Field theoretic study of electron-electron interaction effects in Dirac liquids

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Habilitation à Diriger des Recherches

Brief presentation

I am a condensed matter physicist working on low-dimensional electronic systems (interactions and/or disorder, phenomenology/theory)

Around 2011: study of electron-electron interaction effects in Dirac liquids

- motivations from condensed matter physics
- strong overlap with issues found in high energy physics
- field theory as a thread between low and high energy physics

First publication on the subject: ST, "Electromagnetic current correlations in reduced quantum electrodynamics," PRD (2012).

Since then: beautiful collaboration with Anatoly Kotikov (and Vadim Shilin) from JINR Dubna

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Anatoly Kotikov @ Multi-Loop-2017 (UPMC)

Outline

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Dirac liquids (systems with relativistic-like low-energy excitations)

Emblematic planar Dirac liquid: graphene [Novoselov, Geim et al. '04]

- **One of the most 2D system**
  (thickness 0.5nm = 1 atom)
- **High crystal quality in 2D**
- **Exceptional properties**
  stronger than steel and very stretchable, good conductor of electricity and heat (keep electronics cool), ...
- **Massless Dirac fermions**
  - theory: known for a long time [Wallace '47, Semenoff '84]
  - first experimental evidence from unconventional QHE ([Novoselov et al. '05], theory: [Schakel '91, Gusynin & Sharapov '05])
    $\sigma_{xy} = e^2\nu/h, \qquad \nu = \pm 4\times(n+1/2), \qquad n = 0, 1, \ldots$

Note: by now there are also artificial graphene-like systems + surface states of topological insulators + 3D Dirac and Weyl semimetals

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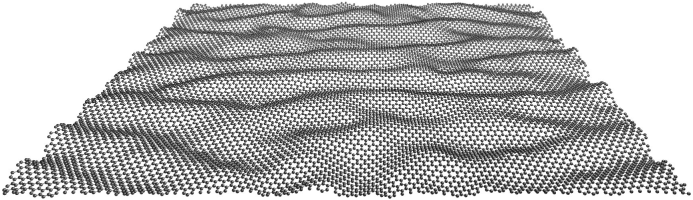
Graphene as a membrane (2-brane)

Stability related to ripples (3 atoms high and 30 atoms long) to overcome the argument of Landau & Peierls



Strong and long-ranged (intrinsic case) electron-electron interactions

$\alpha_g = \frac{e^2}{4\pi\kappa\hbar v} \approx 2.2$ (suspended graphene: $\kappa \approx 1$ and $v = c/300$)

- Our focus: electron-electron interaction effects in an ideally flat, undoped (intrinsic) graphene sheet.
- Our approach: effective gauge field theory which captures universal (IR) properties (valid for other similar planar Dirac liquids)

Outline

Model (I)

General low-energy effective action for planar Dirac liquid:

$$L = \int \mathrm{d}^2 x \left[\bar{\psi}_\sigma \left(\mathrm{i}\gamma^0 \partial_t + \mathrm{i} v \vec{\gamma} \cdot \vec{\nabla} \right) \psi^\sigma - e \bar{\psi}_\sigma \gamma^0 A_0 \psi^\sigma + e \frac{v}{c} \bar{\psi}_\sigma \vec{\gamma} \cdot \vec{A} \psi^\sigma \right] - \frac{1}{4} \int \mathrm{d}^3 x \, F^{\mu\nu} F_{\mu\nu}$$

- simple massless $U(1)$ (abelian) gauge field theory with retardation (relativistic effects in $x = v/c$)
- fermions are in 2 dimensions but interact via exchange of 3 dimensional photons
- spin flavour $\sigma = 1, \cdots, N$ (N=2 for graphene)
- ψ_σ four-component spinor (in sublattice $\otimes$ valley space)
- $U(2N)$ flavour symmetry ($U(4)$ sublattice symmetry for graphene)

Despite apparent simplicity:
model I is difficult to study and few results are available

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$$- \frac{1}{4} \int \mathrm{d}^3x \, F^{\mu\nu} F_{\mu\nu}$$

Important result: proof of an IR Lorentz invariant fixed point (running v)
Early theory: [Gonzalez, Guinea & Vozmediano '94]
Experimental evidence [Elias et al. '11, Siegel et al. '11]

$$\beta_v(\alpha_g) = \begin{cases} -\frac{v\,\alpha_g}{4} \left(1 + \mathrm{O}(x^2)\right) & (\text{case } x = v/c \to 0) \\ -\frac{8(1-x)\,v\,\alpha_g}{5\pi} \left(1 + \mathrm{O}((1-x))\right) & (\text{case } x = v/c \to 1) \end{cases}$$

such that

$$\begin{cases} v(\mu = 200\mathrm{meV}) \approx c/300 & \underset{\mu \to 0}{\longrightarrow} \quad c \\ \alpha_g(\mu = 200\mathrm{meV}) = e^2/(4\pi\hbar v) \approx 2.2 & \underset{\mu \to 0}{\longrightarrow} \quad \alpha_{QED} = 1/137 \end{cases}$$

Model (II)

Most studies focus on a simplified model with instantaneous interactions (non relativistic limit: $x = v/c \to 0$ and $\alpha_g \approx 2.2$):

$$L = \int \mathrm{d}^2 x \, \bar{\psi}_\sigma \left[\gamma^0 \left(\mathrm{i}\partial_t - eA_0 \right) + \mathrm{i}v\vec{\gamma} \cdot \vec{\nabla} \right] \psi^\sigma + \frac{1}{2} \int \mathrm{d}^3 x \, \left(\vec{\nabla} A_0 \right)^2$$

Two important focuses of the last decade related to understanding:

- dynamical flavour (chiral) symmetry breaking (DχSB)
- interaction effects on transport properties (optical conductivity)

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DχSB in planar Dirac liquids

Global $U(2N)$ "chiral" (flavour) symmetry ($U(4)$ for graphene)

Parity-even mass/gap term breaks $U(2N) \to U(N) \times U(N)$

$$\overline{\Psi}_\sigma \, \Psi^\sigma = \bar{\chi}_\sigma \, \chi^\sigma - \bar{\chi}_{N+\sigma} \, \chi^{N+\sigma}$$

Is it possible that "chiral" symmetry is dynamically broken?
(with dynamical generation of a parity-even mass/gap)

To study DχSB one needs to determine:

- critical coupling constant α_c: DχSB for $\alpha_g > \alpha_c$ ($\forall N$)
- critical fermion flavour number N_c: DχSB for $N < N_c$ and $\alpha_g \to \infty$

Accurate computation of numbers (α_c and N_c) required
(understanding phase structure of model / phase diagram of system)

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Experimentally: no gap observed [Elias '11] $\Rightarrow$ expect $\alpha_c > 2.2$

Theoretically: extensive work done [Khveshchenko '01; Gorbar et al. '02; Leal et al. '03; Son '07; Liu et al. '09; Gamayun et al. '09; Drut et al. '08, '09; Gonzalez '12, '15; Buividovich et al. '12, '13; Popovic et al. '13; Katanin '16; Carrington et al. '17; ...] (+ many with Hubbard-like interactions)
Various techniques used (analytical to lattice simulations)
Different levels of approximations made...

Last 10 years: different results obtained for α_c!

α_c (N_c)	**Method**	**Year**
7.65	SD (LO, dynamic RPA, running v)	2013
3.7	FRG, Bethe-Salpeter	2016
$3.2 < \alpha_c < 3.3$	SD (LO, dynamic RPA, running v)	2012
3.1	SD (LO, bare vertex approximation)	2015
2.06	SD (LO, dynamic RPA, running v)	2017
1.62	SD (LO, static RPA)	2002
(3.52)	SD (LO)	2009
1.13 (3.6)	SD (LO, static RPA, running v)	2008
1.11 ± 0.06	Lattice simulations	2008
0.99	RG study	2012
0.92	SD (LO, dynamic RPA)	2009
0.9 ± 0.2	Lattice simulations	2012
0.833	RG study	2008

Optical conductivity of planar Dirac liquids

$$\vec{j}(\omega, \vec{q}\,) = \sigma(\omega, \vec{q}\,)\,\vec{E}(\omega, \vec{q}\,)$$

Optical (collision-less) regime: $\omega \gg T$

- response to a homogeneously applied electric field ($\vec{q} \to 0$)
- photon energies $\omega \approx 1$eV (visible range of the spectrum)

Intrinsic graphene: semi-metal (no state at the Fermi points).

Does it conduct?

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Minimal conductivity of free 2D Dirac fermions (no disorder)

[Ludwig, Fisher, Shankar and Grinstein '94]

$$\sigma_0(\omega) = \frac{e^2}{4\hbar}$$

Non-zero and universal (independent of ω)

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Experimentally: optical conductivity is close to ideal (!?)

$$\sigma(\omega \approx 1\text{eV}) = \sigma_0\,(1.00 \pm 0.02)$$

Data for $\hbar\omega < 1.2$eV from [Mak et al. '08] and for $\hbar\omega > 1.2$eV from [Nair et al. '08]

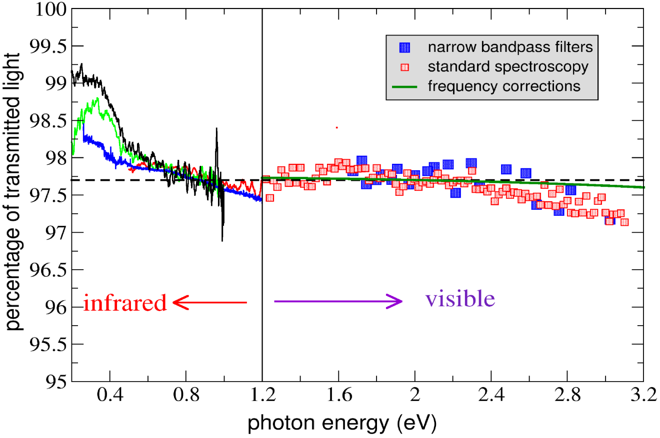

Figure from the review paper of [Peres '10]

Theoretical situation

Optical conductivity from the polarization operator

$$\sigma(q_0) = -\lim_{\vec{q}\to 0} \frac{\mathrm{i}q_0}{|\vec{q}|^2} \Pi^{00}(q_0, \vec{q}\,), \qquad q^\mu = (q^0, v\vec{q})$$

$$\Pi^{00}(q) = \langle T\rho(q)\rho(-q)\rangle, \qquad \rho(q) = e\bar{\Psi}\gamma^0\Psi$$

"Perturbative" expansion: compute interaction correction coefficients

$$\sigma(q_0) = \sigma^{(0)}\left(\mathbf{1} + \mathcal{C}_\mathbf{1}\alpha_r + \mathcal{C}_\mathbf{2}\alpha_r^2 + \cdots\right)$$

Expect $\mathcal{C}_i \neq 0$ because Kohn's theorem does not to apply.

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Extensive theoretical work since 2008 to compute $\mathcal{C}_1$: [Herbut et al. '08; Mishchenko '08; Sheehy and Schmalian '09; Juričić et al. '10; Abedinpour et al. '11; Sodemann and Fogler '12; Rosenstein et al. '13; Gazzola et al. '13; Link et al. '16; Boyda et al. '16; Stauber et al. '17; ...]

Various techniques used: analytical, lattice simulation, ...

Last 10 years: different results obtained for $\mathcal{C}_1$!

$\mathcal{C}_1$	**Method**	**Year**
$\mathcal{C}^{(1)} = \frac{25-6\pi}{12} \approx 0.512$	current + hard cut-off	2008
$\mathcal{C}^{(2)} = \frac{19-6\pi}{12} \approx 0.013$	density + hard cut-off	2008
	current & kinetic equations + soft cut-off	2008
	current + hard cut-off	2009
	density + hard cut-off	2011
	density + hard cut-off	2012
	current + hard cut-off & implicit regularization	2013
	lattice (tight-binding) simulations	2016
	Quantum Monte Carlo calculations	2016
(0.05)	Hartree-Fock simulations (self-screened)	2017
$\mathcal{C}^{(3)} = \frac{11-3\pi}{6} \approx 0.263$	kinetic equations, hard cut-off	2008
	current & density, dim. reg.	2010
	lattice (tight-binding) simulations	2013
$(1/4 = 0.25)$	Hartree-Fock simulations (unscreened)	2017

Model (III)

Alternative (non-conventional) approach: [ST '12]

Study interaction effects starting from the IR Lorentz invariant fixed point ($v/c \to 1$ fully retarded interaction and $\alpha_g \to \alpha = 1/137$)

$$S = \int \mathrm{d}^3x \, \bar{\psi}_\sigma \mathrm{i} \not{D} \psi^\sigma + \int \mathrm{d}^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right]$$

Boundary effective Lagrangian (in $2+1$ dimensions): non-local

$$L = \int \mathrm{d}^3x \left[\bar{\psi}_\sigma \mathrm{i} \left(\not{\partial} + \mathrm{i}e\tilde{\not{A}} \right) \psi^\sigma - \frac{1}{4} \tilde{F}^{\mu\nu} \frac{2}{[-\Box]^{1/2}} \tilde{F}_{\mu\nu} + \frac{1}{2\tilde{\xi}} \tilde{A}^\mu \frac{2\,\partial_\mu \partial_\nu}{[-\Box]^{1/2}} \tilde{A}^\nu \right]$$

- $\tilde{\xi} = (1+\xi)/2$: (non-local) gauge fixing parameter associated to reduced gauge field $\tilde{A}^\mu$
- fractional power $(1/2)$ in the reduced photon propagator

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Model known in some form or the other for a long time (under different names): reduced QED [Gorbar et al. '01], pseudo QED [Marino '93], ... and more recently: mixed-dimensional QED [Son '15]

Generalization to arbitrary dimensions d_e and d_γ (reduced QED$_{d_e,d_\gamma}$):

$$S = \int \mathrm{d}^{d_e}x\, \bar{\psi}_\sigma \mathrm{i}\not{D}\psi^\sigma + \int \mathrm{d}^{d_\gamma}x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right]$$

- case relevant to planar Dirac liquids: $d_e = 3$ and $d_\gamma = 4$ (QED$_{4,3}$)
- case $d_e = d_\gamma = d$: usual QED$_d$ (QED$_4$, QED$_3$, QED$_2$)

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Initial motivations:

- (lower-dimensional) branes (DχSB) [Gorbar et al. '01]
- applications to planar systems (dSC, QHE) [Marino '93; Dorey & Mavromatos '92; Kovner & Rosenstein '92]

Recent burst of activity on reduced QED$_{4,3}$: quantum Hall physics [Marino et al. '14, '15], optical properties [Raya et al. '15, '16], 1/2-filled FQHE systems [Son '15], LKF [Ahmad et al. '16], duality [Hsiao & Son '17], bCFT [Herzog & Huang '17]...

In relation with interaction effects in Dirac liquids

- more symmetry: simpler model to study interaction corrections
- powerful techniques of relativistic Feynman diagram calculations apply

Ideal framework to accurately compute numbers

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$\Rightarrow$ compute: $\mathcal{C}$, α_c, N_c in the ultra-relativistic limit ($x = v/c \to 1$)

$\Rightarrow$ reach a quantitative understanding of the effect of electron-electron interactions at the IR fixed point

$\Rightarrow$ adapt the formalism to the non-relativistic limit ($x = v/c \to 0$)

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Study of reduced QED$_{d_e,d_\gamma}$ in arbitrary dimensions also allows:

- a control over computations, *e.g.*, recover known results in QED$_4$
- to reconnect with long-standing issues, *e.g.*, DχSB in large-N QED$_3$

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DχSB in $(2+1)$-dimensional QED (QED$_3$)

Extensive studies for more that three decades now:

- **Original interest:** [Pisarski '84; Appelquist et al. '84] similarities to $(3+1)$-dimensional QCD and toy model to study systematically DχSB
- **Also:** [Marston & Affleck '89, Ioffe & Larkin '89] effective field theory for condensed matter physics systems (underdoped phase of d-wave superconductors) with relativistic-like low-energy excitations
- **Since then:** many studies using different approaches [Pisarski '91] [Appelquist et al. '88, '99] [Nash '89] [Kotikov '93] [Pennington et al. '91, '92] [Atkinson et al. '90] [Dagotto et al. '89, '90] [Gusynin et al. '96, '01] [Kubota et al. '01] [Bashir, Raya et al. '07, '09]...

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- **Very recently:** burst of activity [Di Pietro et al. '16] [Giombi et al. '16] [Chester & Pufu '16] [Karthik & Narayanan '16] [Janssen '16] [Herbut '16] [Gusynin & Pyatkovskiy '16] ...

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In 30 years: very different results obtained for N_c!

N_c	**Method**	**Year**
∞	SD (LO in $1/N$)	1984
∞	SD (non-perturbative)	1990, 1992
∞	RG study	1991
∞	lattice simulations	1993, 1996
< 4.4	F-theorem	2015
3.5 ± 0.5	lattice simulations	1988, 1989
$32/\pi^2 \approx 3.24$	SD (LO, Landau gauge)	1988
2.89	RG study (one-loop)	2016
$1 + \sqrt{2} = 2.41$	F-theorem	2016
$< 9/4 = 2.25$	RG study (one-loop)	2015
$< 3/2$	Free energy constraint	1999
0	SD (non-perturbative)	1990
0	lattice simulations	2015, 2016

Table: Values of N_c obtained over the years with different methods (at LO).

Outline

Regularization & Renormalization

Dimensional regularization in the $\overline{\mathrm{MS}}$-scheme where $\overline{\mu}^2 = 4\pi\, e^{-\gamma_E}\, \mu^2$

well known regulator which preserves symmetries (gauge invariance, Lorentz, ...) and power counting (mass independent)

For reduced $\mathrm{QED}_{d_e,d_\gamma}$, use a double ε prescription [ST '12]:

$$d_e = 4 - 2\varepsilon_e - 2\varepsilon_\gamma, \qquad d_\gamma = 4 - 2\varepsilon_\gamma$$

ε_e: dimensionality of the fermion field, ε_γ: regulates UV/IR singularities

Case of $\mathrm{QED}_{4,3}$: expansion in vicinity of odd dimension: $d_e = 3 - 2\varepsilon_\gamma$

$\varepsilon_e \to 1/2$ and $\varepsilon_\gamma \to 0$ (limits commute for $\mathrm{QED}_{4,3}$)

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Renormalization: BPHZ prescription
(recursive form: disjoint divergent subrgaphs only)

$$\mathcal{R}\, G \;\; = \;\; (1-\mathcal{K})\, \mathcal{R}'\, G\,, \quad \mathcal{R}' G = G + \sum_{\bar{\Gamma}_d \neq \emptyset} \prod_{\gamma \in \bar{\Gamma}_d} \Big(-\mathcal{K}\mathcal{R}'\gamma \Big) \star G/\bar{\Gamma}_d$$

Compute renormalization constants:
($\psi = Z_\psi^{1/2}\psi_r$, $A = Z_A^{1/2} A_r$, $\alpha = Z_e^2 \alpha_r \mu^{2\varepsilon}$, ...)

$$Z_x = 1 + \sum_{l=1}^{\infty} \sum_{j=1}^{l} Z_x^{(l,j)} \, \frac{\alpha_r^l}{\varepsilon_\gamma^j} \qquad (x \in \{\psi, A, e, \xi, \Gamma, g\})$$

Deduce β-functions and anomalous dimensions of fields:

$$\beta(\alpha_r) = \mu \, \frac{\partial \alpha_r}{\partial \mu} \bigg|_B, \quad \gamma_x(\alpha_r, \xi_r) = -\mu \, \frac{\mathrm{d} \log Z_x(\alpha_r, \xi_r)}{\mathrm{d}\mu} \bigg|_B \quad (x \in \{\psi, A\})$$

Multi-loop techniques (dim. reg. in the $\overline{\mathrm{MS}}$-scheme) + BPHZ prescription:
very powerful and effecient calculational framework

For the year 2016-2017, breakthrough achievements (+ beautiful mathematics):

- 4-loop β-function Gross-Neveu [Gracey et al. '16]
- 5-loop β-function QCD [Baikov et al. '16; Luthe et al. '17; Chetyrkin et al. '17]
- 6-loop Φ^4 [Batkovich et al. '16; Kompaniets & Panzer '17]
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- 7-loop planar $\mathcal{N} = 4$ SYM (+ integrability) [Marboe & Velizhanin '16]

These achievements all concern 4-dimensional models!

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Extend this to odd-dimensional models

(highly non-trivial but largely open: start at 2-loop)

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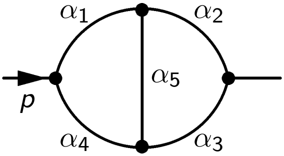
(non-perturbative: implement within a Schwinger-Dyson approach)

Massless propagator type 2-loop diagram (basic building block)

$$J(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \int\int \frac{[\mathrm{d}^D k_1]\,[\mathrm{d}^D k_2]}{k_1^{2\alpha_1}\,k_2^{2\alpha_2}\,(k_2-p)^{2\alpha_3}\,(k_1-p)^{2\alpha_4}\,(k_2-k_1)^{2\alpha_5}}$$

Arbitrary indices α_i and external momentum p in Euclidean space (D)



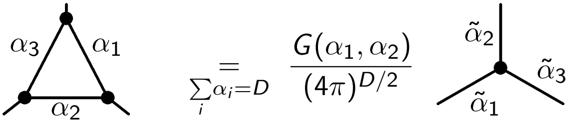
Goal: compute (dimensionless) coefficient function (Laurent series in ε_γ)

$$G(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \frac{(4\pi)^D}{(p^2)^{D-\sum_{i=1}^5 \alpha_i}}\,J(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$$

Long history: exact computations are crucial

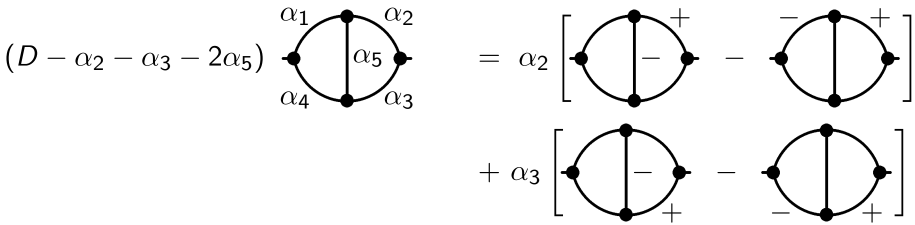
Rules for massless Feynman diagrams [Kazakov '83]

- Uniqueness relation ($\tilde{\alpha} = D/2 - \alpha$):



(Note: unique triangle has index $\sum_i \alpha_i = D$)

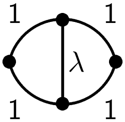
- Integration by Parts (IBP) relation:



(Note: $\pm$ corresponds to add or subtract 1 to index α_i)

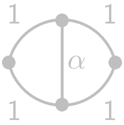
Result (useful in reduced QED for the polarization operator): this allows to compute [Vasil'ev et al. '81] [Vasil'ev et al. '93] [Kivel et al. '94] [Kotikov & ST '13] ($\lambda = D/2 - 1$, $\lambda = 1/2$ in $D = 3$)

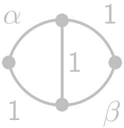
$G(1,1,1,1,\lambda) \quad =$

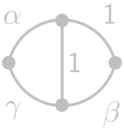


$$= 3\,\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)}\left[\Psi'(\lambda) - \Psi'(1)\right]$$

But the following diagrams are beyond IBP and uniqueness:







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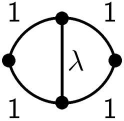
$G(1,1,1,1,\lambda) \quad =$ 

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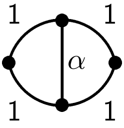
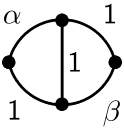
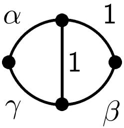
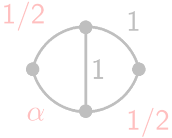
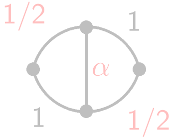






Example: need to compute masters such as:

$\tilde{I}_1(\alpha) \quad =$ 

$\tilde{I}_2(\alpha) \quad =$ 

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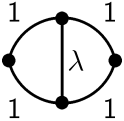
$G(1,1,1,1,\lambda) =$ 

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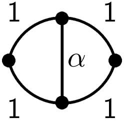

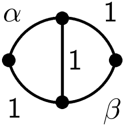

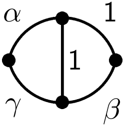

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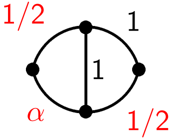
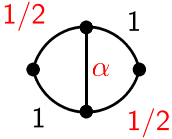
$\tilde{I}_1(\alpha) =$ 

$\tilde{I}_2(\alpha) =$ 

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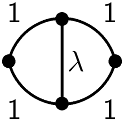

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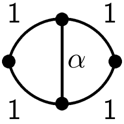

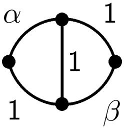

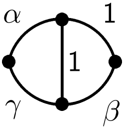

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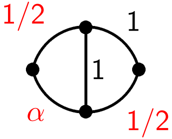

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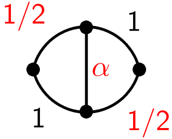
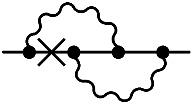
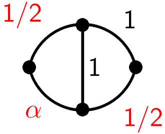


... for which no exact solution is still available.

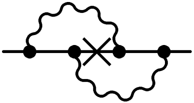
Result (useful for DχSB in QED$_3$): functional relations for the most complicated integrals [Kotikov, Shilin & ST '16] (some similarity to functional relations obtained by [Kazakov '84] but more complicated)

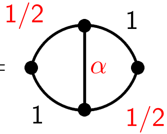
$\Sigma_2 =$ 

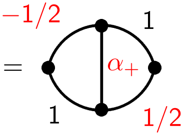
$\tilde{I}_1(\alpha) =$ 

$= \dfrac{(4\pi)^3}{(p^2)^{-\alpha}}\, I_1(\alpha)$

$$\tilde{I}_1(\alpha+1) = \frac{(\alpha-1/2)^2}{\alpha^2}\tilde{I}_1(\alpha) - \frac{1}{\pi\alpha^2}\Big[\Psi'(\alpha) - \Psi'(1/2-\alpha)\Big]$$

$\Sigma_3 =$ 

$\tilde{I}_2(\alpha) =$ 

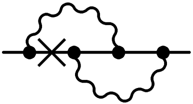
$\tilde{I}_3(\alpha) =$ 

$$\tilde{I}_2(\alpha) = \tilde{I}_2(3/2-\alpha), \quad \tilde{I}_3(\alpha) = \frac{2}{4\alpha-1}\Big(\alpha\tilde{I}_2(1+\alpha) - (1/2-\alpha)\tilde{I}_2(\alpha)\Big) - \frac{\beta^2}{\pi}$$

Other example of result (useful for DχSB in QED$_3$): series representation for the most complicated integrals [Kotikov, Shilin & ST '16] (using rules of [Kotikov '96] obtained via the Gegenbauer polynomial technique)

For Σ_2, related master integral represented in terms of a two-fold series

$\Sigma_2 =$



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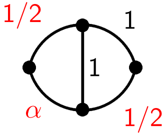

$$= \frac{(4\pi)^3}{(p^2)^{-\alpha}} \, I_1(\alpha)$$

$$\tilde{I}_1(\alpha) = \frac{(4\pi)^3}{(p^2)^{-\alpha}} \int \frac{[\mathrm{d}^3 k_1][\mathrm{d}^3 k_2]}{|p-k_1| k_1^{2\alpha} (k_1-k_2)^2 (p-k_2)^2 |k_2|} = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{B(l,n,1,1/2)}{(n+1/2)\,\Gamma(1/2)}$$

$$\times \left[\frac{2}{n+1/2} \left(\frac{1}{l+n+\alpha} + \frac{1}{l+n+3/2-\alpha} \right) + \frac{1}{(l+n+\alpha)^2} + \frac{1}{(l+n+3/2-\alpha)^2} \right]$$

Outline

Reduced $\mathrm{QED}_{4,3}$ (model III)

Boundary effective model in $(2+1)$-dimensions (non-local)

$$S(p) = \frac{\mathrm{i}}{\not p}, \qquad D_{\mu\nu}(p) = \frac{P^{\xi/2}_{\mu\nu}(p)}{2\,|p|}, \quad P^{\xi}_{\mu\nu}(p) = g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}$$

Power counting similar to 4-dimensional theories

- renormalizable (power counting: $[e] = \varepsilon_\gamma \rightarrow 0, \ \forall \varepsilon_e$)
- UV singular fermion self-energy $\Rightarrow$ anomalous fermion dimension
- finite photon self-energy (divergent subgraphs) $\Rightarrow$ no charge renormalization (non-local)

conformal invariance ($\beta(\alpha) = 0$ Tomonaga-Luttinger like)

Thorough investigation of the perturbative structure of the model
[ST '12, '14; Kotikov & ST '13, '14]

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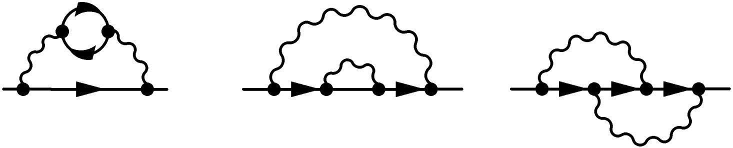
Reduced $\text{QED}_{4,3}$: computation of γ_ψ [Kotikov & ST '14]



Anomalous dimension of the fermion field:

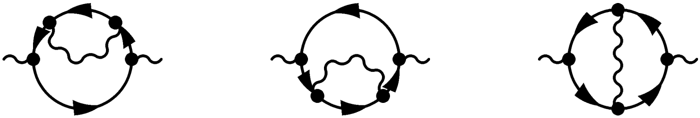
$$\boxed{\gamma_\psi(\bar{\alpha}_r, \xi_r) = 2\bar{\alpha}_r \, \frac{1 - 3\xi_r}{3} - 16 \left(\zeta_2 N_F + \frac{4}{27}\right) \bar{\alpha}_r^2 + \mathrm{O}(\bar{\alpha}_r^3)}$$

Two-loop part is:

- gauge invariant (and similarly for QED_4 and QED_3)
- contains $\zeta_2 = \pi^2/6$ (unusual in $\overline{\text{MS}}$-scheme)
  does not come from the ε_γ-expansion but from ε_e term:

$$K_1 = \frac{\Gamma^3(1 - \varepsilon_e)\Gamma(\varepsilon_e)}{\Gamma(2 - 2\varepsilon_e)} = \pi^2 \qquad (\varepsilon_e = 1/2)$$

Reduced $QED_{4,3}$: computation of $\mathcal{C}^*$ [ST '12; Kotikov & ST '13]



Self-energy and vertex parts almost cancel eachother ($x = v/c = 1$):

$$\boxed{\mathcal{C}^* = (92 - 9\pi^2)/(18\pi) = 46/(9\pi) - \pi/2 = 0.056}$$

Compare with the results in the non-relativistic limit ($x = v/c = 0$):

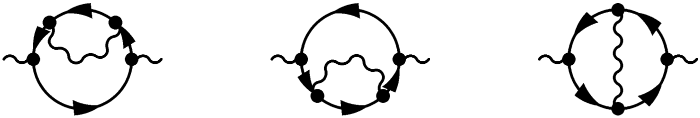
$\mathcal{C}^{(1)} = (25 - 6\pi)/12 = 25/12 - \pi/2 = 0.512$ [Herbut et al. '08]

$\mathcal{C}^{(2)} = (19 - 6\pi)/12 = 19/12 - \pi/2 = 0.013$ [Mishchenko '08]

$\mathcal{C}^{(3)} = (11 - 3\pi)/6 = 11/6 - \pi/2 = 0.263$ [Juričić et al. '10]

Good quantitative agreement with [Mishchenko '08]
(self-energy part encodes relativistic corrections in $x = v/c$)

Reduced QED$_{4,3}$: computation of $\mathcal{C}^*$ [ST '12; Kotikov & ST '13]



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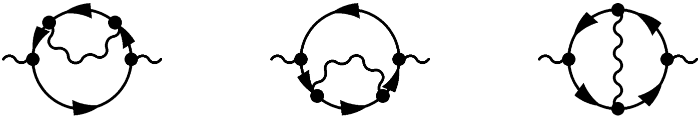
$$\mathcal{C}^{(2)} = (19 - 6\pi)/12 = 19/12 - \pi/2 = 0.013 \qquad \text{[Mishchenko '08]}$$
$$\mathcal{C}^{(3)} = (11 - 3\pi)/6 = 11/6 - \pi/2 = 0.263 \qquad \text{[Juričić et al. '10]}$$

Good quantitative agreement with [Mishchenko '08]
(self-energy part encodes relativistic corrections in $x = v/c$)

Note: number similar to $\mathcal{C}^*$ appeared earlier for QED$_3$ [Gusynin et al. '01].

Reduced QED$_{4,3}$: computation of $\mathcal{C}^*$ [ST '12; Kotikov & ST '13]



Self-energy and vertex parts almost cancel eachother ($x = v/c = 1$):

$$\boxed{\mathcal{C}^* = (92 - 9\pi^2)/(18\pi) = 46/(9\pi) - \pi/2 = 0.056}$$

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Three-dimensional QED

QED_3 is super-renormalizable

- dimensionful coupling constant $a = Ne^2/8$
- loop-expansion plagued by IR singularities (starting from two-loop) [Jackiw & Templeton '81] [Guendelman & Radulovic '83, '84]

Large$-N$ limit of QED_3 ($N \to \infty$ and a fixed): **IR softening**

[Appelquist & Pisarski '81, Appelquist & Heinz '81]

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu}}{p^2\,[1+\Pi(p)]} = \frac{g_{\mu\nu}}{p^2\,[1+a/|p|]} \xrightarrow{p \ll a} \frac{g_{\mu\nu}}{a|p|}$$

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Dynamical chiral symmetry breaking in QED_3

Non-perturbative: cannot take place at any finite order in $1/N$
(30 years of history for the computation of N_c)

Our approach: solve Schwinger-Dyson equation in $1/N$-expansion

Consider the approach of [Appelquist et al. '88] (LO in $1/N$-expansion):

$N_c = 32/\pi^2 = 3.24$ is not large
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Historic papers: [Nash '89] and [Kotikov '93]
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Fermion propagator ($\Sigma(p)$: dynamically generated (parity-conserving) mass, $A(p)$: fermion wave function):

$$S^{-1}(p) = [1 + A(p)]\,(\mathrm{i}\hat{p} + \Sigma(p))$$

Photon propagator (non-local ξ-gauge, $\xi = 0$: Landau gauge):

$$D_{\mu\nu}(p) = \frac{P^{\xi}_{\mu\nu}(p)}{p^2\,[1 + \Pi(p)]}, \quad P^{\xi}_{\mu\nu}(p) = g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\,,$$

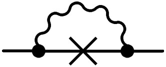
SD equation for the fermion propagator ($\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)]$):

$$\tilde{\Sigma}(p) = \frac{2a}{N}\,\mathrm{Tr}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\gamma^\mu\,D_{\mu\nu}(p-k)\,\Sigma(k)\,\Gamma^\nu(p,k)}{[1 + A(k)]\,(k^2 + \Sigma^2(k))}$$

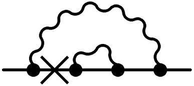
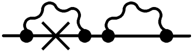
$$A(p)p^2 = -\frac{2a}{N}\,\mathrm{Tr}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{D_{\mu\nu}(p-k)\,\hat{p}\,\gamma^\mu\,\hat{k}\,\Gamma^\nu(p,k)}{[1 + A(k)]\,(k^2 + \Sigma^2(k))}$$

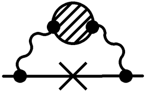
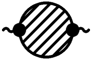
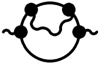
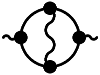
$\Gamma^\nu(p,k)$: vertex function.

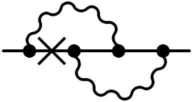
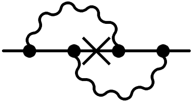
At LO: one diagram contributes to the gap equation (cross = mass insertion)

$\Sigma(p) =$  $= \dfrac{8(2+\xi)a}{N} \displaystyle\int \dfrac{[\mathrm{d}^3 k] \;\; \Sigma(k)}{(k^2 + \Sigma^2(k)) \left[(p-k)^2 + a\,|p-k|\right]}$

At NLO: four contributions

$\Sigma_A =$  $+$ 

$\Sigma_1 =$  ,  $= 2 \times$  $+$ 

$\Sigma_2 =$  , $\Sigma_3 =$ 

Use of advanced multi-loop techniques yields an exact gap equation

Simple form after Nash's resummation (γ_ψ):

$$1 = \frac{8\beta}{3\pi^2 N} + \frac{1}{\pi^4 N^2}\left[8\tilde{S}(\alpha,\xi) - \frac{16}{3}\,\beta\left(\frac{40}{9} + \hat{\Pi}_2\right)\right]$$

with complete (strong) suppression of gauge dependence at LO (NLO)

Implementing [Gusynin & Pyatkovskiy '16]'s prescription yields

$$\boxed{\frac{1}{\beta} = \frac{8}{3\pi^2 N} - \frac{16}{3\pi^4 N^2}\left(\frac{40}{9} + \hat{\Pi}_2\right) \qquad \left(\hat{\Pi}_2 = \frac{92}{9} - \pi^2 = 2\pi\mathcal{C}^*\right)}$$

fully gauge invariant gap equation order by order in the $1/N$-expansion

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N_c	**Method**	**Year**
∞	SD (LO in $1/N$)	1984
∞	SD (non-perturbative)	1990, 1992
∞	lattice simulations	1993, 1996
< 4.4	F-theorem	2015
$(4/3)(32/\pi^2) = 4.32$	SD (LO, resummation)	1989
3.5 ± 0.5	lattice simulations	1988, 1989
3.31	SD (NLO, Landau gauge)	1993
3.29	SD (NLO, Landau gauge)	2016
$32/\pi^2 \approx 3.24$	SD (LO, Landau gauge)	1988
$3.0084 - 3.0844$	SD (NLO, resummation)	2016
2.89	RG study (one-loop)	2016
2.85	SD (NLO, resummation, $\forall\xi$)	2016
$1 + \sqrt{2} = 2.41$	F-theorem	2016
$< 9/4 = 2.25$	RG study (one-loop)	2015
$< 3/2$	Free energy constraint	1999
0	SD (non-perturbative)	1990
0	lattice simulations	2015, 2016

Dynamical chiral symmetry breaking in reduced $QED_{4,3}$

Mapping between large-N QED_3 and $QED_{4,3}$ [Kotikov & ST '16]

$$\frac{1}{\pi^2 N} \to \frac{\alpha}{4\pi} \equiv \frac{e^2}{(4\pi)^2}, \qquad \tilde{\eta} \to \frac{\eta}{2} \quad \left(\tilde{\xi} \to \frac{1+\xi}{2}\right)$$

$$\hat{\Pi}_2 = \frac{92}{9} - \pi^2 \quad \to \quad \hat{\Pi}_1 = \frac{N\pi^2}{2} \quad \text{and} \quad \tilde{\xi}\hat{\Pi}_1 = 0$$

Origin of the mapping: photon propagators have the same form ($\sim 1/p$)

A check: from Gracey's result [Gracey '93] for the NLO fermion anomalous dimension in QED_3 we recover the result of [Kotikov & ST '14]

$$\lambda_\psi = \frac{4}{\pi^2 N}\left(\frac{2}{3} - \tilde{\xi}\right) - \frac{8}{\pi^4 N^2}\left(\frac{8}{27} + \left(\frac{2}{3} - \tilde{\xi}\right)\hat{\Pi}_2\right) + \mathrm{O}(1/N^3)$$

$$\to \gamma_\psi = 2\frac{\alpha}{4\pi}\,\frac{1-3\xi}{3} - 16\left(\zeta_2 N + \frac{4}{27}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \mathrm{O}(\alpha^3)$$

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LO result: early work of [Gorbar, Gusynin & Miransky '01]

$$\alpha_c(N=2) \approx 0.55, \qquad N_c(\alpha \to \infty) = 128/(3\pi^2) \approx 4.32$$

Extend the study of [Gorbar, Gusynin & Miransky '01] to NLO

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NLO result: with Nash's resummation and RPA (fully gauge-invariant)

$$\boxed{\alpha_c(N=2) = 1.22, \qquad N_c(\alpha \to \infty) = 3.04} \quad \alpha_c(N=1) = 0.62$$

Good quantitative agreement with values found in non-relativistic limit...

α_c (N_c)	**Method**	**Year**
7.65	SD (LO, dynamic RPA, running v)	2013
3.7	FRG, Bethe-Salpeter	2016
$3.2 < \alpha_c < 3.3$	SD (LO, dynamic RPA, running v)	2012
3.1	SD (LO, bare vertex approximation)	2015
2.06	SD (LO, dynamic RPA, running v)	2017
1.62	SD (LO, static RPA)	2002
1.22 (3.04)	SD (NLO, RPA, resummation, $v/c \to 1$, $\forall\xi$)	2016
(3.52)	SD (LO)	2009
1.13 (3.6)	SD (LO, static RPA, running v)	2008
1.11 ± 0.06	Lattice simulations	2008
$1.03 < \alpha_c < 1.08$ $3.17 < N_c < 3.24$	SD (NLO, RPA, resummation, $v/c \to 1$)	2016
0.99	RG study	2012
0.92	SD (LO, dynamic RPA)	2009
0.9 ± 0.2	Lattice simulations	2012
0.833	RG study	2008

Non-relativistic reduced QED$_{4,3}$ (model II)

Apply our formalism to the non-relativistic limit ($x = 0$)

NRQED$_{4,3}$ vs QED$_{4,3}$ (broken Lorentz invariance)

$$L = \int \mathrm{d}^2x \, \bar{\psi}_\sigma \left[\gamma^0 \left(\mathrm{i}\partial_t - eA_0\right) + \mathrm{i}v\vec{\gamma}\cdot\vec{\nabla}\right] \psi^\sigma + \frac{1}{2} \int \mathrm{d}^3x \, \left(\vec{\nabla} A_0\right)^2$$

- renormalizable (power counting: $[e] = \varepsilon_\gamma \to 0, \ \ \forall \varepsilon_e$)
- finite photon self-energy (divergent subgraphs) $\Rightarrow$ no charge renormalization (non-local)
- fermion self-energy: $\Sigma(k) = \gamma^0 k_0 \, \Sigma_\omega(k^2) - v\vec{\gamma}\cdot\vec{k} \ \Sigma_k(k^2)$
  At one-loop:
  - $\Sigma_{1\omega} = 0 \Rightarrow$ no wave function renormalization
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Compute $\mathcal{C}$ in the non-relativistic limit
[ST & Kotikov '14, '17]

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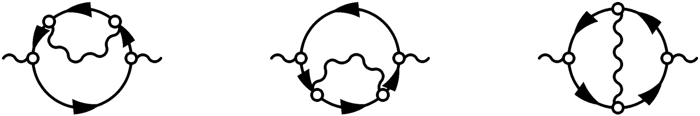
NRQED$_{4,3}$: computation of $\mathcal{C}$ [ST & Kotikov '14, '17]



At the level of multi-loop computations: non-trivial masters

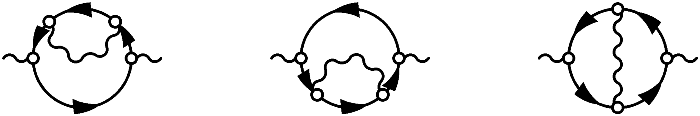
semi-massive 2-loop tadpole integrals $(m = q_{E0}/v,\ D_e = 2 - 2\varepsilon_\gamma)$

$$
\begin{aligned}
I_n(\alpha) &= \int [\mathrm{d}^{D_e} k_1][\mathrm{d}^{D_e} k_2] \, \frac{(\vec{k}_1 \cdot \vec{k}_2)^n [|\vec{k}_1 - \vec{k}_2|^2]^{-1/2}}{[|\vec{k}_1|^2]^\alpha \, [|\vec{k}_1|^2 + m^2] \, [|\vec{k}_2|^2]^\alpha \, [|\vec{k}_2|^2 + m^2]} \\
&= \frac{(m^2)^{D_e + n - 2\alpha - 5/2}}{(4\pi)^{D_e}} \, \tilde{I}_n(\alpha) \,.
\end{aligned}
$$

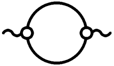
Use rules for massive diagrams [Davydyshev & Boos '91; Kotikov '91]

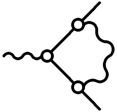
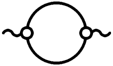
Yields the bare value: $\mathcal{C} = \dfrac{11 - 3\pi}{6} = \mathcal{C}^{(3)}$

$\text{NRQED}_{4,3}$: computation of $\mathcal{C}$ [ST & Kotikov '14, '17]



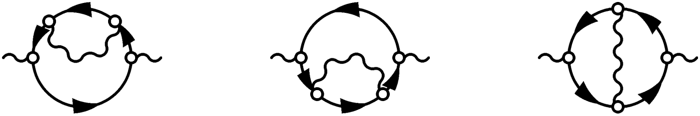
At the level of renormalization: subtract sub-divergent graphs (BPHZ)

$2\Pi^{00}_{2a'} = -2\mathcal{K}\Big[$  $\Big] \star$  $\rightarrow \quad -\frac{1}{4}$

$\Pi^{00}_{2b'} = -2\mathcal{K}\Big[$  $\Big] \star$  $\rightarrow \quad 0$

$$\boxed{\mathcal{C}_r = \frac{11-3\pi}{6} - \frac{1}{4} = \frac{19-6\pi}{12} = \mathcal{C}^{(2)}} \qquad \left(\mathcal{C}^{(3)} = \frac{11-3\pi}{6}\right)$$

NRQED$_{4,3}$: computation of $\mathcal{C}$ [ST & Kotikov '14, '17]



$$\boxed{\mathcal{C}_r = \frac{11-3\pi}{6} - \frac{1}{4} = \frac{19-6\pi}{12} = \mathcal{C}^{(2)}}$$

In perfect agreement with the result of [Mishchenko '08]
(and similarly for the current-current correlation function)

Radiative corrections in the non-relativistic limit are
finite and *well-determined*

Effeciency and power of multi-loop techniques (dim. reg. in $\overline{\mathrm{MS}}$-scheme)
$+$ BPHZ renormalization prescription
in the non-relativistic limit

$\mathcal{C}_1$	**Method**	**Year**
$\mathcal{C}^{(1)} = \frac{25-6\pi}{12} = 0.512$	current + hard cut-off	2008
$\mathcal{C}^{(2)} = \frac{19-6\pi}{12} = 0.013$	density + hard cut-off	2008
	current & kinetic equations + soft cut-off	2008
	current + hard cut-off	2009
	density + hard cut-off	2011
	density + hard cut-off	2012
	current + hard cut-off & implicit regularization	2013
	current & density, DR + CR	2014
	lattice (tight-binding) simulations	2016
	Quantum Monte Carlo calculations	2016
(0.05)	Hartree-Fock simulations (self-screened)	2017
	current & density, DR + BPHZ	2017
$\mathcal{C}^{(3)} = \frac{11-3\pi}{6} = 0.263$	kinetic equations, hard cut-off	2008
	current & density, dim. reg.	2010
	lattice (tight-binding) simulations	2013
$(1/4 = 0.25)$	Hartree-Fock simulations (unscreened)	2017
$\mathcal{C}^* = (92 - 9\pi^2)/(18\pi) = 0.056$	DR + CR ($v/c \to 1$)	2012

Outline

Conclusion

We have:

- developped a field theoretic renormalization study of interaction effects in (planar) Dirac liquids (reduced QED models)
- initiated systematic study of perturbative structure of the models (new master integrals, *cf.*, Loopedia [Bogner, Panzer et al. '17])
- initiated systematic study of non-perturbative structure of the models (dim. reg. within Schwinger-Dyson)
- achieved a quantitative understanding of some of the properties of planar Dirac liquids:
  - ultra-relativistic limit ($v/c = 1$): $\mathcal{C}^*$, α_c, N_c (+ links to QED_3 clarified)
  - non-relativistic limit ($v/c = 0$): $\mathcal{C}$ (closer to experimental reality)

Strong base on which our (quantitative) understanding of odd-dimensional QFTs can be further improved (original approach to many-body physics in cond. matt. physics)

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