# Solving Conformal Theories with the Bootstrap

## Overview and Recent Results

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## Why return to the (conformal) bootstrap?

1. Conformal symmetry very powerful tool that *goes largely unused* in $D > 2$.
2. Completely non-perturbative tool to study field theories
   - Does not require SUSY, large $N$, or weak coupling.
3. In $D = 2$ conformal symmetry enhanced to *Virasoro* symmetry
   - Allows us to *completely solve* some CFTs ($c < 1$).
4. Long term hope: generalize this to $D > 2$?

## Approach

- Use only "global" conformal group, valid in all $D$.
- Study crossing symmetry of a *single scalar correlator* $\langle\sigma\sigma\sigma\sigma\rangle$.*
  (* More recently extended to 4-pt functions of two scalars.)

## Results

- Universal constraints (bounds) on spectrum/couplings in $\sigma \times \sigma$ OPE.
- Ising , $O(N)$ & some susy models seem to saturate these bounds.
- When bounds saturated crossing symmetry fixes full OPE.

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1. Overview & Philosophy
2. CFT basics
3. Conformal Blocks
4. Crossing symmetry
5. Geometry of the solution space
6. Linear Programming
7. "Solving" the 3d Ising
8. Other results
9. Alternative methods: The Gliozzi Method
10. Bootstrapping Theories with Four Supercharges (in $d = 2 - 4$)
11. Our (Modified) Simplex Algorithm
12. How to get Started Bootstrapping…

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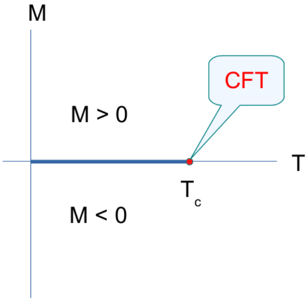
# Overview & Philosophy

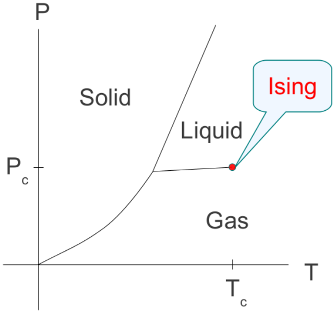
# CFTs: Experimental Perspective

## Overview & Philosophy

What are CFTs and why are they interesting?

- Second order phase transition at the end of a line of first order transitions.
- Same CFT describes many disparate experimental systems.
- e.g. Ising model CFT is universal description of phase transitions with $\mathbb{Z}_2$ symmetry.



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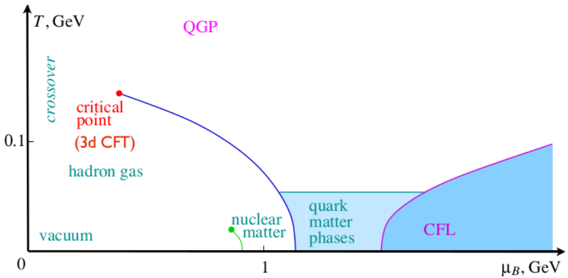



from a talk by Stephanov'2004

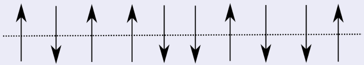
Very different "UV" theories can share the same IR behaviour.

## Example: Ising universality class

- Lattice theory with nearest neighbor interactions

$$H = -J \sum_{<i,j>} s_i s_j$$

with $s_i = \pm 1$ (only discrete translationl or rotational symmetry).



- Has symmetry broken phase $\langle s_i \rangle = \pm 1$ and symmetric phase $\langle s_i \rangle = 0$.
- Ising model CFT describes theory at critical temperature $T_c$ between two phases.

Very different "UV" theories can share the same IR behaviour.

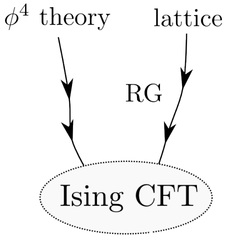
## Example: Ising universality class

- Scalar QFT (and $\sigma(x) \in \mathbb{R}$)

$$S = \int d^D x \left[ (\nabla \sigma(x))^2 + t\, \sigma(x)^2 + \lambda_4\, \sigma(x)^4 + \lambda_6\, \sigma(x)^6 + \dots \right]$$

- $\mathbb{Z}_2$ symmetry: $\sigma \to -\sigma$
- Theory has full rotational/translational invariance.
- Also has symmetric and symmetric broken phase ($\langle \sigma \rangle \neq 0$).
- Mass is related to reduced temperature $t \sim T - T_c$.
- In the IR flows to *same* CFT as lattice model!

## CFTs are fixed points of RG flow*




- Couplings $\lambda_i$ *flow* under rescalings $x \to \Lambda\, x$.
- Flow described by $\beta_i(\lambda_i)$ functions.

$$\beta_i(\lambda_i) = \frac{\partial\, \lambda_i}{\partial\, \log \Lambda}$$

- CFTs correspond to fixed points

$$\beta_i(\lambda_i^*) = 0 \tag{1}$$

- Eqns (1) strongy contrain couplings

  $\Rightarrow$ CFTs very non-generic

(* More generally fixed points have *scale invariance* but generically this leads to *conformal invariance*.)

# Correlation functions & Observables

## Overview & Philosophy

### QFT

- In a QFT we may have (asymptotic) observables, $\mathcal{O} \sim \phi_{\vec{k}}$.
- We compute/observer scattering amplitudes:

$$\langle \mathcal{O}_{\vec{k}_1} \mathcal{O}_{\vec{k}_2} \mathcal{O}_{\vec{k}_3} \mathcal{O}_{\vec{k}_4} \rangle \sim f(t, \lambda_i, \vec{k}_a, \Lambda)$$

- Observables depend on many continuous parameters.

### CFT

- Observables are not asymptotic $\mathcal{O} \sim \phi(x), : \phi(x)^2 :, T_{\mu\nu}$.
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$\Rightarrow$ far fewer parameters!
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## Question

CFTs are:

- Universal: realized as RG limits of many disperate theories.
- Strongly constrained by symmetry.

Can we describe their physics intrinsically (i.e. without picking a particular UV realization)?

## (Partial) Answer

Yes!

- In $d = 2$ infinite classes of CFT can be solved using symmetry alone.
  $\Rightarrow$ solved means compute correlators of all local operators.
- In $d > 2$ no full solution but powerful numerical methods:
  1. Can compute low-lying spectrum and couplings using symmetry alone.
  2. Method is fully non-perturbative: works for strongly coupled theories!
- Method is conformal bootstrap and will be focus of these lectures.

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# Example: the (3d) Ising Model

Intermezzo

## $\mathcal{E}$-expansion

Wilson-Fisher set $D = 4 - \mathcal{E}$ and study critical point of $\sigma^4$ perturbatively.

Setting $\mathcal{E} = 1$ can compute anomolous dimensions in $D = 3$:

$$[\sigma] = 0.5 \to 0.518\dots$$
$$[\epsilon] := [\sigma^2] = 1 \to 1.41\dots$$
$$[\epsilon'] := [\sigma^4] = 2 \to 3.8\dots$$

Using $\mathcal{E}$-expansion, Monte Carlo and other techniques find partial spectrum:

| Field: | $\sigma$ | $\epsilon$ | $\epsilon'$ | $T_{\mu\nu}$ | $C_{\mu\nu\rho\lambda}$ |
|---|---|---|---|---|---|
| Dim ($\Delta$): | 0.518135(50) | 1.41275(25) | 3.832(6) | 3 | 5.0208(12) |
| Spin (l): | 0 | 0 | 0 | 2 | 4 |

Only 5 operators and no OPE coefficients known for 3d Ising...

Lots of room for improvement!

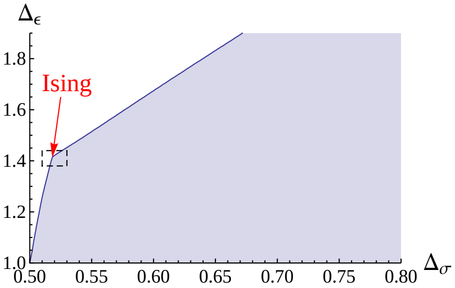
## New (Conformal) Perspective

At fixed point conformal symmetry emerges:

- Strongly constrains data of theory.
- Can we use symmetry to fix e.g. $[\sigma], [\epsilon], [\epsilon'], \ldots$ ?
- Can we also fix interactions this way?

Our first goal: a completley general exclusion plot for $\Delta_\sigma$ & $\Delta_\epsilon$.

Allowed values of $\sigma$-$\epsilon$




Blue = CFT may exists.

White = No CFT exists.

- This exclusion bound applies to **any** conformal theories.
- It is completley non-perturbative.
- In generating it we use no Lagrangian or any data specifying a particular theory.
- Exclusion plots "knows" about Ising model!
- Using bootstrap can compute spectrum & interactions of many operators for any theory on the boundary of exclusion bound.

Generating such a plot and using it to (partially) "solve" conformal theories will be the main focus of this lecture.

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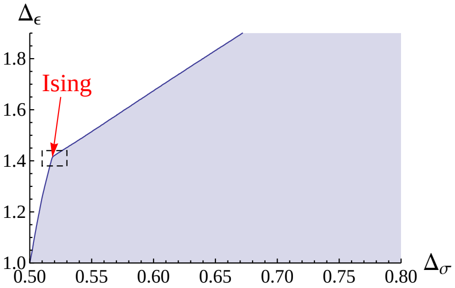


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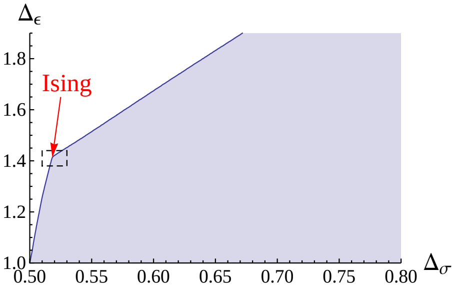


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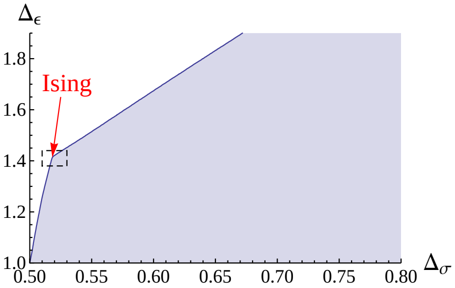


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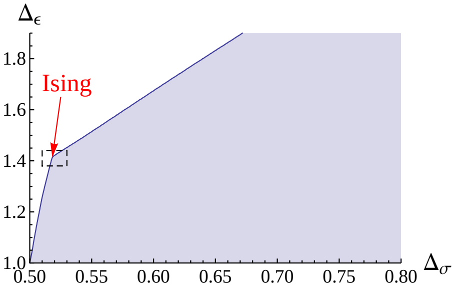


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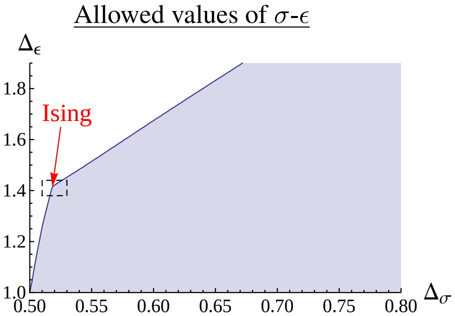


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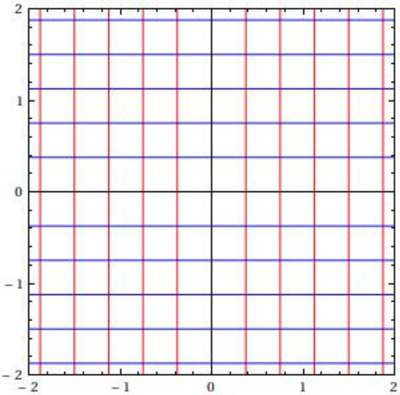
# CFT Basics

- Slava Rychkov's $d > 2$ CFT lecture notes:
  `https://sites.google.com/site/slavarychkov/CFT_LECTURES_Rychkov.pdf`
- Alessandro Vichi's PhD thesis:
  `infoscience.epfl.ch/record/167898/files/EPFL_TH5116.pdf`
- Numerical bootstrap:
  `http://arxiv.org/abs/0807.0004`
- OPE and conformal blocks:
  `http://arxiv.org/abs/1208.6449`

Conformal transformations are angle preserving:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x)$$

They generalizes (constant) scale transformations: $x \to \lambda x$.



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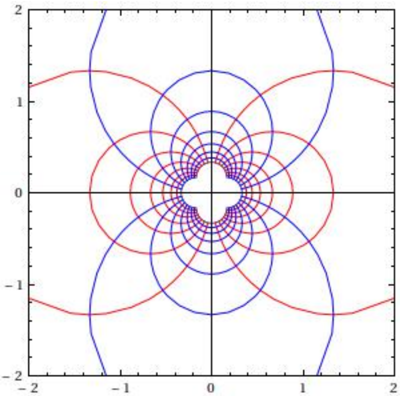

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## Conformal Algebra

Extension of Poincaré:

$$\underbrace{SO(1, D-1) \times \mathbb{R}^{1,D-1}}_{Poincare} \quad + D \quad \text{(Dilatations)} + K_\mu \quad \text{(Special conformal)}$$

Momentum ($P_\mu$) and special conformal ($K_\mu$) raise/lower level:

$$[D, P_\mu] = i P_\mu, \quad [D, K_\mu] = -i K_\mu$$

Act like ladder operators to generate new states in a representation.

In a CFT should organize operators into representations of Conformal Group.

## Operators

Operators should have definite scaling dimension $\Delta$ and spin $l$

$$[D, \mathcal{O}] = i\,\Delta\mathcal{O}, \quad [M_{\mu\nu}, \mathcal{O}] = i\,R_M \cdot \mathcal{O}$$

with $R_M$ a spin-$l$ representation of rotation $M_{\mu\nu}$.

Primary Operators

Highest weight states, $\mathcal{O}$, are called primary operators. Satisfy:

$$\text{Primary operators:} \qquad [K_\mu, \mathcal{O}] = 0$$

From a highest weight state can construct an infinite number of descendants:

$$\text{Descendents:} \qquad \mathcal{O}_n := P_{\mu_1} \ldots P_{\mu_n} \mathcal{O}$$

Correlators of descendants fixed by conformal symmetry (in terms of primaries).

# Representation Theory

CFT Basics

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- From definition primary operator $\mathcal{O}$ transforms under $x \to \lambda x$:

$$\mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x)$$

- Two Point Functions
  Requiring correlators to be invariant under conf group gives:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \xrightarrow[scale]{} \frac{a}{x^{\Delta_i+\Delta_j}} \xrightarrow[special]{} \frac{a\,\delta_{ij}}{x^{2\Delta_i}}$$

  Set $a = 1$ as to fix normalization.
- Three Point Functions

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\rangle \xrightarrow[scale]{} \sum_{a+b+c=\Delta_1+\Delta_2+\Delta_3} \frac{C_{ijk}}{x_{ij}^a x_{ik}^b x_{jk}^c} \xrightarrow[special]{} \frac{C_{ijk}}{x_{ij}^{\delta_{ij}} x_{ik}^{\delta_{ik}} x_{jk}^{\delta_{jk}}}$$

  with $\delta_{ij} = \Delta_i + \Delta_j - \Delta_k$ and $x_{ij} = |\vec{x}_{ij}|$.
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A more physical way to think about what a primary operator is via RG:

1. CFT should be invariant under RG step: $x \to x' = \lambda x$
2. In scale/conf invariant theory Hamiltonian is invariant: $\hat{H} \to \hat{H}' = \hat{H}$.
3. Kinetic term in Lagrangian transforms like:

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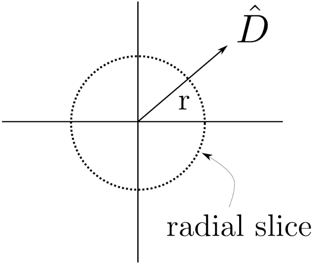


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- Operators defined on radial slices.

**Operator-State Correspondence**

- Each operator inserted at origin defines a state:

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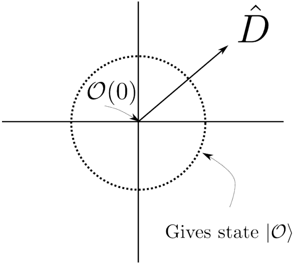


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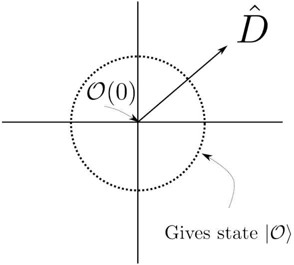
## Operator Product Expansion




- Acting on primary state $|\mathcal{O}_j\rangle$ with primary operator $\mathcal{O}_i(x)$ gives new state that can be decomposed in conformal reps:

$$\mathcal{O}_i(x)|\mathcal{O}_j\rangle = \sum_\alpha c_{ij\alpha}|\Psi_\alpha\rangle$$

  $\alpha$ runs over all eigenstates of $\hat{D}$.
- Contribution of descendents fixed by symmetry.
- Using Operator-State correspondence this gives operator product expansion.
- Repackage using diff op $D(x, \partial)$ (fixed by conf symmetry).
- $C_{ijk}$ are dynamical data of theory (along with $\Delta_i, l_i$).
- OPE only holds if no operator inserted at $|y| < |x|$.

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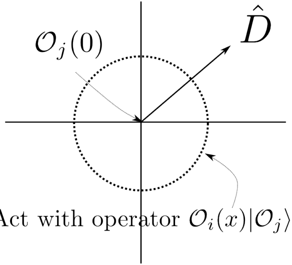


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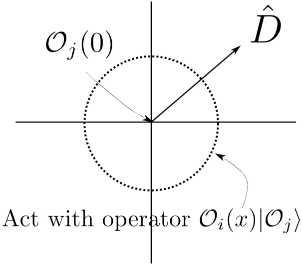
# Operator Product Expansion (OPE)

CFT Basics

## Operator Product Expansion

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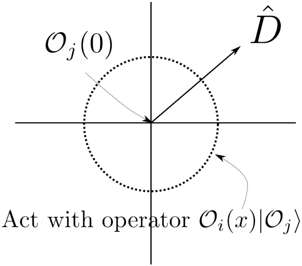


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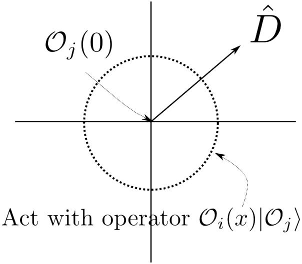


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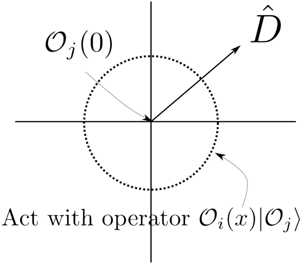


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   Use OPE to reduce 3-pt function to sum over 2-pt function

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2. Now we know everything about OPE and 2/3-pt functions.

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- Spectrum $\mathcal{S} = \{\mathcal{O}_i\}$ of primary operators dimensions, spins: $(\Delta_i, l_i)$
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## Axiomatic Formulation of a CFT



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What about central charge?

- In $d=2$ the central charge, $c$, is a defining property of CFT.
- In $d>2$ no canonical definition of central charge.
- Candidates:

$$\langle T(z)T(0)\rangle \sim \frac{c_1}{z^4}, \qquad \langle T\rangle \sim c_2\, R, \qquad S = 2\pi\sqrt{c_3\,\Delta}$$

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- $c_3$ holds in thermal state.
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- So we will take $c_1$ as "central charge" for $d>2$.
- Appears in $\phi\times\phi$ OPE:

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CFT Basics

- We will restrict attention to "unitary" theories.
- Generally we consider Euclidean theories so actually mean reflection positivity.
- In reflection-positive theory norms of states must be positive.

Combined with conformal algebra unitarity gives constraints on $\Delta$.

$$||P^\mu P_\mu|\mathcal{O}\rangle|| = \langle\mathcal{O}|K_\nu K^\nu P^\mu P_\mu|\mathcal{O}\rangle$$
$$\propto \Delta\left(\Delta - \frac{d-2}{2}\right)$$

Here we use $K^\dagger = P$ (in radial quantization) and assumed $\mathcal{O}$ is a scalar.

Similar arguments give:

$$\Delta \geq \frac{d-2}{2} \qquad (l = 0)$$
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# Conformal Blocks

# Definition

## Conformal Blocks

Recall 4-pt function can be expressed in terms of $g(u,v)$:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}} g(u,v)$$

Can also use OPE to decompose 4-pt into sum:

$$\langle \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}_{\sum_k C_{12}^k D(x_{12},\partial_{x_2})\mathcal{O}_k(x_2)} \underbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}_{\sum_l C_{34}^l D(x_{34},\partial_{x_4})(x_3)\mathcal{O}_l(x_4)} \rangle = \sum_k C_{12}^k C_{34}^k D(x_{12}, x_{34}, \partial_{x_2}, \partial_{x_4}) \left(\frac{1}{x_{24}^{2\Delta_k}}\right)$$

1. The $D$'s acting on 2-pt function is called Conformal Block.
2. Convenient to pull out trivial pre-factor and then express $g(u,v)$ in terms of CB decomposition.
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- Conformal blocks depend on two variables $u, v$.
- It turns out a more natural coordinate system is $z, \bar{z}$ defined via

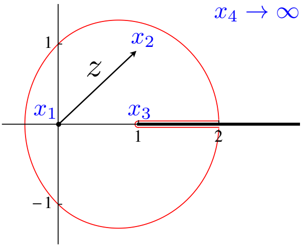
$$u = z\bar{z}, \qquad v = (1-z)(1-\bar{z})$$

- In $d = 2$ these are just standard complex coords.

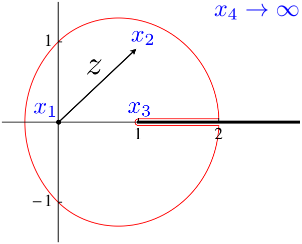
In *all* $d$ conf symm can fix $x_1, \ldots, x_4$ to be co-planar.

- Use conf symm to fix $\vec{x}_1 = 0, \vec{x}_3 = 1, \vec{x}_4 = \infty$.
- $z, \bar{z}$ complex coords on plane spanned by unfixed $\vec{x}_2$.
- From definition follows

$$\begin{aligned} z, \bar{z} \to 0 \quad &\Rightarrow \quad x_{12}, x_{34} \to 0 \\ z, \bar{z} \to 1 \quad &\Rightarrow \quad x_{14}, x_{23} \to 0 \end{aligned}$$



1. From OPE convergence might not expect $(1-2), (3-4)$ CB to converge if $|z| > 1$.
2. Actually conf blocks more convergent than OPE:
   - Treat e.g. $x_1 \leftrightarrow x_2$ symmetrically.
   - Reflects freedom to choose origin in raidal quantization.
3. Conf blocks $G(z,\bar{z})$ can be extended to full $z, \bar{z}$ plane by analytic continuation.
4. Analytic continuation has branch cut for $z > 1$ on real axis.
5. Explicit expressions exist in $d = 2, 4$.



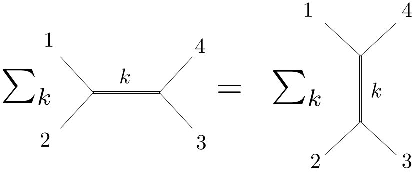

Example: conf blocks in $d = 4$ (for equal external dimensions):

$$G(z,\bar{z}) = \frac{z\,\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right]$$

$$k_\beta(x) = x^{\beta/2}\,{}_2F_1(\beta/2, \beta/2, \beta; x)$$

# Crossing Symmetry

OPE decomposition of 4-pt function into CBs is not unique: $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$




Consistency requires equivalence of two different contractions

$$\sum_k C^k_{12} C^k_{34}\, G^{12;34}_{\Delta_k, l_k}(u, v) = \sum_k C^k_{14} C^k_{23}\, G^{14;23}_{\Delta_k, l_k}(u, v)$$

When operators in correlator identical $G^{12;34}$ and $G^{14;23}$ simply related:

- Crossing means exchanging $x_1 \leftrightarrow x_3$, $x_2 \leftrightarrow x_4$ or equivilently $u \leftrightarrow v$ implying

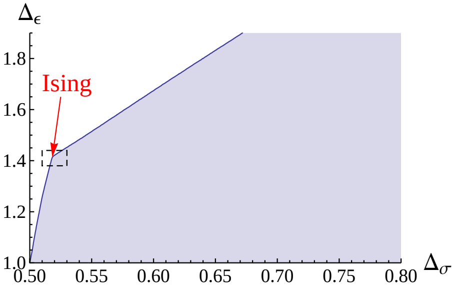
$$G^{12;34}(u, v) = G^{14;23}(v, u)$$

- Crossing symmetry give non-perturbative constraints on $(\Delta_k, C^k_{ij})$.

## Is crossing symmetry consistent with a gap?

$$\sigma\text{-OPE:} \quad \sigma \times \sigma \sim 1 + \epsilon + \dots$$

Crossing symmetric values of $\sigma$-$\epsilon$




Blue = solution may exists.
White = No solution exists.

- Assuming above OPE study crossing symmetry of:

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$$

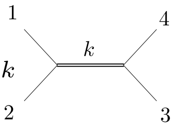
- Certain values of $\Delta_\sigma$, $\Delta_\epsilon$ inconsistent with crossing symmetry.
- Solutions to crossing:
  1. white region $\Rightarrow$ 0 solutions.
  2. blue region $\Rightarrow$ $\infty$ solutions.
  3. boundary $\Rightarrow$ 1 solution (unique)!
- Exclusion plots "knows" about Ising model!

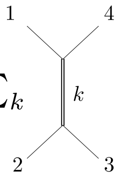
So how do we check crossing symmetry in practice?

Consider four *identical* scalars: $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \qquad \dim(\phi) = \Delta_\phi$

Recall crossing symmetry constraint:

$$\frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 \, G_{\Delta_k, l_k}^{12;34}(u, v) = \frac{1}{x_{14}^{2\Delta_\phi} x_{23}^{2\Delta_\phi}} \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 \, G_{\Delta_k, l_k}^{14;23}(u, v)$$

$\sum_k$ 

$=$ $\sum_k$ 

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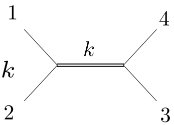
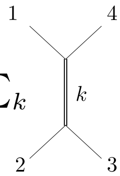
Express everything in terms of $u$, $v$:

$$\left(\frac{v}{u}\right)^{\Delta_\phi} \sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2\, G^{12;34}_{\Delta_k, l_k}(u,v) = \sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2\, G^{14;23}_{\Delta_k, l_k}(u,v)$$

$\sum_k$  $=$ $\sum_k$ 

# A (More) Practical Formulation

Crossing Symmetry

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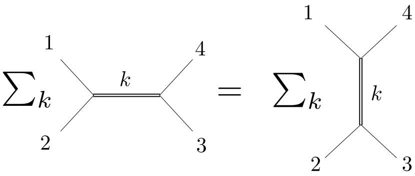
Move everything to LHS:

$$v^{\Delta_\phi} \sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2 \, G^{12;34}_{\Delta_k, l_k}(u,v) - u^{\Delta_\phi} \sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2 \, G^{14;23}_{\Delta_k, l_k}(u,v) = 0$$



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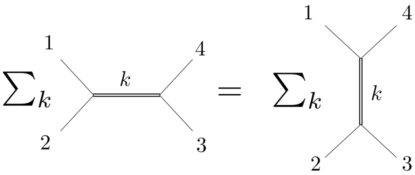
Express as sum of functions with positive coefficients:

$$\sum_{\mathcal{O}_k} \underbrace{(C^k_{\phi\phi})^2}_{p_k} \underbrace{\left[v^{\Delta_\phi} G_{\Delta_k,l_k}(u,v) - u^{\Delta_\phi} G_{\Delta_k,l_l}(v,u)\right]}_{F_k(u,v)} = 0$$



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Functions $F_k(u,v)$ are formally infinite dimensional vectors.

$$p_1 \underbrace{(F_1, F_1', F_1'', \ldots)}_{\vec{v}_1} + p_2 \underbrace{(F_2, F_2', F_2'', \ldots)}_{\vec{v}_2} + p_3 \underbrace{(F_3, F_3', F_3'', \ldots)}_{\vec{v}_3} + \cdots = \vec{0}$$

1. Each compnent is a deriv at a the same point: e.g. $F' := F^{(2,0)}(z^*, \bar{z}^*)$.
2. Each vector $\vec{v}_k$ represents the contribution of an operator $\mathcal{O}_k$.
3. Labels $k = (\Delta, l)$ are continuous (because of $\Delta$).
4. If $\{\vec{v}_1, \vec{v}_2, \ldots\}$ span a positive cone there is no solution.
5. Efficient numerical methods to check if set of vectors $\{\vec{v}_k\}$ span a cone.

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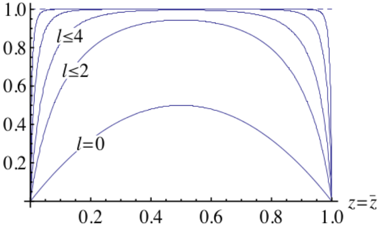
# Convergence

## Crossing Symmetry

Our goal will be to study/constrain spectrum $\mathcal{S}$ of CFTs using sum rule.

Before continuing check: how quickly does conf. block expansion converge?




- Consider free scalar $\phi \times \phi \sim 1 + \phi^2 + T_{\mu\nu} + \ldots$.
- Move contribution of identity, $\mathbb{I}$, to LHS:

$$F_{\mathbb{I}} = -\sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2 F_k(u,v)$$

  (can normalize so $F_{\mathbb{I}} = 1$)
- Plot how quickly sum converges for free theory (along $z = \bar{z}$).
- **Note:** convergence best around $z = \bar{z} = \frac{1}{2}$ so choose this point for Taylor expansion.

Can prove that asymptotically tail of sum rule cut off at $\Delta^*$ is bounded by $e^{-\Delta^*}$.

# Geometry of the Solution Space

# Truncated Sum Rule

## Geometry of Solution Space

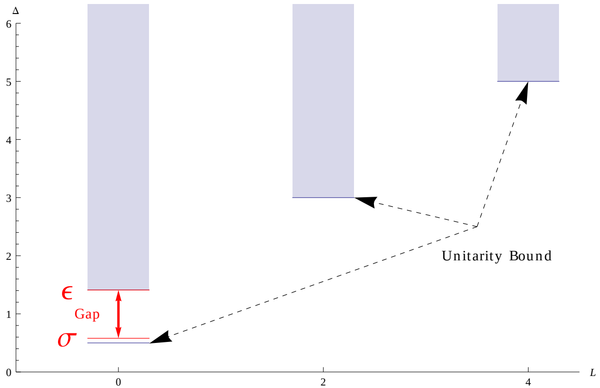
- To get some insight lets consider a very truncated problem.
- We truncate to $\Delta < \Delta^*$, $l < l^*$ (this is a justifiable approximation).
- Each order in Taylor expansion (around $z = \bar{z} = \frac{1}{2}$) of sum rule gives a necassary condition for crossing.
- We consider only two Taylor coefficients $F^{(1,1)}$ & $F^{(3,0)}$ (this is not an approximation)!
- Truncated sum rule becomes:

$$\sum_{\Delta<\Delta^*, l<l^*} p_{\Delta,l} \underbrace{\begin{pmatrix} F^{(3,0)}_{\Delta,l} \\ F^{(1,1)}_{\Delta,l} \end{pmatrix}}_{\vec{v}_{\Delta,l}} = 0 \tag{3}$$

- Recall $p_k = (C^k_{\phi\phi})^2 > 0$ so if $\{\vec{v}_{\Delta,l}\}$ form a positive cone we're doomed (i.e. can't solve eqn(3)).

## Constraining the spectrum

Figure : $\mathcal{S}$: a putative spectrum in $D = 3$




- Unitarity implies:

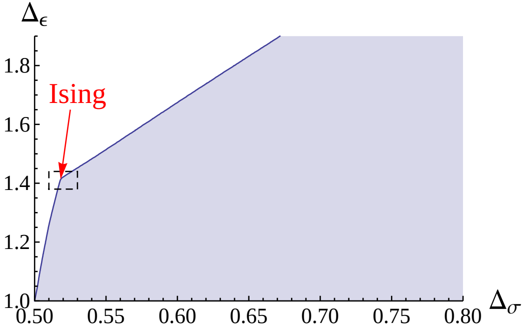
$$\Delta \geq \frac{D-2}{2} \quad (l = 0),$$
$$\Delta \geq l + D - 2 \quad (l \geq 0)$$

- “Carve” landscape of CFTs by imposing gap in scalar sector.
- Fix lightest scalar: $\sigma$.
- Vary next scalar: $\epsilon$.
- Spectrum otherwise *unconstrained*: allow any other operators.

## Is (truncated) crossing symmetry consistent with a gap?

$\sigma$ four-point function: $\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle$

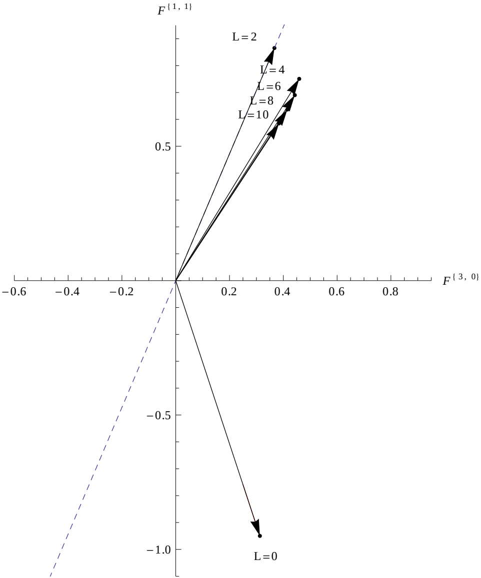
Crossing symmetric values of $\sigma$-$\epsilon$




Blue = solution may exists.
White = No solution exists.

# Cones in Derivative Space

## Geometry of Solution Space




### Two-derivative truncation

- Consider $\langle \sigma\sigma\sigma\sigma \rangle$.
- Fix $\Delta(\sigma) = 0.515$.
- We plot e.g. $(F^{(1,1)}, F^{(3,0)})$.

- Consider putative spectrum $\{\Delta_k, l_k\}$

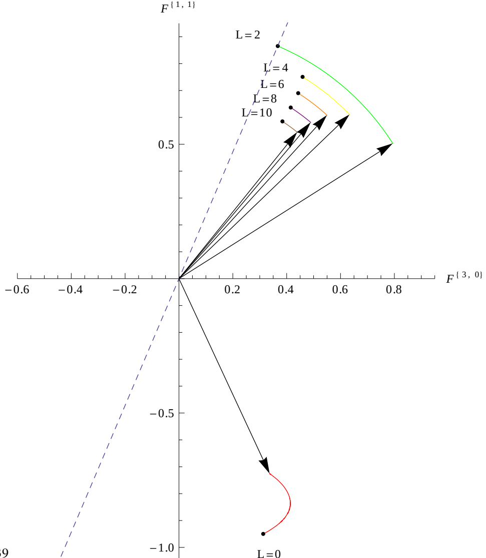
$$\Delta = \Delta_{unitarity}$$
$$l = 0 \text{ to } 10$$

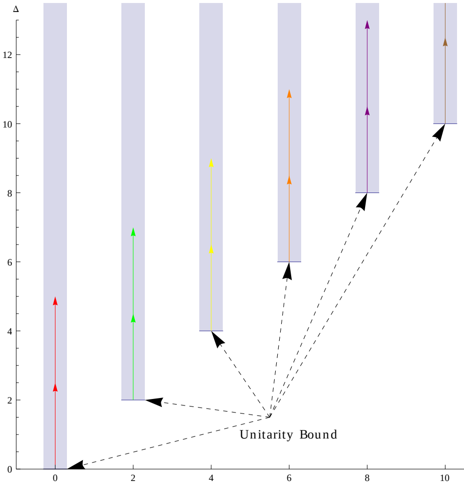
- Vectors *represent operators*.
- All vectors lie inside cone
  $\Rightarrow$ Inconsistent spectrum!

# Cones in Derivative Space

## Geometry of Solution Space

Derivatives $\Longleftrightarrow$ Putative Spectrum



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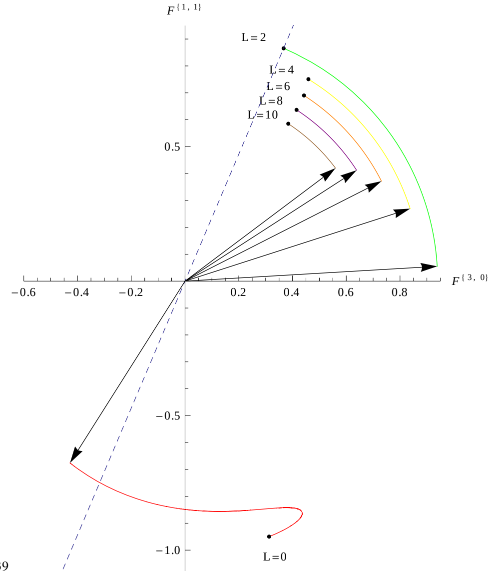



- Allow even more operators in putative spectrum.
- *Scalar channel* plays essential role.
  $\Rightarrow$ vectors *span* plane.
  $\Rightarrow$ In particular can find $p_k \geq 0$

$$\sum_k p_k\, F_{\Delta_k, l_k} = 0$$

$\Rightarrow$ crossing sym. can be satisfied.

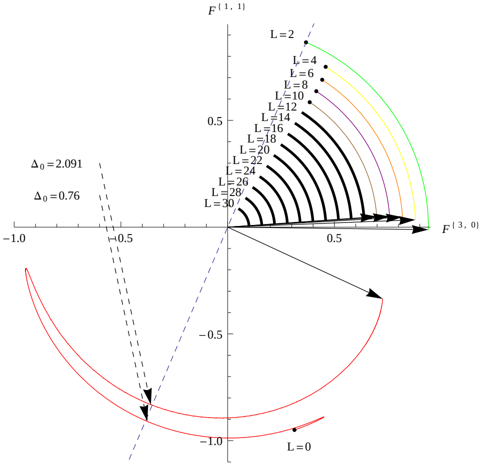
### Why does this work?

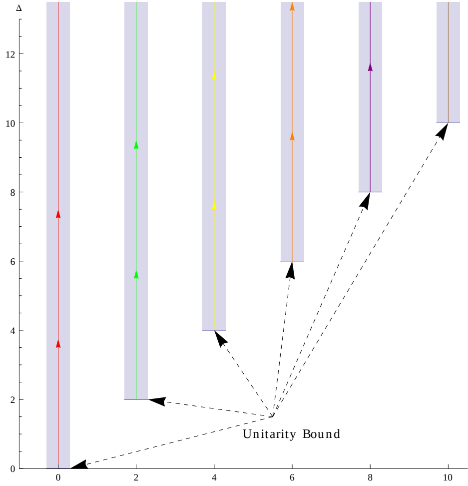
- Cone boundary defined by low-lying operators.
- Higher $\Delta, l$ operators less important.
- Follows from convergence of CB expansion.

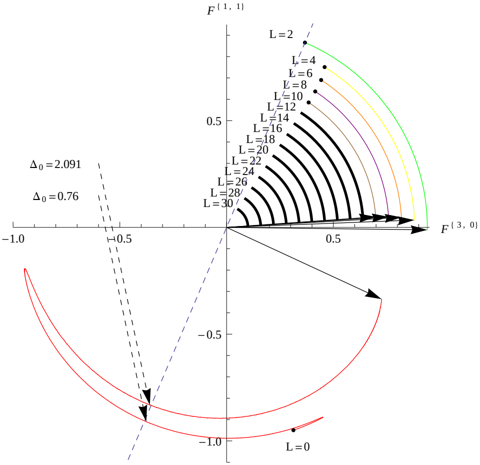
# Cones in Derivative Space

Geometry of Solution Space

Derivatives $\Longleftrightarrow$ Putative Spectrum





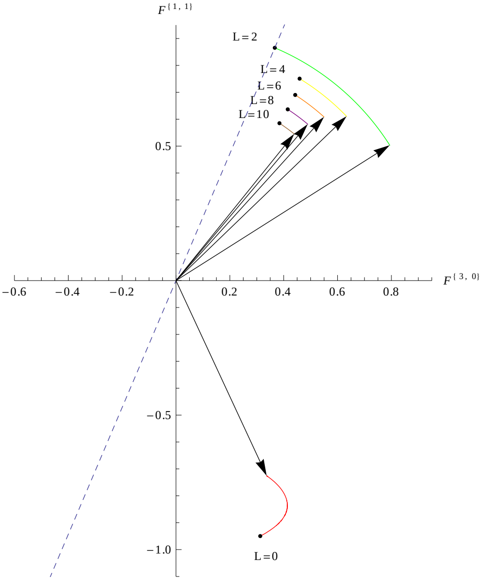

## Carving the Landscape of CFTs

1. Plot imposes necessary conditions.
2. Carve "landscape" via exclusion.

Any CFT in $D = 3$ with $\dim(\sigma) = 0.515$ must have another scalar with $0.76 \leq \Delta \leq 2.091$.

# The "Extremal Solution"

Geometry of Solution Space




## Uniqueness of "Boundary Solution"

– Consider $\Delta_0 < 0.76$

- No combination of vecs give a zero.

$$\sum_i p_i \vec{F}_i \neq 0 \quad \text{for} \quad p_i > 0$$

– Consider $\Delta_0 > 0.76$

- Many ways to form vects to give zero.
- Families of possible $\{p_i\}$.
- Neither spectrum nor OPE fixed.

– Consider $\Delta_0 = 0.76$

- Only one way to form zero.
- Non-zero $p_i$ fixed $\Rightarrow$ unique spectrum.
- Value of $p_i := (C_{ii}^k)^2$ fixed $\Rightarrow$ unique OPE.
- Non-zero $p_i$: $\begin{array}{ll} \Delta \sim 0.76, & L = 0 \\ \Delta = 3, & L = 2 \end{array}$

– NOTE: Num operators $\sim$ num components of $\vec{F}_i$

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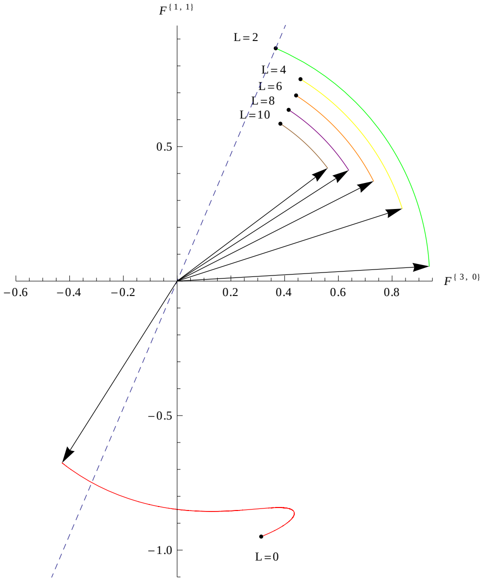


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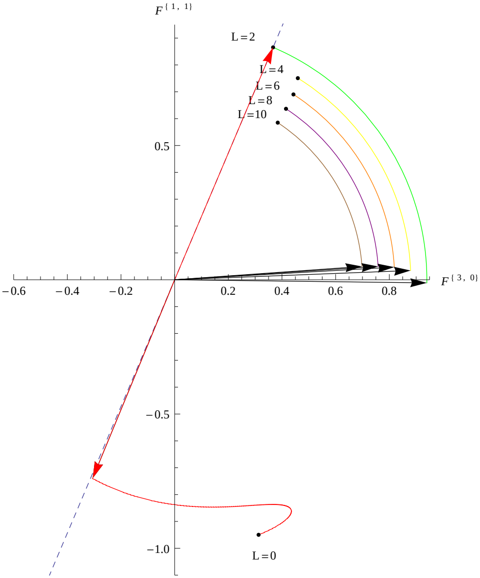


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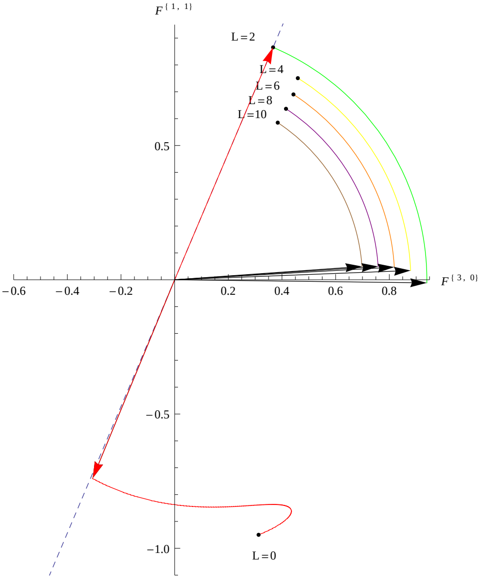


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General properties of the approach:

1. We are never proving that a CFT exists.
   - We only check a *subset* of the constraints coming from a single correlator.
   - In $d > 2$ we do not even know what a sufficient criteria is for CFT to exist!
2. On the other hand we can prove that CFTs cannot exist with certain properties.
3. By adding more vectors (i.e. allowing more operators in the spectrum) we can transition from having no solutions to having many possible solutions.
4. In the boundary between these regions we get a unique solution.
5. This "visual" proof was nice for intuition but does not generalize well when we consider many ($\gg 2$) constraints/Taylor coefficients at once.

The next step is to formulate the bootstrap in a more abstract way and apply standard numerical algorithms (e.g. Linear Programming).

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# Lessons Learned

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# Linear Programming

- The problem we want to solve is:

$$\sum_{\Delta,l} p_{\Delta,l} F_{\Delta,l}(z,\bar{z}) = 0$$

- Taylor expanding around $z = \bar{z} = 1/2$ and requiring each order to vanish gives a matrix:

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which we must solve subject to $p_i \geq 0$.

- Rows of matrix $M_{\mathcal{S}}$ are Taylor coefficeints (labelled by derivaties: $\partial_z^m \partial_{\bar{z}}^n$).
- Columns are operators $\mathcal{O}_k$ allowed in spectrum (continuous label $k = \{\Delta, l\} \in \mathcal{S}$)
- "Matrix" $M_{\mathcal{S}}$ depends on the choice of allowed spectrum.
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# Direct Method

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Where $\sim$ can mean either $=$, $\geq$, or $\leq$.

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- In simplest case $\vec{c} = 0$, $\vec{b} = 0$ and we take $M_{\mathcal{S}} \cdot \vec{p} = 0$.

Some issues with this:

- $\infty$ number of derivs $\Rightarrow$ truncate to $n$.
- Couplings $p_{\Delta,l}$ have a continuous label $\Delta$.
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# Bounding Operator Dimensions

## Linear Programming

How do we produce a plot like this?

1. Fix $\Delta_\sigma$ to some value (e.g. 0.6).
2. Fix gap $\Delta_\epsilon = \Delta_1$ (e.g. $= 1.6$).
3. This defines a putative spectrum

$$\mathcal{S}_1 = \{\Delta_{l=0} \geq \Delta_1; \quad \Delta_{l>0} \geq \Delta_{\text{unitarity}}\}$$

4. Use LP to find $\vec{p}$ with $M = M_{\mathcal{S}_1}$.
5. If LP find non-zero $\vec{p}$:
   $\Rightarrow$ CFT can exist
   increase $\Delta_1$ and try again.
6. If LP finds no $\vec{p}$:
   $\Rightarrow$ no CFT exists with $\Delta_\epsilon \geq \Delta_1$.
   decrease $\Delta_1$ and try again.
7. Bisecting in $\Delta_1$ until we find max value $\Delta_\epsilon^{max}$ (to some resolution).
8. Repeat for the next value of $\Delta_\sigma$.

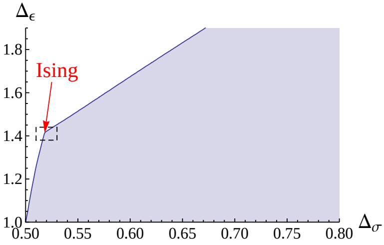
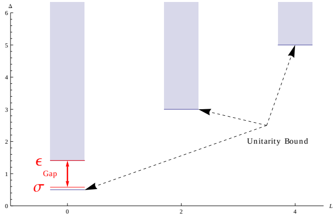



Figure : a putative spectrum $\mathcal{S}$



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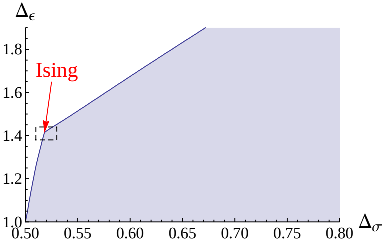
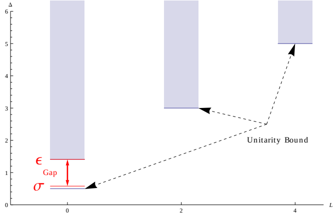



Figure : a putative spectrum $\mathcal{S}$



# Bounding Operator Dimensions

## Linear Programming

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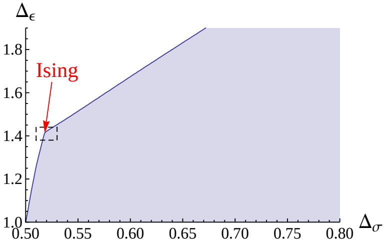


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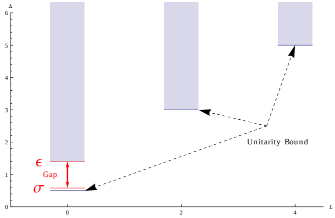

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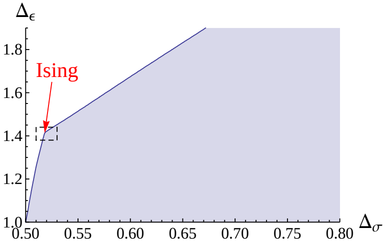


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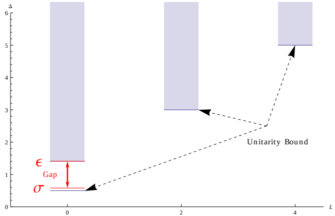

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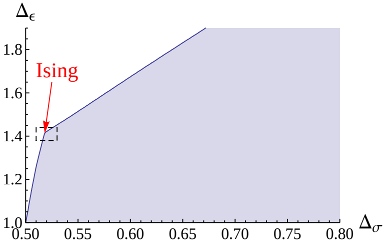


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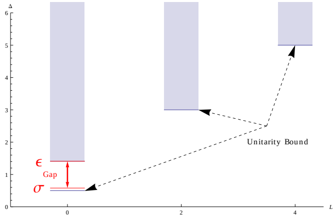

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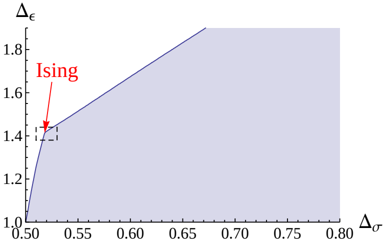


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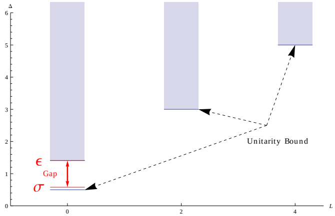

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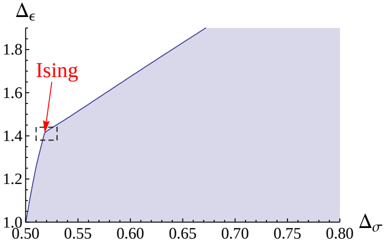


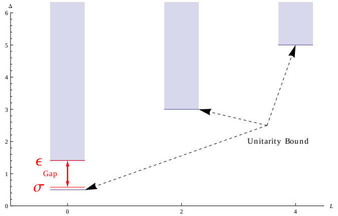


Figure : a putative spectrum $\mathcal{S}$

## What else can we bound?

Bootstrap allows us to:

- Consider arbitrary CFT data $\mathcal{S} = \{(\Delta_i, \ell_i), C_{ijk}\}$.
- Check if this $\mathcal{S}$ is consistent with crossing sym of $\langle\sigma\sigma\sigma\sigma\rangle$.

Any time a bound is saturated can compute full OPE.

Kinds of bounds we can place on $\mathcal{S}$:

- Maximize a gap $\Delta_\epsilon$ in $\sigma \times \sigma$ OPE.
- Maximize coefficient of operator

$$\sigma \times \sigma \sim 1 + \lambda_{\sigma\sigma}^{\epsilon} \boxed{\epsilon} + \cdots + \lambda_{\sigma\sigma}^{T} T_{\mu\nu} + \ldots$$

Can bound dimension of first scalar on $\Delta_\epsilon$ (or any $\ell$).

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Can bound (maximize) OPE coefficient of any operator.

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$$\sigma \times \sigma \sim 1 + \lambda_{\sigma\sigma}^{\epsilon}\,\epsilon + \cdots + \left(\frac{\Delta_\sigma}{\sqrt{c}}\right) T_{\mu\nu} + \ldots$$

If operator e.g. $T_{\mu\nu}$ get *lower bound* on $c$.

# Bounding OPE Coefficients

## Linear Programming

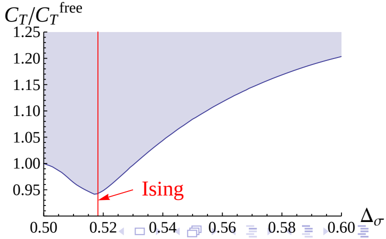
- Instead of bounding $\Delta_\epsilon$ we can formulate a different LP and bound OPE coefficients.
- Recall LP forumulation:

$$\begin{aligned} \text{Minimize:} \quad & \vec{c}\cdot\vec{p}, \\ \text{subject to:} \quad & M_{\mathcal{S}}\cdot\vec{p} = \vec{b}, \qquad \vec{p} \geq 0 \end{aligned}$$

- Previously we took $\vec{c} = 0$ and $\vec{b} = 0$ and varied $M_{\mathcal{S}}$.
- Instead we can fix $M_{\mathcal{S}}$ (i.e. fix the spectrum) but take non-zero $\vec{c}$.
- E.g. if we take $c_{\Delta,l} = -1$ then LP will maximize OPE coefficient of $\mathcal{O}_{\Delta,l}$.

When is this useful?

- Maximizing $T_{\mu\nu}$ OPE coeff.
- $p_{T_{\mu\nu}} = p_{D,l} = \left(\frac{\Delta_\sigma^2}{c}\right)$ so equiv to $c$-minimization.



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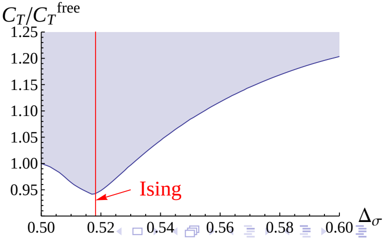


# “Solving” the 3d Ising Model?

## 3d Ising model

Using $\mathcal{E}$-expansion, Monte Carlo and other techniques find partial spectrum:

| Field: | $\sigma$ | $\epsilon$ | $\epsilon'$ | $T_{\mu\nu}$ | $C_{\mu\nu\rho\lambda}$ |
|---|---|---|---|---|---|
| Dim ($\Delta$): | 0.518135(50) | 1.41275(25) | 3.832(6) | 3 | 5.0208(12) |
| Spin (l): | 0 | 0 | 0 | 2 | 4 |

Only 5 operators and no OPE coefficients known for 3d Ising!

Lots of room for improvement!

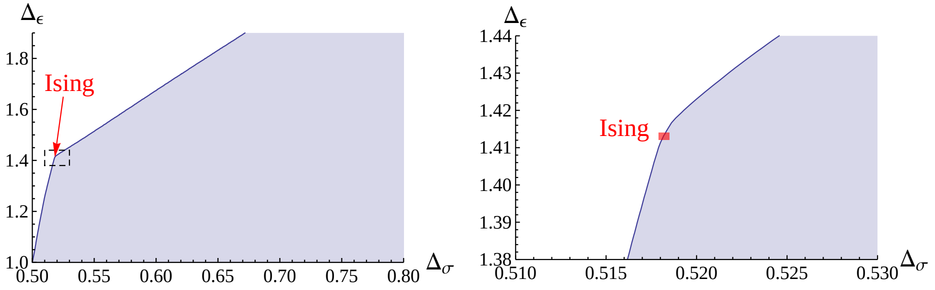
## Our Goal

Compute these anomolous dimensions (and many more) and OPE coefficients using the bootstrap applied along the boundary curve (i.e. the EFM).

# Spectrum of the 3d Ising Model

"Solving" the 3d Ising Model?

A first problem: what point on the boundary? what is correct value of $\sigma$?




1. "Kink" is not so sharp when we zoom in.
2. Gets sharper as we add more constraints.

Is there a better way to compute $\Delta_\sigma$ for 3d Ising?

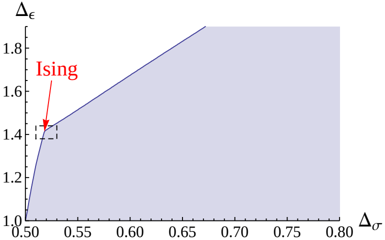
# $c$-minimization

"Solving" the 3d Ising Model?

Our first bound plot was made by maximizing $\Delta_\epsilon$.

$$\sigma \times \sigma \sim 1 + \lambda^{\epsilon}_{\sigma\sigma} \boxed{\epsilon} + \cdots + \lambda^{T}_{\sigma\sigma} T_{\mu\nu} + \ldots$$

- Look for solutions to crossing that maximize e.g. $C^T_{\sigma\sigma}$.
- OPE coefficient of stress tensor fixed by conformal symmetry (in terms of $c$).
- $c$ canonical stress-tensor normalization: $\langle T_{\mu\nu} T_{\rho\sigma} \rangle \sim c$.
- $T_{\mu\nu}$ OPE max $\Rightarrow$ $c$ minimization.



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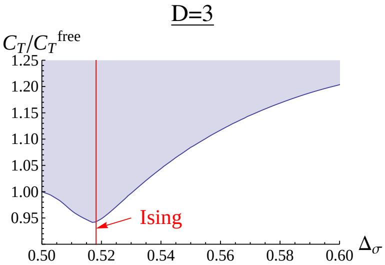

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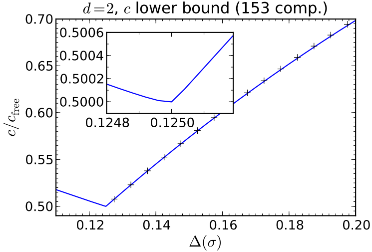
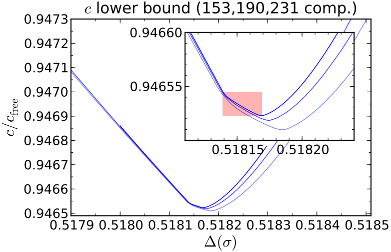
In both $d = 2, 3$ Ising model *minimizes* $c$.

Reproduces $c = \frac{1}{2}$ in $d = 2$ to high precision!

d=2




d=3



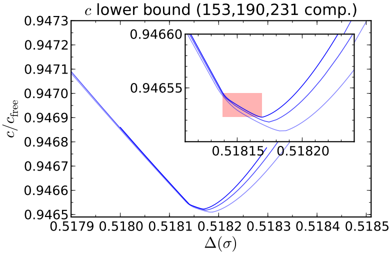
# 3d Critical Exponents from $c$-minimization

"Solving" the 3d Ising Model?

Why minimize $c$?

- $c$-minimization also maximizes $\Delta_\epsilon \Rightarrow$ equivalent approaches.
- Location of "minimum" well defined while "kink" is somewhat ambgious.
- $c$-minimzation is more numerically stable and philosophically palatable.

Fixing $\Delta_\sigma$ to be at min of $c$ we compute exponents using extremizing solution:




- Our estimates are 2-3$\times$ better than nearest competition.
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- In practice technical difficulties due to *approximately conserved currents*.

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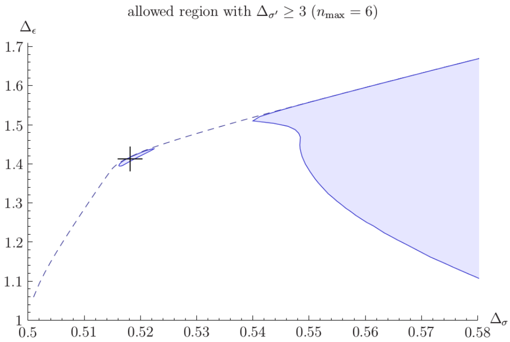
# Multi-correlator Bootstrap [Kos, Poland, Simmons-Duffin]

"Solving" the 3d Ising Model?

Similar to old bootstrap bounds but now consider $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\epsilon\sigma\epsilon\rangle$, and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$.



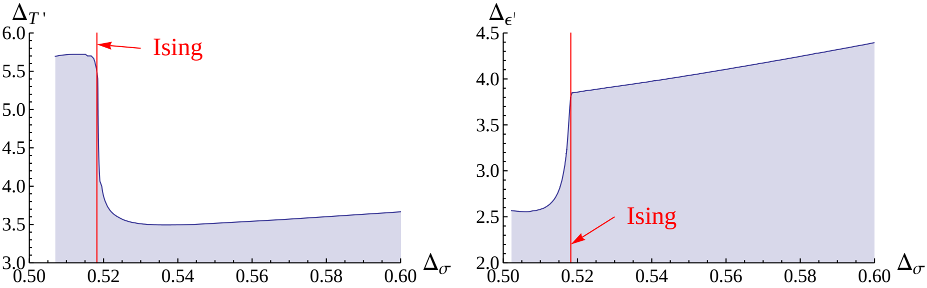

1. Provides access to $\mathbb{Z}_2$ odd spectrum.
2. Only assumption: $\sigma$ is only *relevant* $\mathbb{Z}_2$ scalar.
3. 3d Ising model now approx *only* solution for small $\sigma$!

# Origin of the Kink?

(wild speculation)

Spectrum near the kink undergoes rapid re-arrangement.

Plots for next Scalar and Spin 2 Field




1. "Kink" in $(\epsilon, \sigma)$ plot due to rapid rearrangement of *higher dim spectrum*.
2. This is why its important to determine $\sigma$ to high precision.
3. Does re-arrangement hint at some analytic structure we can use?

Can we find a nice explanation of the kink in $d = 2$?

- In $d = 2$ Virasoro strongly constrains spectrum.
- Minimal models ($c < 1$) have few (Virasoro) primaries in short representations of Virasoro.
- Ising model has only two Virasoro primaries: $|\sigma\rangle$ and $|\epsilon\rangle$.
- Virasoro decendent

  $$T' = (L_{-2} + \eta L_{-1}^2)|\epsilon\rangle$$

  is a spin 2 SL(2,R) primary for certain values of $\eta$.
- Correct value of $\eta$ depends on $c$.
- Norm of $T'$ fixed by Virasoro.
- $T'$ becomes null at $c = 1/2$ (or $\Delta_\sigma = 1/8$)

  $$\langle T'|T'\rangle = 0$$

- EFM shows operator decoupling exactly at Ising point.

# Kinematic Origin of Kinks: Null States?



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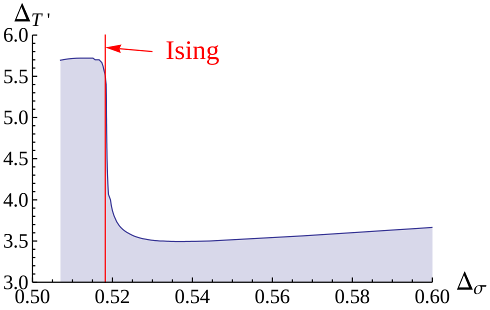
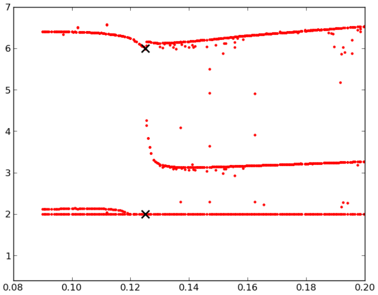
Plot: spin 2 spectrum in $d = 2, 3$.
Curve shows maximal gap in spin 2 sector (above stress tensor).

Spin 2 Bound in d=3




Spin 2 Spectrum in d=2



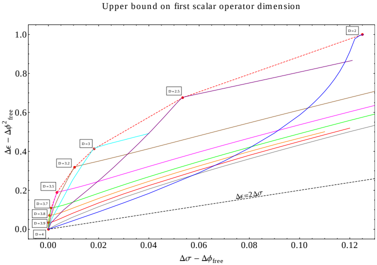

Is d=3 kink also related to a null state decoupling?

How can we understand 3d structures in terms of 2d?

## Fractional spacetime dimension

- Conformal blocks analytic function of spacetime dimension $d$.
- Special structures emerge in $d = 1, 2, 4$

$\boxed{\Rightarrow \text{lessons for } d = 3?}$

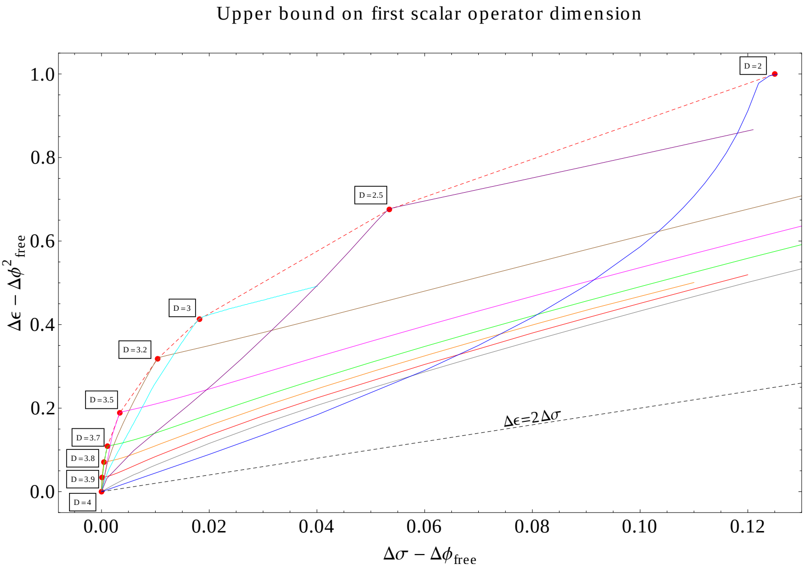



1. Follow "breaking" of Virasoro from $d = 2$?
2. Compare with $\mathcal{E}$-expansion.
3. Find kinks for all $1 < d < 4$
   $\Rightarrow$ easier to extract kinematics

$\boxed{\text{Track spectrum from } d = 2 \text{ to } d = 3?}$

# Ising Model in Fractional Dimensions

## Origin of the Kink



Constraints from Higher Spin symmetry

1. Anomalous dimension of higher spin currents bounded

$$\delta\Delta \leq 2(\Delta_\sigma - \Delta_{\text{free}}) \sim 0.037$$

   E.g. spin 4 has dim 5.02 (conserved current is $\Delta = 5$).

   [Nachtmann, Komargodski & Zhiboedov]

2. Higher spin symmetry only weakly-broken $\Rightarrow$ approx conserved currents.
   $\Rightarrow$ Origin of numerical difficulties for $L > 2$ spectrum
3. In $d = 2, 4$ higher spin fields conserved $\Rightarrow$ is $d = 3$ maximal breaking?

"Hidden" Virasoro-like Symmetry in $d = 3$

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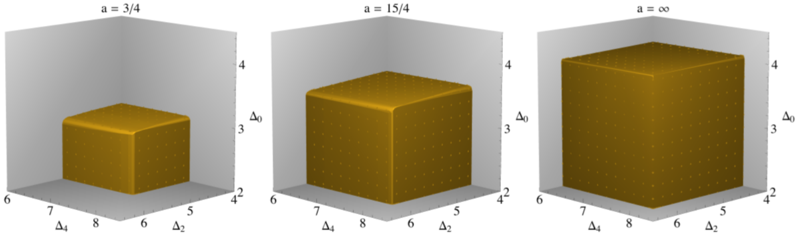
# Other Applications

# $\mathcal{N} = 4$ bounds

Other Applications [Liendo, Rastelli, van Rees]

Setup $\mathcal{N} = 4$ bootstrap. Bounds on leading twist dimension $\Delta_\ell$ for $\ell = 0, 2, 4$.
(holds for all values of $g_{YM}$)




1. Conjecture “corner” corresponds to a self-dual point.
2. Matches some checks computed by resumming perturbative results.
3. Non-perturbative bounds/results at $g \sim \mathcal{O}(1)$.
4. [Alday, Bissi] also conjectured analytic results for OPE.

# Analytic results/conjectures for 6d $\mathcal{N} = (0,2)$ theories

Other Applications [Liendo, Rastelli, van Rees]

Bootstrap in $\mathcal{N} = (0,2)$ theories requires first solving "mini-bootstrap" in a BPS subsector.

1. BPS subsector defined via cohomology of some charge.
2. Novel twist involving conformal generator.
3. Cohomology be projected onto 2d subspace of $\mathbb{R}^6$.
4. Operators in cohomology are chiral, depend only on $z \in \mathbb{C}$.
5. This sector has a 2d $\mathcal{W}$-algebra symmetry.
6. $\mathcal{W}$-algebra fixes three point functions of operators in cohomology.
7. By conjecturing algebra for $A_n$ theories get 3-pt functions for all valus of $N$!

# Liberation at large spin

Other Applications [Fitzpatrick, Kaplan, Poland, Simmons-Duffin] , [Komargodski, Zhiboedov]

In limit $|u| \ll |v| < 1$ leading twist field dominate bootstrap equations.

## Bootstrap equation in $u \to 0$ limit

For every scalar $\phi$ of dimension $\Delta_\phi$

- CFT must contain an infinite tower of operators with dimension $\tau \to 2\Delta_\phi + \ell$.
- Asymtotic estimates holds in $\ell \to \infty$ limit.
- Morally these operators are $\phi(\partial^2)^n \partial_{\mu_1} \ldots \partial_{\mu_\ell} \phi$.

Another interesting constraint, suggested by Nachtmann ('70s), but derived more thoroughly by [Komargodski, Zhiboedov] :

## Convexity (Nachtmann's theorem)

Twist, $\tau_\ell = \Delta - \ell$, of leading twist operator

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# Pros/Cons of our Method

New Methods

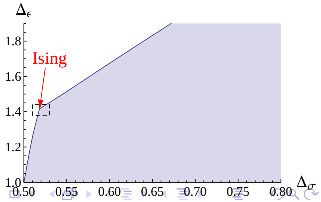
Methods explained so far have several good properties:

1. Can make rigorous statements about non-existance of CFTs for various spectra, $\mathcal{S}$.
2. We have control over our sources of error so can provide systematic error bounds.

But there are several un-features (i.e. bad things):

1. The method is a "blunt tool": cannot easy "pick" which theory to study.
2. If theory we're interested in is not at boundary of solution can't get unique spectrum.
3. Computationally intensive: depending on desired accuracy might require computer cluster.
4. Only applies to unitary CFTs.

In particular if we want to add more correlators (e.g. $\langle\sigma\epsilon\sigma\epsilon\rangle$) our method becomes much more computationally intensive.



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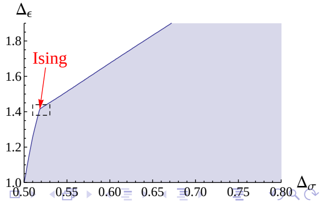

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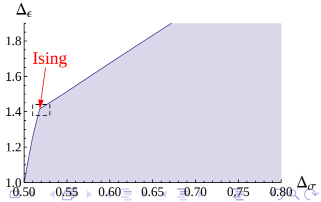

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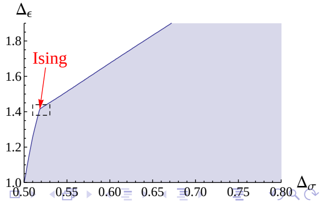

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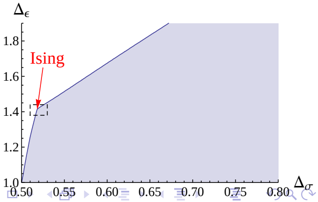

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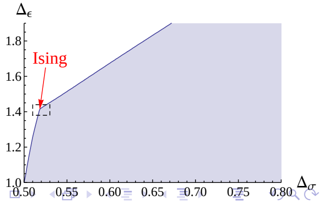

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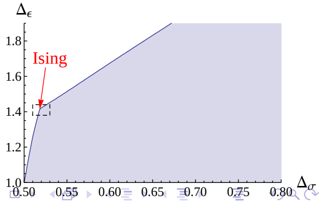

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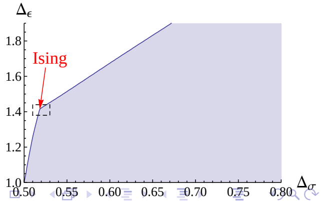

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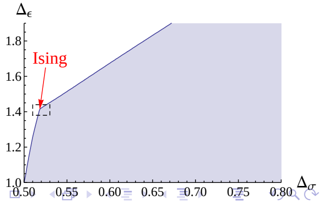

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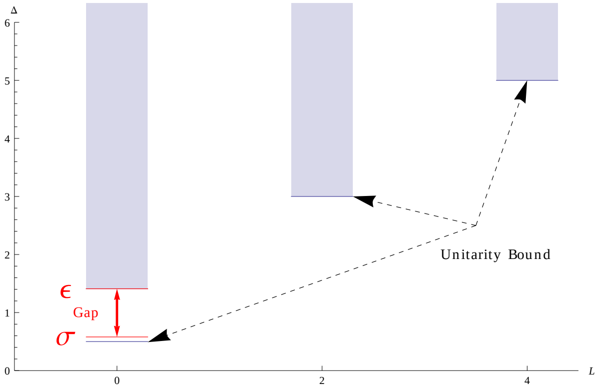
Recall in our method we make the following sort of assumptions about the spectrum:

Figure : a putative spectrum $\mathcal{S}$



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This yields the following equation we solve (with $M_{\mathcal{S}}$ depending on $\mathcal{S}$):

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Note:

- The columns of the matrix are labeleld by $(\Delta, l)$ and there is a continuous infinity of them.
- This system is generally very under-determined.
- Generically there are $\infty$-many solutions to this equation (i.e. for $\mathcal{S}$ deep in allowed region).
- This is because we made very weak assumptions about spectrum (e.g. just gap $\Delta_\epsilon$).

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1. Generically a CFT has a discrete (even sparse) spectrum.
2. Also we might expect that generically there should be unique solution to crossing (if we know enough about the spectrum).

So he proposes the following:

**Our assumptions**

$$\sigma \times \sigma \sim 1 + \epsilon + \dots$$

And we try to maximize $\Delta_\epsilon$ while allowing anything in '...'.

Gliozzi wants to make the following much stronger assumptions:

**Gliozzi's assumptions**

$$\sigma \times \sigma \sim 1 + \epsilon + \epsilon' + T_{\mu\nu} + C_{\mu\nu\rho\lambda} + \dots$$

Solve for $\Delta_\epsilon, \Delta_{\epsilon'}, \Delta_{C_{\mu\nu\rho\lambda}}$ by requiring uniqueness and dropping '...'.

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How do we solve this system?

1. Gliozzi claims crossing symmetry should have a unique solution.
2. If $M = N$ (so $\mathcal{M}$ square) uniqueness means $\det(\mathcal{M}) = 0$.
3. More generally take $M > N$ and require $\det(\mathcal{M}_i) = 0$ with $\mathcal{M}_i$ square sub-matrices.
4. Example: $M = N + 1$.
   - $\mathcal{M}_i$ are $M$ square sub-matrices given by removing $i$'th row.

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for $n = 4$ unknowns (because the $\det(\mathcal{M}_i)$ depends on $\Delta_\sigma, \Delta_\epsilon, \Delta_{\epsilon'}, \Delta_{C_4}$).

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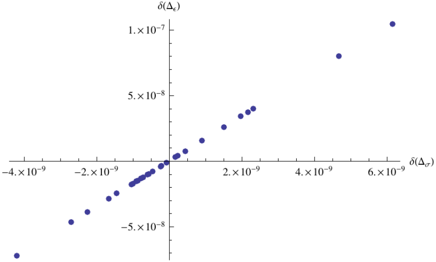

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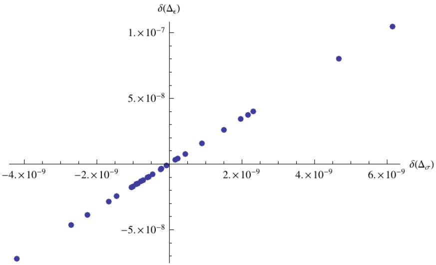

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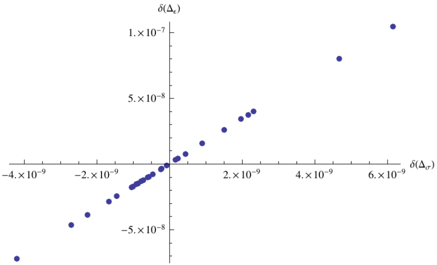

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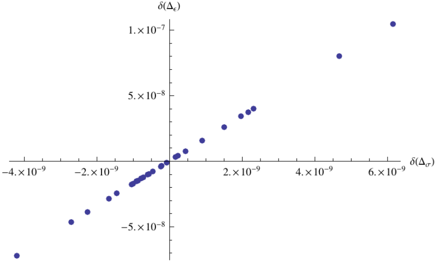

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3. Getting these quantities requires much less computation than our method.

**Problem:** by throwing away high-dim operators he's making an error $O(e^{-\Delta_*})$ with $\Delta_* \sim 10$. So real equation is:

$$\underbrace{\begin{pmatrix} F_\epsilon^{(0,0)} & F_{\epsilon'}^{(0,0)} & F_{T_{\mu\nu}}^{(0,0)} & F_{C_4}^{(0,0)} \\ F_\epsilon^{(2,0)} & F_{\epsilon'}^{(2,0)} & F_{T_{\mu\nu}}^{(2,0)} & F_{C_4}^{(2,0)} \\ F_\epsilon^{(0,2)} & F_{\epsilon'}^{(0,2)} & F_{T_{\mu\nu}}^{(0,2)} & F_{C_4}^{(0,2)} \\ F_\epsilon^{(4,0)} & F_{\epsilon'}^{(4,0)} & F_{T_{\mu\nu}}^{(4,0)} & F_{C_4}^{(4,0)} \end{pmatrix}}_{\mathcal{M}_{\epsilon,\epsilon',C}} \underbrace{\begin{pmatrix} p_\epsilon \\ p_{\epsilon'} \\ p_{T_{\mu\nu}} \\ p_{C_4} \end{pmatrix}}_{\vec{p}} = \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}}_{\vec{\alpha}}$$

with $||\vec{\alpha}|| < e^{-10}$.
We have no idea how varying $\vec{\alpha}$ affects values of $\Delta_\epsilon$, $\Delta'_\epsilon$, etc…

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6. Also no control over the (relatively) larger error.
7. Solving $\det \mathcal{M}_i = 0$ non-trivial as we increase $M, N$ and $n$.

But

1. Can be done with Mathematica on a laptop.
2. Doesn't assume unitarity at all (he checks it in non-unitary theories).
3. Can easily incorporate additional correlators (just add more rows to $\mathcal{M}$).

So

1. Definitely worth investigating more.
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# The Competition: Gliozzi Method

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# Bootstrapping Theories with Four Supercharges (in $d = 2 - 4$)

How can we include SUSY constraints in bootstrap?

1. SUSY relates OPE coefficients of components of SUSY multiplets in some correlators.
2. This yields superconformal blocks for whole SUSY multiplet:

$$\mathcal{G}_{\Delta,l} = G_{\Delta,l} + c_1 G_{\Delta+1,l+1} + c_2 G_{\Delta+1,l-1} + c_3 G_{\Delta+2,l}$$

   with $c_1, c_2, c_3$ fixed by SUSY.
3. SUSY fixes dimensions of protected operators by e.g. $a$-maximization.
4. SUSY imposes stronger unitarity bounds in terms of R-charge:

$$\Delta \geq \frac{d-1}{2} R$$

   (for scalars in theories with 4 supercharges)

Can now try to bootstrap using these additional constraints.

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In theories with 4 supercharges dimensions of some operators constrained by SUSY.

- Chiral operator is annihilated by half supercharges.
- SUSY algebra contains $U(1)$ R-charge and $\Delta$ of chiral operator depends on its R-charge:

$$\Delta = \left(\frac{d-1}{2}\right) R$$

- When only one kind of field superpotential can fix R-charge uniquely since superpotential $W$ must have $R$-charge 2. E.g.:

$$W = X^3$$

  implies superfield $X$ has R-charge 2/3.
- If more than one field (e.g. $XY^2$) can use $a$-maximization to compute R-charge.

# The Benefits of SUSY

## Bootstrapping Theories with Four Supercharges

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# SUSY Constraints



Lets consider theories with four supercharges in $d = 2 - 4$.

- Let $\Phi$ be complex chiral scalar field so $\Delta = \left(\frac{d-1}{2}\right) R$.
- To get strong constraints we consider

$$\Phi \times \Phi \text{ and } \Phi \times \bar{\Phi} \text{ OPE in } \quad \langle \Phi\bar{\Phi}\Phi\bar{\Phi} \rangle$$

- $\Phi$ carries $R$-charge so can decompose OPE in reps of $R$-charge:

$$\Phi \times \bar{\Phi} \sim \text{singlets}$$
$$\Phi \times \Phi \sim \text{R-charge } 2$$

- CB decomposition gives different constraints in each $R$-charge channel.
- Because $\Phi$, $\bar{\Phi}$ not identical can get odd-spin contributions in $\langle \Phi\bar{\Phi}\Phi\bar{\Phi} \rangle$.
- SUSY (+R-charge) fixes dims of some contributions in $\Phi \times \Phi$ OPE:

$$\Phi \times \Phi \sim 1 + \Psi_{d-2\Delta_\Phi,0} + \Phi^2 + \dots$$

with $\Delta \geq |2\Delta_\Phi - (d-1)| + l + (d-1)$ for '…'.

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# Results

## Bootstrapping Theories with Four Supercharges

So what kind of bounds to we get?

- Bounds for $d = 2 - 4$.
- Multiple kinks!!
- Horizontal dashed line: $\Delta_\Phi$ in Wess-Zumino model
- This is SUSY version of $\phi^4$ theory!
- $\Delta_\Phi$ fixed because superpotential

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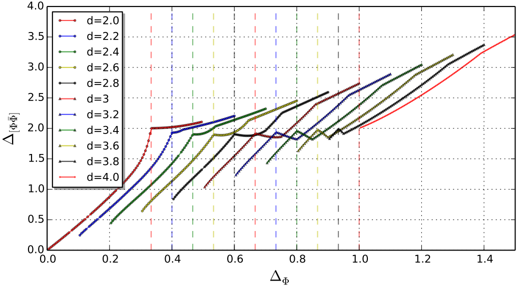
$\Delta_{[\Phi\bar{\Phi}]}$ Bound



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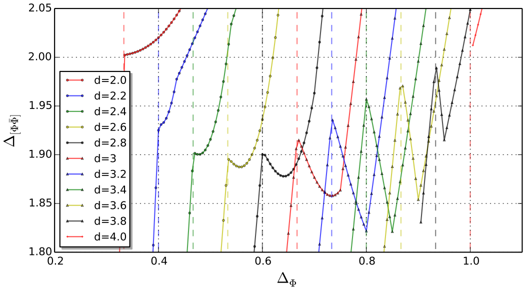
$\Delta_{[\Phi\bar{\Phi}]}$ Bound (zoomed)



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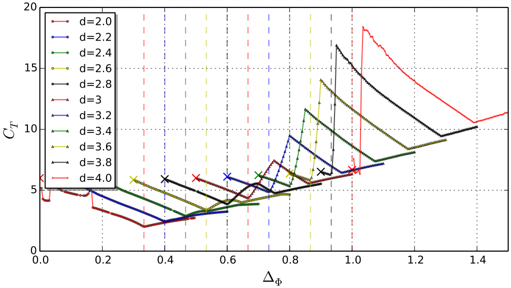

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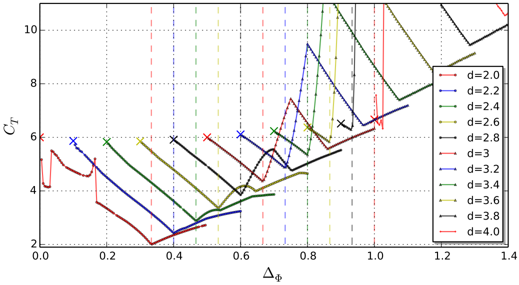
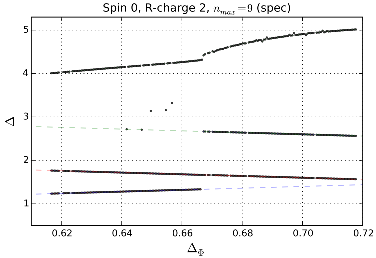
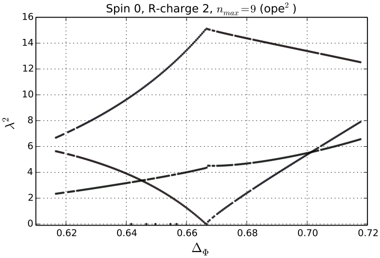


SUSY also imposes interesting dynamical constraints on theory

- Consider Wess-Zumino model: chiral superfield $X = \Phi + \ldots$ with cubic superpotential:
$$W = X^3$$
- Implies superfield $X$ has R-charge 2/3 and (in $d = 3$) $\Delta_\Phi = R$.
- The fact that in e.g. WZ model can compute $\Delta_\Phi$ exaclty means can extract spectrum easily.
- SUSY eqns $\frac{\partial W}{\partial X} = 0$ implies $\Phi^2$ should decouple in theory.

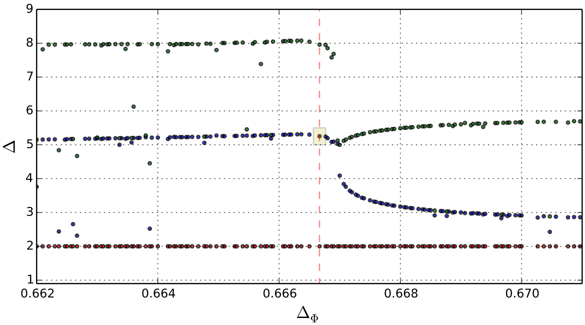


In non-SUSY 3d Ising found interesting (surprising) kinematical structure.

What about SUSY case?

Bounds on Spin 1*




(*Because of susy $T_{\mu\nu}$ and $T'_{\mu\nu}$ are actually SUSY descendents in spin 1 multiplet.)

SUSY analog of 3d null states!!

# New Structure

## Bootstrapping Theories with Four Supercharges

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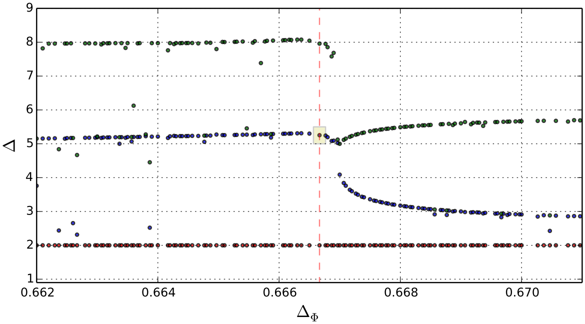


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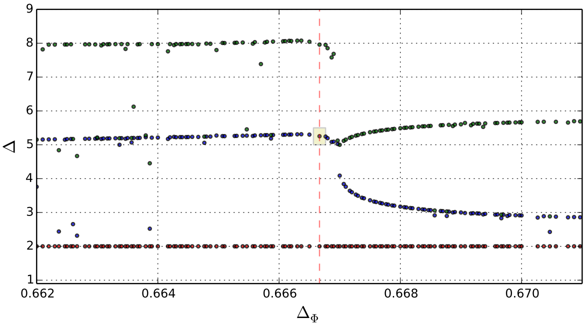


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SUSY analog of 3d null states!!

# Our (Modified) Simplex Algorithm

Recall the original problem we want to solve

$$\text{Minimize:} \quad \vec{c}^T \cdot \vec{x},$$
$$\text{subject to:} \quad M \cdot \vec{x} = \vec{b}, \qquad \vec{x} \geq 0$$

Trivial solution is $\vec{x} = 0$ and $\vec{y} = A^{-1}\vec{b}$ but now must reduce cost:

1. Turning on component $x_\alpha$ reduces cost via
$$\vec{c}_y^T \cdot \vec{y} = \vec{c}_y^T A^{-1}(\vec{b} - x_\alpha \vec{v}_\alpha)$$
2. Choose component $\alpha$ by maximizing $\vec{c}_y^T \cdot A^{-1}\vec{v}_\alpha$ (contribution to cost).
3. Increase $x_\alpha$ until some component of $\vec{y}$ becomes some zero. e.g.
$$x_\alpha v_\alpha^i = b^i \qquad \Rightarrow y^i = 0$$
4. "Swap in" $\vec{v}_\alpha$ into $i$'th column of $A$.
5. Repeat with new $A$ until all components of $\vec{y}$ set to zero.
6. This yields "feasable" $\vec{x}$; can then turn on $c_x$ and continue.

Introduce $n$ "slack" variables $y$ with costs $\vec{c}_y^T = (1, \ldots, 1)$ and solve:

$$\text{Minimize:} \quad \vec{c}_y^T \cdot \vec{y},$$
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Modifications for $x$ continuously labelled, $\alpha \to \Delta$:

- $\vec{y}$ is $n$-vector and $A$ is $n \times n$ matrix so remains discrete.
- $x_\Delta$ only appears in minimzation stage so we need to maximize:

$$\rho(\Delta) = c_y^T A^{-1} \vec{v}_\Delta$$

- Various approaches:
  1. Branch & bound on $\rho(\Delta)$.
  2. Local quadratic or cubic approximation.
- Technical Issues:
  1. $\rho(\Delta)$ approximated by very high-order polynomial $O(\Delta^{60})$.
  2. Matrix $A$ becomes ill-conditioned in most physically interesting cases.
  3. Numerical precision insufficient $\Rightarrow$ MPFR/GMP

How to get Started Bootstrapping…

# Some Resources (code)



So here are some ways to start:

Conformal Blocks

1. In 2d/4d can use exact expressions (see e.g. arXiv:0807.0004).
2. In general $d$ (or just faster):
   - See JuliaBootS (below).
   - Use method described in arXiv:1305.1321.

Bootstrapping

1. Mathematica's LinearProgramming function (good luck!)
2. Mathematica + IBM's CPLEX LP (email me for mathematica plugin)
3. Mathematica + SDPA (probably the most standard solution now)
4. JuliBootS (open-source Julia implementation of Bootstrap - arXiv:1412.4127)
5. Our python implementation (to be released soon ?!?!)
6. Roll your own…

Thanks

# Dual Method

Bootstrap Formulation

- We can also formulate the problem in a "dual" way.
- Instead of solving for $\lambda$ we can look for a diff op $\alpha$ such that:

$$\alpha(F_i) > 0 \qquad \forall F_i \qquad \left( \alpha = \sum_{n,m} \alpha_{n,m} \partial_z^n \partial_{\bar{z}}^m \right)$$

- Taylor expanding around $z = \bar{z} = 1/2$ and requiring each order to vanish gives a matrix:

$$\underbrace{\begin{pmatrix} F_1^{(0,0)} & F_2^{(0,0)} & F_3^{(0,0)} & \xrightarrow{\Delta} \\ F_1^{(2,0)} & F_2^{(2,0)} & F_3^{(2,0)} & \cdots \\ F_1^{(0,2)} & F_2^{(0,2)} & F_3^{(0,2)} & \cdots \\ \downarrow \partial & \vdots & \vdots & \ddots \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \\ \vdots \end{pmatrix}}_{\vec{\lambda}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

which we must solve subject to $\lambda_i^2 \geq 0$.
- Solving for $\lambda_i^2$ is called the **Direct Problem**.