_{12} + \frac{1287\,\varepsilon^5}{2}\,\zeta_{14} - \frac{109395\,\varepsilon^7}{8}\,\zeta_{16} + 376805\,\varepsilon^9\,\zeta_{18} - \frac{26113581\,\varepsilon^{11}}{2}\,\zeta_{20} + \cdots$$

$$\hat{\zeta}_{11} = \zeta_{11} + \frac{11\,\varepsilon}{2}\,\zeta_{12} - \frac{143\,\varepsilon^3}{2}\,\zeta_{14} + \frac{3003\,\varepsilon^5}{2}\,\zeta_{16} - 41327\,\varepsilon^7\,\zeta_{18} + 1431859\,\varepsilon^9\,\zeta_{20} - 60931689\,\varepsilon^{11}\,\zeta_{22} + \cdots$$

$$\hat{\zeta}_{13} = \zeta_{13} + \frac{13\,\varepsilon}{2}\,\zeta_{14} - \frac{455\,\varepsilon^3}{4}\,\zeta_{16} + 3094\,\varepsilon^5\,\zeta_{18} - \frac{214149\,\varepsilon^7}{2}\,\zeta_{20} + 4555915\,\varepsilon^9\,\zeta_{22} - \frac{467142949\,\varepsilon^{11}}{2}\,\zeta_{24} + \cdots$$

$$\hat{\zeta}_{15} = \zeta_{15} + \frac{15\,\varepsilon}{2}\,\zeta_{16} - 170\,\varepsilon^3\,\zeta_{18} + 5814\,\varepsilon^5\,\zeta_{20} - 247095\,\varepsilon^7\,\zeta_{22} + 12666445\,\varepsilon^9\,\zeta_{24} - 770015850\,\varepsilon^{11}\,\zeta_{26} + \cdots$$

Outline

Conclusion

1. our work clarifies the ζ-structure of gauge field theories (proof of "no-π theorem", exact formula valid to all orders of perturbation theory),
2. provides stringent constraints (allowing important checks) on present and future multi-loop results.
3. minimal Vladimirov-scheme should prove itself of very convenient use in multi-loop calculations.

Multi-$\hat{\zeta}$ values: open issue!

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