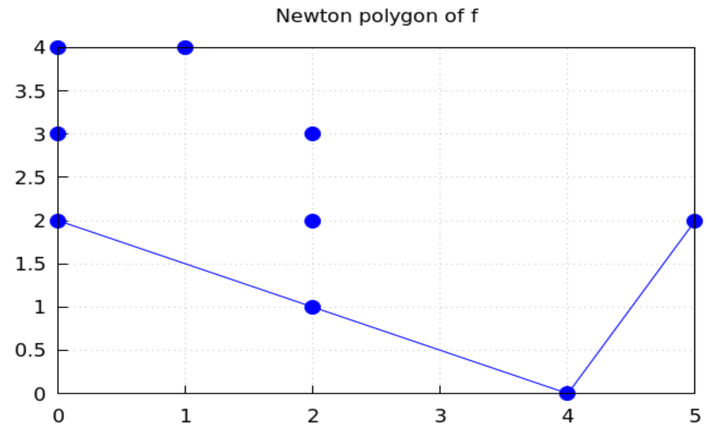
Puiseux expansions.

23 March 2020

Thez discussion below is directly inspired by the book “Algebraic curves” by R. Walker, especially the Chapter 4, “Formal power series”, and its sections 3.2 3.3 and 4. The aim is to “resolve the singularity” of an algebraic curve $f(x, y) = 0$ assuming f and its derivatives vanish at (0,0). One first draws the “Newton polygon”: for each monomial $ax^{\alpha_n}y^n$ one draws a point $P(n, \alpha_n)$. In fact for each value of n the only point of interest is the point with minimal value of exponent α_n above $(n, 0)$. If for some n there is no term $ax^{\alpha_n}y^n$, then no point is drawn above $(n, 0)$. Finally one draws the convex envelope from below of these points, that is all points are above or on the envelope. This is called the Newton polygon. For the example: $f = x^2 + 4x^3 + 6x^4 - 4x^4 * y + (-2x - 4x^2 - 2x^3)y^2 + y^4 + x^2y^5$



The points on the Newton polygon are denoted P_0, P_1, ... where P_n is above $(n, 0)$.The edges of the polygon are lines L_1 starting from P_0 ending at some P_l, on which there are 2 or more points, then L_1, etc. A line L from $P_j = (j, \alpha_j)$ to $P_k = (k, \alpha_k)$ has the equation $\alpha + \gamma\eta = \beta$ where γ is minus its slope:

$$\gamma = -\frac{\alpha_k - \alpha_j}{k - j}, \quad \beta = \alpha_k + \gamma k$$

Its characteristic property (convexity) is that for any point $P = (n, \alpha)$ in the Newton diagram one has $\alpha + \gamma n > \beta$ except if P is on L in which case there is equality.

This is directly related to the usefulness of the Newton polygon for our problem. One tries solutions of $f(x,y)=0$ of the form $y=Cx^\gamma$, valid to lowest order in x. Inserting into f, a monomial $ax^{\alpha_n}y^n$ becomes $aC^nx^{\alpha_n+n\gamma}$ so the exponent is minimized by taking points on an edge of the Newton polygon and keeps a constant value β on this edge, provided γ is minus the slope of the edge.

Concentrate on some edge L from P_j to P_k and let γ be minus its slope. Then $f(x,y)=(\sum_l a_lC^l)x^\beta+o(x^\beta)$ for $y=Cx^\gamma$ where the terms $o(x^\beta)$ vanish faster than x^β when $x\to 0$. Here $\sum_l a_lC^l$ is a non trivial polynomial in C of the form $C^j\sum_{h=0}^{k-j}a_{j+h}C^h$. Thus C is determined by an equation $\phi(C)=0$ where ϕ is a polynomial of degree $(k-j)$. Note that if one used a line containing only one point P it would not be possible to find an appropriate C. Hence the line must contain at least two points P (in this case ϕ is of degree 1), or more. This justifies that only edges of the Newton polygon are to be considered.

Finally the strategy is then, for each line L, to choose a solution C of $\phi(C)=0$ and then define y_1 so that $y=x^\gamma(C+y_1)$. Inserting into $f(x,y)=0$ this will produce an equation between x and y_1, that one renormalises for reasons which will appear below, by defining:

$$f_1(x,y_1)=x^{-\beta}f(x,x^\gamma(C+y_1))$$

Due to the choice of C terms of lowest order will disappear in $f_1(x,y_1)$ and one proceeds recursively, constructing the Newton polygon of f_1 etc.

However it is very important to notice that, even if all exponents are integers in the initial equation $f=0$, the exponent γ has a denominator $(k-j)$ hence fractional exponents will appear. To cope with this situation one introduces the field of fractional formal power series. Such a series is a semi-infinite formal series of the form

$$\sum_{n=-N}^{+\infty}a_n(x^{1/m})^n=a_{-N}x^{-N/m}(1+\sum_{n=1}^{\infty}b_nx^{n/m})$$

Note here that the summation is finite on the left, limited to a $-N$ (which may be positive) and that the denominator m is *fixed* for all fractional exponents of a given expansion, but is generally arbitrary. Hence given two so-called "Puiseux expansions" one can find a common denominator and write them as ordinary formal series of a given variable $t=x^{1/m}$. It is then obvious that one can formally multiply or divide series of the form $(1+\sum_{n=1}^\infty b_nt^n)$, i.e. at each order one has only a finite number of operations to perform. For example $1/(1+b_1t+b-t^2+\cdots)=1+c_1t+c_2t^2+\cdots$ requires

$$b_1+c_1=0,\quad b_2+b_1c_1+c_2=0,\quad \cdots$$

so c_1 then c_2 etc are successively determined. It is then clear that these semi-infinite formal series form a field. The Newton-Puiseux construction ensures that in fact this field is algebraically closed, that is for any polynomial in y with coefficients Puiseux series in x: $a_0(x)+a_1(x)y+\cdots+a_n(x)y^n$ one can find a root y=y(x) which is a Puiseux series in x, and thus decompose completely the polynomial as $\prod_k (y - y_k(x))$ (by recursion). Hence the polynomial equation $f(x,y) = 0$ of degree n in y has exactly n solutions $y = y_k(x)$ where the y_k are Puiseux expansions. This is what we show below. Note that since the polynomial $\sum_k a_k(x)y^k$ has only a finite number of coefficients $a_k(x)$ one can assume a common denominator m for all the fractional exponents appearing in f.

Let us come back to the line L of the Newton polygon of minus slope γ. Since α_j and α_k are of the form n_j/m and n_k/m it is clear that γ is rational of the form p/q where p and q are relatively prime. Hence for any P_l on line L one has $q(\alpha_l - \alpha_j) = -p(l - j)$ thus q divides $(l - j)$ so l is of the form $l = j + sq$ for some integer s. This implies that in the above equation for C, $C^j \sum_{h=0}^{k-j} a_{j+h}C^h = 0$, h is in fact of the form sq, so that only c^q appears in the equation. So $\phi(C) = \phi_1(C^q)$ where ϕ_1 is a polynomial of degree $(k - j)/q$. Instances of this situation occur in the examples below. As long as $q > 1$ reduction of the degree of the equation on C occurs, but also potentially the fractional exponents appearing in the computation may get an extra factor q (since γ has it) so it is capital that at some point in the chain $f \to f_1 \to f_2 \to \cdots$ the denominator q becomes equal to 1 and stays there, as we see below. This implies that the series we construct has fractional exponents with bounded denominators i.e. is a Puiseux series.

Another remark is that we are looking for a solution of the form

$$y = C_0 x^{\gamma_0} + C_1 x^{\gamma_0 + \gamma_1} + \cdots$$

where the exponents are increasing, so that γ_0 may be negative but all other exponents are positive. But γ_1 is precisely the exponent that we find analyzing f_1 since then $y_1 = x^{\gamma - 1}(C_1 + y_2)$, etc. This implies that when analyzing the Newton diagrams of f_1, f_2, etc. one must keep only the lines of negative slope. But by convexity these are the lines starting from P_0 and ending when the lines become flat, because afterwards they will be of positive slope.

In fact the set of lines starting from some P_0 becomes flat on the horizontal axis for the following reason. Consider what happens to a term $ax^{\alpha_n}y^n$ when going from f to f_1. Of course it produces a term $ax^{\alpha_n + \gamma n - \beta}(y_1 + C)^n$ in f_1. Expanding $(y_1 + C)^n$ produces points to the left of the point with y_1^n. Importantly these points have a strictly positive exponent in x, except if P is on the line L in which case the exponent vanishes. This means that points on L produce terms on the horizontal axis of the form $(m, 0)$ for all values $0 <= m <= n$, except of course if their coefficient vanishes. This occurs certainly for $m = 0$ for in this case we are looking at the term ax^0C^n in the above equation, and the sum of all such terms for points on L is precisely the equation $\phi(C) = 0$.

It is possible to find precisely the value r such that the Newton polygon of f_1 has a first point $(r, 0)$, so that the only relevant edges are $P_0, P_1, \cdots, P_{r-1}P_r$ (or edges with several such points on them). To do that, suppose that C corresponds to a multiple root of order r of ϕ (perhaps $r = 1$) so that $\phi(\xi) = (\xi - C)^r\psi(\xi)$ and $\psi(C) \neq 0$. Then the sum of the above terms for points of L is clearly $(y_1 + C)^j\phi(y_1 + C) = y_1^r(y_1 + C)^j\psi(y_1 + C)$ which starts from y_1^r with a non vanishing coefficient $C^j\psi(C)$ and produce other terms with higher powers of y_1 which may vanish. This implies that the first point P_n on the horizontal axis is just P_r. In particular if the root is simple, $r = 1$ and there is just one relevant edge of horizontal width 1. We see below that in this case one can solve $f_1(x, y_1) = 0$ in a much simpler way.

Note that r is the multiplicity of a root of ϕ and the degree of ϕ is $(k-j)$ so $r <= (k-j)$. Hence the horizontal width of an edge of f_1 is smaller than the horizontal width of the edge of f from which it is derived. In the successive steps f, f_1,f_2 etc. such widths may only decrease, so at some point they must become stationnary, and one gets $r = r_0$. In this case ϕ can only be of the form $(\xi - C)^{r_0}$ otherwise if the exponent is smaller than r_0 one would get a point at the left of $(r, 0)$ at the next step. This implies that the first edge of the Newton polygon is just P_0P_r, and that the denominator q in γ is 1. Otherwise we have $\phi(\xi) = \phi_1(\xi^q)$ which is incompatible with $\phi(x) = (\xi - C)^r$ since the right hand side expands on all powers of ξ not only on powers multiple of q. Since $q = 1$, γ is an integer and no new denominator can appear in the fractional exponents. From there one can iterate the Puiseux construction indefinitely from the unique edge that appears at the leftmost position of the Newton polygon at each step, and so get the Puiseux expansion of a root of $f(x, y) = 0$ as claimed above. This finally proves the theorem that the field of fractional formal series is algebraically closed.

However there is a shortcut when we get at a stage where there is an edge P_0P_1 of horizontal width 1. In this case one can simply insert a power series development of the variable $x^{1/m}$ in the equation $f_j(x, y) = 0$ and the terms are successively determined. Replacing $x^{1/m}$ by a variable t one can assume $m = 1$. Suppose for example that one wants to solve $f(x, y) = 0$ where $f = a_0(x) + a_1(x)y + \cdots$ where by hypothesis $a_0(x) = a^0_n x^n + a^0_{n+1}x^{n+1} + \cdots$ and $a_1(x) = 1 + a^1_1 x + a^1_2 x^2 + \cdots$ etc. This implies that the points $P_0 = (0, n)$ and $P_1 = (1, 0)$ are as assumed, and the coeff 1 is chosen by normalisation of f. Then at lowest order one must have $y = -a^0_n x^n$ and then expanding $y = -a^0_n x^n + b_1 x^{n+1} + b_2 x^{n+2} + \cdots$ one gets successively equations $b_1 + a^0_{n+1} - a^0_n a^1_1 = 0$, $b_2 + a^0_{n+2} - a^0_n a^2_1 + b_1 a^1_1 + \cdots = 0$. We see that b_n is determined in terms of the previous $b_1, \cdots, b_{n-1}$ and the coefficients a^k_j.

From the geometric viewpoint consider a solution y expanded as a formal series in $x^{1/m}$ with m minimal. Then one can introduce a new variable t such that $x = t^m$ and $y = \sum_k a_k t^k$. A priori this is a formal power series but it can be shown that it is in fact a convergent power series defining an analytic function in some disk of convergence. This is because $f(x, y) = 0$ has solutions $y(x)$ analytic in x for $x \neq 0$, and thus $y(t)$ can be analytically continued at $t = 0$. The point $p(t) = (t^m, y(t))$ depends analytically of the local parameter t and describes a smooth branch of the curve $f(x, y) = 0$. This is also called a “place” in a more algebraic context. The point $(0, 0)$ is called the center of the place. What we have proved shows that each point on the curve is the center of at least one place. Moreover for each place like above there are exactly m roots $y(x)$ of $f(x, y) = 0$ which exchange between themselves by performing rotations around the origin. Conversely each root belongs to exactly one place. Finally at the singular point $(0, 0)$ we have an intersection of several different places. If each one has an associated m_j one must have $\sum_j m_j = n$ the degree in y of f.

```
➔ ;

➔ load(draw);
(%o1) /usr/share/maxima/5.42.2/share/draw/draw.lisp

➔ kill(all);
(%o0) done
```

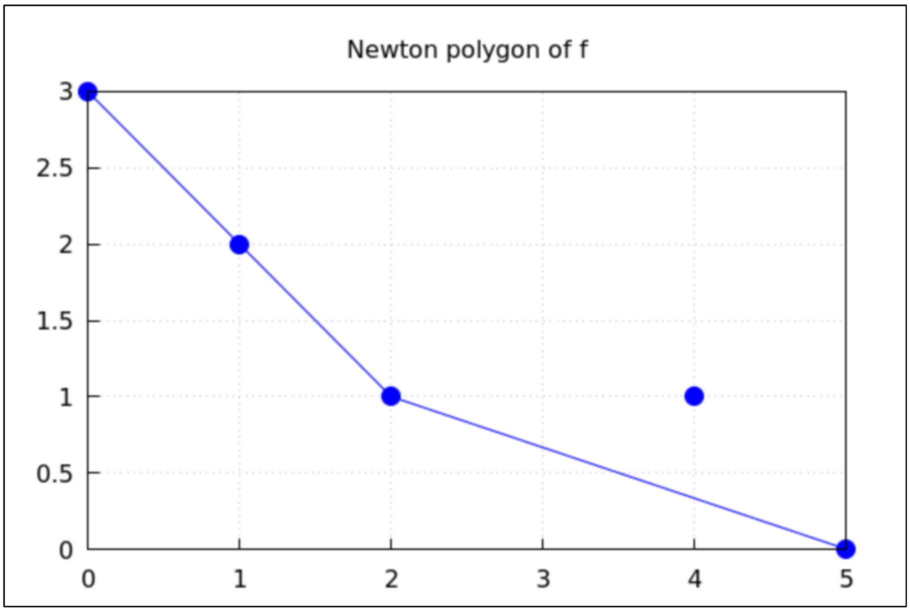
How to find the branches of f=0 around (0,0).

```
➔ f:-x^3+x^4-2*x^2*y-x*y^2+2*x*y^4+y^5;
```

(f) $y^5+2\,x\,y^4-x\,y^2-2\,x^2\,y+x^4-x^3$

```
➔ wxdraw2d(title="Newton polygon of f",
    grid=true,point_type=filled_circle,
    point_size=2, points([4],[1]),
    points_joined=true,
    points([0,1,2,5],[3,2,1,0]));
```

(%t2)



(%o2)

P_0=(0,3), P_1=(1,2), P_2=(2,1), P_3 absent, P_4=(4,1), P_5=(5,0)
Newton polygon has 2 edges, P_0,P_1,P_2 and P_2,P_5.
First edge has (minus) slope = γ =1, and equation α_j+γ*j=β for points (j,α_j) on it, here j=0,1,2, so β=3. One tries solution y=C*x^γ to lowest order in x,
-x*y^2-2*x^2*y-x^3=-x^3*(1+2*C+C^2)
so C=-1 double root. Then one sets y=x^γ*(C+y_1) and define f_1=x^(-β)*f expressed as function of x and y_1. Then iterate.

```
➔ y:x*(-1+y_1);
```

(y) $x\,(y_1-1)$

```
➔ f_1: subst(y,'y,x^(-3)*f);
```

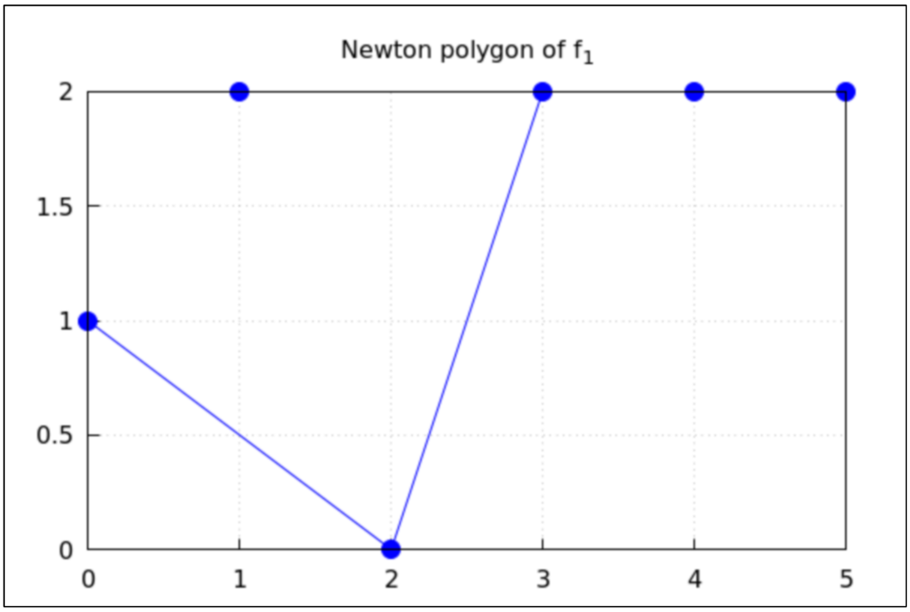
(f_1) $$\frac{-2\,x^3\,(y_1-1)+x^5\,(y_1-1)^5+2\,x^5\,(y_1-1)^4-x^3\,(y_1-1)^2+x^4-x^3}{x^3}$$

```
➔ f_1:rat(f_1,y_1);
```

(f_1) $$x^2\,{y_1}^5-3\,x^2\,{y_1}^4+2\,x^2\,{y_1}^3+(2\,x^2-1)\,{y_1}^2-3\,x^2\,y_1+x^2+x$$

```
➔ wxdraw2d(title="Newton polygon of f_1",
    grid=true,point_type=filled_circle,
    point_size=2, points([1],[2]),
    points_joined=true,
    points([0,2,3,4,5],[1,0,2,2,2]));
```

(%t6)



(%o6)

From now on only look at the first edge, so γ=1/2,β=1. C is determined

by y_1=C*√x in -y_1^2+x=0 so C=±1. Take C=1.

```
➔ y_1:x^(1/2)*(1+y_2);
```

(y_1) $$\sqrt{x}\,(y_2+1)$$

```
➔ f_2: subst(y_1,'y_1,x^(-1)*f_1);
```

(f_2) $$(x^{9/2}\,(y_2+1)^5-3\,x^4\,(y_2+1)^4+2\,x^{7/2}\,(y_2+1)^3+x\,(2\,x^2-1)\,(y_2+1)^2-3\,x^{5/2}\,(y_2+1)+x^2+x)/x$$

```
➔ f_2:collectterms(expand(f_2),y_2);
```

(f_2) $$x^{7/2}\,{y_2}^5+(5\,x^{7/2}-3\,x^3)\,{y_2}^4+(10\,x^{7/2}-12\,x^3+2\,x^{5/2})\,{y_2}^3+(10\,x^{7/2}-18\,x^3+6\,x^{5/2}+2\,x^2-1)\,{y_2}^2+(5\,x^{7/2}-12\,x^3+6\,x^{5/2}+4\,x^2-3\,x^{3/2}-2)\,y_2+x^{7/2}-3\,x^3+2\,x^{5/2}+2\,x^2-3\,x^{3/2}+x$$

At this point we know that the first edge of the Newton polygon of f_2 is of horizontal width 1, the lowest terms being -2*y_2+x, so one can develop y_2 in powers of x^(1/2) beginning by y_2=x/2+... Due to

this structure the coefficients are successively determined.

```
➔ yy:x/2+sum(c[i]*x^(1+i/2),i,1,4);
```

(yy) $$c_4\, x^3+c_3\, x^{5/2}+c_2\, x^2+c_1\, x^{3/2}+\frac{x}{2}$$

```
➔ f_2:subst(yy,y_2,f_2)$
```

```
➔ ff:taylor(f_2,x,0,3);
```

(ff) $$(-2\, c_1-3)\, x^{3/2}-\frac{(8\, c_2-7)\, x^2}{4}-\frac{(4\, c_3+2\, c_1-1)\, x^{5/2}}{2}+(-2\, c_4-c_2-c_1^2-3\, c_1-1)\, x^3+\ldots$$

```
➔ eqs: makelist(coeff(ff,x^(1+1/2*n)),n,1,4);
```

(eqs) $$\left[-2\, c_1-3, -\frac{8\, c_2-7}{4}, -\frac{4\, c_3+2\, c_1-1}{2}, -2\, c_4-c_2-c_1^2-3\, c_1-1\right]$$

```
➔ sol:solve(eqs,makelist(c[i],i,1,4));
```

(sol) $$\left[\left[c_1=-\frac{3}{2}, c_2=\frac{7}{8}, c_3=1, c_4=\frac{3}{16}\right]\right]$$

```
➔ y_2:expand(subst(sol[1],yy));
```

(y_2) $$\frac{3\, x^3}{16}+x^{5/2}+\frac{7\, x^2}{8}-\frac{3\, x^{3/2}}{2}+\frac{x}{2}$$

From this we compute back y_1 and y and check that f vanishes at order x^7.

```
➔ y_1:expand(''y_1);
```

(y_1) $$\frac{3\, x^{7/2}}{16}+x^3+\frac{7\, x^{5/2}}{8}-\frac{3\, x^2}{2}+\frac{x^{3/2}}{2}+\sqrt{x}$$

```
➔ y:expand(''y);
```

(y) $$\frac{3\, x^{9/2}}{16}+x^4+\frac{7\, x^{7/2}}{8}-\frac{3\, x^3}{2}+\frac{x^{5/2}}{2}+x^{3/2}-x$$

```
➔ taylor(''f,x,0,8);
```

(%o18)/T/ $$-3\, x^{15/2}-\frac{765\, x^8}{64}+\ldots$$

The second solution with C=-1 obviously consists in doing
√x -> -√x. It is the analytic continuation doing a circle around 0.
Both define one "place" or "branch" on the curve, of parametric equation
(x=t^2,y=-t+t^3+t^5/2-3*t^6/2+...)

Now take care of the second edge of the Newton polygon
γ = 1/3 and β = 5/3 so β = α_j + γ*j for P_2 and P_5.
C is given by y=C*x^γ in -x*y^2+y^5 =y^2(C^3*x -x)=0
hence C^3=1, giving C=1, C=ω, C=ω^2 where
ω=exp(2*%i*%pi/3).
Take C=1.

```
➔ kill(all);
```

(%o0) $$done$$

```
➔ f:-x^3+x^4-2*x^2*y-x*y^2+2*x*y^4+y^5;
```

(f) $$y^5+2\,x\,y^4-x\,y^2-2\,x^2\,y+x^4-x^3$$

```
➔ y:x^(1/3)*(1+y_1);
```

(y) $$x^{1/3}\,(y_1+1)$$

```
➔ f_1:subst(y,'y,x^(-5/3)*f);
```

(f_1) $$\frac{x^{5/3}\,(y_1+1)^5+2\,x^{7/3}\,(y_1+1)^4-x^{5/3}\,(y_1+1)^2-2\,x^{7/3}\,(y_1+1)+x^4-x^3}{x^{5/3}}$$

```
➔ f_1:collectterms(expand(f_1),y_1);
```

(f_1) $$y_1{}^5+(2\,x^{2/3}+5)\,y_1{}^4+(8\,x^{2/3}+10)\,y_1{}^3+(12\,x^{2/3}+9)\,y_1{}^2+(6\,x^{2/3}+3)\,y_1+x^{7/3}-x^{4/3}$$

The first edge of the Newton polygon of f_1 is of horizontal width 1
the relevant terms are 3*y_1-x^(4/3) so f_1 expands in series of x^(1/3)
beginning by y_1=x^(4/3)/3+...

```
➔ yy: x^(4/3)/3+sum(c[i]*x^(4/3+i/3),i,1,6);
```

(yy) $$c_6\,x^{10/3}+c_5\,x^3+c_4\,x^{8/3}+c_3\,x^{7/3}+c_2\,x^2+c_1\,x^{5/3}+\frac{x^{4/3}}{3}$$

```
➔ f_1:subst(yy,y_1,f_1)$
```

```
➔ ff:taylor(f_1,x,0,10/3);
```

(ff)
$$3\;c_1\;x^{5/3}+(3\;c_2+2)\;x^2+(3\;c_3+6\;c_1+1)\;x^{7/3}+(3\;c_4+6\;c_2+1)\;x^{8/3}+(3\;c_5+6\;c_3+6\;c_1)\;x^3+\frac{(9\;c_6+18\;c_4+18\;c_2+27\;c_1^2+4)\;x^{10/3}}{3}+\ldots$$

```
➔ eqs: makelist(coeff(ff,x^(1/3*n)),n,5,10);
```

(eqs)
$$[3\;c_1,3\;c_2+2,3\;c_3+6\;c_1+1,3\;c_4+6\;c_2+1,3\;c_5+6\;c_3+6\;c_1,\frac{9\;c_6+18\;c_4+18\;c_2+27\;c_1^2+4}{3}]$$

```
➔ sol:solve(eqs,makelist(c[i],i,1,6));
```

(sol)
$$[[c_1=0,c_2=-\frac{2}{3},c_3=-\frac{1}{3},c_4=1,c_5=\frac{2}{3},c_6=-\frac{10}{9}]]$$

```
➔ y_1:expand(subst(sol[1],yy));
```

(y_1)
$$-\frac{10\;x^{10/3}}{9}+\frac{2\;x^3}{3}+x^{8/3}-\frac{x^{7/3}}{3}-\frac{2\;x^2}{3}+\frac{x^{4/3}}{3}$$

```
➔ y:expand(''y);
```

(y)
$$-\frac{10\;x^{11/3}}{9}+\frac{2\;x^{10/3}}{3}+x^3-\frac{x^{8/3}}{3}-\frac{2\;x^{7/3}}{3}+\frac{x^{5/3}}{3}+x^{1/3}$$

```
➔ taylor(''f,x,0,6);
```

(%o12)/T/
$$2\;x^{16/3}-\frac{44\;x^{17/3}}{27}+\frac{16\;x^6}{3}+\ldots$$

This vanishes to order x^5, we see that for γ=1/3 many terms have to be computed for a modest gain in precision relative to f. Of course the solution for C=ω C=ω^2 are obtained by the substitution x -> ω*x and x -> ω^2*x and are the analytic continuations of the above

doing 1 and 2 circles around 0. Thse 3 solutions define one "place" of parametric equation (x=t^3,y=t+t^5/3-2*t^7/3-t^8/3+t^9+...).

```
➔ kill(all);
```

(%o0) *done*

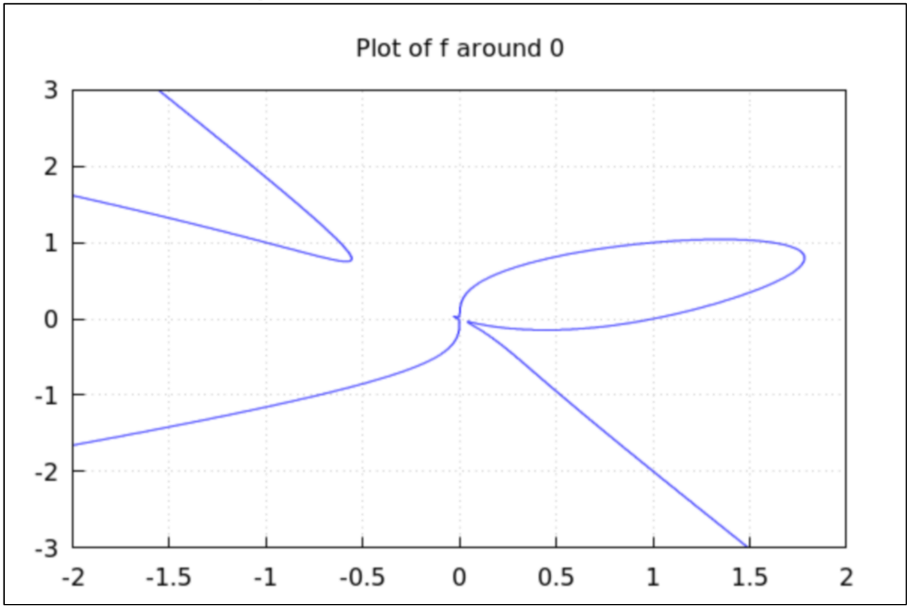
```
➔ f:-x^3+x^4-2*x^2*y-x*y^2+2*x*y^4+y^5;
```

(f)
$$y^5+2\;x\;y^4-x\;y^2-2\;x^2\;y+x^4-x^3$$

```
➔ wxdraw2d(title="Plot of f around 0",
      grid=true,ip_grid=[100,100],
      ip_grid_in=[10,10],
      implicit(f=-1e-16,x,-2,2,y,-3,3));
```

```
rat: replaced 1.0e−16 by 1/10000000000000000 = 1.0e−16
```

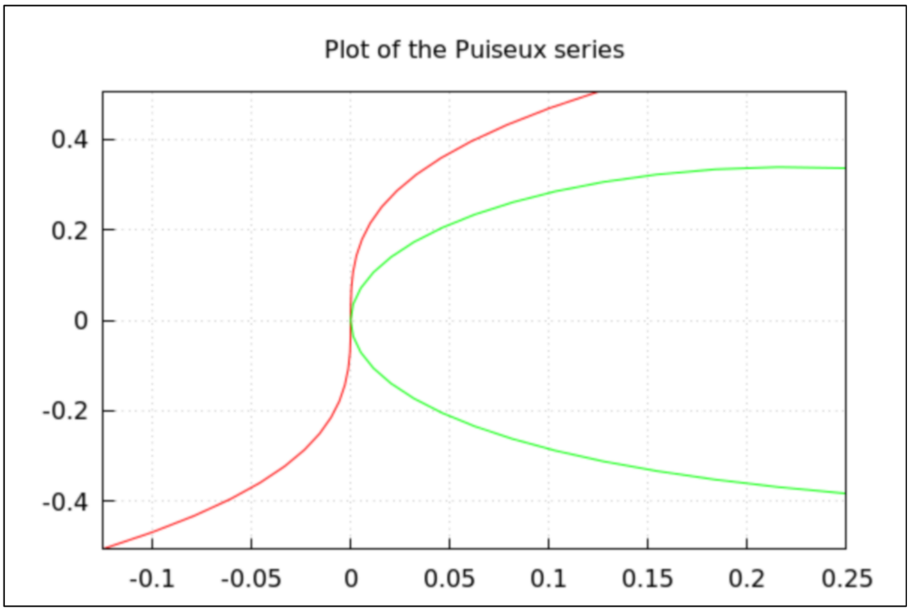
(%t2)



(%o2)

```
➔ wxdraw2d(title="Plot of the Puiseux series",
      grid=true,ip_grid=[100,100],
      ip_grid_in=[10,10],color=red,
  parametric(t^3,t+t^5/3-2*t^7/3,t,-.5,.5),
  color=green,
  parametric(t^2,-t+t^3+t^5/2-3*t^6/2,t,-.5,.5)
  );
```

(%t3)



(%o3)

We see that once more the implicit plotter has a problem near the singularity.

```
➔        kill(all);
```

(%o0) $done$

```
(%i2)    f:x^4-x^3*y+3*x^2*y^3-3*x*y^5+y^7;
```

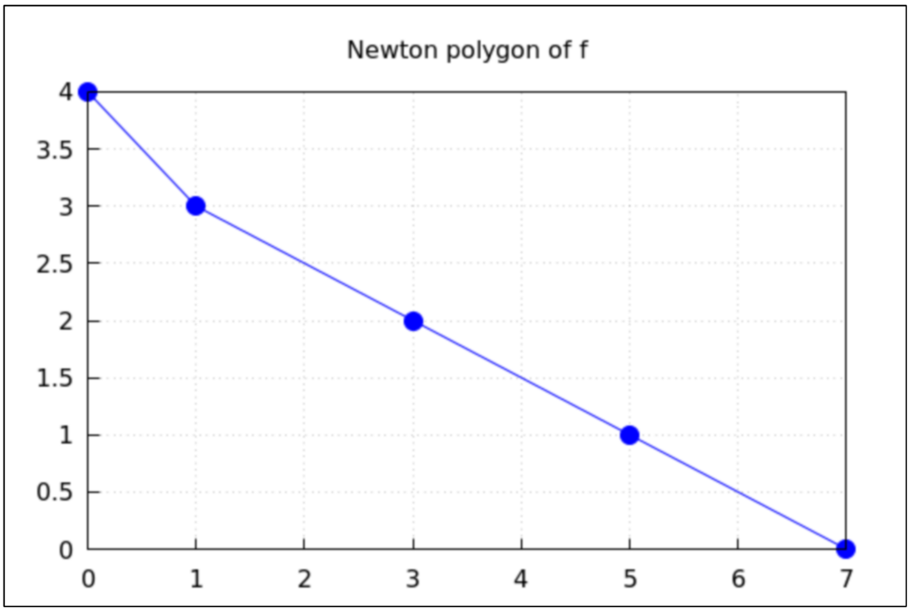
(f) $y^7-3\ x\ y^5+3\ x^2\ y^3-x^3\ y+x^4$

```
(%i3)    load(draw);
```

(%o3) $/usr/share/maxima/5.42.2/share/draw/draw.lisp$

```
(%i5)    wxdraw2d(title="Newton polygon of f",
            grid=true,point_type=filled_circle,
            point_size=2,
            points_joined=true,points([0,1,3,5,7],[4,3,2,1,0]));
```

(%t5)



(%o5)

First side γ=1,β=4, horizontal width1, y=Cx with
0=-x^3*y+x^4=x^4*(1-C)
so C=1. Next step will have width=1 and can be solved directly.

```
(%i9)    y:x*(1+y_1);
```

(y) $x\ (y_1+1\ )$

```
(%i10)   f_1:subst(y,'y,x^(-4)*f);
```

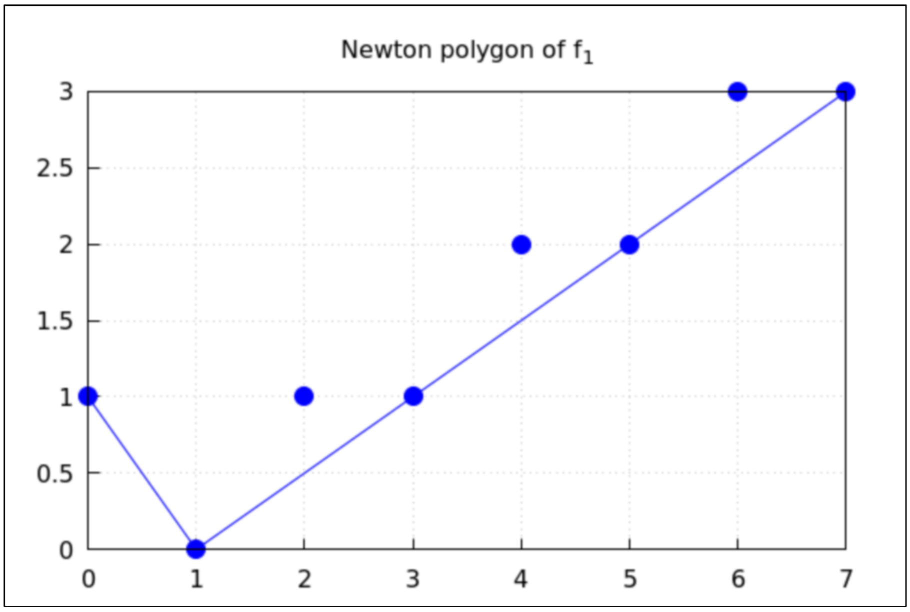
(f_1)
$$\frac{x^7\ (y_1+1\ )^7-3\ x^6\ (y_1+1\ )^5+3\ x^5\ (y_1+1\ )^3-x^4\ (y_1+1\ )+x^4}{x^4}$$

```
(%i11)   f_1:rat(f_1,y_1);
```

(f_1)
$$x^3\ {y_1}^7+7\ x^3\ {y_1}^6+(\,21\ x^3-3\ x^2\,)\ {y_1}^5+(\,35\ x^3-15\ x^2\,)\ {y_1}^4+(\,35\ x^3-30\ x^2+3\ x\,)\ {y_1}^3+(\,21\ x^3-30\ x^2+9\ x\,)\ {y_1}^2+(\,7\ x^3-15\ x^2+9\ x-1\,)\ y_1+x^3-3\ x^2+3\ x$$

```
(%i47)  wxdraw2d(title="Newton polygon of f_1",
            grid=true,point_type=filled_circle,
            point_size=2,points([2,4,6],[1,2,3]),
            points_joined=true,points([0,1,3,5,7],[1,0,1,2,3]));
```

(%t47)



(%o47)

```
(%i12)  yy:3*x+c_2*x^2+c_3*x^3+c_4*x^4+c_5*x^5;
```

(yy) $c_5\, x^5 + c_4\, x^4 + c_3\, x^3 + c_2\, x^2 + 3\, x$

```
(%i17)  ff:subst(yy,'y_1,f_1)$
```

```
(%i18)  ff:taylor(ff,x,0,5);
```

(ff) $(-c_2+24)\, x^2 + (-c_3+9\, c_2+37)\, x^3 + (-c_4+9\, c_3+39\, c_2-168)\, x^4 + (-c_5+9\, c_4+39\, c_3+9\, {c_2}^2-92\, c_2-621)\, x^5 + \ldots$

```
(%i19)  eqs:makelist(coeff(ff,x^n),n,2,5);
```

(eqs) $[-c_2+24, -c_3+9\, c_2+37, -c_4+9\, c_3+39\, c_2-168, -c_5+9\, c_4+39\, c_3+9\, {c_2}^2-92\, c_2-621]$

```
(%i20)  sol:solve(eqs,[c_2,c_3,c_4,c_5]);
```

(sol) $[[c_2=24, c_3=253, c_4=3045, c_5=39627]]$

```
(%i21)  y_1:subst(sol[1],yy);
```

(y_1) $39627\, x^5 + 3045\, x^4 + 253\, x^3 + 24\, x^2 + 3\, x$

```
(%i23)  y:expand(''y);
```

(y) $39627\, x^6 + 3045\, x^5 + 253\, x^4 + 24\, x^3 + 3\, x^2 + x$

```
(%i24)  taylor(''f,x,0,11);
```

(%o24)/T/ $543004\, x^{10} + 2831304\, x^{11} + \ldots$

Showing that f vanishes to order 9 with this value of y. Cope with next line P_1,P_3,P_5,P_7 and γ=p/q=1/2. Note the points are spaced by q=2. β=7/2. Equation for C is of the form φ(C^2)=0 with deg φ=3. One finds (C^2-1)^3=0. So C=±1 each being triple. First edge of next Newton polygon will be of width 3. So take C=1.

```
(%i17) kill(allbut(f));
```

(%o0) $$done$$

```
(%i1) y:x^(1/2)*(1+y_1);
```

(y) $$\sqrt{x}\,(y_1+1)$$

```
(%i2) f_1:expand(subst(y,'y,x^(-7/2)*f));
```

(f_1) $$y_1^7+7\,y_1^6+18\,y_1^5+20\,y_1^4+8\,y_1^3+\sqrt{x}$$

Edge with γ=1/6, β=1/2, width=3, 8*C^3+1=0 so C=-1/2 or -ω/2 or -ω^2/2. These are simple roots so next stage will be solved directly. Take -1/2

```
(%i21) y_1:x^(1/6)*(-1/2+y_2);
```

(y_1) $$x^{1/6}\left(y_2-\frac{1}{2}\right)$$

```
(%i22) f_2:subst(y_1,'y_1,x^(-1/2)*f_1)$
```

```
(%i23) f_2:collectterms(expand(f_2),y_2);
```

(f_2) $$x^{2/3}\,y_2^7+\left(7\,\sqrt{x}-\frac{7\,x^{2/3}}{2}\right)y_2^6+\left(\frac{21\,x^{2/3}}{4}-21\,\sqrt{x}+18\,x^{1/3}\right)y_2^5+\left(-\frac{35\,x^{2/3}}{8}+\frac{105\,\sqrt{x}}{4}-45\,x^{1/3}+20\,x^{1/6}\right)y_2^4+\left(\frac{35\,x^{2/3}}{16}-\frac{35\,\sqrt{x}}{2}+45\,x^{1/3}-40\,x^{1/6}+8\right)y_2^3+\left(-\frac{21\,x^{2/3}}{32}+\frac{105\,\sqrt{x}}{16}-\frac{45\,x^{1/3}}{2}+30\,x^{1/6}-12\right)y_2^2+\left(\frac{7\,x^{2/3}}{64}-\frac{21\,\sqrt{x}}{16}+\frac{45\,x^{1/3}}{8}-10\,x^{1/6}+6\right)y_2-\frac{x^{2/3}}{128}+\frac{7\,\sqrt{x}}{64}-\frac{9\,x^{1/3}}{16}+\frac{5\,x^{1/6}}{4}$$

This can be expanded in powers of x^(1/6) starting with y_2=-5*x^(1/6)/24.

```
(%i24) yy:-5*x^(1/6)/24+sum(c[i]*x^(1/6+i/6),i,1,5);
```

(yy)
$$c_5\,x+c_4\,x^{5/6}+c_3\,x^{2/3}+c_2\,\sqrt{x}+c_1\,x^{1/3}-\frac{5\,x^{1/6}}{24}$$

```
(%i25) ff:subst(yy,'y_2,f_2)$
```

```
(%i26) ff:taylor(ff,x,0,1);
```

(ff)
$$(6\,c_1+1)\,x^{1/3}+\frac{(10368\,c_2-8640\,c_1+289)\,\sqrt{x}}{1728}+\frac{(20736\,c_3-17280\,c_2-41472\,c_1^2-20160\,c_1-1207)\,x^{2/3}}{3456}+\frac{(248832\,c_4-207360\,c_3+(-995328\,c_1-241920)\,c_2+1036800\,c_1^3+118368\,c_1-4445)\,x^{5/6}}{41472}$$
$$+((663552\,c_5-552960\,c_4+(-2654208\,c_1-645120)\,c_3-1327104\,c_2^2+(5529600\,c_1+315648)\,c_2+884736\,c_1^3+276480\,c_1^2+277696\,c_1+4975)\,x)/110592+\ldots$$

```
(%i29) eqs: makelist(coeff(ff,x^(n/6)),n,2,6);
```

(eqs)
$$\left[6\,c_1+1,\frac{10368\,c_2-8640\,c_1+289}{1728},\frac{20736\,c_3-17280\,c_2-41472\,c_1^2-20160\,c_1-1207}{3456},\frac{248832\,c_4-207360\,c_3+(-995328\,c_1-241920)\,c_2+1036800\,c_1^3+118368\,c_1-4445}{41472},(663552\,c_5-552960\,c_4+(-2654208\,c_1-645120)\,c_3-1327104\,c_2^2+(5529600\,c_1+315648)\,c_2+884736\,c_1^3+276480\,c_1^2+277696\,c_1+4975)/110592\right]$$

```
(%i30) sol:solve(eqs,makelist(c[i],i,1,5));
```

(sol)
$$\left[\left[c_1=-\frac{1}{6},c_2=-\frac{1729}{10368},c_3=-\frac{91}{486},c_4=-\frac{231}{1024},c_5=-\frac{1870}{6561}\right]\right]$$

```
(%i31) y_2:expand(subst(sol[1],yy));
```

(y_2)
$$-\frac{1870\,x}{6561}-\frac{231\,x^{5/6}}{1024}-\frac{91\,x^{2/3}}{486}-\frac{1729\,\sqrt{x}}{10368}-\frac{x^{1/3}}{6}-\frac{5\,x^{1/6}}{24}$$

```
(%i32) y_1:expand(''y_1);
```

(y_1)
$$-\frac{1870\,x^{7/6}}{6561}-\frac{231\,x}{1024}-\frac{91\,x^{5/6}}{486}-\frac{1729\,x^{2/3}}{10368}-\frac{\sqrt{x}}{6}-\frac{5\,x^{1/3}}{24}-\frac{x^{1/6}}{2}$$

```
(%i33)  y:expand(''y);
```

(y)
$$-\frac{1870\,x^{5/3}}{6561}-\frac{231\,x^{3/2}}{1024}-\frac{91\,x^{4/3}}{486}-\frac{1729\,x^{7/6}}{10368}-\frac{x}{6}-\frac{5\,x^{5/6}}{24}-\frac{x^{2/3}}{2}+\sqrt{x}$$

```
(%i45)  taylor(expand(''f),x,0,16/3)
        ;
```

(%o45)/T/
$$\frac{240352783\,x^{31/6}}{107495424}+\frac{733153717\,x^{16/3}}{644972544}+\ldots$$

So we get a solution up to order x^5. Writing 1/6=1/2-1/3 each term in the expansion is of the form (x^(1/2))^n*(x^(1/3))^m. Now the solutions with C=-ω/2 and ω^2/2 are obtained by replacing x^(1/3) -> ω*x^(1/3) or x^(1/3) -> ω^2*x^(1/3) in each such term. Also in the first step we could chose triple roots C=±1, and the other choice leads to the replacement x^(1/2) -> -x^(1/2). So in total we have 6 analytic continuations obtained by rotations around x=0 of the same place (x=t^6,y=t^3-t^4/2-5t^5/24-t^6/6+...), plus the first solution, so all the 7 roots of the equation, but only 2 real traces.

```
(%i49)  kill(all);
```

(%o0) *done*

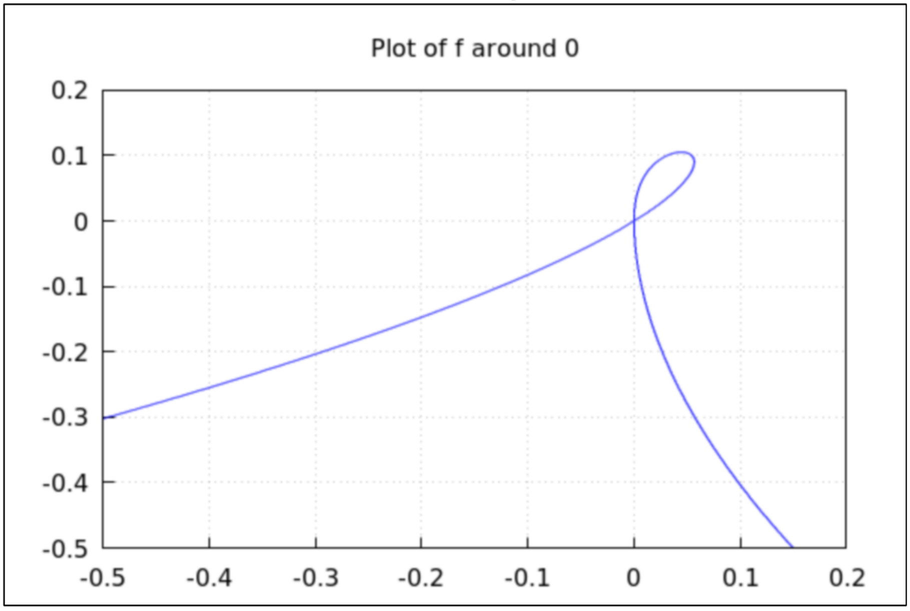
```
(%i1)   f:x^4-x^3*y+3*x^2*y^3-3*x*y^5+y^7;
```

(f)
$$y^7-3\,x\,y^5+3\,x^2\,y^3-x^3\,y+x^4$$

```
(%i5)   wxdraw2d(title="Plot of f around 0",
            grid=true,ip_grid=[100,100],
            ip_grid_in=[10,10],
            implicit(f=-1e-6,x,-.5,.2,y,-.5,.2));
```

```
rat: replaced 9.999999999999999e-7 by 1/1000000 = 9.999999999999999e-7
```

(%t5)



(%o5)

```
→        ;
```

```
→        kill(all);
```

(%o0) $done$

```
→        f:2*x^5-x^3*y+2*x^2*y^2-x*y^3+2*y^5;
```

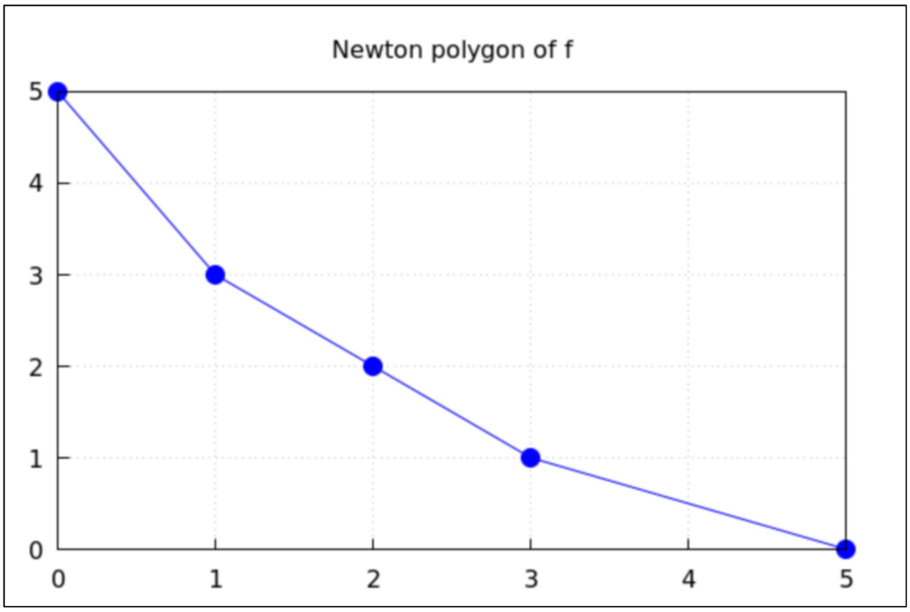
(f) $2\,y^5-x\,y^3+2\,x^2\,y^2-x^3\,y+2\,x^5$

```
→        load(draw);
```

(%o2) $/usr/share/maxima/5.42.2/share/draw/draw.lisp$

```
→        wxdraw2d(title="Newton polygon of f",
            grid=true,point_type=filled_circle,
            point_size=2,
            points_joined=true,points([0,1,2,3,5],[5,3,2,1,0]));
```

(%t3)



(%o3)

First side, P_0,P_1, γ=2,β=5,C=2, since x^3(2x^2-y)=0 gives C=2 for y=C*x^2

```
→        y:x^2*(2+y_1);
```

(y) $x^2\,(y_1+2)$

```
→        f_1:subst(y,'y,x^(-5)*f);
```

(f_1) $$\frac{2\,x^{10}\,(y_1+2)^5-x^7\,(y_1+2)^3+2\,x^6\,(y_1+2)^2-x^5\,(y_1+2)+2\,x^5}{x^5}$$

```
→        f_1:rat(f_1,y_1);
```

(f_1) $2\,x^5\,{y_1}^5+20\,x^5\,{y_1}^4+(80\,x^5-x^2)\,{y_1}^3+(160\,x^5-6\,x^2+2\,x)\,{y_1}^2+(160\,x^5-12\,x^2+8\,x-1)\,y_1+64\,x^5-8\,x^2+8\,x$

First side of width 1, f_1 develops in powers of x. starting with y_1=8*x

```
➔ yy:8*x+c_2*x^2+c_3*x^3+c_4*x^4;
```

(yy) $c_4\,x^4+c_3\,x^3+c_2\,x^2+8\,x$

```
➔ ff:subst(yy,'y_1,f_1)$
```

```
➔ ff:taylor(ff,x,0,4);
```

(ff) $(-c_2+56)\,x^2+(-c_3+8\,c_2+32)\,x^3+(-c_4+8\,c_3+20\,c_2-384)\,x^4+\ldots$

```
➔ eqs:makelist(coeff(ff,x,n),n,2,4);
```

(eqs) $[-c_2+56,-c_3+8\,c_2+32,-c_4+8\,c_3+20\,c_2-384]$

```
➔ sol:solve(eqs,[c_2,c_3,c_4]);
```

(sol) $[[c_2=56,c_3=480,c_4=4576]]$

```
➔ y_1:expand(subst(sol[1],yy));
```

(y_1) $4576\,x^4+480\,x^3+56\,x^2+8\,x$

```
➔ y:expand(''y);
```

(y) $4576\,x^6+480\,x^5+56\,x^4+8\,x^3+2\,x^2$

```
➔ taylor(expand(''f),x,0,11);
```

(%o14)/T/ $46656\,x^{10}+124672\,x^{11}+\ldots$

We see that f vanishes to order x^9 for this value of y.

Then second side of the Newton polygon, P_1,P_2,P_3, γ=1,β=4.
Equation for C is x^3*y-2*x^2*y^2+x*y^3=x^4*(C-1)^2=0.
So C=1 double root and next edge is of width 2.

```
➔ kill(allbut(f));
```

(%o0) $done$

```
➔ y:x*(1+y_1);
```

(y) $x\,(y_1+1)$

```
➔ f_1:subst(y,'y,x^(-4)*f);
```

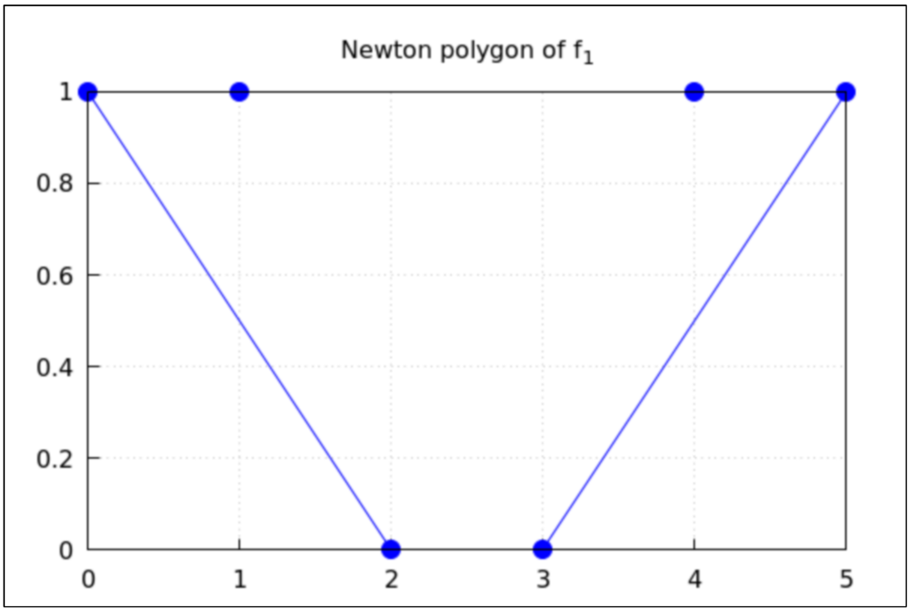
(f_1) $$\frac{2\,x^5\,(y_1+1)^5-x^4\,(y_1+1)^3+2\,x^4\,(y_1+1)^2-x^4\,(y_1+1)+2\,x^5}{x^4}$$

```
➔ f_1:rat(f_1);
```

(f_1) $2\,x\,{y_1}^5+10\,x\,{y_1}^4+(20\,x-1)\,{y_1}^3+(20\,x-1)\,{y_1}^2+10\,x\,y_1+4\,x$

```
➔ wxdraw2d(title="Newton polygon of f_1",
     grid=true,point_type=filled_circle,
     point_size=2,points([1,4],[1,1]),
     points_joined=true,points([0,2,3,5],[1,0,0,1]));
```

(%t4)



(%o4)

First side is P_0,P_2, with γ=1/2,β=1, equation -y_1^2+4*x=0 so C=±1.
Take C=1.
Simple racine so next step has width 1.

```
➔ y_1:x^(1/2)*(2+y_2);
```

(y_1) $\sqrt{x}\,(y_2+2)$

```
➔ f_2:subst(y_1,'y_1,x^(-1)*f_1)$
```

```
➔ f_2:collectterms(expand(f_2),y_2);
```

(f_2) $2\,x^{5/2}\,y_2^5+(20\,x^{5/2}+10\,x^2)\,y_2^4+(80\,x^{5/2}+80\,x^2+20\,x^{3/2}-\sqrt{x})\,y_2^3+(160\,x^{5/2}+240\,x^2+120\,x^{3/2}+20\,x-6\,\sqrt{x}-1)\,y_2^2+(160\,x^{5/2}+320\,x^2+240\,x^{3/2}+80\,x-2\,\sqrt{x}-4)\,y_2+64\,x^{5/2}+160\,x^2+160\,x^{3/2}+80\,x+12\,\sqrt{x}$

y_2 develops in series of x^(1/2) starting with 3*x^(1/2).

```
➔ yy:3*x^(1/2)+c_2*x+c_3*x^(3/2)+c_4*x^2;
```

(yy) $c_4\,x^2+c_3\,x^{3/2}+c_2\,x+3\,\sqrt{x}$

```
➔ ff:subst(yy,y_2,f_2)$
```

```
➔ ff:taylor(ff,x,0,2);
```

(ff) $(-4\,c_2+65)\,x+(-4\,c_3-8\,c_2+346)\,x^{3/2}+(-4\,c_4-8\,c_3-c_2^2+44\,c_2+1033)\,x^2+\ldots$

```
→ eqs:makelist(coeff(ff,x,n/2),n,2,4);
```

(eqs) $$[-4\,c_2+65,-4\,c_3-8\,c_2+346,-4\,c_4-8\,c_3-{c_2}^2+44\,c_2+1033]$$

```
→ sol:solve(eqs,[c_2,c_3,c_4]);
```

(sol) $$[[c_2=\frac{65}{4},c_3=54,c_4=\frac{16831}{64}]]$$

```
→ y_2:collectterms(subst(sol[1],yy),x);
```

(y_2) $$\frac{16831\,x^2}{64}+54\,x^{3/2}+\frac{65\,x}{4}+3\,\sqrt{x}$$

```
→ y_1:expand(''y_1);
```

(y_1) $$\frac{16831\,x^{5/2}}{64}+54\,x^2+\frac{65\,x^{3/2}}{4}+3\,x+2\,\sqrt{x}$$

```
→ y:expand(''y);
```

(y) $$\frac{16831\,x^{7/2}}{64}+54\,x^3+\frac{65\,x^{5/2}}{4}+3\,x^2+2\,x^{3/2}+x$$

```
→ taylor(expand(''f),x,0,8);
```

(%o16)/T/ $$4448\,x^{15/2}+\frac{3909761\,x^8}{128}+\ldots$$

So this y is a correct solution up to order x^7. Of course the second solution with C=-2 is obtained by x^(1/2)-> -x^(1/2). Now to the third side P_3,P_5 with γ=1/2,β=5/2. Equation for C is 2*y^5-x*y^3=y^3(2*x*C^2-x)=0 so C=±1/√2. Take C=1/√2.

```
→ kill(allbut(f));
```

(%o0) *done*

```
→ y:x^(1/2)*(1/sqrt(2)+y_1);
```

(y) $$\sqrt{x}\left(y_1+\frac{1}{\sqrt{2}}\right)$$

```
→ f_1:subst(y,'y,x^(-5/2)*f)$
```

```
→ f_1:collectterms(expand(f_1),y_1);
```

(f_1) $$2\,{y_1}^5+5\,\sqrt{2}\,{y_1}^4+9\,{y_1}^3+\left(2\,\sqrt{x}+\frac{7}{\sqrt{2}}\right){y_1}^2+\left(-x+2^{3/2}\,\sqrt{x}+1\right)y_1+2\,x^{5/2}-\frac{x}{\sqrt{2}}+\sqrt{x}$$

First side of the Newton polygon of f_1 has width 1 so y_1 develops in powers of x^(1/2) beginning by y_1=-x^(1/2).

```
➔ yy:-x^(1/2)+c_2*x+c_3*x^(3/2)+c_4*x^2;
```

(yy) $$c_4\,x^2+c_3\,x^{3/2}+c_2\,x-\sqrt{x}$$

```
➔ ff:subst(yy,'y_1,f_1)$
```

```
➔ ff:taylor(expand(ff),x,0,2);
```

(ff) $$\left(c_2+\sqrt{2}\right)x+\left(c_3-5\,\sqrt{2}\,c_2-6\right)x^{3/2}+\frac{\left(\sqrt{2}\,c_4-10\,c_3+7\,{c_2}^2+11\,\sqrt{2^3}\,c_2+10\right)x^2}{\sqrt{2}}+\ldots$$

```
➔ eqs:makelist(coeff(ff,x,n/2),n,2,4);
```

(eqs) $$\left[c_2+\sqrt{2},c_3-5\,\sqrt{2}\,c_2-6,\frac{\sqrt{2}\,c_4-10\,c_3+7\,{c_2}^2+11\,\sqrt{2^3}\,c_2+10}{\sqrt{2}}\right]$$

```
➔ sol:solve(eqs,[c_2,c_3,c_4]);
```

(sol) $$\left[\left[c_2=-\sqrt{2},c_3=-4,c_4=-5\,2^{3/2}\right]\right]$$

```
➔ y_1:collectterms(subst(sol[1],yy),x);
```

(y_1) $$-5\,2^{3/2}\,x^2-4\,x^{3/2}-\sqrt{2}\,x-\sqrt{x}$$

```
➔ y:expand(''y);
```

(y) $$-5\,2^{3/2}\,x^{5/2}-4\,x^2-\sqrt{2}\,x^{3/2}-x+\frac{\sqrt{x}}{\sqrt{2}}$$

```
➔ taylor(expand(''f),x,0,6);
```

(%o11)/T/ $$58\,x^5-7\,\sqrt{2^9}\,x^{11/2}-176\,x^6+\ldots$$

Solution y is correct up to order 9/2. The solution for C=-1/√2 is obtained by sqrt(x/2) -> -sqrt(x/2). In total we have all the five roots of the y^5 equation. Solutions in ±x^(1/2) are analytic continuations of each over and correspond to the same "place" or "branch" of the curve. So (0,0) is the center of 3 places:

1) (x=t,y=2*t^2+8*t^3+...)
2) (x=t^2,y=t^2+2*t^3+...)
3) (x=t^2,y=t/√2 -t^2+...)

```
➔ kill(allbut(f));
```

(%o0) *done*

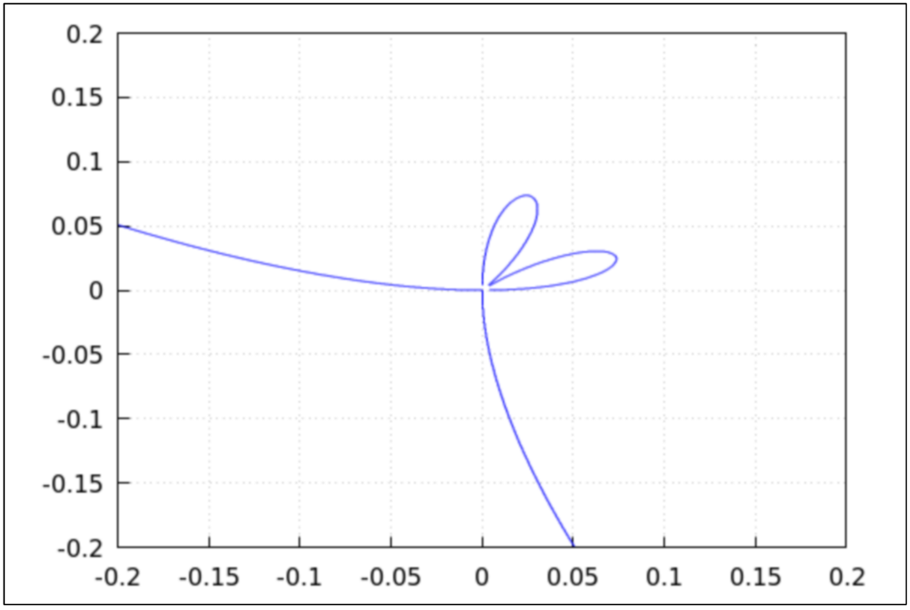
```
➔ load(draw);
```

(%o1) */usr/share/maxima/5.42.2/share/draw/draw.lisp*

```
→       wxdraw2d(
            ip_grid=[100,100],
            ip_grid_in=[10,10],
            implicit(
                2*x^5-x^3*y+2*x^2*y^2-x*y^3+2*y^5=-1e-6,
                x,-.2,.2,
                y,-.2,.2
            ),
            grid=true
        );
```

```
rat: replaced 9.999999999999999e−7 by 1/1000000 = 9.999999999999999e−7
```

(%t2)



(%o2)

Plot the Puiseux series. Note the cusp at (0,0) of (t^2,t/sqrt(2)-t^2) which is an artefact
of only considering real t, since (x(t),y(t)) is perfectly regular at t=0.

See how the different branches at (0,0) interconnect providing a regular

travel along the curve, and how there are indeed 5 real roots of f(x,y)=0

for a range of values of x, 3 elsewhere and 1 in other parts.

```
→ ;
```

```
→ f:x^2+4*x^3+6*x^4 -4*x^4*y+(-2*x-4*x^2-2*x^3)*y^2+y^4;
```

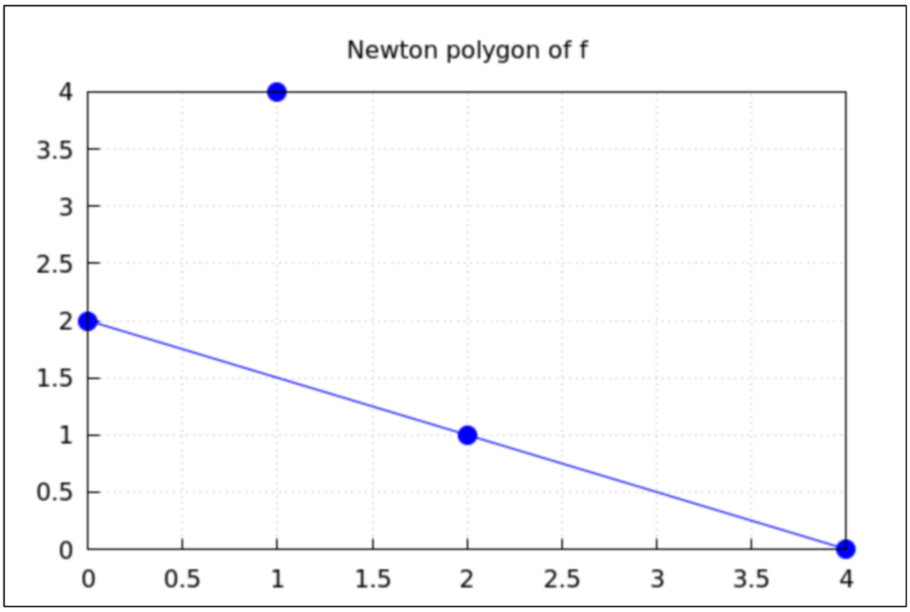
(f) $$y^4+(-2\,x^3-4\,x^2-2\,x)\,y^2-4\,x^4\,y+6\,x^4+4\,x^3+x^2$$

```
→ load(draw);
```

(%o2) $$/usr/share/maxima/5.42.2/share/draw/draw.lisp$$

```
→ wxdraw2d(title="Newton polygon of f",
     grid=true,point_type=filled_circle,
     point_size=2,points([1],[4]),
     points_joined=true,points([0,2,4],[2,1,0]));
```

(%t3)



(%o3)

Just one edge. γ=1/2,β=2, try y=C*x^γ gives
y^4-2*x*y^2+x^2=
x^2*(C^4-2*C^2+1)=0 hence C=±1 each double root. Take C=1.

```
→ y:x^(1/2)*(1+y_1);
```

(y) $$\sqrt{x}\,(y_1+1)$$

```
→ f_1:x^(-2)*subst(y,'y,f);
```

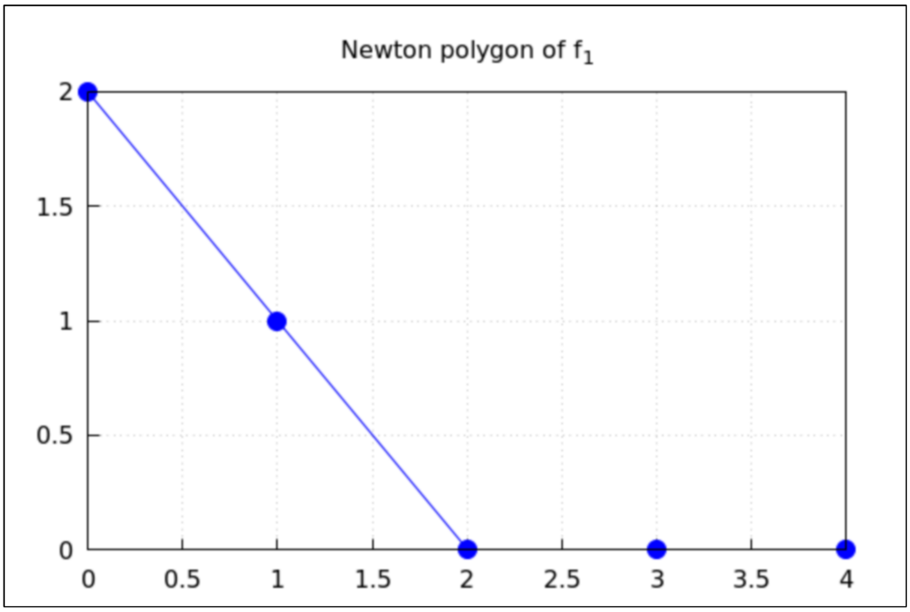
(f_1) $$\frac{x^2\,(y_1+1)^4+x\,(-2\,x^3-4\,x^2-2\,x)\,(y_1+1)^2-4\,x^{9/2}\,(y_1+1)+6\,x^4+4\,x^3+x^2}{x^2}$$

```
→ f_1:collectterms(expand(f_1),y_1);
```

(f_1) $$y_1{}^4+4\,y_1{}^3+(-2\,x^2-4\,x+4)\,y_1{}^2+(-4\,x^{5/2}-4\,x^2-8\,x)\,y_1-4\,x^{5/2}+4\,x^2$$

```
➔ wxdraw2d(title="Newton polygon of f_1",
     grid=true,point_type=filled_circle,
     point_size=2,
     points_joined=true,points([0,1,2,3,4],[2,1,0,0,0]));
```

(%t7)



(%o7)

From now on, one only looks at the first edge. γ=1,β=2, C=1 double root.

```
➔ y_1:x*(1+y_2);
```

(y_1) $x\,(y_2+1)$

```
➔ f_2:x^(-2)*subst(y_1,'y_1,f_1);
```

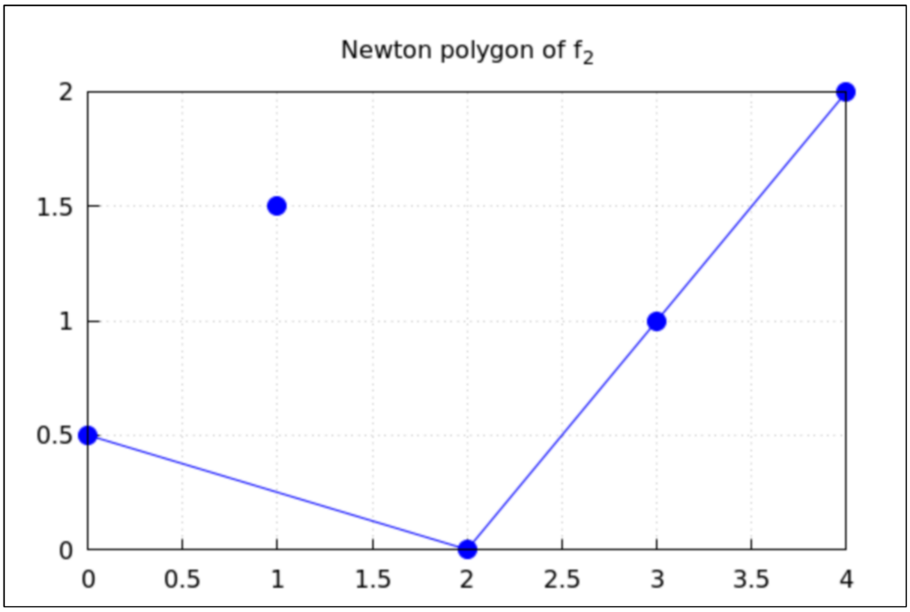
(f_2) $(x^4\,(y_2+1)^4+4\,x^3\,(y_2+1)^3+x^2\,(-2\,x^2-4\,x+4)\,(y_2+1)^2+x\,(-4\,x^{5/2}-4\,x^2-8\,x)\,(y_2+1)-4\,x^{5/2}+4\,x^2)/x^2$

```
➔ f_2:collectterms(expand(f_2),y_2);
```

(f_2) $x^2\,{y_2}^4+(4\,x^2+4\,x)\,{y_2}^3+(4\,x^2+8\,x+4)\,{y_2}^2-4\,x^{3/2}\,y_2-x^2-4\,x^{3/2}-4\,x-4\,\sqrt{x}$

```
➔ wxdraw2d(title="Newton polygon of f_2",
    grid=true,point_type=filled_circle,
    point_size=2,points([1,3],[3/2,1]),
    points_joined=true,points([0,2,4],[1/2,0,2]));
```

(%t11)



(%o11)

Edge of width 2, γ=1/4,β=1/2, C=±1 simple roots. Take C=1

```
➔ y_2:x^(1/4)*(1+y_3);
```

(y_2) $x^{1/4}\,(y_3+1)$

```
➔ f_3:x^(-1/2)*subst(y_2,'y_2,f_2);
```

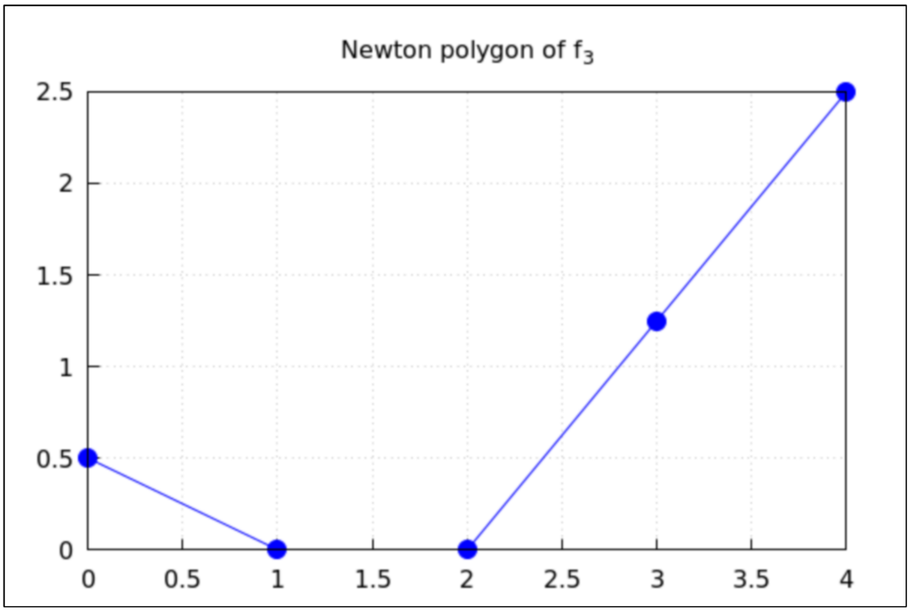
(f_3) $\left(x^3\,(y_3+1)^4+x^{3/4}\,(4\,x^2+4\,x)\,(y_3+1)^3+\sqrt{x}\,(4\,x^2+8\,x+4)\,(y_3+1)^2-4\,x^{7/4}\,(y_3+1)-x^2-4\,x^{3/2}-4\,x-4\,\sqrt{x}\right)/\sqrt{x}$

```
➔ f_3:collectterms(expand(f_3),y_3);
```

(f_3) $x^{5/2}\,{y_3}^4+(4\,x^{5/2}+4\,x^{9/4}+4\,x^{5/4})\,{y_3}^3+(6\,x^{5/2}+12\,x^{9/4}+4\,x^2+12\,x^{5/4}+8\,x+4)\,{y_3}^2+(4\,x^{5/2}+12\,x^{9/4}+8\,x^2+8\,x^{5/4}+16\,x+8)\,y_3+x^{5/2}+4\,x^{9/4}+4\,x^2-x^{3/2}+4\,x-4\,\sqrt{x}$

```
➔ wxdraw2d(title="Newton polygon of f_3",
    grid=true,point_type=filled_circle,
    point_size=2, points_joined=true,
    points([0,1,2,3,4],[1/2,0,0,5/4,5/2]));
```

(%t15)



(%o15)

Finally first edge has width 1, so f_3 expands on x^(1/4) starting from 8*y_3-4*√x =0 hence y_3=1/2*x^(1/2)+... We see below that

the coefficients are successively determined.

```
➔ yy:1/2*x^(1/2)+sum(c[i]*x^(i/4),i,3,8);
```

(yy) $$c_8\,x^2+c_7\,x^{7/4}+c_6\,x^{3/2}+c_5\,x^{5/4}+c_4\,x+c_3\,x^{3/4}+\frac{\sqrt{x}}{2}$$

```
➔ ff:taylor(subst(yy,y_3,f_3),x,0,2);
```

(ff) $$8\,c_3\,x^{3/4}+(8\,c_4+5)\,x+(8\,c_5+4\,c_3)\,x^{5/4}+(8\,c_6+4\,c_4+4\,c_3^2+7)\,x^{3/2}+(8\,c_7+4\,c_5+8\,c_3\,c_4+16\,c_3+4)\,x^{7/4}+(8\,c_8+4\,c_6+8\,c_3\,c_5+4\,c_4^2+16\,c_4+8\,c_3+6)\,x^2+\ldots$$

```
➔ eqs:makelist(coeff(ff,x,n/4),n,3,8);
```

(eqs) $$[8\,c_3,8\,c_4+5,8\,c_5+4\,c_3,8\,c_6+4\,c_4+4\,c_3^2+7,8\,c_7+4\,c_5+8\,c_3\,c_4+16\,c_3+4,8\,c_8+4\,c_6+8\,c_3\,c_5+4\,c_4^2+16\,c_4+8\,c_3+6]$$

```
➔ sol:solve(eqs,makelist(c[i],i,3,8));
```

(sol) $$[[c_3=0,c_4=-\frac{5}{8},c_5=0,c_6=-\frac{9}{16},c_7=-\frac{1}{2},c_8=\frac{75}{128}]]$$

```
➔ y_3:subst(sol[1],yy);
```

(y_3) $$\frac{75\,x^2}{128}-\frac{x^{7/4}}{2}-\frac{9\,x^{3/2}}{16}-\frac{5\,x}{8}+\frac{\sqrt{x}}{2}$$

Now recover up to y and check if f=0. This vanishes up to order 27/4.

```
➔ y_2:expand(''y_2);
```

(y_2)
$$\frac{75\ x^{9/4}}{128}-\frac{x^{2}}{2}-\frac{9\ x^{7/4}}{16}-\frac{5\ x^{5/4}}{8}+\frac{x^{3/4}}{2}+x^{1/4}$$

```
➔ y_1:expand(''y_1);
```

(y_1)
$$\frac{75\ x^{13/4}}{128}-\frac{x^{3}}{2}-\frac{9\ x^{11/4}}{16}-\frac{5\ x^{9/4}}{8}+\frac{x^{7/4}}{2}+x^{5/4}+x$$

```
➔ y:expand(''y);
```

(y)
$$\frac{75\ x^{15/4}}{128}-\frac{x^{7/2}}{2}-\frac{9\ x^{13/4}}{16}-\frac{5\ x^{11/4}}{8}+\frac{x^{9/4}}{2}+x^{7/4}+x^{3/2}+\sqrt{x}$$

```
➔ f:expand(''f);
```

(f)
$$\frac{31640625\ x^{15}}{268435456}-\frac{421875\ x^{59/4}}{1048576}+\frac{523125\ x^{29/2}}{8388608}+\frac{113475\ x^{57/4}}{131072}-\frac{3260681\ x^{14}}{4194304}+\frac{29907\ x^{55/4}}{65536}+\frac{848889\ x^{27/2}}{1048576}-\frac{46423\ x^{53/4}}{16384}+\frac{286749\ x^{13}}{262144}+\frac{149683\ x^{51/4}}{524288}-\frac{34017\ x^{25/2}}{8192}+\frac{357949\ x^{49/4}}{65536}+\frac{53743\ x^{12}}{16384}-\frac{648509\ x^{47/4}}{524288}+\frac{20379\ x^{23/2}}{4096}-\frac{275595\ x^{45/4}}{65536}-\frac{49895\ x^{11}}{8192}-\frac{50667\ x^{43/4}}{32768}-\frac{36287\ x^{21/2}}{4096}-\frac{8969\ x^{41/4}}{8192}+\frac{63\ x^{10}}{16}+\frac{26813\ x^{39/4}}{4096}+\frac{18761\ x^{19/2}}{2048}+\frac{229\ x^{37/4}}{256}+\frac{753\ x^{9}}{128}+\frac{461\ x^{35/4}}{128}-\frac{14151\ x^{17/2}}{4096}+\frac{33\ x^{33/4}}{32}-\frac{1363\ x^{8}}{256}-\frac{11\ x^{31/4}}{2}+\frac{43\ x^{15/2}}{128}-11\ x^{29/4}-\frac{123\ x^{7}}{32}$$

We must redo the computation for the 4 choices of sign for C. One gets the following

```
➔ Y1:(75*x^(15/4))/128-x^(7/2)/2-(9*x^(13/4))/16-(5*x^(11/4))/8+x^(9/4)/2+
  x^(7/4)+x^(3/2)+sqrt(x);
```

(Y1)
$$\frac{75\ x^{15/4}}{128}-\frac{x^{7/2}}{2}-\frac{9\ x^{13/4}}{16}-\frac{5\ x^{11/4}}{8}+\frac{x^{9/4}}{2}+x^{7/4}+x^{3/2}+\sqrt{x}$$

```
➔ Y2: (-(75*x^(15/4))/128)-x^(7/2)/2+(9*x^(13/4))/16+(5*x^(11/4))/8-x^(9/4)/2
                      -x^(7/4)+x^(3/2)+sqrt(x);
```

(Y2)
$$-\frac{75\ x^{15/4}}{128}-\frac{x^{7/2}}{2}+\frac{9\ x^{13/4}}{16}+\frac{5\ x^{11/4}}{8}-\frac{x^{9/4}}{2}-x^{7/4}+x^{3/2}+\sqrt{x}$$

→
```
Y3:(75*%i*x^(15/4))/128+x^(7/2)/2+(9*%i*x^(13/4))/16-
(5*%i*x^(11/4))/8-(%i*x^(9/4))/2+%i*x^(7/4)-x^(3/2)-sqrt(x);
```

(Y3)
$$\frac{75\ \%i\ x^{15/4}}{128}+\frac{x^{7/2}}{2}+\frac{9\ \%i\ x^{13/4}}{16}-\frac{5\ \%i\ x^{11/4}}{8}-\frac{\%i\ x^{9/4}}{2}+\%i\ x^{7/4}-x^{3/2}-\sqrt{x}$$

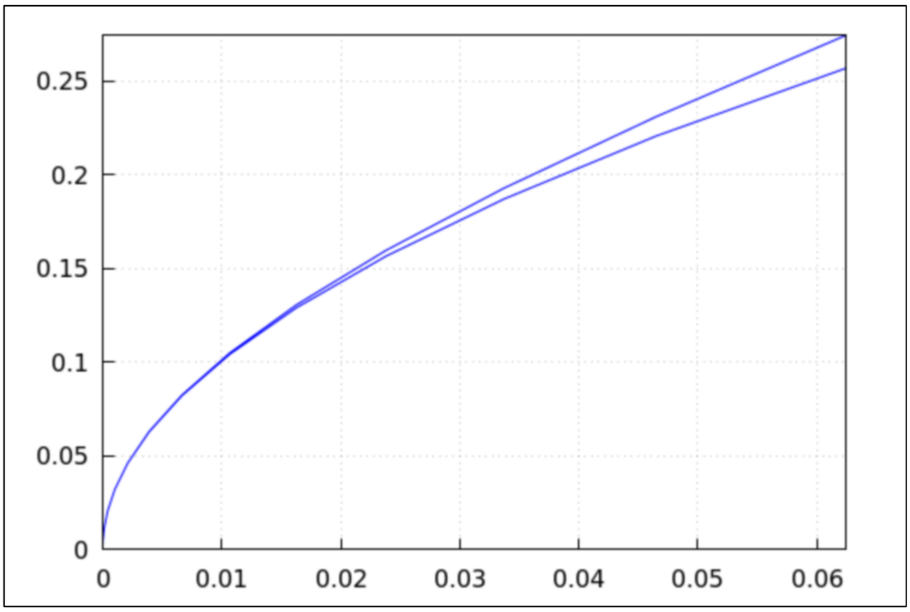
→
```
Y4:-(75*%i*x^(15/4))/128+x^(7/2)/2-(9*%i*x^(13/4))/16+
(5*%i*x^(11/4))/8+(%i*x^(9/4))/2-%i*x^(7/4)-x^(3/2)-sqrt(x);
```

(Y4)
$$-\frac{75\ \%i\ x^{15/4}}{128}+\frac{x^{7/2}}{2}-\frac{9\ \%i\ x^{13/4}}{16}+\frac{5\ \%i\ x^{11/4}}{8}+\frac{\%i\ x^{9/4}}{2}-\%i\ x^{7/4}-x^{3/2}-\sqrt{x}$$

They are all analytic continuations of the first one. Doing one turn around x=0, √x -> -√x and x^(±1/4) -> ±%i x^(±1/4), so Y1 -> Y4.
Doing 2 turns √x -> √x and x^(±1/4) -> -x^(±1/4), so Y1 -> Y2, finally with 3 turns Y1 -> Y3. So these 4 solutions belong to just one place of parametric equation
(x=t^4,y=t^2+t^6+t^7+1/2*t^9+...)
For the real slice of the curve, only Y1,Y2 are relevant.

→
```
wxdraw2d(grid=true,parametric(t^4,t^2+t^6+t^7+1/2*t^9,t,-.5,.5));
```

(%t29)



(%o29)

→
```
kill(all);
```

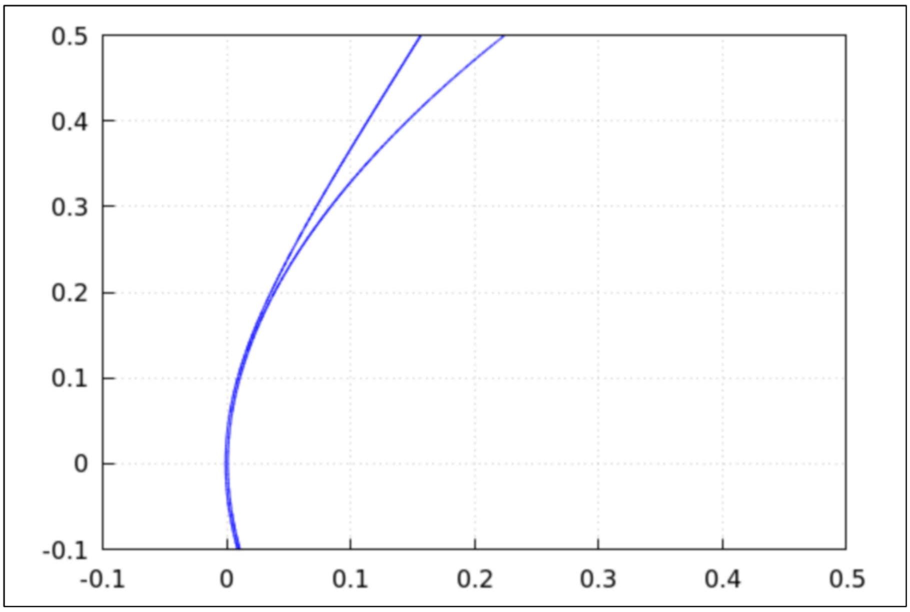
(%o0) *done*

→
```
f:x^2+4*x^3+6*x^4 -4*x^4*y+(-2*x-4*x^2-2*x^3)*y^2+y^4;
```

(f)
$$y^4+(-2\ x^3-4\ x^2-2\ x)\ y^2-4\ x^4\ y+6\ x^4+4\ x^3+x^2$$

```
➔ wxdraw2d(grid=true,ip_grid=[500,500],
    ip_grid_in=[10,10],
    implicit(f=0,x,-.1,.5,y,-.1,.5));
```

(%t2)



(%o2)

➔

➔