New results for the critical fermion flavour number of three-dimensional QED

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Based on works with Anatoly V. Kotikov and Vadim I. Shilin

Kotikov, Shilin and ST, PRD **94** (2016) 056009 [arXiv:1602.01962 [hep-th]]
Kotikov and ST, PRD **94** (2016) no.11, 114011 [arXiv:1605.01911 [hep-th]]

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Outline

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Extensive studies for more that three decades now:

- **Original interest:** [Pisarski '84; Appelquist et al. '84]
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"Chiral" (flavour) symmetry in QED_3

Massless QED_3 with N flavours of 4-component fermions

$$L = \overline{\Psi}_\sigma (i\hat{\partial} - e\hat{A})\Psi^\sigma - \frac{1}{4}F_{\mu\nu}^2 \qquad (\sigma = 1, \cdots, N)$$

Global $U(2N)$ "chiral" (flavour) symmetry

Only possible with 4-component spinors because in this case it is possible to add γ^3 and γ^5 anticommuting with γ^0, γ^1 and γ^2

Parity-even mass term breaks $U(2N) \to U(N) \times U(N)$

$$\overline{\Psi}_\sigma \Psi^\sigma = \bar{\chi}_\sigma \chi^\sigma - \bar{\chi}_{N+\sigma} \chi^{N+\sigma}$$

where χ_i is a 2-component spinor $(i = 1, \cdots, 2N)$

Question (parity-odd mass term neglected): [Pisarski '84]

Is it possible that "chiral" symmetry is dynamically broken in QED_3?

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Some properties of the model

QED_3 is super-renormalizable

- dimensionful coupling constant $a = Ne^2/8$
- loop-expansion plagued by IR singularities (starting from two-loop) [Jackiw & Templeton '81] [Guendelman & Radulovic '83, '84]

Large$-N$ limit of QED_3 ($N \to \infty$ and a fixed): **IR softening**

[Appelquist & Pisarski '81, Appelquist & Heinz '81]

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu}}{p^2\,[1+\Pi(p)]} = \frac{g_{\mu\nu}}{p^2\,[1+a/|p|]} \xrightarrow{p \ll a} \frac{g_{\mu\nu}}{a|p|}$$

- Gauge propagator $\sim 1/p$ and dimensionless coupling $\sim 1/\sqrt{N}$ ($\sim 1/p$ similar to reduced QED [ST '12, Kotikov & ST '13])
- Power counting in large-N QED_3 similar to 4-dimensional theories
  - model becomes IR finite
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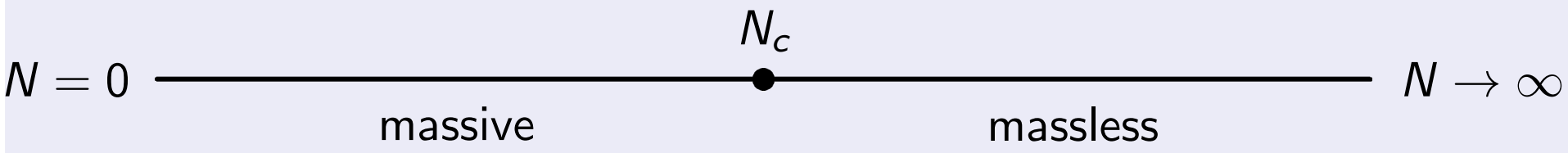
Dynamical chiral symmetry breaking in QED_3

- should take place at momentum scales $p \ll a$
  (breaks scale invariance)
- cannot take place at any finite order in $1/N$
  (requires a non-perturbative approach)
- may take place below some critical fermion flavour number N_c



Challenge: determine the value of N_c

Value of N_c crucial to understand the phase structure of QED_3
(important consequences for particle/condensed matter physics)

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  - Method: solves Schwinger-Dyson (SD) gap equation at leading order (LO) of $1/N$-expansion
  - Support: RG study [Pisarski '91], further SD studies [Pennington et al. '91, '92], lattice simulation [Azcoiti '93, '96]
- [Appelquist et al. '88] DχSB for $N < N_c = 32/\pi^2 = 3.24$ (Landau gauge)
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In 30 years: very different results obtained for N_c!

N_c	**Method**	**Year**
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< 4.4	F-theorem	2015
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Table: Values of N_c obtained over the years with different methods (at LO).

Beyond leading order

All these very different results reflect a poor understanding of the problem

[Pisarski '91]: "difficult to presume we understand χSB in QCD_4 when we do not fully understand flavour-symmetry breaking in QED_3"

Important question: stability of the critical point

Consider the approach of [Appelquist et al. '88] (LO in $1/N$-expansion):

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First and most well-known study: [Nash '89]

- non-local ξ-gauge
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$$\text{LO} + \text{resummation:}\ N_c = (4/3)(32/\pi^2) = 4.32$$

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- attempt to compute NLO corrections
  - approximate calculation of diagrams
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Fermion propagator ($\Sigma(p)$: dynamically generated (parity-conserving) mass, $A(p)$: fermion wave function):

$$S^{-1}(p) = [1 + A(p)]\ (\mathrm{i}\hat{p} + \Sigma(p))$$

Photon propagator (non-local ξ-gauge, $\xi = 0$: Landau gauge):

$$D_{\mu\nu}(p) = \frac{P^{\xi}_{\mu\nu}(p)}{p^2\,[1 + \Pi(p)]}, \quad P^{\xi}_{\mu\nu}(p) = g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\ ,$$

SD equation for the fermion propagator ($\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)]$):

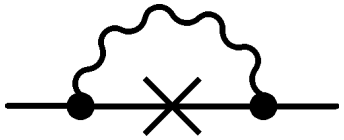
$$\tilde{\Sigma}(p) = \frac{2a}{N}\,\mathrm{Tr}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\gamma^\mu\,D_{\mu\nu}(p-k)\,\Sigma(k)\,\Gamma^\nu(p,k)}{[1 + A(k)]\ (k^2 + \Sigma^2(k))}$$

$$A(p)p^2 = -\frac{2a}{N}\,\mathrm{Tr}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{D_{\mu\nu}(p-k)\,\hat{p}\,\gamma^\mu\,\hat{k}\,\Gamma^\nu(p,k)}{[1 + A(k)]\ (k^2 + \Sigma^2(k))}$$

$\Gamma^\nu(p,k)$: vertex function.

At leading order ($a = Ne^2/8$): $A(p) = 0, \quad \Pi(p) = \frac{a}{|p|}, \quad \Gamma^\nu(p,k) = \gamma^\nu$

A single diagram contributes to the gap equation (cross = mass insertion):

$\Sigma(p) =$  $= \dfrac{8(2+\xi)a}{N} \displaystyle\int \dfrac{[\mathrm{d}^3 k] \;\; \Sigma(k)}{(k^2 + \Sigma^2(k)) \left[(p-k)^2 + a\,|p-k|\right]}$

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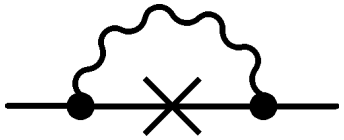
$\Sigma(p) =$  $= \dfrac{8(2+\xi)a}{N} \displaystyle\int \frac{[\mathrm{d}^3 k] \;\; \Sigma(k)}{(k^2 + \Sigma^2(k)) \left[(p-k)^2 + a\,|p-k|\right]}$

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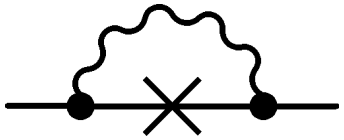
$\Sigma(p) =$  $= \dfrac{8(2+\xi)a}{N} \displaystyle\int \frac{[\mathrm{d}^3 k] \;\; \Sigma(k)}{(k^2 + \Sigma^2(k)) \left[(p-k)^2 + a\,|p-k|\right]}$

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Outline

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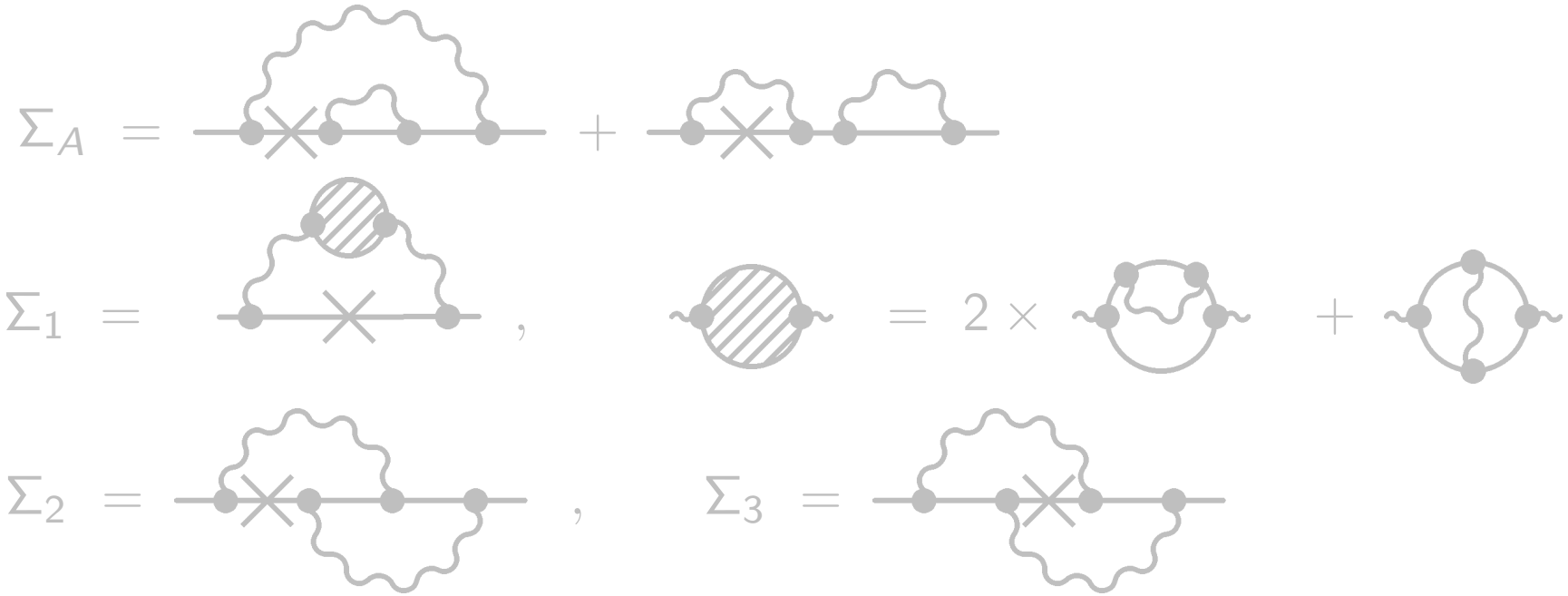

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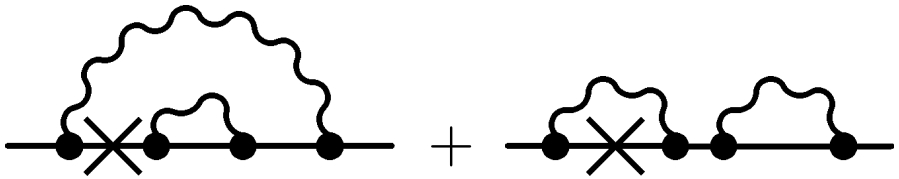
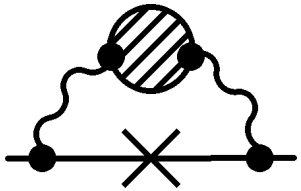
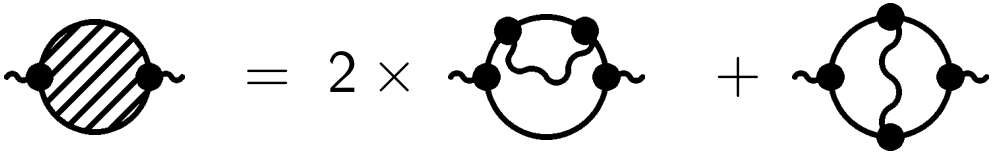
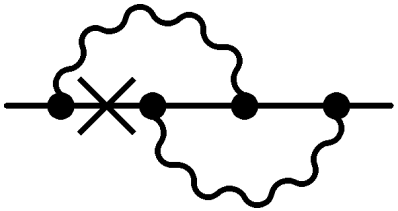
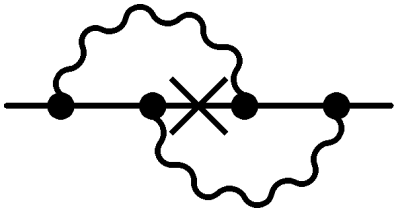
$\Sigma_A =$ 

$\Sigma_1 =$  , 

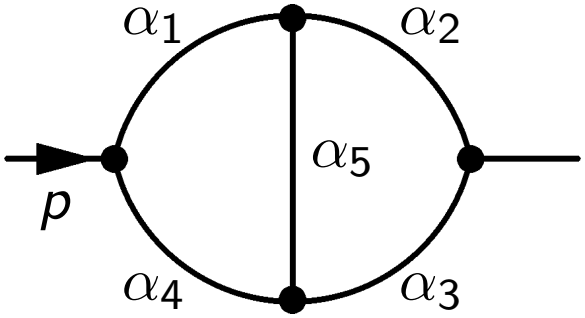
$\Sigma_2 =$  , $\Sigma_3 =$ 

Massless propagator type 2-loop diagram

Basic building block of multi-loop calculations ($[\mathrm{d}^D k] = \mathrm{d}^D k/(2\pi)^D$):

$$J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \int\int \frac{[\mathrm{d}^D k_1]\,[\mathrm{d}^D k_2]}{k_1^{2\alpha_1}\, k_2^{2\alpha_2}\,(k_2 - p)^{2\alpha_3}\,(k_1 - p)^{2\alpha_4}\,(k_2 - k_1)^{2\alpha_5}}$$

Arbitrary indices α_i and external momentum p in Euclidean space (D)



Coefficient function (dimensionless as our Σ_i):

$$G(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \frac{(4\pi)^D}{(p^2)^{D - \sum_{i=1}^5 \alpha_i}}\, J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

Goal of multi-loop computation:

in $D = n - 2\varepsilon$ ($n = 3$), compute $G(\{\alpha_i\})$ as a Laurent series in $\varepsilon \to 0$

Long history: exact computations are crucial

- all indices integers: well-known and easy to compute, *e.g.* IBP [Vasil'ev, Pismak & Khonkonen '81] [Tkachov '81] [Chetyrkin & Tkachov '81]

- $\alpha_i = n_i + a_i\varepsilon$ $(\forall i)$: already non-trivial [Kazakov '83, '84, '85] [Broadhurst '86] Automated in [Bierenbaum & Weinzierl '03] ($D = 4$: MZV)

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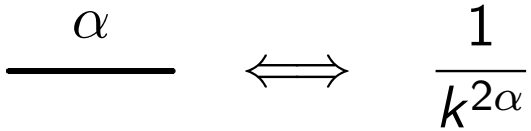

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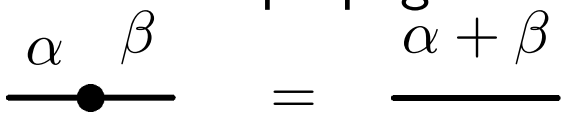

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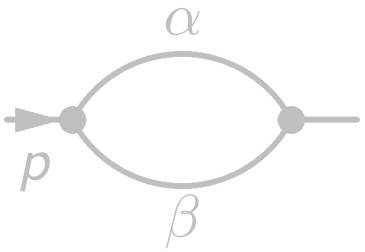

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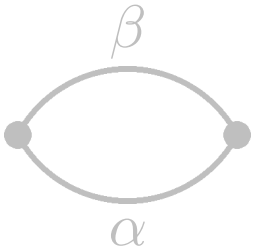

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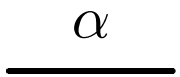

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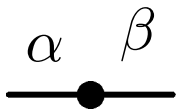

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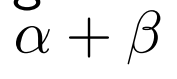

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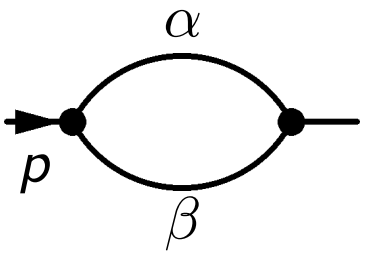

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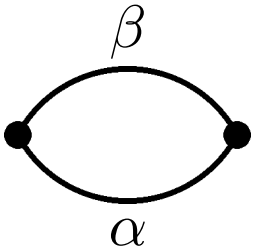

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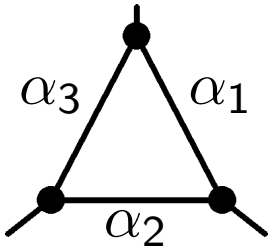
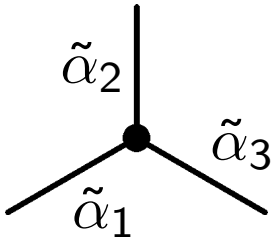
- Uniqueness relation ($\tilde{\alpha} = D/2 - \alpha$): [Polyakov '70] [D'Eramo, Parisi & Peliti '71] [Vasil'ev, Pismak & Khonkonen '81] [Usyukina '83] [Kazakov '83]



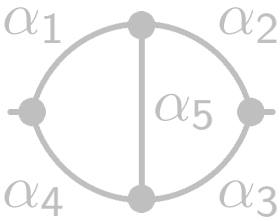
$$\underset{\sum_i \alpha_i = D}{=} \frac{G(\alpha_1, \alpha_2)}{(4\pi)^{D/2}}$$



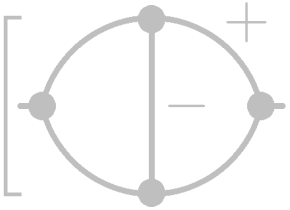
(Note: unique triangle has index $\sum_i \alpha_i = D$)

- IBP relation: [Vasil'ev, Pismak & Khonkonen '81] [Tkachov '81] [Chetyrkin & Tkachov '81]

$(D - \alpha_2 - \alpha_3 - 2\alpha_5)$



$= \alpha_2 \Big[$



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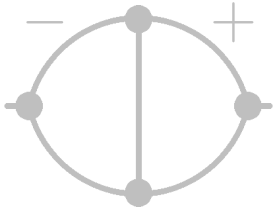
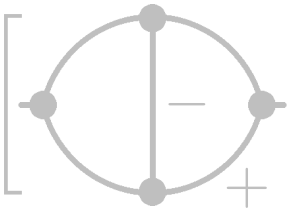


$\Big]$

$+ \alpha_3 \Big[$



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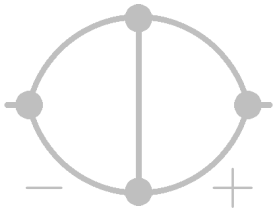
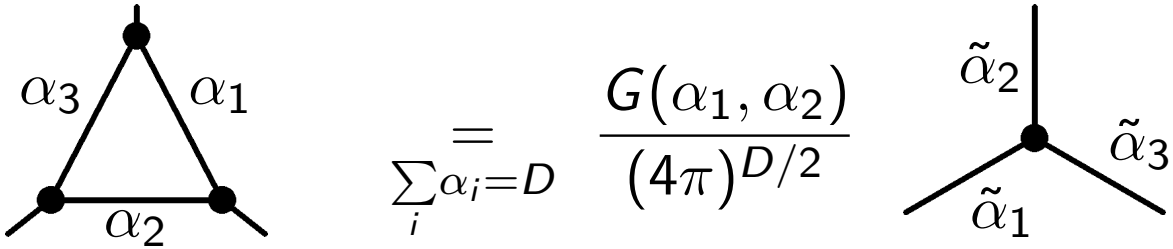


$\Big]$

(Note: $\pm$ corresponds to add or subtract 1 to index α_i)

- Uniqueness relation ($\tilde{\alpha} = D/2 - \alpha$): [Polyakov '70] [D'Eramo, Parisi & Peliti '71] [Vasil'ev, Pismak & Khonkonen '81] [Usyukina '83] [Kazakov '83]



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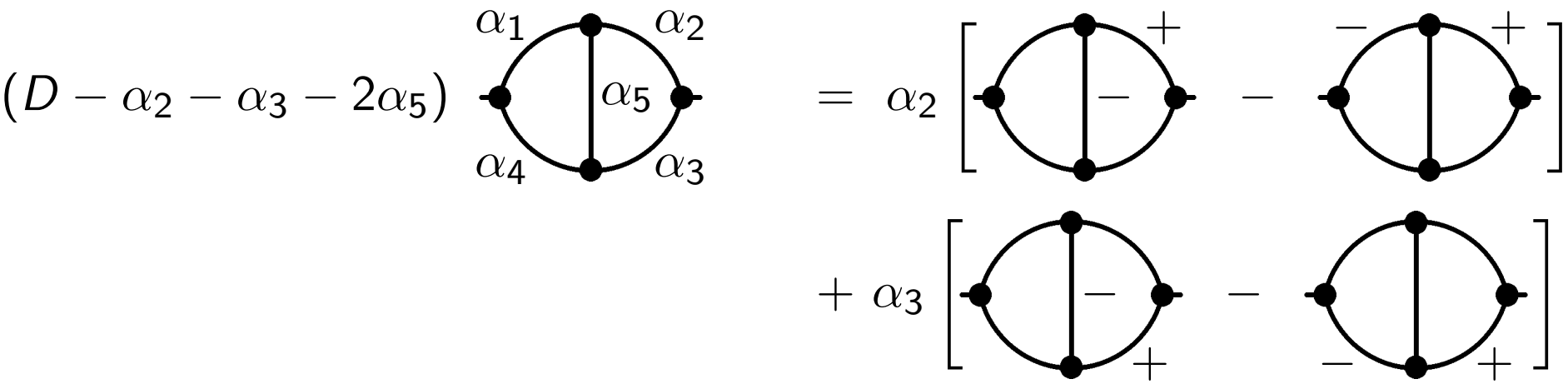
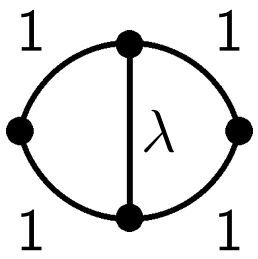


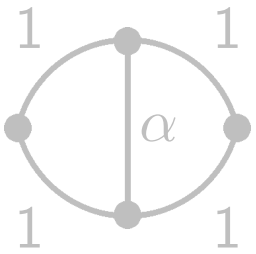
(Note: $\pm$ corresponds to add or subtract 1 to index α_i)

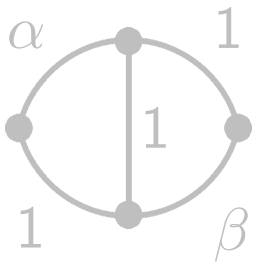
Example: this allows to compute [Vasil'ev et al. '81] [Vasil'ev et al. '93] [Kivel et al. '94] [Kotikov & ST '13] ($\lambda = D/2 - 1$, $\lambda = 1/2$ in $D = 3$)

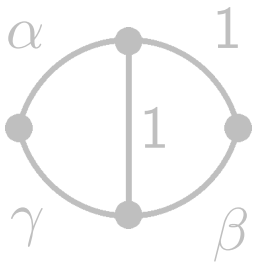
$G(1,1,1,1,\lambda) \quad =$ 

$= 3\,\dfrac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)}\Big[\Psi'(\lambda) - \Psi'(1)\Big]$

But the following diagrams are beyond IBP and uniqueness:







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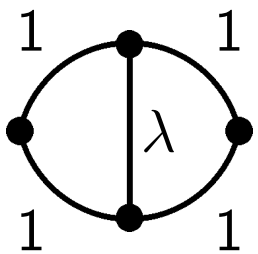
$G(1,1,1,1,\lambda) \quad =$ 

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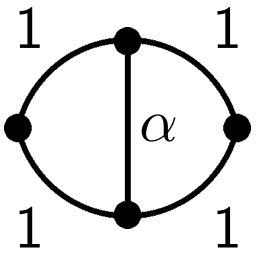
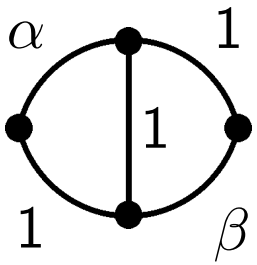
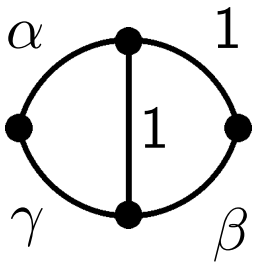
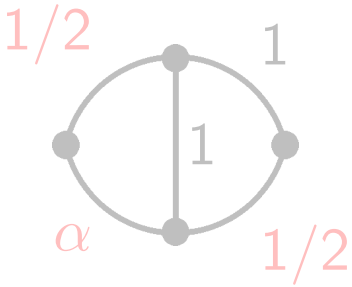
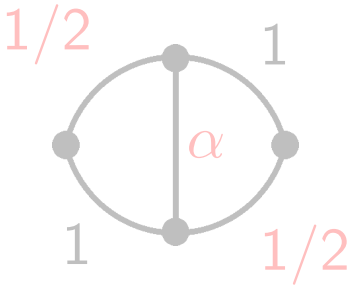






... and our calculations require the evaluations of masters such as:

$\tilde{I}_1(\alpha) \quad =$ 

$\tilde{I}_2(\alpha) \quad =$ 

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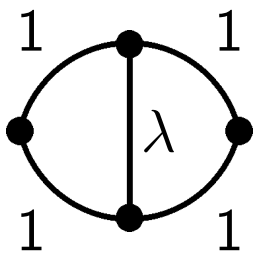
$G(1,1,1,1,\lambda) \quad =$ 

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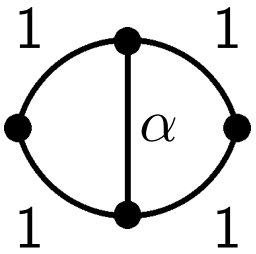

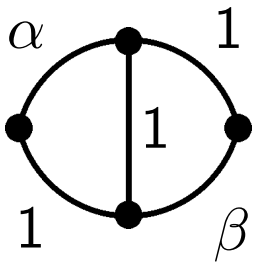

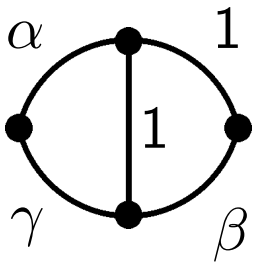

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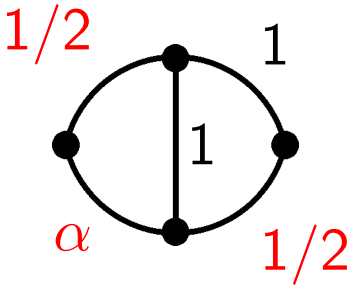
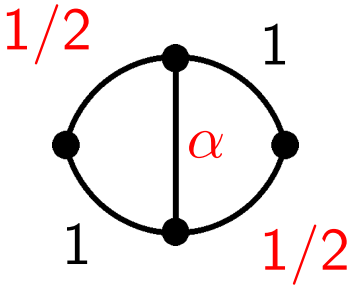
$\tilde{I}_1(\alpha) \quad =$ 

$\tilde{I}_2(\alpha) \quad =$ 

... for which no exact solution is still available.

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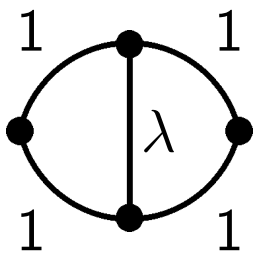
$G(1,1,1,1,\lambda) \quad =$ 

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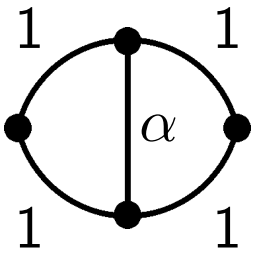

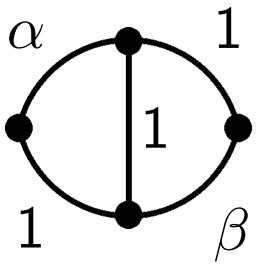

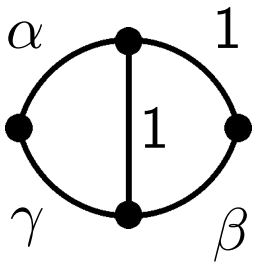

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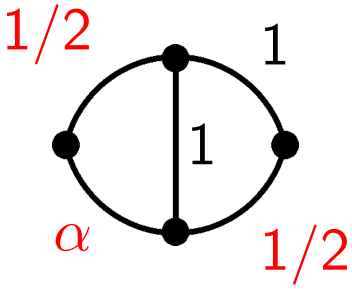
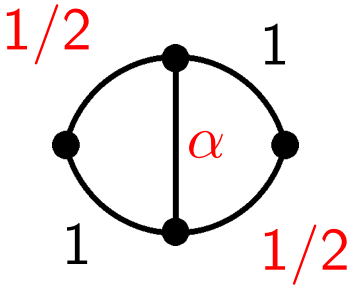
$\tilde{I}_1(\alpha) \quad =$ 

$\tilde{I}_2(\alpha) \quad =$ 

... for which no exact solution is still available.

Technicalities (2)

The Gegenbauer polynomial x-space technique
[Chetyrkin, Kataev & Tkachov '80] [Kotikov '96]

Gegenbauer polynomial C_n^β of degree n and index β defined as:

$$\frac{1}{(1-2xw+w^2)^\beta} = \sum_{k=0}^{\infty} C_k^\beta(x)\, w^k \qquad C_n^\beta(1) = \frac{\Gamma(n+2\beta)}{\Gamma(2\beta)\, n!}$$

Orthogonality relation on the unit D-dimensional sphere ($\hat{x} = x/\sqrt{x^2}$):

$$\frac{1}{\Omega_D} \int \mathrm{d}_D\, \hat{x}\, C_n^\lambda(\hat{z}\cdot\hat{x}) C_m^\lambda(\hat{x}\cdot\hat{z}) = \delta_{n,m}\, \frac{\lambda\, \Gamma(n+2\lambda)}{\Gamma(2\lambda)\,(n+\lambda)\, n!}, \qquad \lambda = \frac{D}{2} - 1\,,$$

$$\mathrm{d}^D x = \frac{1}{2}\, x^{2\lambda}\, \mathrm{d}x^2\, \mathrm{d}_D \hat{x} \qquad \Omega_D = 2\pi^{D/2}/\Gamma(D/2)$$

They allow to generalize the multi-pole expansion to arbitrary dimension D

For a propagator with arbitrary power β ($\Theta(x) \equiv$ Heaviside (step) function):

$$\frac{1}{(x_1 - x_2)^{2\beta}} = \sum_{n=0}^{\infty} C_n^{\beta}(\hat{x}_1 \cdot \hat{x}_2) \left[\frac{(x_1^2)^{n/2}}{(x_2^2)^{n/2+\beta}} \Theta(x_2^2 - x_1^2) + (x_1^2 \longleftrightarrow x_2^2) \right],$$

where:

$$C_n^{\delta}(x) = \sum_{k=0}^{[n/2]} C_{n-2k}^{\lambda}(x) \frac{(n-2k+\lambda)\Gamma(\lambda)}{k!\,\Gamma(\delta)} \frac{\Gamma(n+\delta-k)\Gamma(k+\delta-\lambda)}{\Gamma(n-k+\lambda+1)\Gamma(\delta-\lambda)}$$

Rules for integrating diagrams with Heaviside functions [Kotikov '96]

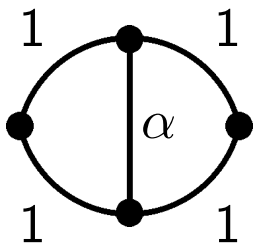
$$\int \frac{\mathrm{d}^D x}{x^{2\alpha}(x-y)^{2\beta}} \Theta(x^2 - y^2) = \frac{\pi^{D/2}}{(y^2)^{\alpha+\beta-\lambda-1}} \sum_{m=0}^{\infty} \frac{B(m,n|\beta,\lambda)}{m+\alpha+\beta-1-\lambda}$$

$$\overset{(\beta=\lambda)}{=} \frac{\pi^{D/2}}{(y^2)^{\alpha-1}} \frac{1}{\Gamma(\lambda)} \frac{1}{(\alpha-1)(n+\lambda)} \qquad \text{(as one example)}$$

$$B(m,n|\beta,\lambda) = \frac{\Gamma(m+n+\beta)}{m!\Gamma(m+n+\lambda+1)\Gamma(\beta)} \frac{\Gamma(m+\beta-\lambda)}{\Gamma(\beta-\lambda)}.$$

With these rules, one-fold series ($_3F_2$-hypergeometric function of argument 1) obtained [Kotikov '96]:

$$G(1,1,1,1,\alpha) \quad = \quad$$



$$= -2\,\Gamma(\lambda)\Gamma(\lambda-\alpha)\Gamma(1-2\lambda+\alpha)\times$$

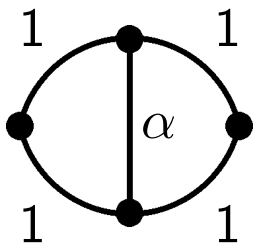
$$\times\left[\frac{\Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(3\lambda-\alpha-1)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)\Gamma(n+1)}{n!\,\Gamma(n+1+\alpha)}\,\frac{1}{n+1-\lambda+\alpha}+\frac{\pi\cot\pi(2\lambda-\alpha)}{\Gamma(2\lambda)}\right]$$

One-fold series (two $_3F_2$-hypergeometric functions of argument -1) obtained earlier by [Kazakov '84] (using functional relations):

$$G(1,1,1,1,\alpha) = -2\,\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)\Gamma(-\varepsilon-\alpha)\Gamma(\alpha+2\varepsilon)}{\Gamma(2-2\varepsilon)}\left[\frac{1}{\Gamma(1+\alpha)\Gamma(1-3\varepsilon-\alpha)}\right.$$

$$\left.\times\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)}\left(\frac{1}{n+\alpha+\varepsilon}+\frac{1}{n-\alpha-2\varepsilon}\right)+\cos[\pi\varepsilon]\right]$$

With these rules, one-fold series ($_3F_2$-hypergeometric function of argument 1) obtained [Kotikov '96]:

$G(1,1,1,1,\alpha) \quad =$ 

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Recently, the two results were proved to be equal [Kotikov & ST '16]

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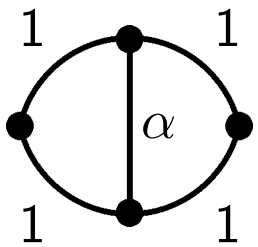
$G(1,1,1,1,\alpha) \quad =$ 

$= -2\,\Gamma(\lambda)\Gamma(\lambda-\alpha)\Gamma(1-2\lambda+\alpha)\times$

$$\times\left[\frac{\Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(3\lambda-\alpha-1)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)\Gamma(n+1)}{n!\,\Gamma(n+1+\alpha)}\,\frac{1}{n+1-\lambda+\alpha}+\frac{\pi\cot\pi(2\lambda-\alpha)}{\Gamma(2\lambda)}\right]$$

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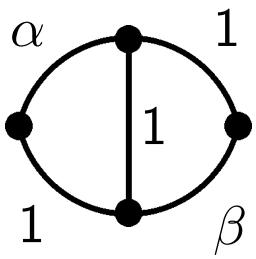
Recently, the two results were proved to be equal [Kotikov & ST '16]

In a more complicated case, one-fold series (as a combination of two ${}_3F_2$-hypergeometric functions of argument 1) were also obtained in [Kotikov & ST '14] from the rules of [Kotikov '96]:

$I(\alpha,1,\beta,1,1) \quad =$



$$= \frac{1}{\pi^D}\frac{1}{\tilde{\alpha}-1}\frac{1}{1-\tilde{\beta}} \times$$

$$\times \frac{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta})\Gamma(3-\tilde{\alpha}-\tilde{\beta})}{\Gamma(\alpha)\Gamma(\lambda-2+\tilde{\alpha}+\tilde{\beta})}\frac{\Gamma(\lambda)}{\Gamma(2\lambda)}\, I(\tilde{\alpha},\tilde{\beta})$$

where:

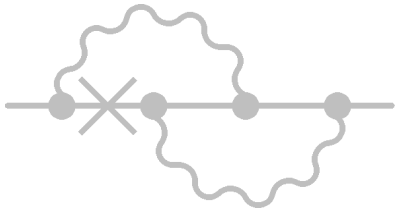
$$I(\tilde{\alpha},\tilde{\beta}) = \frac{\Gamma(1+\lambda-\tilde{\alpha})}{\Gamma(3-\tilde{\alpha}-\tilde{\beta})}\frac{\pi\sin[\pi\tilde{\alpha}]}{\sin[\pi(\lambda-1+\tilde{\beta})]\sin[\pi(\tilde{\alpha}+\tilde{\beta}+\lambda-1)]}$$

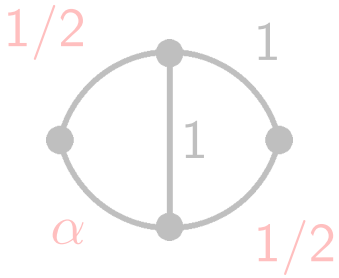
$$+\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)}{n!}\left(\frac{1}{n+\lambda+\tilde{\alpha}-1}\frac{\Gamma(n+1)}{\Gamma(n+2+\lambda-\tilde{\beta})}+\frac{1}{n+\lambda+1-\tilde{\alpha}}\times\right.$$

$$\left.\times\frac{\Gamma(n+2-\tilde{\alpha})\Gamma(2-\tilde{\beta})\Gamma(\lambda)}{\Gamma(n+3+\lambda-\tilde{\alpha}-\tilde{\beta})\Gamma(3-\tilde{\alpha}-\tilde{\beta})\Gamma(\lambda+\tilde{\alpha}-1)}\frac{\sin[\pi(\tilde{\beta}+\lambda-1)]}{\sin[\pi(\tilde{\alpha}+\tilde{\beta}+\lambda-1)]}\right)$$

Back to QED$_3$: for the most complicated integrals, the rules [Kotikov '96] yield multiple-series representations [Kotikov, Shilin & ST '16]

For Σ_2, related master integral represented in terms of a two-fold series

$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

$= \dfrac{(4\pi)^3}{(p^2)^{-\alpha}} I_1(\alpha)$

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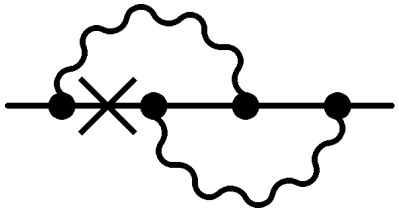
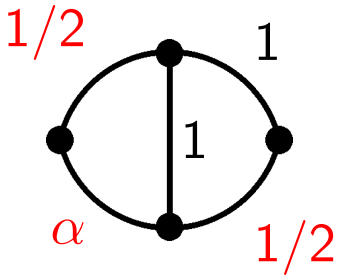
$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

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$$\times \left[\frac{2}{n+1/2} \left(\frac{1}{l+n+\alpha} + \frac{1}{l+n+3/2-\alpha} \right) + \frac{1}{(l+n+\alpha)^2} + \frac{1}{(l+n+3/2-\alpha)^2} \right]$$

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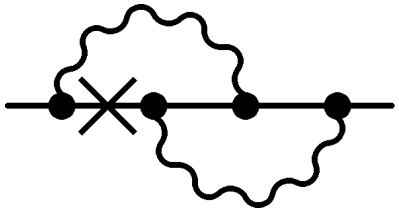
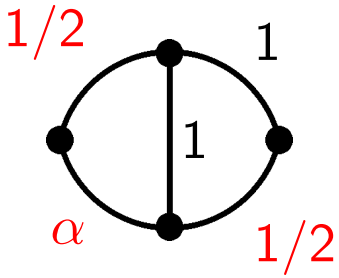
$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

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Moreover, it obeys the following functional relation:

$$\tilde{I}_1(\alpha+1) = \frac{(\alpha-1/2)^2}{\alpha^2}\tilde{I}_1(\alpha) - \frac{1}{\pi\alpha^2}\Big[\Psi'(\alpha) - \Psi'(1/2-\alpha)\Big].$$

(obtained by analogy with the ones in [Kazakov '84])

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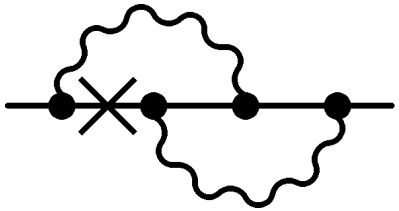
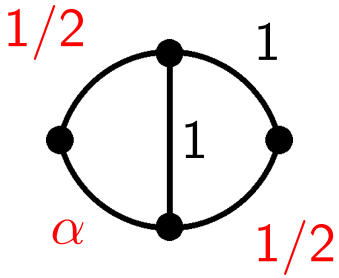
$\Sigma_2 =$ 

$\tilde{I}_1(\alpha) =$ 

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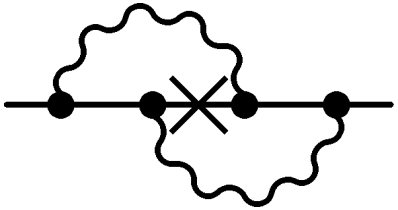
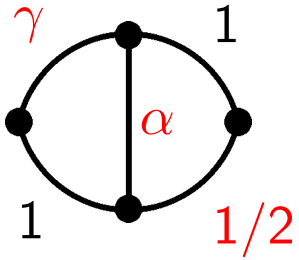
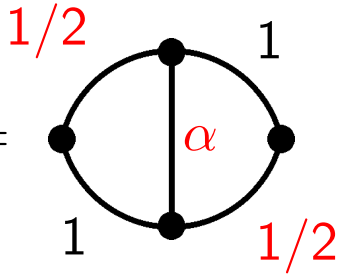
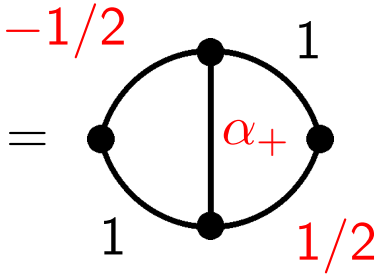
$\Sigma_3 =$ 

$\tilde{I}(\alpha,\gamma) =$ 

$= \dfrac{(4\pi)^3}{(p^2)^{-\alpha-\gamma+1/2}}\, I(\alpha,\gamma)$

$$\tilde{I}(\alpha,\gamma) = \frac{(4\pi)^3}{(p^2)^{-\alpha-\gamma+1/2}} \int \frac{[\mathrm{d}^3k_1][\mathrm{d}^3k_2]}{(p-k_1)^{2\gamma}\, k_1^{2\alpha}\, (k_1-k_2)^2\, (p-k_2)^2\, |k_2|}$$

$\tilde{I}_2(\alpha) =$ 

$\tilde{I}_3(\alpha) =$ 

$(\alpha_+ = 1+\alpha)$

Only one is independent. Functional relations:

$$\tilde{I}_2(\alpha) = \tilde{I}_2(3/2-\alpha), \quad \tilde{I}_3(\alpha) = \frac{2}{4\alpha-1}\Big(\alpha\tilde{I}_2(1+\alpha) - (1/2-\alpha)\tilde{I}_2(\alpha)\Big) - \frac{\beta^2}{\pi}$$

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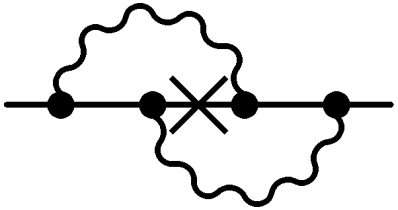
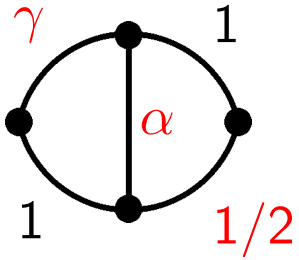
$\Sigma_3 =$ 

$\tilde{I}(\alpha,\gamma) =$ 

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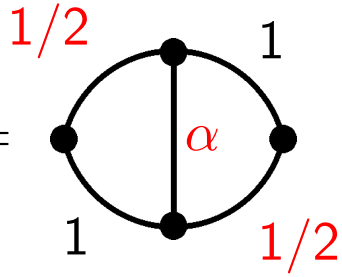
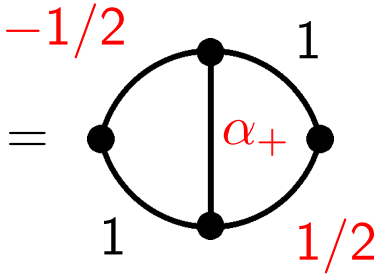
$\tilde{I}_2(\alpha) =$ 

$\tilde{I}_3(\alpha) =$ 

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Representation of $\tilde{I}_2(\alpha)$ in terms of a three-fold series

$$\tilde{I}_2(\alpha) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B(m,n,\beta,1/2)\sum_{l=0}^{\infty} B(l,n,1,1/2)\times C(n,m,l,\alpha)\,,$$

$$C(n,m,l,\alpha) = \frac{1}{(m+n+\alpha)(l+n+\alpha)} + \frac{1}{(m+n+\alpha)(l+m+n+1)}$$
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$$+\frac{1}{(n+l+\alpha)(l+m+n+\alpha)} + \frac{1}{(l+n+3/2-\alpha)(l+n+m+\alpha)}\,.$$

Back to the gap equation at NLO

$$1 = \frac{(2+\xi)\beta}{L} + \frac{\overline{\Sigma}_A(\xi) + \overline{\Sigma}_1(\xi) + 2\,\overline{\Sigma}_2(\xi) + \overline{\Sigma}_3(\xi)}{L^2}$$

All diagrams can be computed exactly

Contribution of $\overline{\Sigma}_A$ originates from LO $A(p)$ (singular):

$$\overline{\Sigma}_A(\xi) = 4\,\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\,\beta\left[\left(\frac{4}{3}(1-\xi)-\xi^2\right)\left[\frac{1}{\varepsilon}+\Psi_1-\frac{\beta}{4}\right]+\left(\frac{16}{9}-\frac{4}{9}\xi-2\xi^2\right)\right]$$

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$$\overline{\Sigma}_1(\xi) = -2(2+\xi)\,\beta\,\hat{\Pi}, \qquad \hat{\Pi} = \frac{92}{9}-\pi^2\,,$$

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$$\left.+\frac{28}{9}+\frac{8}{9}\xi-4\xi^2\right]+(1-\xi)\,\hat{\Sigma}_2\,,$$

where $\hat{\Sigma}_2$ is the "complicated" part (depending on $\tilde{I}_1(\alpha)$):

$$\hat{\Sigma}_2(\alpha) = (4\alpha-1)\beta\Big[\Psi'(\alpha)-\Psi'(1/2-\alpha)\Big]+\frac{\pi\,\tilde{I}_1(\alpha)}{2\alpha}+\frac{\pi\,\tilde{I}_1(\alpha+1)}{2(1/2-\alpha)}\,.$$

Singularities in $\overline{\Sigma}_A(\xi)$ and $\overline{\Sigma}_2(\xi)$ cancel each other and the sum is finite:

$$\overline{\Sigma}_{2A}(\xi) = \overline{\Sigma}_A(\xi)+2\overline{\Sigma}_2(\xi)\,,$$
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Finally, the contribution of $\overline{\Sigma}_3$ is finite too:

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Notice: while $\hat{\Sigma}_2$ is gauge-invariant, $\hat{\Sigma}_3$ depends on ξ

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Gap equation (1)

Combing all previous results yields an explicit form for the gap equation:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8S(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta - \left(\frac{5}{3} - \frac{26}{3}\xi + 3\xi^2\right)\beta^2 - 8\beta\left(\frac{2}{3}(1-\xi) - \xi^2\right)\right],$$

where $S(\alpha,\xi)$ contains all the "complicated" parts:

$$S(\alpha,\xi) = \left(\hat{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\hat{\Sigma}_2(\alpha)\right)/8\,.$$

At the critical point $\alpha = 1/4$ ($\beta = 16$): ($L = \pi^2 N$)

$$\boxed{L_c^2 - 16(2+\xi)L_c - 8\left[S(\xi) - 4(2+\xi)\hat{\Pi} - 16\left(4 - 50\xi/3 + 5\xi^2\right)\right] = 0}$$

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Solving the gap equation yields two standard solutions:

$$L_{c,\pm} = 8(2+\xi) \pm \sqrt{d_1(\xi)}\,,$$
$$d_1(\xi) = 8\left[S(\xi) - 8\left(4 - \frac{112}{3}\xi + 9\xi^2 + \frac{2+\xi}{2}\hat{\Pi}\right)\right].$$

To get a numerical estimate for N_c, we use the series representations to evaluate the integrals:

$$\pi\tilde{I}_1(\alpha = 1/4) \equiv R_1, \quad \pi\tilde{I}_2(\alpha = 1/4 + \mathrm{i}\delta) \equiv R_2 - \mathrm{i}P_2\delta + O(\delta^2)$$

where $\delta \to 0$ regulates an artificial singularity in $\pi\tilde{I}_3(1/4) = R_2 + P_2/4$.

With 10000 iterations for each series, the following numerical estimates are obtained:

$$R_1 = 163.7428, \quad R_2 = 209.175, \quad P_2 = 1260.720\,,$$

from which the complicated part of the self-energies can be evaluated:

$$\hat{\Sigma}_2 = 4R_1, \quad \hat{\Sigma}_3(\xi) = (\xi^2 - 1)R_2 - (7 + 16\xi - 3\xi^2)\,P_2/16\,.$$

Results: exact NLO (without resummation)

Combining these values with the one of $\hat{\Pi}$, yields:

$$L_c(\xi = 2/3) = 30.51, \qquad L_c(\xi = 0) = 32.45$$
$$N_c(\xi = 2/3) = 3.09, \qquad N_c(\xi = 0) = 3.29$$

"$-$" solutions are unphysical
and there is **no** solution in the Feynman gauge ($\xi = 1$)

The range of ξ-values for which there is a solution corresponds to:

$$-2.36 \leq \xi \leq 0.88$$

Gap equation (2)

Previous results show that LO $\sim \beta$ while NLO has $\sim \beta$ and $\sim \beta^2$ contribution:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8S(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta - \left(\frac{5}{3} - \frac{26}{3}\xi + 3\xi^2\right)\beta^2 - 8\beta\left(\frac{2}{3}(1-\xi) - \xi^2\right)\right]$$

Extracting terms $\sim \beta$ and $\sim \beta^2$ from the "complicated" part:

$$S(\alpha,\xi) = \left(\hat{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\hat{\Sigma}_2(\alpha)\right)/8\,.$$

yields another, equivalent, gap equation:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\beta^2 + 4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\right],$$

where $\tilde{S}(\alpha,\xi) = \left(\tilde{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\tilde{\Sigma}_2(\alpha)\right)/8$ is "the rest".

How "the rest" looks like:

$$\hat{\Sigma}_2(\alpha) = \beta\big(3\beta - 8\big) + \tilde{\Sigma}_2(\alpha)\,,$$
$$\tilde{\Sigma}_2 = \tilde{\Sigma}_2(\alpha = 1/4) = 4\tilde{R}_1, \qquad \tilde{R}_1 = 3.7428\,.$$

$$\overline{\Sigma}_3(\xi) = -4\xi(4+\xi)\beta + \tilde{\Sigma}_3(\alpha,\xi)\,,$$
$$\tilde{\Sigma}_3(\alpha,\xi) = \frac{1}{4}\big(1 + 8\xi + \xi^2 + 2\alpha(1-\xi^2)\big)\pi\tilde{J}_2(\alpha)$$
$$+\frac{1}{2}\big(1 + 4\xi - \alpha(1-\xi^2)\big)\pi\tilde{J}_2(1+\alpha) - \frac{1}{4}\big(-7 - 16\xi + 3\xi^2\big)\pi\tilde{J}_3(\alpha)\,,$$
$$\tilde{\Sigma}_3(\xi) = \tilde{\Sigma}_3(\alpha = 1/4, \xi) = \big(\xi^2 - 1\big)\tilde{R}_2 - \big(7 + 16\xi - 3\xi^2\big)\,\frac{\tilde{P}_2}{16}\,,$$
$$\pi\tilde{J}_2(\alpha = 1/4) = \pi\tilde{J}_2(\alpha = 5/4) = \tilde{R}_2^N = 17.175, \qquad \tilde{R}_2 = \tilde{R}_2^N - 16 = 1.175\,,$$
$$\pi\tilde{J}_3(\alpha = 1/4) = \tilde{R}_2^N + \frac{\tilde{P}_2^N}{4}, \qquad \tilde{P}_2 = \tilde{P}_2^N = -19.28\,.$$

Nash's resummation

The last form of the gap equation

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\Big[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\,\beta^2 + 4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\Big],$$

is a convenient starting point to implement a resummation of the wave-function renormalization constant [Nash '89]

Recall that ($\lambda^{(1)}$ at LO and $\lambda^{(2)}$ at NLO from [Gracey '93]):

$$\lambda_A = \frac{\lambda^{(1)}}{L} + \frac{\lambda^{(2)}}{L^2} + \cdots, \quad \lambda^{(1)} = 4\left(\frac{2}{3} - \xi\right), \quad \lambda^{(2)} = -8\left(\frac{8}{27} + \left(\frac{2}{3} - \xi\right)\hat{\Pi}\right)$$

Crucial observation: the NLO term $\sim \beta^2$ is proportional to $\lambda^{(1)}$

(in the gap equation the LO and NLO contain, respectively, the zeroth and first-order terms in λ_A)

$\Rightarrow$ resum the full expansion of λ_A at the level of the gap equation!

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For more details, the beautiful observation of [Nash '89] is that the gap equation

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\Big[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\,\beta^2 + 4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\Big],$$

can be re-written (in Appelquist form [Appelquist et al. '88]):

$$1 = \frac{4(2+\xi)}{L\Sigma(p)}\int_0^a \frac{\mathrm{d}|k|\,\Sigma(|k|)}{\mathrm{Max}(|k|,|p|)}\left\{1 + \frac{4(2-3\xi)}{3L}\ln\left[\frac{\mathrm{Max}(|k|,|p|)}{\mathrm{Min}(|k|,|p|)}\right]\right\} + \frac{\Delta(\alpha,\xi)}{L^2},$$

where $\Delta(\alpha,\xi) = 8\tilde{S}(\alpha,\xi) - 4\beta\left(\xi^2 + 4\xi + \frac{8}{3} + \frac{2+\xi}{2}\hat{\Pi}\right)$.

For resummation:

$$\int_0^a \frac{\mathrm{d}|k|\,\Sigma(|k|)}{\mathrm{Max}(|k|,|p|)}\left\{1 + \frac{\lambda^{(1)}}{L}\ln\left[\frac{\mathrm{Max}(|k|,|p|)}{\mathrm{Min}(|k|,|p|)}\right]\right\} \to \int_0^a \frac{\mathrm{d}|k|\,\Sigma(|k|)}{\mathrm{Max}(|k|,|p|)}\left[\frac{\mathrm{Max}(|k|,|p|)}{\mathrm{Min}(|k|,|p|)}\right]^{\lambda_A}$$

After resummation, the gap equation reads:

$$1 = \frac{8\beta}{3L} + \frac{\beta}{4L^2}\left(\lambda^{(2)} - 4\lambda^{(1)}\left(\frac{14}{3} + \xi\right)\right) + \frac{\Delta(\alpha,\xi)}{L^2},$$

where the LO term is now gauge independent [Nash '89].

More explicitly, our careful analysis shows that:

$$1 = \frac{8\beta}{3L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - \frac{16}{3}\beta\left(\frac{40}{9} + \hat{\Pi}\right)\right],$$

there is a strong suppression of the gauge dependence even at NLO

ξ-dependent terms do exist but they enter the gap equation only through the rest, $\tilde{S}$, which is very small numerically

At the critical point ($\alpha = 1/4$ or $\beta = 16$): ($L = \pi^2 N$)

$$\boxed{L_c^2 - \frac{128}{3}L_c - \left[8\tilde{S}(\xi) - \frac{256}{3}\left(\frac{40}{9} + \hat{\Pi}\right)\right] = 0\,.}$$

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We have again two standard solutions:

$$L_{c,\pm} = \frac{64}{3} \pm \sqrt{d_2(\xi)}, \quad d_2(\xi) = \left(\frac{64}{3}\right)^2 + \left[8\tilde{S}(\xi) - \frac{256}{3}\left(\frac{40}{9} + \hat{\Pi}\right)\right].$$

The numerical estimate for N_c, is obtained by using the values of $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{P}_2$ obtained from the series representation together with:

$$\tilde{S}(\xi = 0) = \tilde{R}_1 - \frac{\tilde{R}_2}{8} - \frac{7\tilde{P}_2}{128},$$

$$\tilde{S}(\xi = 1) = -\frac{5\tilde{P}_2}{32},$$

$$\tilde{S}(\xi = 2/3) = \frac{\tilde{R}_1}{3} - \frac{5\tilde{R}_2}{72} - \frac{49\tilde{P}_2}{384},$$

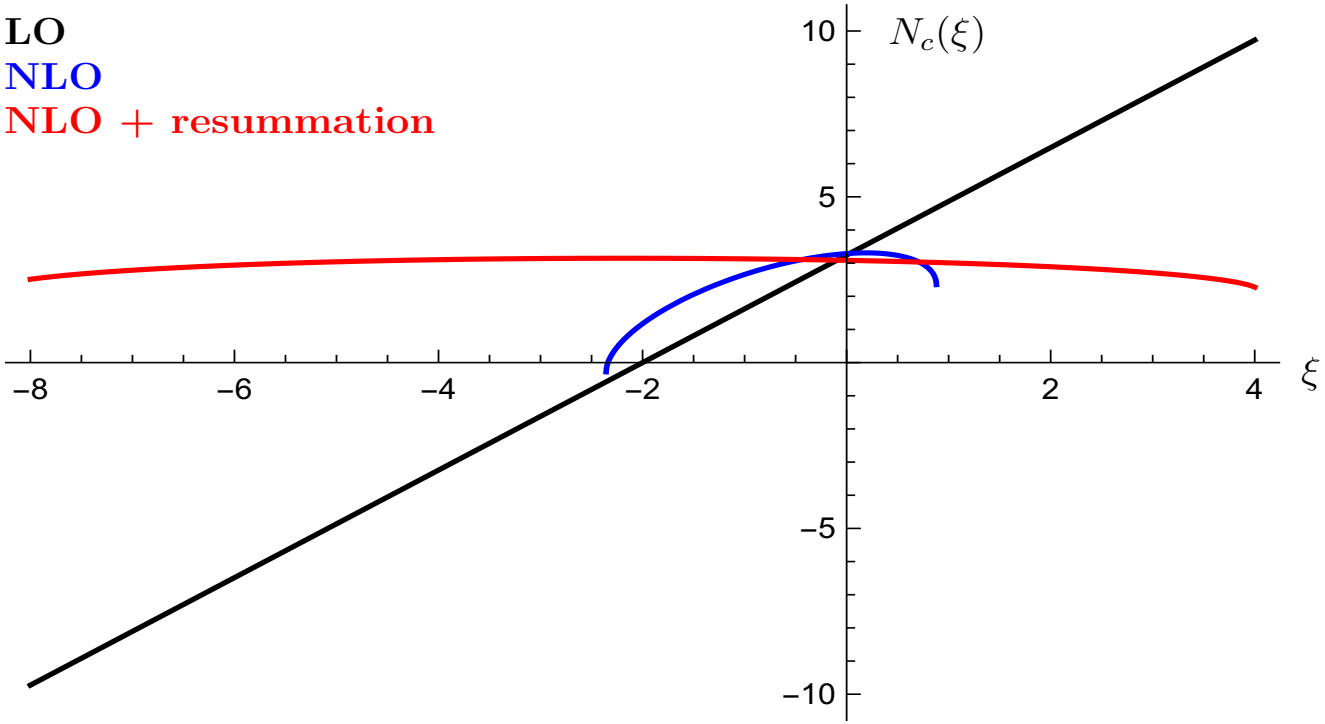
together with the value of $\hat{\Pi}$.

Results: exact NLO with Nash's resummation

$$L_c(1) = 29.69, \qquad L_c(2/3) = 29.98, \qquad L_c(0) = 30.44\,,$$
$$N_c(1) = 3.0084, \qquad N_c(2/3) = 3.0377, \qquad N_c(0) = 3.0844\,.$$

Remarks:

- "$-$" solutions unphysical but there is a solution in the Feynman gauge
- solutions exist for a broad range of values of ξ: $-8.412 \leq \xi \leq 4.042$ (consistent with the weak ξ-dependence of the gap equation)
- for $\xi = 2/3$ the value of N_c is very stable as it decreases only by 1-2% during resummation
  ($\xi = 2/3$ could be the "right(est)" gauge in agreement with [Gorbar, Gusynin & Miransky '01])
- if we neglect the rest ($\tilde{S}(\xi) = 0$) the gap equation becomes ξ-independent and we have: $\overline{L}_c = 28.0981$, $\overline{N}_c = 2.85$
  (in full agreement with recent results of [Gusynin & Pyatkovskiy '16])



Remarkable crossing in the vicinity of the Landau gauge $N_c(0) = 3.0844$

NLO and NLO+resummation cross at:

$$N_c(\xi_1 = -0.4367) = 3.1074, \qquad N_c(\xi_2 = 0.7092) = 3.0342\,.$$

NLO and NLO+resummation have a maximum at $(\mathrm{d}N_c(\xi_0)/\mathrm{d}\xi = 0)$:

$$N_c^{(NLO)}(\xi_0 = 0.1903) = 3.3095, \qquad N_c^{(NLO+R)}(\xi_0 = -2.1849) = 3.1471\,.$$

Outline

Reduced $\mathrm{QED}_{4,3}$

Fermion field in $2+1$-dimensions and photon field in $3+1$-dimensions:

$$S = \int \mathrm{d}^3x\, \bar{\psi}_\sigma \mathrm{i} \not{D} \psi^\sigma + \int \mathrm{d}^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right]$$

Boundary effective Lagrangian (in 3 dimensions): non-local

$$L = \bar{\psi}_\sigma \mathrm{i} \left(\not{\partial} + \mathrm{i}e \tilde{\not{A}} \right) \psi^\sigma - \frac{1}{4} \tilde{F}^{\mu\nu} \frac{2}{[-\Box]^{1/2}} \tilde{F}_{\mu\nu} + \frac{1}{2\tilde{\xi}} \tilde{A}^\mu \frac{2\, \partial_\mu \partial_\nu}{[-\Box]^{1/2}} \tilde{A}^\nu$$

$\tilde{\xi} = (1+\xi)/2$: gf parameter associated to reduced gauge field $\tilde{A}^\mu$

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Reduced $QED_{4,3}$ is the relativistic effective field theory describing planar Dirac-Weyl liquids at the IR Lorentz-invariant fixed point [ST '12]

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Mapping between
large-N QED_3 (coupling $\sim 1/N$) and $\mathrm{QED}_{4,3}$ (coupling $\alpha = e^2/(4\pi)$)

Origin of the mapping: photon propagators have the same form

$$D^{\mu\nu}_{\mathrm{RQED}}(p) = \frac{d^{\mu\nu}(\eta/2)}{2\,|p|}, \quad D^{\mu\nu}_{\mathrm{QED3}}(p) = \frac{d^{\mu\nu}(\tilde{\eta})}{a|p|} \quad d^{\mu\nu}(\eta) = g^{\mu\nu} - \eta\frac{p^\mu p^\nu}{p^2}$$

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Transformations (from large-N QED to reduced $\mathrm{QED}_{4,3}$)

$$\frac{1}{L} \equiv \frac{1}{\pi^2 N} \to \frac{\alpha}{4\pi} \equiv \frac{e^2}{(4\pi)^2}, \qquad \tilde{\eta} \to \frac{\eta}{2} \quad \left(\tilde{\xi} \to \frac{1+\xi}{2}\right)$$

$$\hat{\Pi}_2 = \frac{92}{9} - \pi^2 \quad \to \quad \hat{\Pi}_1 = \frac{N\pi^2}{2} \quad \text{and} \quad \tilde{\xi}\hat{\Pi}_1 = 0$$

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DχSB planar Dirac-Weyl materials (dynamical gap generation)

- Critical coupling constant α_c: DχSB for $\alpha > \alpha_c$ ($\forall N$)
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α_c (N_c)	**Method**	**Year**
7.65	SD (LO, dynamic RPA, running v)	2013
3.7	FRG, Bethe-Salpeter	2016
$3.2 < \alpha_c < 3.3$	SD (LO, dynamic RPA, running v)	2012
3.1	SD (LO, bare vertex approximation)	2015
1.62	SD (LO, static RPA)	2002
(3.52)	SD (LO)	2009
1.13 (3.6)	SD (LO, static RPA, running v)	2008
1.11 ± 0.06	Lattice simulations	2008
$1.03 < \alpha_c < 1.08$ $3.17 < N_c < 3.24$	SD (NLO, RPA, resummation, $v/c \to 1$)	2016
$0.94 < \alpha_c < 1.02$ $3.24 < N_c < 3.36$	SD (NLO, RPA, $v/c \to 1$)	2016
0.99	RG study	2012
0.92	SD (LO, dynamic RPA)	2009
0.9 ± 0.2	Lattice simulations	2012
0.833	RG study	2008

Graphene: $N = 2$ and $\alpha_g = 2.2$. But $\alpha = 1/137$ ($v/c \to 1$).

Outline

Conclusion

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- all calculations were carried out in an arbitrary non-local gauge
- Nash's resummation of the wave function renormalization at the level of the gap equation was implemented
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Definite and complete solution to NLO computation of N_c in QED_3

Final result "dangerously" close to 3... what about NNLO?

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Final result "dangerously" close to 3... what about NNLO?

Thank you for your attention

Conclusion

Study of dynamical "chiral" (flavour) symmetry breaking in QED_3 solving the Schwinger-Dyson gap equation in the large-N limit (+ extension to $QED_{4,3}$ via a new mapping)

- the LO and NLO in the $1/N$-expansion were computed exactly
- all calculations were carried out in an arbitrary non-local gauge
- Nash's resummation of the wave function renormalization at the level of the gap equation was implemented
- a strong (but not complete) suppression of the gauge dependence of N_c at NLO was proved
  - exact NLO + resummation: $3.0084 < N_c < 3.0844$
  - increasing support for the stability of the critical point
  - suggests that DχSB should take place for integer values $N \leq 3$

Definite and complete solution to NLO computation of N_c in QED_3

Final result "dangerously" close to 3... what about NNLO?

Thank you for your attention