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SOLVING THE BIRTH AND DEATH PROCESSES WITH QUADRATIC ASYMPTOTICALLY SYMMETRIC TRANSITION RATES*

BERTRAND ROEHNER† AND GALLIANO VALENT‡

Abstract. Birth and death processes with quadratic transition rates are considered with the constraint $\lim_{n\to\infty} \lambda(n)/\mu(n) = 1$. The partial differential equation governing the generating function is solved quite generally and gives an integral representation for it. In the process a generalized Mehler-Fock transform is defined and inverted. From the generating function we extract the transition probabilities which are shown to be the solution of the Chapman-Kolmogorov equations. Various approximation schemes are then devised to deal with large populations and to extract the large time behavior for the transition probabilities.

1. Introduction. In the field of stochastic birth and death processes, little attention has been given to the case where the transition rates become quadratic with respect to the population number. To our knowledge the only case completely solved is given in Karlin (1962) and corresponds to a finite population.

The principal aim of this work is to exhibit the general solution for the transition rates:

$$\lambda_n = \alpha(n^2 + bn + c), \qquad \mu_n = \alpha(n^2 + \tilde{b}n). \tag{0}$$

These processes will be called quadratic asymptotically symmetric (QAS), and include the finite population as a particular case.

Let us make some comments on our motivation for studying these processes.

Practical applications. a) Birth and death processes have long been used as a stochastic model of coupled chemical reactions (McQuarrie (1967)). In recent years, the so-called "Brussels group" gave considerable attention to exact and approximate resolution of some nonlinear birth and death processes related to nonequilibrium phase transitions in chemical reactions (Lemarchand (1976), Nicolis (1978)).

The most general form of QAS processes corresponds to the following set of reactions:

$$1A + 2X \underset{k_1}{\overset{k_2}{\leftrightarrows}} B + X,$$

$$C + 2X \underset{k_3}{\rightarrow} D + 3X,$$

$$X \underset{k_4}{\overset{k_5}{\leftrightarrows}} E.$$

Only the concentration of X is variable. The concentration of A and C should satisfy $k_3[C] = k_1[A]$. Of course some reactions could be absent, which corresponds to particular cases of the QAS process.

b) The particular finite process obtained when λ_n vanishes for some value $n = N$ has already proved useful in genetics for the modeling of fertilization and mutation phenomena in a finite set of gametes. It has been discussed in a different framework by Karlin (1962). We discuss this point further in § 4.1.

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† Université Paris VII, 2, place Jussieu, 75251 Paris, CEDEX 05, France.
‡ Laboratoire de Physique Théorique et Hautes Energies, Université Paris VII, 2, place Jussieu 75251 Paris, CEDEX 05 France. The laboratory is associated with the Centre National de Recherche Scientifique.

Mathematical considerations. Our motivations for the study of QAS processes are mathematical ones as well. Indeed, infinite nonlinear birth and death processes are in general extremely difficult to solve. But we have here several favorable circumstances which are worthy of mention:

1) The spectrum of the process is here continuous. This could not be deduced from some general theorem (Lederman (1954)). One may conjecture that the same is true for all the processes such that:

$$\lambda_n \underset{n\to\infty}{\sim} \mu_n \underset{n\to\infty}{\sim} an^\alpha, \qquad 0<\alpha\leqq 2.$$

In other cases, in particular if the degree of λ_n and μ_n is greater than two (even if these degrees are different), the eigenvalues are discrete. They are then likely to be the roots of some difficult (probably transcendental) equation. This is in particular the case for the Schlögel process which is of interest in the study of bifurcation phenomena (Nicolis (1979)).

2) The infinite set of equations for the moments exhibits in our case a remarkable property: it is a closed system for QAS processes, and only for them (see § 1.4). Thus they may (in principle) be computed recursively. In our approach these moments can be obtained in the following directly from the generating function.

3) From the point of view of partial differential equations, the problem to be solved in this paper is the resolution of the equation

$$\frac{\partial G}{\partial t}(x_1t) = L_x G(x_1t)$$

where L_x is a generalized hypergeometric differential operator:

$$L_x = \alpha(1-x)\left[x(1-x)\frac{\partial^2}{\partial x^2} + (1+\tilde{b}-(1+b)x)\frac{\partial}{\partial x} - c\right]$$

on the finite interval (0, 1) and with boundary conditions:

$$G(x, 0) = A(x), \qquad G(1, t) < \infty.$$

Let us recall that this equation frequently occurs in one-dimensional physics: it is the heat equation or the Schrödinger equation when t becomes imaginary.

Since an exact solution is possible, we get further insight into the structure of the generating function expansion for a continuous spectrum.

Furthermore, in the course of the resolution a new integral transform emerges which is a generalization of the Mehler–Fock transform and may be interesting by itself.

Asymptotic approximation of the solutions are given for

$$\alpha t \gg 1 \quad \text{and} \quad n \gg 1.$$

Such formulas are of frequent use in practical applications of birth and death processes, since in the real world n is usually very large. It is possible here to check them against the exact solutions. Let us also notice that these results are rather out of reach for numerical computer resolution of the Chapman–Kolmogorov equations.

This article has the following structure:

In § 1 we give the necessary definitions and then discuss under what conditions on the parameters b, $\tilde{b}$ (more precisely $1+\tilde{b}-b\geqq 0$) the existence and uniqueness of the transition probabilities is ensured for all times. At a formal level, the time dependence of the momenta is discussed, and the generating function technique is considered: this transforms the original problem into a partial differential equation.

The main point is to solve this equation by a separation of variables technique, and expand the generating function on the eigenvectors of some second order differential operator. The boundary conditions emerge after a heuristic discussion of the conservation of probabilities at all times.

In § 2 we give the eigenvectors and eigenvalues in closed form.

In § 3 we explain how to get the generating function. This is not a completely trivial question since it is necessary first to invert a new kind of integral transform whose kernel is a hypergeometric function, and second to make a process of analytic continuation.

This gives only a formal answer for the generating function $G(x, t)$. It is then proved with full mathematical rigor, in Appendices A, B and C, that the generating function which is so found is indeed a solution of the original partial differential equation with the appropriate boundary conditions. This proof leaves out of reach only the limiting case where $1+\tilde{b}-b=0$, but works for $1+\tilde{b}-b>0$.

From the generating function we extract the transition probabilities for which it is easy to prove that they are a solution of the Chapman–Kolmogorov equations with prescribed initial population and probability conservation at all times.

From the generating function we compute easily the first moment and obtain rigorously the same answer as in the formal discussion of § 1.

In § 4, we give a detailed analysis of two simple processes: these are chosen for their intrinsic simplicity and we show explicitly how the technique of variables separation works for these cases.

In § 5 we have devised several approximation schemes since our final answer for the transition probabilities involves an integral of a complicated structure.

In many places we have given only sketchy arguments. The interested reader will find all the details in the more extended version of Roehner (1980).

1.1. Definitions and hypotheses. We shall deal with homogeneous birth and death processes (BD) evolving over a denumerable set of states labelled by a positive integer $n=0, 1, 2, \cdots$.

Let $p_{n_0 \to n}(t)$ be the transition probabilities from an initial state n_0 at time 0 to a final state n at time t. In what follows we shall often omit n_0 and write simply $p_n(t)$.

These probabilities must satisfy (see Bailey (1964)) the Chapman–Kolmogorov equations:

$$(1) \qquad \dot{p}_n(t) = -(\lambda_n+\mu_n)p_n(t)+\lambda_{n-1}p_{n-1}(t)+\mu_{n+1}p_{n+1}(t), \qquad \dot{p}_n(t)=\frac{d}{dt}p_n(t),$$

with the conventional definition

$$p_n(t)=0 \quad \text{if } n<0.$$

We take as boundary conditions for $t=0$

$$(2) \qquad p_n(t=0)=\delta_{n,n_0}.$$

In order that $p_n(t)$ be interpreted as probabilities at all times, we must have

$$(3) \qquad \sum_{n=0}^{\infty} p_n(t)=1, \qquad t \geqq 0.$$

Except for the case of finite processes to be considered later on, we shall suppose that the transition rates are positive functions of n in its whole range, and furthermore we suppose that $\mu_0=0$. In this way $n=0$ becomes the natural limit of the process.

Since we require the positivity of μ_n we must have

$$\tilde{b} \geqq -1$$

and when λ_n has two real roots these must be negative.

1.2. Existence and uniqueness. As is known from Bailey (1964 p. 102), a sufficient condition which ensures the existence and uniqueness of a solution to (1) with (3) fulfilled, is the divergence of the series

$$\sum_{n=\nu}^{\infty} \prod_{i=i_0}^{n} \frac{\mu_i}{\lambda_i} \tag{4}$$

where ν, i_0 are positive integers.

This condition is clearly fulfilled for symmetric processes. For QAS processes we must study the divergence of the series (4). This can be done using Raabe's test which gives the following constraint:

$$a = \frac{1+\tilde{b}-b}{2} > 0.$$

For $a = 0$ Raabe's test cannot be used. Nevertheless, Gauss's convergence test (for these tests see Gradshteyn (1965)) ensures uniqueness even for $a = 0$.

Hence the existence and uniqueness of the solution of (1) is ensured for $a \geqq 0$.

1.3. Time dependence of the moments. The moments are defined by

$$m_k(t) = \sum_{n=0}^{\infty} n^k p_n(t), \qquad k = 0, 1, 2, \cdots.$$

We suppose their existence. Using (1) we get for the moments:[1]

$$\mathring{m}_1(t) = \sum_{n=0}^{\infty} (\lambda_n - \mu_n) p_n(t),$$

$$\mathring{m}_2(t) = 2 \sum_{n=0}^{\infty} n(\lambda_n - \mu_n) p_n(t) + \sum_{n=0}^{\infty} (\lambda_n + \mu_n) p_n(t),$$

$$\vdots \qquad\qquad \vdots$$

$$\mathring{m}_k(t) = \sum_{n=0}^{\infty} \sum_{i=1}^{k} \binom{k}{k-i} n^{k-i} [\lambda_n + (-1)^i \mu_n] p_n(t).$$

Consider the system of the first two equations: this system is closed if all powers of n higher than two are absent. This may happen if:

i) $\lambda_n + \mu_n$ is at most of degree two: a quadratic process;

ii) $\lambda_n - \mu_n$ is at most of degree one: a QAS process.

Now the expression of $m_k(t)$ makes obvious the closedness of the equations for the first k moments. This is a characteristic property of QAS processes. The equation for $m_1(t)$ closes and is easily integrated:

$$m_1(t) = \begin{cases} \dfrac{c}{2a-1} + \left(n_0 - \dfrac{c}{2a-1}\right) e^{-(2a-1)\alpha t}, & a \neq \frac{1}{2}, \\ n_0 + c\alpha t, & a = \frac{1}{2} \end{cases}$$

For a symmetric process $m_1(t) = n_0$ is time independent.

[1] This is somewhat heuristic; we shall prove later on that it indeed gives the right answer.

Let us observe that these expressions are identical to the solution of the deterministic equations:

$$\mathring{x}(t) = -\alpha(2a-1)x(t) + \alpha c.$$

Another interesting feature is displayed by the processes with transition rates:

$$\lambda_n = \alpha(n^2 + bn), \quad b > 0, \qquad \mu_n = \alpha n^2.$$

In this case, $m_1(t) = n_0 e^{b\alpha t}$, although we know that

$$\lim_{t\to+\infty} p_0(t) = 1$$

from a general theorem (Karlin (1966 p. 205)). This exhibits a drastic difference between the time evolution of the mean value and the most probable value of the population.

1.4. Generating function technique.

i) *Principle.* We define:

(5)
$$G(x,t) = \sum_{n=0}^{\infty} p_n(t) x^n, \qquad 0 \leqq x \leqq 1.$$

It is easy to show formally that $G(x, t)$ must obey the partial differential equation

$$\frac{\partial G}{\partial t} = (1-x)\left[\frac{1}{x}\mu\left(x\frac{\partial}{\partial x}\right) - \lambda\left(x\frac{\partial}{\partial x}\right)\right]G,$$

where

$$\mu\left(x\frac{\partial}{\partial x}\right) \equiv \mu\left(n \to x\frac{\partial}{\partial x}\right).$$

The boundary condition (2) becomes $G(x, t=0) = x^{n_0}$.

The simplest way to solve this equation is to use a separation of variables method. We define the eigenvectors $y_s(x)$ and the eigenvalues s by

(6)
$$(1-x)\left[\frac{1}{x}\mu\left(x\frac{d}{dx}\right) - \lambda\left(x\frac{d}{dx}\right)\right]y_s(x) = s y_s(x).$$

Once these are known, the generating function is given symbolically:

(7)
$$G(x,t) = \sum_s c_s e^{st} y_s(x)$$

where the coefficients c_s can be obtained expanding x^{n_0} on the eigenvectors:

$$G(x, t=0) = x^{n_0} = \sum_s c_s y_s(x).$$

ii) *Boundary conditions.* The eigenvalue problem (6) is defined precisely only when definite boundary conditions are given.

The main point is relation (3) which expresses probability conservation:

$$\sum_{n=0}^{\infty} p_n(t) = 1, \qquad t \geqq 0,$$

which implies

(8)
$$\lim_{x\to 1} G(x,t) = 1, \qquad t \geqq 0.$$

This proves that the series (5) has a unit radius of convergence and defines an analytic function[2] for $|x|<1$ (uniformly for $t \geqq 0$). Since $G(x, t)$ is given by the sum of formula (7):

$$G(x,t)=\sum_s c_s e^{st} y_s(x),$$

we are led to impose that $y_s(x)$ must be analytic when $|x|<1$ for all s.

From (7) and (8) we get also

$$1=c_0 y_0(x=1)+\sum_{s\neq 0} c_s e^{st} y_0(x=1) \quad \text{for } t\geqq 0,$$

and if the eigenvalues are all different we must impose

$$y_s(x=1)=0 \quad \text{if } s\neq 0.$$

This is also suggested by the factor $(1-x)$ appearing in (6), but this argument is compelling only if the various derivatives at $x=1$ exist, which is not obviously true.

We shall take the eigenvalue problem (6) supplemented by two boundary conditions:

i) $y_s(x)$ analytic for[3] $|x|<1$;

ii) $y_s(x=1)=0$ if $s\neq 0$.

Clearly the eigenstates corresponding to $s\neq 0$ span a linear vector space.

2. Eigenvalue problem for QAS processes. As explained in § 1.4, we want to solve

$$\frac{\partial G}{\partial t}=L\left(x,\frac{\partial}{\partial x}\right)G,$$

with

(9) $$L=\alpha(1-x)\left[x(1-x)\frac{\partial^2}{\partial x^2}+(1+\tilde{b}-(1+b)x)\frac{\partial}{\partial x}-c\right].$$

For these processes the eigenvalue equation (6) becomes

(10) $$x(1-x)^2y''+(1-x)[1+\tilde{b}-(1+b)x]y'+(s'-x(1-x))y=0, \qquad y'=\frac{dy}{dx},$$

where $s'=-s/\alpha$.

We look for a solution of the form

$$y(x)=(1-x)^r u(x).$$

If we impose

(11) $$r^2-2ar+s'=0,$$

(10) reduces to

(12) $$x(1-x)u''+(1+\tilde{b}-(1+b+2r)x)u'+(s'-c-r(1+\tilde{b}))u=0,$$

whose solutions are hypergeometric functions.

At this stage we must recall the positivity condition on μ_n given in § 1.1, $\tilde{b}\geqq -1$. For $\tilde{b}=-1$ (and more generally a negative integer) we shall give later on a separate detailed analysis (see § 3.3).

[2] Here x is taken as a complex variable.

[3] If x is taken as a real variable, $y_s(x)$ must be C^∞ for $0\leqq x\leqq R<1$.

In the following we shall take the condition $\tilde{b} > -1$. Inspection of the linearly independent solutions of (12) (see, e.g., Gradshteyn (1965, p. 1046)) gives only one solution analytic at $x = 0$. It is

$$y_r(x) = (1-x)^r F\left(r + \frac{b}{2} + \delta, r + \frac{b}{2} - \delta; 1 + \tilde{b}; x\right).$$

In order to make transparent the following formulas we introduce

$$\nu_\pm = -\frac{b}{2} \pm \delta, \quad a = \frac{1 + \tilde{b} - b}{2} = \frac{1 + \tilde{b} + \nu_+ + \nu_-}{2}, \quad \delta = \sqrt{(b/2)^2 - c^2}$$

which are the zeros of λ_n:

$$\lambda_n = \alpha(n - \nu_+)(n - \nu_-).$$

Using this notation we obtain

$$y_r(x) = (1-x)^r F(r - \nu_-, r - \nu_+; 1 + \tilde{b}; x).$$

As a side remark, when $\tilde{b} = -p$ with $p = 1, 2, \cdots$, the eigenvectors are given by Gradshteyn (1965, p. 1047)

$$y_r(x) = x^p (1-x)^r F(r + p - \nu_-, r + p - \nu_+; 1 + p; x).$$

(See also § 3.3 for a further discussion of this case at the level of the generating function.)

At this stage we have imposed only one part of the boundary conditions. It remains to ensure:

$$y_s(1) = 0 \quad \text{if } s \neq 0.$$

The eigenvectors behavior at $x = 1$ depends on whether $r = a$ or not.

Case 1. $r \neq a$. The behavior for $x \to 1$ of the hypergeometric function is given in Whittaker (1965, p. 291):

$$\begin{aligned} y_r(x) = {} & \frac{\Gamma(1+\tilde{b})\Gamma(2a-2x)}{\Gamma(1+\tilde{b}+\nu_- - r)\Gamma(1+\tilde{b}+\nu_+ - r)}(1-x)^r F(r-\nu_-, r-\nu_+; -\nu_- - \nu_+ - \tilde{b}; 1-x) \\ & + \frac{\Gamma(1+\tilde{b})\Gamma(2r-2a)}{\Gamma(r-\nu_+)\Gamma(r-\nu_-)}(1-x)^{2a-r} F(1+\tilde{b}+\nu_- - r, 1+\tilde{b}+\nu_+ - r; 2a-r+1; 1-x); \end{aligned} \tag{13}$$

hence if we take

$$\operatorname{Re} r > 0 \quad \text{and} \quad \operatorname{Re}(2a - r) > 0,$$

the eigenvectors vanish for $x = 1$.

These conditions imply $a > 0$, which makes contact with the discussion on unicity in § 1.2 even if we do not recover exactly $a \geqq 0$. This result strongly supports the choice made earlier for the boundary conditions (see § 1.4).

Due to the symmetry $r \to 2a - r$ of the eigenvectors, we can constrain the spectral parameter r to

$$0 < \operatorname{Re} r < a,$$

and the eigenvalue is given by (11)

$$s' = r(2a - r).$$

Case 2. $r=a$. The behavior at $x=1$ of the hypergeometric function is given in Whittaker (1965, p. 299):

$$y(x)=(1-x)^r\left\{\frac{\Gamma(2r-\nu_+-\nu_-)}{\Gamma(r-\nu_-)\Gamma(r-\nu_+)}\ln\frac{1}{1-x}+\text{regular terms}\right\};$$

hence for $r=a>0$ we have $\lim_{x\to 1} y(x)=0$.

Let us notice that $r=a=0$ is indeed admissable because this corresponds to $s'=0$. So we obtain for the spectral parameter:

$$0\leqq \operatorname{Re} r\leqq a$$

as final constraint.

We have obtained the following eigenvectors:

$$\begin{aligned}&0\leqq r\leqq a, \\ &s=-\alpha r(2a-r);\end{aligned}\qquad y_r^{(1)}(x)=(1-x)^r F(r-\nu_-, r-\nu_+; 1+\tilde{b}; x),$$

$$\begin{aligned}&r=a+iu, \\ &u\geqq 0,\quad a>0, \\ &s=-\alpha(a^2+u^2),\end{aligned}\qquad y_u^{(2)}(x)=(1-x)^{a+iu}F(a-\nu_-+iu, a-\nu_++iu; 1+\tilde{b}; x),$$

with the condition $\tilde{b}>-1$.

Let us mention that these eigenstates are indeed real even if $\nu_\pm$ become complex. This can be proved using elementary relations involving the hypergeometric function.

3. Transition probabilities for QAS processes. Once the eigenvectors and the eigenvalues are obtained, we have to solve the following problem. Given

$$A(x)=x^{n_0}$$

we must find $h_{n_0}(u)$ such that:[4]

$$\int_0^\infty h_{n_0}(u)y_u^{(2)}(x)\,du=A(x). \tag{14}$$

This is a new kind of integral transform for which we need an inversion theorem.

Anticipating this inversion formula, we give the general structure of this integral transform without any reference to the actual framework. Define:

$$K_{ab}(u\,|\,x)=(1-x)^{iu}F(a+iu, b+iu; a+b; x);$$

then we have the reciprocity formulas

$$\hat{h}(x)=\int_0^\infty h(u)K_{ab}(u\,|\,x)\,du,$$

$$h(u)=\frac{\Gamma(a+iu)\Gamma(a-iu)\Gamma(b+iu)\Gamma(b-iu)}{\Gamma^2(a+b)}\,\frac{u\,\mathrm{sh}\,2\pi u}{\pi^2}\int_0^1\frac{x^{a+b-1}}{1-x}\hat{h}(x)K_{ab}(u\,|\,x)\,dx.$$

These relations are only formal: it would be quite interesting to set up by itself the general theory of this new integral transform.

Let us simply notice that if

$$b=\tfrac{1}{2},\quad a=\tfrac{1}{2}-\mu,\quad \mu<0,$$

[4] We shall explain later on how the eigenvectors $y_r^{(1)}(x)$ appear once equation (14) is inverted.

the nucleus $K_{ab}(u\,|\,x)$ reduces to Legendre functions if we use the relation (Erdélyi et al. (1953, vol. I, p. 124))

$$K_{1/2-\mu,1/2}(u\,|\,x)=\Gamma(-\mu)x^{\mu/2}(1-x)^{-1/2}P^{\mu}_{-1/2+iu}\left(\frac{1+x}{1-x}\right). \tag{15}$$

In this case (14) is nothing but a Mehler–Fock transform of order μ (see Sneddon (1972, p. 414) for the definition when μ is an integer).

In this work we shall proceed in the following way:

i) We give a formal treatment of the inversion problem for general $A(x)$. This will be done without any discussions of the necessary restrictions for $A(x)$.

ii) This gives an answer for $h_{n_0}(u)$ when $A(x)=x^{n_0}$. We then prove, with due respect to mathematical rigor, in Appendix A that relation (14) is indeed true for this particular case.

iii) In fact, step ii) requires the restrictions

$$\tilde{b}>-1,\quad \mathrm{Re}\,(a-\nu_{\pm})>0,\quad a<0. \tag{16}$$

But we have seen in the discussion in § 1.2 that the "physical" world corresponds to positive values of a. At this stage a process of analytic continuation with respect to a will give the general decomposition of x^{n_0} on the whole eigenvectors basis, and we shall see how the eigenvectors $y_r^{(1)}(x)$ creep in naturally during this process of analytic continuation.

3.1. Formal inversion of the integral transform. From a general point of view the inversion of an integral transform is a rather difficult problem. The basic idea for solving this problem is to find an "adapted" integral representation for $y_u^{(2)}(x)$ such that (14) can be considered as the composition of several classical integral transforms (by classical we intend those integral transforms available, e.g., in Sneddon (1972)).

This is possible for $y_u^{(2)}(x)$: we obtain the composition of a Hankel transform of order $\tilde{b}$ with a Kontorovich–Lebedev transform. Using the known inversion theorems will lead to the inversion of (14) in terms of $y_u^{(2)}(x)$.

We start from the eigenvectors given in § 2 and use formula (3) (Erdélyi et al. (1953, vol. I, p. 105)) to write them as

$$y_u^{(2)}(x)=(1-x)^{\nu_-}F\left(a-\nu_-+iu,a-\nu_--iu;1+\tilde{b};-\frac{x}{1-x}\right),\qquad \tilde{b}>-1.$$

Putting

$$x=\frac{\xi^2}{1+\xi^2},\qquad \xi\geqq 0,$$

the eigenvector becomes

$$y_u^{(2)}\left(\frac{\xi^2}{1+\xi^2}\right)=(1+\xi^2)^{-\nu_-}F(a-\nu_-+iu,a-\nu_--iu;1+\tilde{b};-\xi^2).$$

We use formula (31) (Erdélyi et al. (1953, vol. II, p. 52)):

$$F(a-\nu_-+iu,a-\nu_--iu;1+\tilde{b};-\xi^2)=N_u\xi^{-\tilde{b}}\int_0^\infty t^{\nu_+-\nu_-}K_{2iu}(t)J_{\tilde{b}}(\xi t)\,dt \tag{17}$$

valid with

$$N_u=\frac{2^{1-\nu_++\nu_-}\Gamma(1+\tilde{b})}{\Gamma^2(a-\nu_-\pm iu)},\qquad \mathrm{Re}\,(a-\nu_-)>0,$$

$$\Gamma^2(a-\nu_-\pm iu)\equiv\Gamma(a-\nu_-+iu)\cdot\Gamma(a-\nu_--iu).$$

Inserting this integral representation into (14) gives

$$\int_0^\infty tJ_{\bar b}(\xi t)t^{-1+\nu_+-\nu_-}\,dt\int_0^\infty h(u)N_uK_{2iu}(t)\,du=\xi^{\bar b}(1+\xi^2)^{\nu_-}A\left(\frac{\xi^2}{1+\xi^2}\right),$$

where we have already interchanged the order of integration. One recognizes in the integral over u a Kontorovich–Lebedev transform, and over t a Hankel transform.

Proceeding formally we use the Hankel inversion theorem in Sneddon (1972, p. 309):

$$\int_0^\infty h(n)N_uK_{2iu}(t)\,du=t^{1-\nu_++\nu_-}\int_0^\infty d\xi\,\xi^{1+\bar b}(1+\xi^2)^{\nu_-}J_{\bar b}(\xi t)A\left(\frac{\xi^2}{1+\xi^2}\right)d\xi.$$

We then apply Kontorovich–Lebedev inversion theorem in Sneddon (1972, p. 361):

$$h(u)=\frac{8ush2\pi u}{\pi^2N_u}\int_0^\infty t^{-\nu_++\nu_-}K_{2iu}(t)\,dt\int_0^\infty \xi^{1+\bar b}A\left(\frac{\xi^2}{1+\xi^2}\right)J_{\bar b}(\xi t)(1+\xi^2)^{\nu_-}\,d\xi.$$

We invert the order of the integrations:

$$h(u)=\frac{8ush2\pi u}{\pi^2N_u}\int_0^\infty \xi^{1+\bar b}(1+\xi^2)^{\nu_-}A\left(\frac{\xi^2}{1+\xi^2}\right)d\xi\int_0^\infty t^{-\nu_++\nu_-}K_{2iu}(t)J_{\bar b}(\xi t)\,dt$$

and use (17) to get $y_u^{(2)}(x)$ again:

$$h(u)=\frac{8ush2\pi u}{\pi^2N_uN'_u}\int_0^\infty \xi^{2\bar b+1}(1+\xi^2)^{\nu_++\nu_-}A\left(\frac{\xi^2}{1+\xi^2}\right)y_u^{(2)}\left(\frac{\xi^2}{1+\xi^2}\right)d\xi$$

with

$$N'_u=N_u(\nu_+\leftrightarrow\nu_-),\qquad \operatorname{Re}(a-\nu_+)>0.$$

We return to the original x variable:

$$h(u)=ush2\pi u\frac{\Gamma^2(a-\nu_+\pm iu)\Gamma^2(a-\nu_-\pm iu)}{\pi^2\Gamma^2(1+\bar b)}\int_0^1 x^{\bar b}(1-x)^{-2a-1}A(x)y_u^{(2)}(x)\,dx. \tag{18}$$

If $A(x)=x^{n_0}$ this integral converges onlý if $a<0$, hence we have obtained along this discussion all the restrictions (16).

We shall see in § 3.2 how it is possible to escape to the constraint $a<0$. The integral (18) for $A(x)=x^{n_0}$ can be written:

$$h_{n_0}(u)=ush2\pi u\frac{\Gamma^2(a-\nu_+\pm iu)\Gamma^2(a-\nu_-\pm iu)}{\pi^2\Gamma^2(1+\bar b)}$$
$$\times\int_0^\infty x^{\bar b}(1-x)^{-2a-1}y_u^{(2)}(x)\sum_{r=0}^{n_0}(-1)^r\binom{n_0}{r}(1-x)^r\,dx$$

and each term in the sum is evaluated using Gradshteyn (1965, p. 849):

$$h_{n_0}(u)=ush2\pi u\frac{\Gamma^2(a-\nu_+\pm iu)\Gamma^2(a-\nu_-\pm iu)}{\pi^2\Gamma(1+\bar b)}\sum_{r=0}^{n_0}(-1)^r\binom{n_0}{r}\frac{\Gamma^2(r-a\pm iu)}{\Gamma(r-\nu_+)\Gamma(r-\nu_-)} \tag{19}$$

Of course in this way we have obtained at best an "educated guess" for $h_{n_0}(u)$. The proof that with the expression (19) for $h_{n_0}(u)$ we have

$$x^{n_0}=\int_0^\infty h_{n_0}(u)y_u^{(2)}(x)\,du,$$

if (16) are satisfied, is given with full mathematical details in Appendix A.

3.2. Process of analytic continuation to the physical region $a \geqq 0$. From Appendix B, we know that if conditions (16) are fulfilled then (14) is valid with $h_{n_0}(u)$ given by (19). Putting $z = iu$,

$$H'_{n_0}(z) = \frac{-z \sin 2\pi z}{2\pi^2 \Gamma(1+\tilde{b})} \Gamma(a-\nu_+ + z)\Gamma(a-\nu_+ - z)\Gamma(a-\nu_- + z)\Gamma(a-\nu_- - z)$$

$$\times \sum_{r=0}^{n_0} (-1)^r \binom{n_0}{r} \frac{\Gamma(r-a+z)\Gamma(r-a-z)}{\Gamma(r-\nu_+)\Gamma(r-\nu_-)}$$

and

$$Y_z(x) = (1-x)^{\nu_-} F\left(a-\nu_- + z, a-\nu_- - z; 1+\tilde{b}; -\frac{x}{1-x}\right),$$

we transform (14) to a contour integral:

$$x^{n_0} = \int_{-i\infty}^{+i\infty} H'_{n_0}(z) Y_z(x) \frac{dz}{i}.$$

Let us suppose, for the moment, that $a < 0$ (such that $|a|$ is small with respect to 1) and Re $\nu_\pm$ is negative and sufficiently large to ensure Re $(a - \nu_\pm) > 0$).

The integrand is an even function of z; $Y_z(x)$ is analytic in z and $H'_{n_0}(z)$ has the following poles (see Fig. 1):

$$z = a - \nu_+ + p,$$
$$z = a - \nu_- + p, \qquad p = 0, 1, 2, \cdots.$$
$$z = -a + p,$$

All the other poles are of opposite sign.

When a increases to 0 from small negative values, there is a pinching of the integration contour between the poles $\pm a$ which destroys the convergence of the integral.

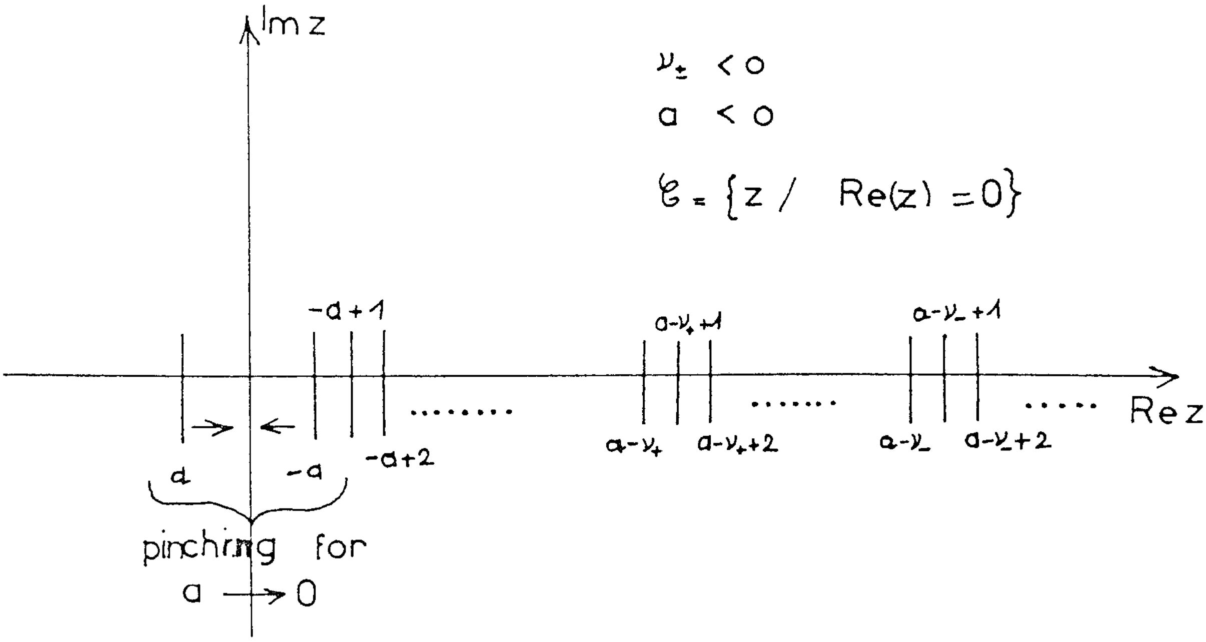

FIG. 1

Hence, before taking the $a \to 0$ limit, we push the contour to the right in order to avoid the pole $z=-a$ (see Fig. 2). This will give by Cauchy's theorem a supplementary contribution related to the residue at the simple pole $z=-a$.

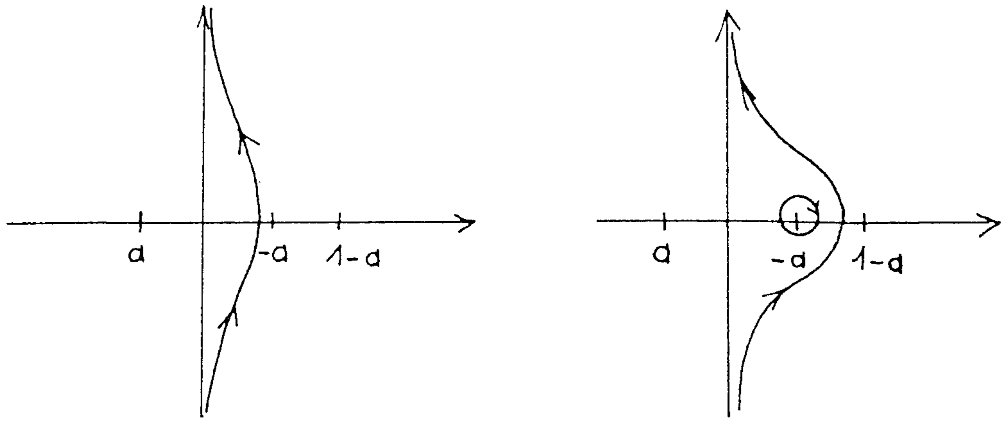

FIG. 2

Now the $a \to 0$ limit can be taken (because the residue is continuous for $a \to 0$). One can even increase a to positive values.

But if $a \to \frac{1}{2}$, the poles a and $1-a$ pinch the contour again. In order to avoid this new pinching, we push back the contour to $\operatorname{Re} z = 0$ (see Fig. 3).

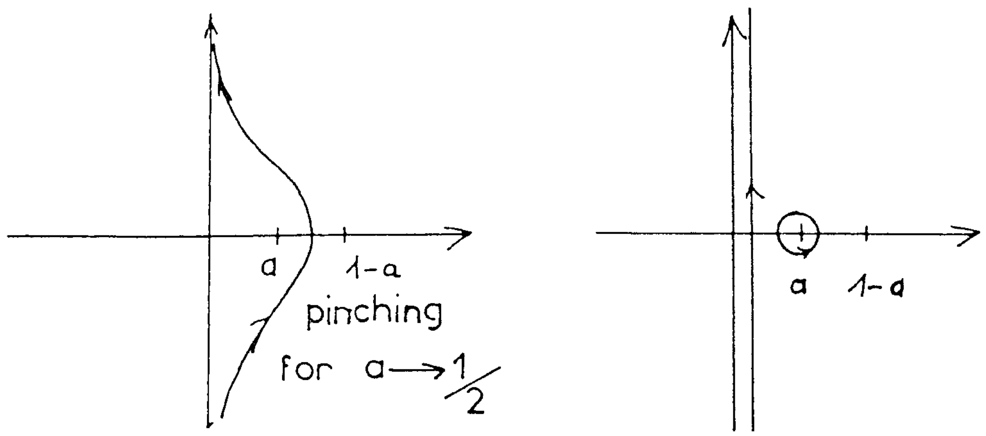

FIG. 3

We obtain in this way the residues at the poles $z=\pm a$, and taking into account all signs these terms double and give

$$x^{n_0} = \frac{-2a \sin 2\pi a}{\pi \Gamma(1+b)} \Gamma(2a-\nu_+)\Gamma(2a-\nu_-)\Gamma(-2a)\, Y_{-a}(x) + \int_{-i\infty}^{+i\infty} H_{n_0}(z)\, Y_z(x) \frac{dz}{i},$$

a formula valid for $0<a<1$.

For $a=0$ the right formula is obtained by the limit $a \to 0$ but with a contour pushed slightly to the right of $a=0$.

The relation

$$2a \sin 2\pi a \Gamma(-2a) = \frac{\pi}{\Gamma(2a)}$$

exhibits the well-behaved dependence on a of the residue, even for positive integer values of $2a$.

When a is increased further, the poles at $z=\pm a$ keep away from the integration contour while the poles $z=\pm(1-a)$ produce a new pinching for $a\to 1$.

The technique is the same as for $a\to 0$. In this previous case only the $r=0$ term in $H'_{n_0}(z)$ had a pole, when $a\to 1$ the $r=0$ and $r=1$ terms have a simple pole.

It is an easy matter to compute the residues, and to obtain

$$x^{n_0}=\sum_{l=0}^{[a]} H_l Y_{l-a}(z)+\int_{-i\infty}^{+i\infty} H_{n_0}(z)\,Y_z(x)\frac{dz}{i},$$

where $[a]$ = integer part of a, with

$$\begin{aligned}H_l=(-1)^{l+1}&\frac{2(l-a)\sin 2\pi(l-a)}{\pi\Gamma(1+\tilde{b})}\Gamma(l-\nu_+)\Gamma(l-\nu_-)\Gamma(2a-l-\nu_+)\Gamma(2a-l-\nu_-)\\ &\times\sum_{r=0}^{\inf(n_0,l)}\frac{\binom{n_0}{r}}{(l-r)!}\,\frac{\Gamma(l+r-2a)}{\Gamma(r-\nu_+)\Gamma(r-\nu_-)}\end{aligned}\tag{20}$$

which is valid when a is not some positive integer.

We observe that if $a\to p$ (integer), the residues in H_l are well defined, and before taking this limit one must push the integration contour slightly to the right of Re $z=0$ as previously explained.

Now we can further enlarge the validity domain of (20) to positive values of Re $\nu_\pm$. This is a consequence of analytic continuation, provided $\tilde{b}>-1$ and Re $(a-\nu_\pm)>0$. Hence if Re $\nu_\pm$ is positive, one must take large-enough values of a.

Coming back to $y=z/i$ we can write:

$$x^{n_0}=\sum_{l=0}^{[a]} H_l y_l^{(1)}(x)+\int_0^\infty h_{n_0}(u)y_u^{(2)}(x)\,du.$$

Since the eigenvalues (see § 2) are:

$$y_l^{(1)}(x)\to -\alpha l(2a-l),\qquad y_u^{(2)}(x)\to -\alpha(u^2+a^2),$$

the generating function will be given by

$$G(x,t)=\sum_{l=0}^{[a]} H_l y_l^{(1)}(x)\,e^{-l(2a-l)\alpha t}+\int_0^\infty h_{n_0}(u)y_u^{(2)}(x)\,e^{-(u^2+a^2)\alpha t}\,du\tag{21}$$

provided that $\tilde{b}>-1$, Re $(a-\nu_\pm)>0$.

The proof that $G(x,t)$ is indeed a solution of the partial differential equation (9)

$$\frac{\partial G}{\partial t}=LG,\qquad t\geqq 0,\quad 0\leqq x\leqq R<1,$$

with the boundary condition

$$G(x,0)=x^{n_0},\qquad 0\leqq x\leqq R<1$$

and such that

$$\lim_{x\to 1} G(x,t)=1,\qquad t\geqq 0$$

is given in Appendices A and B. The proof is valid only for $a>0$.

3.3. The case where $\tilde{b}$ is a negative integer. If $\tilde{b}=-p$ for $p=1,2,3,\cdots$, despite the lack of positivity of μ_n for $n<p$, one can imagine a population evolving only with

$n \geqq p$. This corresponds to a process whose natural limit (for which μ vanishes) is no more $n=0$ but rather $n=p$.

It is easy to show that the probabilities (and also the generating function and the eigenfunctions) can be deduced from the corresponding ones for an associated process with, as usual, $\mu_0=0$. The process $p=(\lambda_n, \mu_n)$, $n \geqq p$, will have as associated process $p'=(\lambda_{n+p}, \mu_{n+p})$, $n \geqq 0$, with the obvious relations:

$$y_s[p]=x^p y_s[p'], \qquad G[p]=x^p G[p'].$$

The first relation was indeed observed in our discussion of the eigenvectors in § 2.

3.4. Mean value and transition probabilities. Once the generating function is known, we can extract from it the moments:

$$m_k(t)=\sum_{n=0}^{\infty} n^k pn(t)=\lim_{x\to 1}\left(x\frac{\partial}{\partial x}\right)^k G(x,t).$$

Since the computation for general k is tedious, we shall content ourselves with the first moment and check that the result agrees with the expression already given in § 1.4.

We start from:

$$G(x,t)=\sum_{l=0}^{[a]} H_l y_l^{(1)}(x)\, e^{-l(2a-l)\alpha t}+\int_0^{\infty} h_{n_0}(u) y_u^{(2)}(x)\, e^{-(a^2+u^2)\alpha t}\, du.$$

In order to compute safely the derivative for $x=1$, we suppose $a>1$ since from Lemma 4 in Appendix B we can bring the x derivative inside the integral over u.

Using formula 20 in (Erdélyi et al. (1953, vol. I, p. 102)) gives for the first moment:

$$m_1(t)=\frac{c}{2a-1}+\left(n_0-\frac{c}{2a-1}\right)e^{-(2a-1)\alpha t}, \qquad a>1,$$

as expected.

The transition probabilities are given by:

$$p_{n_0\to n}(t)=\frac{1}{n!}\lim_{x\to 0}\frac{\partial^n}{\partial x^n}G(x,t).$$

Using Lemma 3 in Appendix B, we can bring the x derivatives inside the integral over u. Hence to get the transition probabilities, it is sufficient to expand the eigenstates around $x=0$. We define:

$$y_l^{(1)}(x)=\sum_{n=0}^{\infty} G_{l,n}x^n, \qquad y_u^{(2)}(x)=\sum_{n=0}^{\infty} g_n(u)x^n.$$

This gives for the transition probabilities:[5]

$$p_{n_0\to n}(t)=\sum_{l=0}^{[a]} H_l G_{l,n}\, e^{-l(2a-l)\alpha t}+\int_0^{\infty} h_{n_0}(u) g_n(u)\, e^{-(a^2+u^2)\alpha t}\, du. \tag{22}$$

In order to express the coefficients $g_n(u)$, we start from the eigenvectors as given in § 3.1:

$$y_u^{(2)}(x)=(1-x)^{\nu_-}F\left(a-\nu_-+iu, a-\nu_--iu; 1+\tilde{b}; \frac{x}{x-1}\right).$$

[5] With $h_{n_0}(u)$ and H_l given by (19), (20) and $G_{l,n}$, $g_n(u)$ given by (19a), (20a).

Using the known series for the hypergeometric function, we get:

$$y_u^{(2)}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{(a-\nu_-+iu)_k(a-\nu_--iu)_k}{(1+\tilde{b})_k}x^k(1-x)^{\nu_--k}$$

with

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)},\quad (0)_k=0,\quad (0)_0=1.$$

Then we expand $(1-x)^{\nu_--k}$ using the binomial series:

$$(1-x)^{\nu_--k}=\sum_{j=0}^{\infty}(-1)^j\binom{\nu_--k}{j}\cdot x^j=\sum_{j=0}^{\infty}\binom{j+k-1-\nu_-}{j}\cdot x^j.$$

Collecting all terms we get:

$$g_n(u)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n-1-\nu_-}{n-k}\frac{(a-\nu_-+iu)_k(a-\nu_--iu)_k}{(1+\tilde{b})_k}. \tag{20a}$$

Similarly we obtain:

$$G_{l,n}=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n-1-\nu_-}{n-k}\frac{(l-\nu_-)_k(2a-\nu_--l)_k}{(1+\tilde{b})_k}. \tag{19a}$$

When $\nu_\pm$ are complex, the reality of $g_n(u)$ and $G_{l,n}$ is not obvious. Since the eigenstates are real even if $\nu_\pm$ are complex we know a priori that g_n and $G_{l,n}$ are indeed real. Nevertheless, the expressions we have given are not *manifestly* real. In order to have a manifestly real formulation, one must start from the eigenstates in the form:

$$y_u^{(2)}(x)=\frac{1}{2}\left\{(1-x)^{\nu_-}F\left(a-\nu_-+iu,\,a-\nu_--iu;\,1+\tilde{b};\,\frac{x}{x-1}\right)+(\nu_-\leftrightarrow\nu_+)\right\}$$

when $\nu_\pm$ are complex. The manifestly real form of the coefficients g_n and $G_{l,n}$ is simply obtained by the same substitution in $g_n(u)$ and $G_{l,n}$.

Let us now prove that the transition probabilities $p_{n_0\to n}(t)$ given by (22) are indeed the solution of the Chapman–Kolmogorov equation (1) with boundary condition (2) and such that (3) holds.

We have proved in Appendix B that $G(x,t)$ is C^∞ for $0\leqq x\leqq R<1$ uniformly for $t\geqq 0$. Hence in the partial differential equation (9):

$$\frac{\partial G}{\partial t}=L\left(x,\frac{\partial}{\partial x}\right)G,$$

we can expand each member in powers of x, and as already explained in § 1.4, we recover (1) for $p_n(t)$. Since we work with $x\leqq R<1$, the partial differential equation is valid for $a\geqq 0$.

Then expanding the relation proved in Appendix A,

$$G(x,0)=x^{n_0}$$

in powers of x gives the boundary condition (2).

The probabilities conservation $\sum_{n=0}^{\infty}p_n(t)=1$ for $t\geqq 0$ is simply a consequence of

$$\lim_{x\to 1}G(x,t)=1,\qquad t\geqq 0,$$

which is valid (see Appendix B) only for $a>0$.

The limiting case $a=0$ is out of reach of our analysis and deserves further study. We have explored numerically the case $\lambda_n=\alpha(n^2+n)\mu_u=\alpha n^2$. The agreement is good for all probabilities $p_n(t)$.

From formula (22) and if $\lambda_0=\mu_0=0$, one can easily check that:

$$\lim_{t\to\infty} p_{n_0\to 0}(t)=1,$$

which is in agreement with a general theorem in Karlin (1966).

4. Detailed analysis of two typical processes. In order to illustrate the preceding general analysis, we shall consider two simple cases of transition rates which exhibit interesting peculiarities. Even though the generating function can be obtained by taking the appropriate limits in the general form given in § 3, we think it instructive to give a direct analysis for these cases.

4.1. Finite processes.

1) $\lambda_n=\alpha(n-N)^2$, $\mu_n=\alpha n^2$. We take for N some positive integer. If $n_0 \leqq N$ the population can only envolve between the levels 0 and N. As a consequence of (5) and (7), we know a priori that the eigenvectors are polynomials. Indeed, from § 2 we have (Gradshteyn (1965, p. 1010)):

$$0\leqq r\leqq N+\tfrac{1}{2},\qquad y_r^{(1)}(x)=(1-x)^N P_{N-r}\left(\frac{1+x}{1-x}\right),$$

$$0\leqq u,\qquad y_u^{(2)}(x)=(1-x)^N P_{-1/2+iu}\left(\frac{1+x}{1-x}\right)$$

on which we must expand x^{n_0}. Since the eigenvectors

$$(1-x)^N P_n\left(\frac{1+x}{1-x}\right),\qquad n=0,1,\cdots,N$$

are polynomials, they are sufficient for the expansion of x^{n_0}. We must find the coefficients such that

$$x^{n_0}=\sum_{l=0}^{N} c_l(1-x)^N P_l\left(\frac{1+x}{1-x}\right);$$

going to the variable $z=(1+x)/(1-x)$ this becomes

$$(z-1)^{n_0}(z+1)^{N-n_0}=2^N\sum_{l=0}^{N} c_l P_l(z),\qquad z\geqq 1.$$

The two members of this equality are analytic with respect to z; hence we make an analytic continuation to $-1\leqq z\leqq +1$. In this case we know the scalar product of Legendre polynomials, which gives:

$$c_l=\frac{2l+1}{2^{N+1}}\int_{-1}^{+1}(z-1)^{n_0}(z+1)^{N-n_0}P_l(z)\,dz.$$

We expand the integrand with respect to $z+1$:

$$c_l=\frac{2l+1}{2^{N+1}}\sum_{k=0}^{n_0}(-2)^k\binom{n_0}{k}\int_{-1}^{+1}(z+1)^{N-l}P_l(z)\,dz;$$

this integral is given in Gradshteyn (1965, p. 797):

$$c_l=(2l+1)\sum_{k=0}^{n_0}(-1)^k\binom{n_0}{k}\frac{[(N-k)!]^2}{(N-k-l)!(N-k+l+1)!}.$$

So we obtain the generating function:

$$G(x,t)=\sum_{l=0}c_l(1-x)^N P_l\left(\frac{1+x}{1-x}\right)e^{-(N-l)(N+l+1)\alpha t}$$

which is in agreement with the general formula (21).

The special case treated here suffices to emphasize the great simplicity of the expansion problem in this case, due to the existence of a scalar product.

2) *Relation with Moran processes.* The transition probabilities for these processes have been given in Karlin (1962) in a somewhat different framework.

Their work corresponds to the transition rates:

$$\lambda_n=\alpha(n-N)(n-\nu_+),\qquad \mu_n=\alpha n(n+\tilde{b}).$$

In the general formula (22) $h_{n_0}(u)$ vanishes since we suppose $n_0\leqq N$. Taking into account some compensations between vanishing numerators and denominators gives for the coefficients H_l:

$$H_l=\begin{cases}0, & N<l<[a],\\ \dfrac{2(l-a)\sin 2\pi(l-a)}{\pi\Gamma(1+\tilde{b})}\Gamma(l-\nu_+)\Gamma(2a-l-\nu_+)\Gamma(2a-l-\nu_-) & \\ \times\displaystyle\sum_{r=0}^{\inf(n_0,l)}(-1)^{r+1}\frac{\binom{n_0}{r}}{(l-r)!}\frac{\Gamma(l+r-2a)}{\Gamma(r-\nu_+)}(N-l+1)_{l-r}, & 0\leqq l\leqq N\end{cases}$$

The coefficients $G_{l,n}$ are given by (20a).

The relation with Karlin and McGregor notation is:

$$\alpha=\frac{\lambda(\gamma_1+\gamma_2-1)}{N^2},\qquad \tilde{b}=N\cdot\frac{(1-\gamma_2)}{\gamma_1+\gamma_2-1},$$
$$\nu_-=N,\qquad \nu_+=N\cdot\frac{\gamma_2}{\gamma_1+\gamma_2-1},\qquad \gamma_1+\gamma_2-1>0.$$

The case where $\gamma_1+\gamma_2-1<0$ cannot be deduced directly from our general formula since it corresponds to transition rates

$$\lambda_n=\alpha(N-n)(n-\nu_+),\qquad \nu_+<0,$$
$$\mu_n=\alpha n(n+\tilde{b}),\qquad 0\leqq n\leqq N,$$

and these are not of the form (0).

This is the only case which is not covered by our analysis since it is a genuinely finite population case. Nevertheless one can obtain the transition probabilities for this case: the eigenstates are related to Jacobi polynomials, and as in the previous paragraph, due to the existence of a scalar product for Jacobi polynomials, the expansion problem is easily solved.

4.2. A symmetric process $\lambda_n = \mu_n = \alpha n^2$. From § 2 the eigenvectors are

$$0 \leqq r \leqq \tfrac{1}{2}, \qquad y_r^{(1)}(x) = P_{-r}\left(\frac{1+x}{1-x}\right),$$

$$0 \leqq u, \qquad y_u^{(2)}(x) = P_{-1/2+iu}\left(\frac{1+x}{1-x}\right).$$

The expansion of x^{n_0} over these functions leads to the integral equation:

$$x^{n_0} = \int_0^\infty h_{n_0}(u) P_{-1/2+iu}\left(\frac{1+x}{1-x}\right) du.$$

Going to $z = (1+x)/(1-x)$, this becomes

$$A(z) = \left(\frac{z-1}{z+1}\right)^{n_0} = \int_0^\infty h_{n_0}(u) P_{-1/2+iu}(z)\, du.$$

We recognize a Mehler–Fock transform whose inversion theorem is given, e.g., in Sneddon (1972, p. 390). Unfortunately, for $z \to \infty$:

$$\lim_{z\to\infty} A(z) = 1$$

and the inversion theorem does not work in this case. This simply means that we must subtract some terms to improve the decrease for $z \to \infty$ of $A(z)$. Among the eigenstates $y_r^{(1)}$ we have:

$$y_0^{(1)}(x) = 1.$$

Hence if we consider

$$A'(z) = \left(\frac{z-1}{z+1}\right)^{n_0} - 1,$$

which has a good behavior at infinity we can use the inversion theorem:

$$h_{n_0}(u) = u th \pi u \int_1^\infty P_{-1/2+iu}(z)\left[\left(\frac{z-1}{z+1}\right)^{n_0} - 1\right] dz.$$

We expand the integrand in inverse powers of $(z+1)$:

$$h_{n_0}(u) = u th \pi u \sum_{l=1}^{n_0} (-1)^l \binom{n_0}{l} \int_1^\infty P_{-1/2+iu}(z)\left(\frac{2}{z+1}\right)^l dz.$$

This integral is given in Ditkine (1978):

$$h_{n_0}(u) = 2u th \pi u \sum_{l=1}^{n_0} (-1)^l \frac{\binom{n_0}{l}}{[(l-1)!]^2} \Gamma^2(l - \tfrac{1}{2} \pm iu).$$

The generating function is

$$G(x,t) = 1 + 2\sum_{l=1}^{n_0} \frac{(-1)^l \binom{n_0}{l}}{[(l-1)!]^2} \int_0^\infty u th \pi u\, \Gamma^2(l - \tfrac{1}{2} \pm iu) P_{-1/2+iu}\left(\frac{1+x}{1-x}\right) e^{-(u^2+1/4)\alpha t}\, du.$$

One can check that the appropriate limits in formula (21) give the same result after some tedious algebraic work.

Going to the transition probabilities gives for $n_0 = 1$:

$$P_{1\to n}(t) = \delta_{n,0} - 2\sum_{k=0}^{n} \frac{(-1)^k}{(k!)^2}\binom{n-1}{k-1}\int_0^\infty \pi u th\pi u\, \Gamma^2(k+\tfrac{1}{2}\pm iu)\, e^{-(u^2+1/4)\alpha t}\, du,$$

in agreement with (22). This simplifies for $n = 0$ to:

$$P_{1\to 0}(t) = 1 - 2\, e^{-\alpha t/4}\int_0^\infty \pi u \frac{sh\pi u}{ch^2\pi u}\, e^{-u^2\alpha t}\, du.$$

5. Approximation methods.

5.1. Large time approximation for transition probabilities. The main problem is to obtain the large time behavior of the integral over u appearing in $p_n(t)$.

This can be done by use of Watson's lemma as given in Bleistein (1975).

It is first convenient to change the variable to $\sqrt{v} = u$ which leads for the integral over u to:

$$e^{-a^2\alpha t}\int_0^\infty e^{-v\alpha t}\frac{h_{n_0}(\sqrt{v})}{2\sqrt{v}}\, g_n(\sqrt{v})\, dv \equiv e^{-a^2\alpha t}\int_0^\infty e^{-v\alpha t} f(v)\, dv.$$

It is easy to check that the hypotheses of Watson's lemma are satisfied. For $v \to 0^+$, $f(v)$ has a Taylor series in powers of $\sqrt{v}$:

$$f(v) \underset{v\to 0^+}{\sim} \sum_{l=0}^{\infty} c_l v^{(l+1)/2}.$$

The knowledge of the coefficients c_l gives the asymptotic behavior of the transition probabilities:

$$p_{n_0\to n}(t) \underset{t\to\infty}{\sim} \sum_{l=0}^{[a]} H_l G_{l,n}\, e^{-l(2a-l)\alpha t} + e^{-a^2\alpha t}\sum_{l=0}^{\infty} c_l \frac{\Gamma((l+3)/2)}{(\alpha t)^{(l+3)/2}}$$

The computation of the coefficients c_l is tedious but straightforward. We give only the first coefficient:

$$c_0 = \frac{[\Gamma(a-\nu_+)\Gamma(a-\nu_-)]^2}{\pi\Gamma(1+\tilde{b})}\sum_{r=0}^{n_0}\frac{(-1)^r\binom{n_0}{r}[\Gamma(r-a)]^2}{\Gamma(r-\nu_+)\Gamma(r-\nu_-)}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{[(a-\nu_-)_k]^2}{(1+\tilde{b})_k}\binom{n-1-\nu_-}{n-k}.$$

As in § 3.4, when $\nu_\pm$ are complex, in order to get a manifestly real quantity, one must make the substitution

$$c_0 \to c_0' = \tfrac{1}{2}(c_0 + \nu_- \leftrightarrow \nu_+).$$

5.2. Large n approximation for the transition probabilities. From the explicit formula (22) for the transition probabilities, we observe that for large n, $G_{l,n}$ and $g_n(u)$ are a sum of alternating sign terms whose number increases with n. This is quite unpleasant for computational purposes since the final result is obtained as a sum of large terms among which large compensations occur.

i) *The method.* Let us consider some function of the form

$$H(x) = (x_0 - x)^\alpha F(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where α is some complex number and such that $F(x)$ is analytic near x_0 and 0:

$$F(x) = \sum_{k=0}^{\infty}\varphi_k x^k, \qquad F(x) = \sum_{k=0}^{\infty} F_k (x_0 - x)^k.$$

Expanding in powers of x, $F(x)$ and $(x_0-x)^\alpha$ gives directly:

$$a_n=\sum_{k=0}^{n}\binom{\alpha}{k}x_0^{\alpha-k}(-1)^k\varphi_{n-k}. \tag{23'}$$

But this formula is of little practical use since it is an alternating series whose terms are large. Nevertheless, it is possible to give an asymptotic expansion for a_n when n is large according to Dingle (1973, p. 141).

We again expand $H(x)$:

$$H(x)=\sum_{k=0}^{\infty}(x_0-x)^{\alpha+k}F_k=\sum_{k=0}^{\infty}F_k\sum_{n=0}^{\infty}\binom{\alpha+k}{n}x_0^{\alpha+k-n}(-x)^n,$$

giving

$$a_n=\sum_{k=0}^{\infty}(-1)^n\binom{\alpha+k}{n}x_0^{\alpha+k-n}F_k, \tag{23}$$

$$\begin{aligned} a_n=(-1)^n\binom{\alpha}{n}x_0^{\alpha-n}[F_0&+x_0q_n(\alpha+1)F_1+\cdots\\ &+x_0^kq_n(\alpha+1)\cdots q_n(\alpha+k)F_k+\cdots], \end{aligned} \tag{24}$$

$$q_n(\alpha)=\frac{\alpha}{\alpha-n},\qquad \alpha\neq n.$$

Here a discussion is needed.

Case 1. *α is not a positive integer.* We notice that the second and third terms in (24) are respectively of order $1/n$ and $1/n^2$. However, this is not an asymptotic expansion because we cannot ensure the kth order term to be $(1/n)^k$ when k is of the order of n. Even if it would be the case, since we must then integrate (24) over u, it is not obvious that the integrated expansion would remain an asymptotic one. Indeed we are looking for a simplified approximation and then we will check by a numerical computation that this gives sensible results.

Case 2. *α is a positive integer p.* Relation (23) shows that the terms for $k=1,2,\cdots,n-p-1$ vanish, and the first and the second factor are in the ratio n instead of $1/n$ as previously. In this case, and if p is not too large, we shall use the expression (23').

ii) *Application to the generating function.* We define

$$G(x,t)=G_s(x,t)+G_i(x,t),$$

with

$$G_s(x,t)=\sum_{l=0}^{[a]}H_l\,e^{-l(2a-l)\alpha t}(1-x)^lF(l-\nu_-,l-\nu_+;1+\tilde{b};x)=\sum_{n=0}^{\infty}s_nx^n,$$

$$G_i(x,t)=\int_0^\infty h_{n_0}(u)\,e^{-(a^2+u^2)\alpha t}(1-x)^{a+iu}F(a-\nu_-+iu,a-\nu_++iu;1+\tilde{b};x)\,du$$

$$=\sum_{n=0}^{\infty}i_nx^n.$$

The subtraction terms $G_s(x,t)$ correspond to integer $\alpha=l$; hence for s_n we take the formula given in § 3.4. For practical use it is necessary to suppose that $[a]$ is of the order of some units.

The evaluation of $G_i(x, t)$ is made using (24) with $x_0 = 1$. To obtain the coefficients F_k we use relation (13) which relates the hypergeometric function of variable x and $1-x$:

$$y_u^{(2)}(x) = \frac{\Gamma(1+\tilde{b})\Gamma(-2iu)(1-x)^{a+iu}}{\Gamma(a-\nu_+ - iu)\Gamma(a-\nu_- - iu)} F(a-\nu_- + iu, a-\nu_+ + iu; 1+2iu; 1-x) + \text{c.c.}$$

where c.c. means complex conjugate. Hence we have

$$F_0 = 1,$$

$$F_k = \frac{1}{k!} \frac{(a-\nu_- + iu)_k (a-\nu_+ + iu)_k}{(1+2iu)_k} + \text{c.c.},$$

which gives for the transition probabilities

$$i_n = \int_0^\infty h_{n_0}(u)\Gamma(1+\tilde{b})\, e^{-(a^2+u^2)\alpha t}(-1)^n \binom{a+iu}{n} \frac{\Gamma(-2iu)}{\Gamma(a-\nu_+ - iu)\Gamma(a-\nu_- - iu)} \times [1 + F_1 q_n(a+1+iu) + \cdots]\, du + \text{c.c.} \tag{25}$$

Collecting all terms gives:

$$p_n(t) = s_n(t) + i_n(t).$$

iii) *Numerical check.* We come back to the symmetric process $\lambda_n = \mu_n = \alpha n^2$ and $n_0 = 1$. In this case $a = \frac{1}{2}$, $[a] = 0$ and the only subtraction term is 1. The general formula (25) gives:

$$p_n(t) \underset{n \gg 1}{\simeq} (-2\pi) \int_0^\infty \frac{u\,sh \pi u}{ch^2 \pi u}\, e^{-(u^2+1/4)\alpha t} \frac{\Gamma(-2iu)}{\Gamma^2(1/2-iu)}(-1)^n \binom{1/2+iu}{n} \times \left[1 + \frac{(1/2+iu)^2}{1+2iu} q_n(3/2+iu) + \cdots \right] du + \text{c.c.} \tag{26}$$

The numerical computation results are given in Table 1, and show that already when n is of order 20 or 30 the first few terms give reasonable results.

TABLE 1

Number of terms	$p_{10}(t)$	$p_{20}(t)$	$p_{30}(t)$
1	$1.1\ 10^{-3}$	$6.0\ 10^{-5}$	$9.1\ 10^{-6}$
2	$7.8\ 10^{-4}$	$4.4\ 10^{-5}$	$7.0\ 10^{-6}$
3	$6.7\ 10^{-4}$	$4.23\ 10^{-5}$	$6.86\ 10^{-6}$
4	$7.09\ 10^{-4}$	$4.29\ 10^{-5}$	$6.89\ 10^{-6}$
"exact" value	$7.04\ 10^{-4}$	$4.27\ 10^{-5}$	$6.89\ 10^{-6}$

iv) *Large n and large time approximation.* This case is of particular interest since it is difficult to grasp by a numerical resolution of the equations (1): the computer calculation time increases quickly with both n and t. Nevertheless, we shall be able to write down a rather simple limiting expression which combines the preceding two approximations.

To get rid of the subtraction terms we suppose[6] $0 < a < 1$ and also $n_0 = 1$ (an asymptotic formula for large n_0 remains an open problem). We take only the first

[6] This covers a large class of transition probabilities, among which are $\lambda_n = \mu_n = \alpha(n^2 + bn)$ and $\lambda_n = \alpha(n^2 + c)$, $\mu_n = \alpha n^2$.

term of the $1/n$ expansion (25). In the integral we make the large t approximation, that is, we expand the integrand to first order in $\sqrt{v}=u$. We obtain:

$$p_n(t) \underset{\substack{n\gg 1\\ \alpha t\gg 1}}{\simeq} \frac{(r-\nu_-)_n(r-\nu_+)_n}{(1+b)_n}\frac{1}{n!}+c_n\frac{e^{-a^2\alpha t}}{(\alpha t)^{3/2}}+\cdots$$

with

$$c_n=\frac{\sqrt{\pi}}{2}\frac{\Gamma(a-\nu_+)\Gamma(a-\nu_-)}{\Gamma(1-\nu_+)\Gamma(1-\nu_-)}\frac{\Gamma(n-a)}{n!}(\nu_+\nu_- -a^2)\Gamma(-a)$$
$$\times[\psi(n-a)-\psi(-a)-\psi(a-\nu_+)-\psi(a-\nu_-)-2\gamma],$$
$$\psi(x)=\frac{d}{dx}\ln\Gamma(x), \qquad \gamma=-\psi(1).$$

If we go back to the case

$$\lambda_n=\mu_n=\alpha n^2, \quad \nu_+=\nu_-=0, \quad a=\tfrac{1}{2},$$

we get a remarkably simple formula:

$$p_n(t) \underset{\substack{n\gg 1\\ \alpha t\gg 1}}{\simeq} \frac{\pi}{h}\frac{\Gamma(n-\frac{1}{2})}{n!}(\ln(n-1)+6\ln 2-2+\gamma)\frac{e^{-\alpha t/4}}{(\alpha t)^{3/2}}$$

where we used Stirling's approximation:

$$\psi(n) \underset{n\to\infty}{\sim} \ln(n-\tfrac{1}{2}).$$

Appendix A. We shall prove that if conditions (16) given in § 3 are satisfied, the relation

$$x^{n_0}=\int_0^\infty h_{n_0}(u)y_u^{(2)}(x)\,du, \qquad 0\leqq x<1$$

holds if $h_{n_0}(u)$ is given by formula (19).

LEMMA 1. *The relation*

$$\int_0^\infty u\,\mathrm{sh}\,2\pi u\,\Gamma^2(\lambda\pm iu)\Gamma^2(\mu\pm iu)K_{2iu}(t)\,du=2^{-\mu-\lambda}\pi^2\Gamma(\mu+\lambda)t^{\mu+\lambda}K_{\mu-\lambda}(t)$$

holds if $\operatorname{Re}\lambda>0$, $\operatorname{Re}\mu>0$ *and* $t\geqq 0$.

Proof. Let us denote

$$f(t)=2^{-\mu-\lambda}\pi^2\Gamma(\mu+\lambda)t^{\mu+\lambda}K_{\mu-\lambda}(t).$$

For $t\to 0$ it is easy to see, using its definition, that

$$K_{\mu-\lambda}(t)=\frac{\pi}{2\sin(\mu-\lambda)\pi}\left\{t^{\lambda-\mu}\sum_{n=0}^{\infty}\frac{(t/2)^{2n}}{n!\Gamma(\lambda-\mu+n+1)}+t^{\mu-\lambda}\sum_{n=0}^{\infty}\frac{(t/2)^{2n}}{n!\Gamma(\mu-\lambda+n+1)}\right\}.$$

If we restrict ourselves to $\operatorname{Re}\mu>1$ and $\operatorname{Re}\lambda>1$, then $t^{-1}f(t)$ is continuous for $t\geqq 0$ as well as for its first derivative.[7] When $t\to+\infty$, due to the exponential decrease of $|K_{\mu-\lambda}(t)|$, $tf(t)$ and $t(d/dt)(t^{-1}f(t))$ are absolutely integrable for $t\geqq 0$.

[7] This remains true if $\mu-\lambda=n$ is some integer. In this case we use formula 37 in Bateman II (1953, p. 9) for $K_n(t)$.

According to a theorem due to Lebedev (see Sneddon (1972, p. 361)), we can conclude that

$$\text{(A1)} \qquad f(t)=\frac{2}{\pi^2}\int_0^\infty K_{i\tau}(t)\tau sh\pi\tau\tilde{f}(\tau)\,d\tau$$

where:

$$\tilde{f}(\tau)=\int_0^\infty \frac{1}{t}f(t)K_{i\tau}(t)\,dt=2^{-\mu-\lambda}\pi^2\Gamma(\mu+\lambda)\int_0^\infty t^{\mu+\lambda-1}K_{i\tau}(t)K_{\mu-\lambda}(t)\,dt.$$

This last integral is given by formula 36 in Erdélyi et al. (1953, vol. II, p. 93):

$$\text{(A2)} \qquad \tilde{f}(\tau)=\frac{\pi^2}{8}\Gamma^2\left(\lambda\pm\frac{i\tau}{2}\right)\Gamma^2\left(\mu\pm\frac{i\tau}{2}\right).$$

Inserting A2 into A1 gives the lemma after the change of variable $\tau=2u$, provided that $\mathrm{Re}\,\lambda>1$, $\mathrm{Re}\,\mu>1$.

Analytic continuation extends this formula to $\mathrm{Re}\,\lambda>0$, $\mathrm{Re}\,\mu>0$. □

LEMMA 2. *The integral*

$$I=\int_0^\infty dt\int_0^\infty ush2\pi u|\Gamma^2(\lambda\pm iu)\Gamma^2(\mu\pm iu)|\,|K_{2iu}(t)|t^\sigma\frac{|J_{\tilde{b}}(\xi t)|}{\xi^{\tilde{b}}}\,du$$

converges if $\mathrm{Re}\,\lambda>0$, $\mathrm{Re}\,\mu>0$, $\tilde{b}>-1$, $\sigma\geqq 0$ *and* $\xi\geqq 0$.

Proof. We first deal with the t integral:

$$L(u,\xi)=\int_0^\infty t^\sigma\left|K_{2iu}(t)\cdot\frac{J_{\tilde{b}}(\xi t)}{\xi^{\tilde{b}}}\right|\,dt.$$

From Lemma 3 in Appendix C we know that $L(u,\xi)$ is continuous with u. Since it is multiplied by a function of u which is continuous for $u\geqq 0$ because $\mathrm{Re}\,\lambda>0$, $\mathrm{Re}\,\mu>0$, the integral I will converge provided that the integrand is sufficiently decreasing for $u\to+\infty$.

For large u, Lemma 5 in Appendix C gives:

$$L(u,\xi)\underset{u\to\infty}{\leqq} e^{-\pi u}\begin{cases} AO(u^{k_1})+B\xi^{-1/2-\tilde{b}}O(u^{k_2}), & \xi>0,\\ AO(u^{k_1}), & \xi=0\end{cases}$$

and the Stirling formula:

$$ush2\pi u|\Gamma^2(\lambda\pm iu)\Gamma^2(\mu\pm iu)|\underset{u\to+\infty}{=}u^{2\mathrm{Re}(\lambda+\mu)-1}\left(1+O\left(\frac{1}{u}\right)\right);$$

hence the integrand exhibits an overall exponential decrease ensuring the convergence of I.

THEOREM 1. *If* $a<0$, $\tilde{b}>-1$ *and* $\mathrm{Re}\,(a-\nu_\pm)>0$, *we have*

$$x^{n_0}=\int_0^\infty h_{n_0}(u)y_u^{(2)}(x)\,du, \qquad 0\leqq x<1.$$

Proof. We first suppose that $\nu_\pm$ are real negative numbers. In order to compute

$$\int_0^\infty h_{n_0}(n)y_u^{(2)}\left(\frac{\xi^2}{1+\xi^2}\right)du,$$

we insert the integral representation (17) for $y_u^{(2)}$, which is valid if we suppose $\tilde{b}>-1$

and Re $(a-\nu_-)>0$. Replacing $h_{n_0}(u)$ by its expression we get

$$\frac{2^{1-\nu_++\nu_-}}{\pi^2}(1+\xi^2)^{-\nu_-}\sum_{r=0}^{n_0}\frac{(-1)^r\binom{n_0}{r}}{\Gamma(r-\nu_+)\Gamma(1-\nu_-)}\int_0^\infty u\,sh\,2\pi u|\Gamma(a-\nu_++iu)|^2|\Gamma(r-a+iu)|^2\,du$$
$$\times\int_0^\infty t^{\nu_+-\nu_-}K_{2iu}(t)\frac{J_{\tilde b}(\xi t)}{\xi^{\tilde b}}\,dt.$$

By Lemma 2, this double integral is absolutely convergent (from our definitions Re $(\nu_+-\nu_-)\geqq 0$) for $\xi\geqq 0$ provided that[8] Re $(a-\nu_+)>0$. We can first integrate over u by use of Lemma 1 and obtain

$$(1+\xi^2)^{-\nu_-}\sum_{r=0}^{n_0}\frac{(-1)^r\binom{n_0}{r}}{\Gamma(r-\nu_-)}2^{-r+\nu_-+1}\int_0^\infty t^{r-\nu_-}\frac{J_{\tilde b}(\xi t)}{\xi^{\tilde b}}K_{r-1-\tilde b-\nu_-}(t)\,dt,$$

making use of the relation

$$-2a+\nu_+=-1-\tilde b-\nu_-$$

of § 2. The remaining integral is given in Erdélyi et al. (1953, vol. II, formula 39, p. 93):

$$\sum_{r=0}^{n_0}(-1)^r\binom{n_0}{r}(1+\xi^2)^{-r},$$

whose sum is

$$\left(\frac{\xi^2}{1+\xi^2}\right)^{n_0}=x^{n_0},\qquad 0\leqq x<1.$$

Analytic continuation extends the validity of this relation to Re $(a-\nu_\pm)>0$. In fact, we need this relation only for $0\leqq x\leqq R<1$.

From a more general standpoint, a thorough study of the integral transform (14) would be desirable, though difficult. As an interesting byproduct this would settle at the same time the theory of Mehler–Fock transform of arbitrary order which is included as a special case.

Nevertheless, one could wonder why we have not combined the general inversion theorems for Hankel and Kontorovich–Lebedev transforms, in order to have an inversion theorem with a larger range than $\{x^n\}$, $n\in N$.

The reason for this is that Hankel's inversion theorem requires $\tilde b>-\frac{1}{2}$, which is clearly too restrictive with respect to $\tilde b>-1$. Furthermore, when we use the Kontorovich–Lebedev inversion theorem, the conditions under which it is true are so restrictive that they do not apply to our special case where $A(x)=x^{n_0}$!

Appendix B. We shall prove that $G(x,t)$, given by formula (21) in § 3.2 is, at least for $a>0$, a solution of the partial differential equation:

$$\frac{\partial G}{\partial t}=L\left(x,\frac{\partial}{\partial x}\right)G,\qquad 0\leqq x\leqq R<1,\quad 0\leqq t,$$

where

$$L\left(x,\frac{\partial}{\partial x}\right)=\alpha(1-x)\left[x(1-x)\frac{\partial^2}{\partial x^2}+(1+\tilde b-(1+b)x)\frac{\partial}{\partial x}-c\right],$$

[8] Notice that from our definition of $\nu_\pm$ the condition Re $(a-\nu_+)>0$ implies Re $(a-\nu_-)>0$.

with the boundary condition

$$G(x, 0) = x^{n_0}, \qquad 0 \leqq x \leqq R < 1,$$

subject to the constraint

$$\lim_{x \to 1} G(x, t) = 1, \qquad t \geqq 0.$$

We introduce the notation

$$G(x, t) = G_s(x, t) + G_i(x, t),$$

where

$$G_s(x, t) = \sum_{l=0}^{[a]} H_l y_l^{(1)}(x)\, e^{-l(2a-l)\alpha t},$$

$$G_i(x, t) = \int_0^\infty h_{n_0}(u) y_u^{(2)}(x)\, e^{-(a^2+u^2)\alpha t}\, du.$$

LEMMA 3. *If $a \geqq 0$, $G(x, t)$ is C^∞ with respect to x and t in the domain: $0 \leqq x \leqq R < 1$ and $t \geqq 0$. It is legitimate to bring the x and t derivatives inside the integral over u and we have:*

$$\lim_{t \to 0} G(x, t) = x^{n_0}, \qquad 0 \leqq x \leqq R < 1.$$

Proof. The first part of the lemma is trivial for $G_s(x, t)$ since each term consists of

$$y_l^{(1)}(x)\, e^{-l(2a-l)\alpha t},$$

which is the product of a C^∞ function of $t \geqq 0$ by a function C^∞ when $0 \leqq x \leqq R < 1$ (even for $a = 0$).

Let us now consider $G_i(x, t)$. The eigenvectors

$$y_u^{(2)}(x)\, e^{-(a^2+u^2)\alpha t}$$

are also C^∞ with respect to x and t in the domain we are considering. We want to study

$$\frac{\partial^p}{\partial x^p} \frac{\partial^n}{\partial t^n} h_{n_0}(u) y_u^{(2)}(x)\, e^{-(a^2+u^2)\alpha t} = h_{n_0}(u) \frac{\partial^p}{\partial x^p} y_u^{(2)}(x) P_n(u)\, e^{-(a^2+u^2)\alpha t},$$

where $P_n(u)$ is some nth degree polynomial in u. We have the inequality

$$\left| h_{n_0}(u) \frac{\partial^p}{\partial x^p} y_u^{(2)}(x) P_n(u)\, e^{-(a^2+u^2)\alpha t} \right| \leqq \left| h_{n_0}(u) \frac{\partial^p}{\partial x^p} y_u^{(2)}(x) P_n(u) \right|.$$

Furthermore, since for $0 \leqq x \leqq R < 1 \, |\partial^p (y_u^{(2)}(x)/\partial x^p|$ is a continuous function of x, there exists some x_0 such that:

$$\left| \frac{\partial^p}{\partial x^p} y_u^{(2)}(x) \right| \leqq \left| \frac{\partial^p}{\partial x^p} y_u^{(2)} \right| (x_0), \qquad 0 \leqq x_0 \leqq R.$$

Hence the whole integrand is less than

$$\eta(u) = \left| h_{n_0}(u) \cdot P_n(u) \cdot \frac{\partial^p}{\partial x^p} y_u^{(2)}(x_0) \right|,$$

and this is a continuous function of $u \geqq 0$. Using Lemma 6 in Appendix C we know that

(B1) $$\left| \frac{\partial^p}{\partial x^p} y_u^{(2)} \right| (x_0) \underset{u \to +\infty}{\leqq} O(u^k)$$

for some real constant k. Stirling's formula gives:

(B2) $$|h_{n_0}(u)| \underset{u\to+\infty}{=} e^{-\pi u}O(u^{k'}).$$

We can conclude that $\eta(u)$ is integrable for $u \geqq 0$. From this follows the first part of the lemma.

Since $G(x, t)$ is continuous with respect to t we have:

$$\lim_{t\to 0} G(x, t) = G(x, 0).$$

Using Theorem 1 in Appendix A and the analytic continuation process in § 3.2 we obtain:

$$\lim_{t\to 0} G(x, t) = x^{n_0}, \qquad 0 \leqq x \leqq R < 1. \qquad \square$$

LEMMA 4. *If $a > 0$ $G(x, t)$ is continuous with x for $0 \leqq x \leqq 1$ uniformly for $t \geqq 0$, and we have*

$$\lim_{x\to 1} G(x, t) = 1, \qquad t \geqq 0.$$

Proof. First we consider $G_s(x, t)$. As in Lemma 3, if $a > 0$ $y_l^{(1)}(x)$ is continuous for $0 \leqq x \leqq 1$ and is multiplied by $e^{-l(2a-l)\alpha t}$, which is also continuous for $t \geqq 0$.

In $G_i(x, t)$ we know that $y_{(u)}^{(2)}(x)$ is continuous for $0 \leqq x \leqq 1$. As in Lemma 3 we can write:

$$|h_{n_0}(u) y_u^{(2)}(x)\, e^{-(a^2+u^2)\alpha t}| \leqq |h_{n_0}(u) y_u^{(2)}(x_1)|, \quad 0 \leqq x_1 < 1.$$

The possibility that $x_1 = 1$ is excluded since $y_u^{(2)}(1) = 0$ and $y_u^{(2)}(0) = 1$. This function is continuous for $u \geqq 0$ and integrable by (B1) and (B2).

From the continuity we get:

$$\lim_{x\to 1} G(x, t) = \lim_{x\to 1} G_s(x, t) = H_0 \cdot \lim_{x\to 1} y_0^{(1)}(x) = 1.$$

Remark. For $a > 1$, one can show in the same way that $\partial G(x, t)/\partial x$ is continuous for $0 \leqq x \leqq 1$ and for $a > 2$ that this is also true for $\partial^2 G(x, t)/\partial x^2$ when $0 \leqq x \leqq 1$. Similarly the derivatives can be brought inside the integral over u.

THEOREM 2. *If $a > 0$, $G(x, t)$ given by formula* (21) *is a solution of*:

$$\frac{\partial G}{\partial t} = L\left(x, \frac{\partial}{\partial x}\right) G, \qquad 0 \leqq x \leqq R < 1, \quad 0 \leqq t,$$

with the boundary condition: $G(x, 0) = x^{n_0}$ *and probability conservation*:

$$\lim_{x\to 1} G(x, t) = 1, \qquad t \geqq 0.$$

Proof. The boundary condition and the probability conservation follow from Lemmas 3 and 4.

The proof of the partial differential equation is a straightforward consequence of the definition of the eigenstates $y_l^{(1)}(x)$ and $y_u^{(2)}(x)$ and of Lemmas 3 and 4, which enable us to bring the various x and t derivatives inside the integral over u. In this way the formal proof given in § 1.4 becomes rigorous. $\square$

Appendix C. We give two lemmas which are used in Appendices A and B. The proofs involve technicalities and are omitted for the sake of brevity. The interested reader will find all the details in Roehner (1980).

LEMMA 5. *Consider*

$$L(u,\xi)=\int_0^\infty t^\sigma\left|K_{2iu}(t)\frac{J_{\tilde b}(\xi t)}{\xi^{\tilde b}}\right|dt.$$

For fixed $\xi \geqq 0$ *and if* $\tilde b > -1$, $\sigma \geqq 0$, $L(u,\xi)$ *is continuous with respect to* $u \geqq 0$. *Furthermore, there exist some real constants* k_1, k_2 *such that for* $u \to +\infty$

$$L(u,\xi) \underset{u\to+\infty}{\leqq} e^{-\pi u}\begin{cases} AO(u^{k_1})+B\xi^{-1/2-\tilde b}O(u^{k_2}), & \xi>0,\\ AO(u^{k_1}), & \xi=0.\end{cases}$$

LEMMA 6. *For fixed x such that* $0 \leqq x \leqq R < 1$, *there exists some real constant k for which*

$$\left|\frac{\partial^n}{\partial x^n}y_u^{(2)}(x)\right| \underset{u\to+\infty}{\leqq} O(u^k).$$

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