

APPLICATION OF STIELTJES THEORY FOR S-FRACTIONS TO BIRTH AND DEATH PROCESSES

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Abstract

We are interested in obtaining bounds for the spectrum of the infinite Jacobi matrix of a birth and death process or of any process (with nearest-neighbour interactions) defined by a similar Jacobi matrix.

To this aim we use some results of Stieltjes theory for S-fractions, after reviewing them. We prove a general theorem giving a lower bound of the spectrum. The theorem also gives sufficient conditions for the spectrum to be discrete.

The expression for the lower bound is then worked out explicitly for several, fairly general, classes of birth and death processes. A conjecture about the asymptotic behavior of a special class of birth and death processes is presented.

CONTINUED FRACTIONS; STIELTJES FRACTIONS; SPECTRUM; JACOBI MATRICES

1. Introduction

The Stieltjes theorem for the so-called S-fraction is a celebrated result given by all classical treatises on continued fractions (see, for example, Wall (1967), p. 120).

Actually, Stieltjes theory goes a good deal further than usually stated. In this paper, after reviewing Stieltjes theory (for convenience these results are collected in the appendix), we show that application to birth and death continued fractions gives rather ‘practical’ results such as the nature of the spectrum and lower bounds for the absolute value of the poles.

The same results also apply to any system governed by an infinite set of differential equations similar to the Kolmogorov equations of a birth and death process. Several examples will be given later, but first of all let us state the Kolmogorov equations.

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1.1. *The differential equations.* We consider a birth and death process whose transition rates λ_n, μ_n are such that

$$\lambda_0 > 0, \quad \mu_0 = 0; \quad \lambda_n > 0, \quad \mu_n > 0 \quad n \geq 1.$$

$\mu_0 = 0$ means that $n = 0$ is a natural limit of the process.

A complete description of that Markov process is provided by the transition probabilities from an initial state n_0 to state n at time t :

$$P(n, t/n_0, t_0) \equiv p_n(t).$$

These transition probabilities are the solutions of the Kolmogorov equations

$$(1) \quad \frac{dp_n}{dt} = -(\lambda_n + \mu_n)p_n + \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1} \quad n \geq 0$$

with the convention that p_n should be 0 for $n < 0$.

1.2. *Practical applications.* Although we are primarily interested in birth and death processes, we note that similar sets of differential equations occur also for mechanical systems with an infinite number of degrees of freedom and with nearest-neighbour interactions. In those cases the first derivative of the left-hand side is of course to be replaced by a second derivative. Examples are:

- the longitudinal vibrations of an infinite number of harmonic oscillators;
- small transverse oscillations of a beaded string;
- small oscillations of a multiple pendulum.

In all these situations, the equations are of the form

$$\frac{d^\alpha \vec{p}}{dt^\alpha} + A\vec{p} = \vec{0}, \quad \alpha = 1 \text{ or } 2$$

where A is an infinite (normal) Jacobi matrix (Gantmacher and Krein (1960), p. 77). Obtaining bounds for the spectrum of that matrix will be our main purpose. To motivate such a program, it should be recalled that solving (or even obtaining the complete spectrum of) the Kolmogorov equations (1) is a very difficult task whenever the transition rates λ_n, μ_n are non-linear. Information about the spectrum can be considered as a first step on the way to that solution.

Let us also mention two cases in which knowledge of a bound for the spectrum is of immediate importance. Since the spectrum will be discrete, non-positive and simple, let us denote it by

$$-s_k \quad k = 1, 2, \dots \quad \text{with} \quad 0 \leq s_1 < s_2 < s_3 < \dots$$

1. If $s_1 \neq 0$ the large-time behavior of a birth and death process is given by s_1 . Of course it would be useful also to know the next smallest, i.e. s_2 , which

tells how soon the dominant behavior occurs; for such an estimate in the case of a finite birth and death process, see Dambrine and Moreau (1981).

If $s_1 = 0$, the process has a stationary distribution and s_2 gives an estimate of the speed of convergence to that stationary distribution when time goes to ∞ .

2. In the approximation theory of continued fractions, the pole which is nearest to the origin is directly related to the speed of convergence of the approximants of a continued fraction, as indicated by the following result (Baker (1975), Gilewicz (1977), p. 6. 29).

If a function represented by a continued fraction is analytic in a disc of radius R around the origin, its n th-order approximant converges at least as ψ^n in the whole domain:

$$|\psi| = \left| \frac{(1+z/R)^{1/2} - 1}{(1+z/R)^{1/2} + 1} \left(\frac{z}{R} \right)^{1/2} \right| < 1.$$

The paper is organized as follows. In section 2, we introduce the continued J -fraction as a formal solution of Kolmogorov equations and we show that it can be written in a Stieltjes fraction form. In section 3, we prove the main theorem. In sections 4 and 5, we apply that theorem to asymptotically symmetric and asymptotically proportional processes. Finally, in section 6 we present an example of a non-asymptotically proportional process.

2. The Stieltjes S-fraction

2.1. *The J-fraction.* Going to the Laplace transform $\tilde{p}_n(s)$ of $p_n(t)$ in the Kolmogorov equations changes them into a set of recurrence relations which reads, for $n \neq n_0$,

$$s + \lambda_n + \mu_n = q_{n-1} + \frac{\mu_{n+1} \lambda_n}{q_n}$$

with

$$q_n = \frac{\tilde{p}_n \lambda_n}{\tilde{p}_{n+1}}.$$

If $n_0 = 0$, this set can be solved recursively for $\tilde{p}_0(s)$ in the form of a continued fraction (a theorem by Perron (1929), p. 251, gives full justification for this formal derivation):

$$(2) \quad \tilde{p}_0(s) = 1/s + \sigma_0 - \underline{\gamma_0}/s + \sigma_1 - \underline{\gamma_1}/ - \cdots - \underline{\gamma_{n-1}}/s + \sigma_n - \cdots \quad \sigma_n = \lambda_n + \mu_n \\ \gamma_n = \lambda_n \mu_{n+1}$$

which is the starting point of our study.

2.2. *Check of the Stieltjes criterion.* The J -fraction (2) is the even part of a

continued S-fraction:

$$(3) \quad \underline{1/a_1s} + \underline{1/a_2} + \underline{1/a_3s} + \cdots \quad a_n > 0$$

provided there exist numbers $P_0 = 0$, $P_1 > 0$, $P_2 > 0, \dots$ such that (Stieltjes (1918), p. 405):

$$P_n < \sigma_n; \quad \frac{\gamma_n}{\sigma_n - P_n} \leq P_{n+1} \quad n \geq 0.$$

With $P_n = \mu_n$, $n \geq 0$ the inequalities are easily seen to be satisfied.

2.3. Transformation formulas from S-fraction to J-fraction. Given the S-fraction (3), we introduce the numbers

$$(4) \quad b_0 = \frac{1}{a_1}, \quad b_n = \frac{1}{a_n a_{n+1}} \quad n \geq 1.$$

The fraction (3) transforms to

$$\underline{b_0/s} + \underline{b_1/1} + b_2s + \underline{b_3/} + \cdots$$

which (by a standard transformation) can be rewritten as

$$\underline{b_0/s} + b_1 - \underline{b_1 b_2/s} + b_2 + b_3 - \underline{b_3 b_4/} + \cdots$$

Hence, by comparison with the J-fraction (2) we derive the relations

$$(5) \quad \begin{aligned} 1 &= b_0 \\ \sigma_0 &= b_1 \\ \gamma_{n-1} &= b_{2n-1} b_{2n} \quad n \geq 1 \\ \sigma_n &= b_{2n} + b_{2n+1}. \end{aligned}$$

These formulas are quite general, but for $\mu_0 = 0$, there is a convenient term-to-term correspondence between the λ_n , μ_n and the b_n :

$$\begin{aligned} \lambda_n &= b_{2n+1} = \frac{1}{a_{2n+1} \cdot a_{2n+2}} \quad n \geq 0 \\ \mu_n &= b_{2n} = \frac{1}{a_{2n} \cdot a_{2n+1}} \quad n \geq 1. \end{aligned}$$

2.4. Relation between the Laplace transform $\tilde{p}_0(s)$ of the probability transitions for different processes. Before coming to our main subject, let us notice that the Stieltjes form of the birth and death continued fraction leads quite easily to relations between the $\tilde{p}_0(s)$ of different processes, which would not be obvious

otherwise. (We are grateful to our referee for bringing this property to our attention.)

Proposition 1.

1. Let $(\lambda_n^{(1)}, \mu_n^{(1)})$ be a process such that

$$\lambda_0^{(1)} = 1 \quad (\text{and } \mu_0^{(1)} = 0).$$

Define a second process $(\lambda_n^{(2)}, \mu_n^{(2)})$ by

$$\begin{aligned} \mu_0^{(2)} &= 0, & \lambda_n^{(2)} &= \mu_{n+1}^{(1)} & n \geq 0 \\ \mu_n^{(2)} &= \lambda_n^{(1)} & & & n \geq 1. \end{aligned}$$

The Laplace transforms of the probability transitions to 0 of both processes are related in the following way:

$$\tilde{p}_0^{(1)}(s) = \frac{1}{s(\tilde{p}_0^{(2)}(s) + 1)}.$$

2. Let $(\lambda_n^{(1)}, \mu_n^{(1)})$ be a process such that

$$\lambda_0^{(1)} = \mu_1^{(1)} = 1 \quad (\text{and } \mu_0^{(1)} = 0).$$

Define a second process $(\lambda_n^{(2)}, \mu_n^{(2)})$ by

$$\begin{aligned} \lambda_n^{(2)} &= \lambda_{n+1}^{(1)} & n \geq 0 \\ \mu_n^{(2)} &= 0, & \mu_n^{(2)} &= \mu_{n+1}^{(1)}. \end{aligned}$$

Then

$$\tilde{p}_0^{(1)}(s) = \frac{\tilde{p}_0^{(2)}(s) + 1}{p_0^{(2)}(s)(s + 1) + s}.$$

Proof

1. Let us first show that

$$a_n^{(2)} = a_{n+1}^{(1)} \quad n \geq 1.$$

The inversion of formula (4) gives

$$\begin{aligned} a_2^{(2)} &= \frac{1}{\lambda_0^{(2)}} & a_{2n}^{(2)} &= \frac{\mu_1^{(2)} \cdots \mu_{n-1}^{(2)}}{\lambda_0^{(2)} \lambda_1^{(2)} \cdots \lambda_{n-1}^{(2)}} & n \geq 2 \\ a_1^{(2)} &= 1 & a_{2n+1}^{(2)} &= \frac{\lambda_0^{(2)} \cdots \lambda_{n-1}^{(2)}}{\mu_1^{(2)} \cdots \mu_n^{(2)}} & n \geq 1, \end{aligned}$$

which reads

$$\begin{aligned} a_2^{(2)} &= \frac{1}{\lambda_0^{(1)}} a_3^{(1)} & a_{2n}^{(2)} &= \frac{a_{2n+1}^{(1)}}{\lambda_0^{(1)}} \\ a_1^{(2)} &= \lambda_0^{(1)} a_2^{(1)} & a_{2n+1}^{(2)} &= \lambda_0^{(1)} a_{2(n+1)}^{(1)}. \end{aligned}$$

This reduces to the relation we wish to prove if $\lambda_0^{(1)} = 1$. The Stieltjes form of $\tilde{p}_0^{(2)}(s)$ is

$$\tilde{p}_0^{(2)}(s) = \underline{1}/a_1^{(2)}s + \underline{1}/a_2^{(2)} + \underline{1}/a_3^{(2)}s + \dots$$

This reads here

$$\tilde{p}_0^{(2)}(s) = \underline{1}/a_2^{(1)}s + \underline{1}/a_3^{(1)} + \underline{1}/a_4^{(1)}s + \dots$$

which should be compared with

$$\tilde{p}_0^{(1)}(s) = \underline{1}/a_1^{(1)}s + \underline{1}/a_2^{(1)} + \underline{1}/a_3^{(1)}s + \dots$$

Thus we have (since $a_1^{(1)} = 1$)

$$\tilde{p}_0^{(1)}(s) = \frac{1}{s + s\tilde{p}_0^{(2)}(s)}.$$

2. The proof of the second part of the proposition is very similar. In that case, we have

$$a_n^{(2)} = a_{n+2}^{(1)} \quad n \geq 1.$$

Proposition 2. The processes 1 and 2 considered here are those defined in Part 1 of Proposition 1. If the spectrum, $-s_k^1$, of process 1 is such that

$$s_1^1 > m, \quad m > 0$$

then the spectrum, $-s_k^2$, of process 2 satisfies

$$s_1^2 = 0, \quad s_2^2 > m.$$

Proof. Writing $\tilde{p}_0^{(1)}$ as N/D and inverting the formula giving $\tilde{p}_0^{(1)}$, we obtain

$$\tilde{p}_0^{(2)}(s) = \frac{D - sN}{sN}.$$

Denote the zeros of $\tilde{p}_0^{(1)}(s)$ by

$$-\sigma_k^1 \quad k = 1, 2, \dots, \quad \sigma_1^1 < \sigma_2^1 < \dots$$

Thus

$$\{s_1^2, s_2^2, \dots\} \subset \{0, \sigma_1^1, \sigma_2^1, \dots\}.$$

If there are no common factors in the numerator and denominator of $\tilde{p}_0^{(2)}$, then

$$\{s_1^2, s_2^2, \dots\} = \{0, \sigma_1^1, \sigma_2^1, \dots\}.$$

To complete the proof concerning the bound of the spectrum, we observe that $\sigma_1^1 \geq s_1^1$. In fact, $\tilde{p}_0^1(s)$ is positive in the whole convergence domain of the Laplace integral, i.e. for $s > -s_1^1$.

3. General result for the lower bound of the spectrum

3.1. *Preliminaries.* First we introduce the following three basic series:

$$S = \sum_{n=1}^{\infty} a_n$$

$$B = \sum_{n=1}^{\infty} (a_1 + \cdots + a_{2n-1}) a_{2n}$$

$$C = \sum_{n=1}^{\infty} (a_2 + \cdots + a_{2n}) a_{2n+1}.$$

The convergence of series S on one hand and of series B and C on the other hand are related:

$$S < \infty \Leftrightarrow B < \infty \quad \text{and} \quad C < \infty$$

because $B < S^2$, $C < S^2$, $S < B + C$.

We define furthermore the series

$$D_1 = \sum_{n=1}^{\infty} a_{2n-1} \quad D_2 = \sum_{l=1}^{\infty} a_{2l+1} \sum_{k=1}^l (a_1 + \cdots + a_{2k-1}) a_{2k}.$$

Let us emphasize that, apart from technicalities, the following result is based on two facts:

- the easy correspondence between the coefficients a_n of the S-fraction and the transition rates λ_n, μ_n (section 2.3).
- the existence of a limit for the coefficients of the polynomials in the denominators and numerators of the approximants of the S-fraction (appendix).

3.2. Main theorem

- (a) If any of the series S , B or C converge, the process $(\lambda_n, \mu_n; \lambda_0 > 0, \mu_0 = 0)$ has a discrete, non-positive, simple spectrum:

$$-s_k \quad k = 1, 2, \dots \quad 0 \leq s_1 < s_2 < \cdots < s_k < \cdots$$

which is such that the series

$$\sum_{k=2}^{\infty} \frac{1}{s_k}$$

converges.

- (b) The absolute value of the spectrum has the following lower bound:

1. If S converges (which implies convergence of B and C)

$$s_1 \geq 1/B > 1/S^2.$$

2. If B converges (and S diverges along with C)

$$s_1 \geq 1/B.$$

This process is dishonest, i.e. $\sum_{n=0}^{\infty} p_n(t) < 1$.

3. If C converges (and S diverges along with B)

$$s_1 = 0, \quad s_2 \geq D_1/D_2 > 1/C.$$

This process is honest (i.e. $\sum_{n=0}^{\infty} p_n(t) = 1$) and has a stationary distribution.

Proof

1. Part (a) of the theorem is merely a consequence of the Stieltjes theory reviewed in the appendix. It has been stated here for completeness.
2. Let us first assume that at least B converges (which comprises the case when S converges).

From Equation (A.2) and with due regard to the positivity of the $s_l^{(2n)}$,

$$\frac{1}{s_1^{(2n)}} < \sum_{l=1}^n \frac{1}{s_l^{(2n)}} = B_1^{(2n)}.$$

Now (appendix)

$$s_1^{(2n)} \xrightarrow{n \rightarrow \infty} s_1,$$

and from Equation (A.4)

$$B_1^{(2n)} \xrightarrow{n \rightarrow \infty} B.$$

Hence, going to the limit, we obtain $1/s_1 \leq B$. Moreover, if S converges, $B < S^2$. The last (less accurate) bound is interesting in so far as it may be easier to estimate S than B .

3. Assume now that only C converges. Then, we have to work with $Q_{2n+1}(s)$. But in that case, since S diverges, the odd and even approximants have the same limit. Hence the results for the poles of the odd part will be relevant for our J -fraction too.

From Equation (A.3),

$$\prod_{k=2}^{n+1} \left(1 + \frac{s}{s_k^{(2n+1)}} \right) = 1 + \frac{D_2^{(2n+1)}}{D_1^{(2n+1)}} s + \cdots + \frac{D_{n+1}^{(2n+1)}}{D_1^{(2n+1)}} s^n.$$

Thus

$$\frac{1}{s_2^{(2n+1)}} < \sum_{l=2}^{n+1} \frac{1}{s_l^{(2n+1)}} = \frac{D_2^{(2n+1)}}{D_1^{(2n+1)}}.$$

Going to the limit and taking into account relation (A.5), we get

$$\frac{1}{s_2} \leq \frac{D_2}{D_1}.$$

Finally, the inequality $D_1/D_2 > 1/C$ comes from the fact that

$$D_2 < \sum_{l=1}^{\infty} a_{2l+1} \sum_{k=1}^l a_{2k} \sum_{j=1}^{\infty} a_{2j-1} = C \cdot D_1.$$

4. We come now to the interpretation of series S , B and C in terms of birth and death processes. This will prove the characterizations of the process given in Parts b2 and b3 of the theorem.

Notice first that the inversion of formula (4) gives

$$a_2 = \frac{1}{\lambda_0}, \quad a_{2n} = \frac{\mu_1 \cdots \mu_{n-1}}{\lambda_0 \lambda_1 \cdots \lambda_{n-1}} \quad n \geq 2$$

$$a_1 = 1, \quad a_{2n+1} = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \quad n \geq 1$$

which indicates that series S , B and C , stand respectively for series T , R and S of Reuter (1957). We may thus summarize Reuter's results regarding uniqueness and honesty along with our results for the spectrum in the following classification of birth and death processes. We have also indicated in the table whether a stationary distribution exists or not. For an honest process, the stationary distribution's existence criterion is the convergence of series D_1 (Parzen (1962), p. 280).

Uniqueness of Kolmogorov equation solution	Honesty	Stationary distribution	Spectrum bound given by the theorem	
$S < \infty$	No	Dishonest, ex- cept for one of the solutions	No	$1/B$
$B < \infty$ $C = \infty$	Yes	Dishonest	No	$1/B$
$B = \infty$ $C < \infty$ $B = \infty$ $C = \infty$	Yes	Honest	Yes	D_1/D_2 Theorem does not apply

The most pathological case is when S converges and in that case a problem still remains: is the solution given by the continued fraction (with the \tilde{p}_n , $n \geq 1$ obtained from \tilde{p}_0 through the recurrence relations) the one which is honest or not?

In more intuitive terms, a process is honest if the average transition time of the process from any given state n_0 to ∞ is infinite (John (1957)), i.e. the process does not diverge in finite time.

In the following sections, the theorem will be applied to three classes of birth and death processes (belonging exclusively to Cases 2 and 3). This will at the

same time illustrate, in specific situations, how an estimate of the bound can be gained.

Let us make a general comment about those applications. Since

$$\rho(n) = \frac{\mu_n}{\lambda_n} = \frac{a_{2(n+1)}}{a_{2n}}$$

we have $a_{2n} = \rho(n-1) \cdots \rho(1)a_2$. Thus a_{2n} depends on $\rho(n)$ in a very 'critical' way. The three classes of process we discuss below correspond respectively to

$$1. \quad \rho(n) \underset{n \rightarrow \infty}{\sim} 1.$$

$$2. \quad \rho(n) \underset{n \rightarrow \infty}{\sim} r \neq 1.$$

$$3. \quad \rho(n) \underset{n \rightarrow \infty}{\sim} n^\alpha.$$

It is to be expected that the most accurate bounds will be obtained for the first and second class rather than for the fast-varying series of class 3.

4. Application to asymptotically symmetric processes

The asymptotically symmetric processes which are characterized by the property that

$$\lambda_n \underset{n \rightarrow \infty}{\sim} \mu_n$$

were first introduced for quadratic processes (i.e., when λ_n and μ_n are quadratic functions of n) where they play a very special role since it is the only case where the (infinite) system for the moments $m_k(t)$, $k = 1, 2, \dots$ closes for every k (Roehner and Valent (1982)).

In our framework, asymptotically symmetric processes correspond to the case

$$a_{2n} \underset{n \rightarrow \infty}{\sim} a_{2(n+1)}.$$

The following corollary gives lower bounds of the spectrum for some particular asymptotically symmetric processes.

Remark. In Corollaries 1 to 4 below at least one of the series S , B or C will converge. Hence the spectrum will be discrete and non-positive and such that

$$\sum_{k=2}^{\infty} \frac{1}{s_k}$$

converges. We make that statement once and for all to avoid repetition.

Corollary 1. For the following asymptotically symmetric processes, the spectrum is discrete with lower bounds as indicated in each case.

1.

$$(P1) \quad \begin{aligned} \lambda_n &= (n+1)^\alpha & n \geq 0 & \alpha > 2 \\ \mu_n &= n^\alpha & & \end{aligned}$$

$$s_1 \geq \frac{1}{\zeta(\alpha-1)} \geq \frac{\alpha-2}{\alpha-1},$$

where $\zeta(s)$ is Riemann's zeta function.

2.

$$(P2) \quad \begin{aligned} \lambda_n &= (n+1)^\alpha & n \geq 0 & \alpha > 2 \\ \mu_n &= (n+1)^\alpha & n \geq 1, & \mu_0 = 0 \\ s_1 &= 0; & s_2 \geq \frac{\alpha-1}{\alpha \frac{\zeta(\alpha-1)}{\zeta(\alpha)} - \frac{1-2^{2-\alpha}}{\alpha-2}}. & \end{aligned}$$

3.

$$(P3) \quad \begin{aligned} \lambda_n &= (n+1)\rho^n & n \geq 0 & \rho > 1 \\ \mu_n &= n\rho^n & & \\ s_1 &= 0; & s_2 \geq \frac{(\rho^2-1)^2}{\rho(\rho^2+\rho+1)}. & \end{aligned}$$

Proof

1. The a_n corresponding to the first process are

$$a_{2n+1} = 1, \quad n \geq 0; \quad a_{2n} = \frac{1}{n^\alpha}, \quad n \geq 1.$$

Since α is assumed larger than 2, the series B converges. We are thus using the bound given by Case 2 of the proposition.

$$B = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha-1}} = \zeta(\alpha-1) < \int_1^{\infty} \frac{dx}{n^{\alpha-1}} + 1 = \frac{\alpha-1}{\alpha-2}.$$

2. The processes P_1 and P_2 are related in the way indicated by the first part of Proposition 1 (section 1.4) and Proposition 2 gives the relation between their spectra. However since we have no *a priori* knowledge about possible common factors in the numerator and denominator, let us obtain the bound by direct calculation. It is interesting to notice that this bound will prove sharper than the potential bound $(\alpha-2)/(\alpha-1)$, but only for $\alpha > 2 \cdot 4$.

The a_n corresponding to the second process are

$$a_{2n+1} = \frac{1}{(n+1)^\alpha} \quad a_{2n} = 1.$$

Since α is larger than 2, C converges. Hence the bound is given by Case 3 of the proposition.

Let us obtain an upper bound for D_2 by the following steps.

$$(a_1 + \cdots + a_{2k-1})a_{2k} = \sum_{n=1}^k \frac{1}{n^\alpha} < \int_1^k \frac{dx}{x^\alpha} + 1 = \frac{1}{\alpha-1} \left(\alpha - \frac{1}{k^{\alpha-1}} \right)$$

$$\sum_{k=1}^l (a_1 + \cdots + a_{2k-1})a_{2k} < \frac{1}{\alpha-1} \left[\alpha l - \int_1^{l+1} \frac{dx}{x^{\alpha-1}} \right].$$

Hence

$$D_2 < \frac{1}{\alpha-1} \sum_{l=1}^{\infty} \left[\alpha l - \frac{1}{\alpha-2} \left(1 - \frac{1}{(l+1)^{\alpha-2}} \right) \right] \frac{1}{(l+1)^\alpha}$$

$$D_2 < \frac{1}{\alpha-1} \sum_{l=1}^{\infty} \left[\alpha l - \frac{1}{\alpha-2} \left(1 - \frac{1}{2^{\alpha-2}} \right) \right] \frac{1}{l^\alpha}$$

which gives

$$D_2 < \frac{1}{\alpha-1} \left[\alpha \zeta(\alpha-1) - \frac{1}{\alpha-2} (1 - 2^{2-\alpha}) \zeta(\alpha) \right].$$

Since $D_1 = \zeta(\alpha)$ the claimed result follows.

Remark 1. Note that in both previous cases, the lower bound goes to 0 when $\alpha \rightarrow 2$. This is obvious in Case 1, and in Case 2 it results from (Magnus et al. (1966), p. 19)

$$\zeta(s) \underset{s \rightarrow 1}{\sim} \frac{1}{s-1}$$

This result is in agreement with the fact that the quadratic asymptotically symmetric process has a continuous spectrum extending from 0 to $-\infty$ (Roehner and Valent (1982)).

Remark 2. Ledermann and Reuter (1954) proved the discreteness of the spectrum, (in a matrix framework) for processes of the form

$$\lambda_n = c(n+h)^\alpha + O\left(\frac{1}{n}\right)$$

$$\mu_n = n^\alpha + O\left(\frac{1}{n}\right)$$

in the two cases

$$1. \quad c = 1, \quad h < 1 - \frac{1}{\alpha}, \quad \alpha > 2$$

$$2. \quad 0 < c < 1, \quad \alpha > \frac{1}{2}.$$

The first and second processes of Corollary 1, although bordering on this class, do not belong to it.

Maki (1976) obtained similar results in a framework involving orthogonal polynomials and continued fractions. But he restricted himself to rational transition rates, although his method could be extended (private communication).

3. The a_n corresponding to the third process are

$$a_{2n+1} = \frac{1}{\rho^n} \quad a_{2n} = \frac{1}{n}.$$

In this case also, C converges; indeed,

$$C = \sum_{n=1}^{\infty} \left(1 + \dots + \frac{1}{n}\right) \frac{1}{\rho^n} < \sum_{n=1}^{\infty} \frac{n}{\rho^n}.$$

Let us obtain an upper bound for D_2 by the following steps.

$$(a_1 + \dots + a_{2k-1})a_{2k} = \frac{1}{k} \frac{1 - \frac{1}{\rho^k}}{1 - \frac{1}{\rho}}$$

$$\sum_{k=1}^l (a_1 + \dots + a_{2k-1})a_{2k} < \frac{\rho}{\rho-1} \left(1 - \frac{1}{\rho^l}\right) \left[\int_1^l \frac{dx}{x} + 1 \right].$$

Hence

$$D_2 < \frac{\rho}{\rho-1} \sum_{l=1}^{\infty} (\ln l + 1) \left(1 - \frac{1}{\rho^l}\right) \frac{1}{\rho^l}$$

$$D_2 < \frac{\rho}{\rho-1} \sum_{l=1}^{\infty} l \left(1 - \frac{1}{\rho^l}\right) \frac{1}{\rho^l}$$

which gives

$$D_2 < \frac{\rho}{\rho-1} \left[\frac{\rho}{(\rho-1)^2} - \frac{\rho^2}{(\rho^2-1)^2} \right].$$

Since $D_1 = \rho/(\rho-1)$ the claimed result follows.

5. Application to asymptotically proportional processes

We shall say that a process is asymptotically proportional if

$$\frac{\lambda_n}{\mu_n} \underset{n \rightarrow \infty}{\sim} r \neq 1.$$

The motivation for this definition is the special properties the spectrum seems to have in that case (see conjecture at the end of the section).

Let us first state a lemma which will prove of frequent use.

Lemma. If

$$\rho \geq \left(\frac{3}{2}\right)^{\alpha+1} \quad \text{and} \quad n \geq 2$$

the following upper bound holds:

$$\sum_{k=1}^n \frac{\rho^k}{k^\alpha} < a(\alpha, \rho) \frac{\rho^n}{n^\alpha} + b(\alpha, \rho)$$

$$a(\alpha, \rho) = 1 + \frac{\alpha+1}{\ln \rho} \quad b(\alpha, \rho) = \rho - \frac{\rho^2}{2^\alpha \ln \rho}.$$

Proof. For the sequence

$$\frac{\rho^k}{k^\alpha} \quad k = m, m+1, \dots$$

to be increasing, the condition on ρ is

$$\rho \geq \left(1 + \frac{1}{m}\right)^\alpha.$$

Writing

$$\sum_{k=1}^n \frac{\rho^k}{k^\alpha} = \rho + \sum_{k=2}^n \frac{\rho^k}{k^\alpha}$$

we observe that the terms of the sum in the right-hand side are increasing, provided that $\rho \geq \left(\frac{3}{2}\right)^\alpha$. Hence an estimate can be obtained from

$$\sum_{k=2}^n \frac{\rho^k}{k^\alpha} < \int_2^n \frac{\rho^x}{x^\alpha} dx + \frac{\rho^n}{n^\alpha}.$$

Integrating by parts,

$$\int_2^n \frac{\rho^x}{x^\alpha} dx = \frac{1}{\ln \rho} \left| \frac{\rho^x}{x^\alpha} \right|_2^n + \frac{\alpha}{\ln \rho} \int_2^n \frac{\rho^x}{x^{\alpha+1}} dx.$$

Now for $n \geq 2$

$$\int_2^n \frac{\rho^x}{x^{\alpha+1}} dx \leq (n-2) \frac{\rho^n}{n^{\alpha+1}} < \frac{\rho^n}{n^\alpha}.$$

The first inequality is obvious provided that $\rho \geq \left(\frac{3}{2}\right)^{\alpha+1}$. Collecting all the terms, the lemma follows.

5.1. First example

Corollary 2. The spectrum of the birth and death process

$$(P4) \quad \begin{aligned} \lambda_n &= (n+1)^\alpha & n \geq 0 \\ \mu_n &= \rho n^\alpha \end{aligned}$$

is discrete in the following two cases, with lower bounds as indicated:

1. $\rho < 1, \alpha > 1$.

$$s_1 > \frac{1-\rho}{\zeta(\alpha) + \left(\ln \frac{1}{\rho}\right)^{\alpha-1} \Gamma\left(1-\alpha, \ln \frac{1}{\rho}\right)} = M_1$$

where $\Gamma(a, x)$ is the incomplete gamma function.

2. $\rho > 1, \alpha > 1$. For $\rho \geq (\frac{3}{2})^{\alpha+1}$

$$s_1 = 0 \quad s_2 > \frac{\rho}{\frac{\rho-1}{\rho} + a(\zeta(\alpha)-1) + \frac{b}{\rho(\rho-1)}} = M_2,$$

where a and b are the numbers defined in the lemma.

Proof. This process corresponds to the following a_n :

$$a_{2n+1} = \frac{1}{\rho^n} \quad n \geq 0; \quad a_{2n} = \frac{\rho^{n-1}}{n^\alpha} \quad n \geq 1.$$

1. In the first case, the series B converges; indeed,

$$B = \frac{1}{1-\rho} \left[\sum_{n=1}^{\infty} \frac{1}{n^\alpha} - \sum_{n=1}^{\infty} \frac{\rho^n}{n^\alpha} \right].$$

Thus the bound of the spectrum is given by $1/B$. The sequence ρ^n/n^α is of course decreasing; hence,

$$\sum_{n=1}^{\infty} \frac{\rho^n}{n^\alpha} > \int_1^{\infty} \frac{\rho^x}{x^\alpha} dx.$$

Taking into account the definition of the incomplete gamma function (Magnus et al. (1966), p. 337) we obtain Result 1 of the corollary.

A limiting case. If ρ goes to 0, the process (P4) becomes a pure birth process and the pole with smallest absolute value is $s_1 = \lambda_0 = 1$. On the other hand, the bound M_1 becomes

$$\lim_{\rho \rightarrow 0} M_1 = \frac{1}{\zeta(\alpha)} < 1.$$

As α tends to ∞ , the bound approaches 1. As $\alpha \rightarrow 1$, $\zeta(\alpha) \rightarrow \infty$; this singularity is discussed at the end of this section.

2. The second case is different since now the series B diverges. Since series C converges, we are using the bound D_1/D_2 . In that case the process does not diverge and furthermore has a stationary distribution. D_1 is simply a geometric

series, and for D_2 we have

$$\sum_{k=1}^l (a_1 + \cdots + a_{2k-1}) a_{2k} < \frac{\rho}{\rho-1} \sum_{k=1}^l \frac{\rho^{k-1}}{k^\alpha}.$$

To use the lemma, we must assume that

$$\rho \geq (\frac{3}{2})^{\alpha+1}.$$

Then

$$D_2 < a_1 a_2 a_3 + \frac{1}{\rho-1} \sum_{l=2}^{\infty} \left[\frac{a}{l^\alpha} + \frac{b}{\rho^l} \right].$$

Replacing the Riemann series and the geometric series by their value and collecting all terms gives Result 2 of the corollary.

A limiting case. Divide both transition rates of the previous process by ρ . This leads to the process

$$(P4') \quad \begin{aligned} \lambda_n &= \frac{1}{\rho} (n+1)^\alpha \\ \mu_n &= n^\alpha \end{aligned} \quad n \geq 0; \quad \rho > 1, \quad \alpha > 1.$$

How are both spectra related?

The answer is given by the following scaling rule that follows from the Kolmogorov equations (1).

If the transition rates are multiplied by a constant k , the spectrum (i.e., each pole) is multiplied by the same constant k . Thus the spectrum of process (P4') has the bound

$$s_1 = 0, \quad s_2 > \frac{1}{\frac{\rho-1}{\rho} + a(\zeta(\alpha) - 1) + \frac{b}{\rho(\rho-1)}} = M'_2.$$

If ρ goes to ∞ , the process (P4') reduces to a pure death process, and after $s_1 = \mu_0 = 0$, the smallest s_k is $s_2 = \mu_1 = 1$. On the other hand, the bound M'_2 becomes

$$\lim_{\rho \rightarrow \infty} M'_2 = \frac{1}{\zeta(\alpha)}.$$

The case $\alpha \leq 1$. It is easy to see that neither series B nor C converges if $\alpha \leq 1$. Indeed

$$\begin{aligned} B &= \sum_{n=1}^{\infty} \left(1 + \cdots + \frac{1}{\rho^{n-1}} \right) \frac{\rho^{n-1}}{n^\alpha} > \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \\ C &= \sum_{n=1}^{\infty} \left(1 + \frac{\rho}{2^\alpha} + \cdots + \frac{\rho^{n-1}}{n^\alpha} \right) \frac{1}{\rho^n} > \frac{1}{\rho} \sum_{n=1}^{\infty} \frac{1}{n^\alpha}. \end{aligned}$$

Nevertheless, if $\rho \neq 1$, the spectrum is discrete. For $\alpha = 1$, the spectrum is even known explicitly:

$$s_n = n(1 - \lambda) \quad n = 1, 2, \dots$$

Of course, in that case, the series

$$\sum_{n=1}^{\infty} \frac{1}{s_n}$$

does not converge. Thus the three cases enumerated by Stieltjes do not cover all the situations where the spectrum is discrete.

5.2. Second example. Our second example is a generalization of a remarkable continued fraction (corresponding to the case $\alpha = 2$) given by Stieltjes (1918), p. 550, for which the spectrum is known exactly.

Corollary 3. The spectrum of the process

$$(P5) \quad \begin{aligned} \lambda_n &= (2n+1)^\alpha \\ \mu_n &= \frac{(2n)^\alpha}{\rho} \quad n \geq 0; \quad \alpha > 1, \quad \rho \geq \left(\frac{3}{2}\right)^{(\alpha/2)+1} \end{aligned}$$

is discrete and has the following lower bound:

$$\begin{aligned} s_1 &> \frac{1}{m} \\ m &= 1 + \left(1 + \frac{b(\alpha/2, \rho)}{\pi^{\alpha/2}}\right) \left(\frac{2}{3}\right)^\alpha \frac{1}{\rho^{\alpha-1}} + \frac{a(\alpha/2, \rho)}{\pi^{\alpha/2}} \zeta(\alpha) \end{aligned}$$

with a, b as defined in the lemma.

Proof. The process (P5) is very similar to (P4). However, the a_n are rather different:

$$\begin{aligned} a_1 &= a_2 = 1 \\ a_{2n+1} &= A_{2n+1}^\alpha \rho^n \quad A_{2n+1} = \frac{(2n-1)!!}{(2n)!!} \quad n \geq 1 \\ a_{2n} &= A_{2n}^\alpha \frac{1}{\rho^{n-1}} \quad A_{2n} = \frac{(2n-2)!!}{(2n-1)!!} \quad n \geq 2, \end{aligned}$$

where we have used the notation

$$(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$$

$$(2n)!! = 2 \cdot 4 \cdots (2n).$$

Let us find an upper bound for series B . Actually it is not, *a priori*, obvious that series B converges, but this will become apparent as we proceed.

First let us obtain estimates for the sequences A_{2n+1} and A_{2n} . They are decreasing, but the associate sequences $A_{2n+1} \cdot \sqrt{2n}$ and $A_{2n} \cdot \sqrt{2n-1}$ are both increasing and therefore bounded by their limits. We have (Pólya and Szegö (1972), p. 97)

$$A_{2n+1} \underset{n \rightarrow \infty}{\sim} \frac{1}{\sqrt{n\pi}} \quad A_{2n} \underset{n \rightarrow \infty}{\sim} \frac{1}{2} \sqrt{\frac{\pi}{n}}.$$

Thus

$$A_{2n+1} < \frac{1}{\sqrt{n\pi}} \quad A_{2n} < \frac{1}{2} \sqrt{\frac{\pi}{n-1}}.$$

Then

$$\sum_{k=0}^{n-1} a_{2k+1} = 1 + \sum_{k=1}^{n-1} a_{2k+1} < 1 + \frac{1}{\pi^{\alpha/2}} \sum_{k=1}^{n-1} \frac{\rho^k}{k^{\alpha/2}}.$$

Using the lemma, we obtain, for $\rho \geq (\frac{3}{2})^{\alpha/2+1}$

$$\sum_{k=1}^{n-1} \frac{\rho^k}{k^{\alpha/2}} < a(\alpha/2, \rho) \frac{\rho^{n-1}}{(n-1)^{\alpha/2}} + b(\alpha/2, \rho).$$

This will give rise to two different series in series B (where we omit the first term $a_1 a_2$):

1.

$$\sum_{n=2}^{\infty} \frac{\rho^{n-1}}{(n-1)^{\alpha/2}} a_{2n} = \sum_{n=2}^{\infty} \frac{A_{2n}^{\alpha}}{(n-1)^{\alpha/2}} < \frac{\sqrt{\pi}}{2} \sum_{n=2}^{\infty} \frac{1}{(n-1)^{\alpha}}.$$

Hence

$$\sum_{n=2}^{\infty} \frac{\rho^{n-1}}{(n-1)^{\alpha/2}} \cdot a_{2n} < \frac{\sqrt{\pi}}{2} \zeta(\alpha).$$

2.

$$\sum_{n=2}^{\infty} a_{2n} = \sum_{n=2}^{\infty} \frac{A_{2n}^{\alpha}}{\rho^{n-1}} < A_4^{\alpha} \sum_{n=1}^{\infty} \frac{1}{\rho^n} = (\frac{2}{3})^{\alpha} \frac{1}{\rho-1}.$$

Collecting all the terms gives Corollary 3.

Application to $\alpha = 2$. In this case, the poles have the following expression (Stieltjes (1918), p. 551):

$$s_n = \left[\frac{(2n-1)\pi}{2K(\rho)} \right]^2 \quad n = 1, 2, \dots$$

where

$$K(\rho) = \int_0^{\pi/2} \left[1 - \frac{\sin^2 \theta}{\rho} \right]^{-\frac{1}{2}} d\theta.$$

Let us compare s_1 with our bound, for $\rho = 2$. We obtain

$$s_1 = \frac{4\pi^3}{\Gamma^4(1/4)} \Rightarrow s_1 \approx 0.72$$

and on the other hand

$$\frac{1}{m} \approx 0.30.$$

To conclude this section, let us propose the following conjecture.

Conjecture. If $\lambda_n \sim_{n \rightarrow \infty} n^\alpha$
 $\mu_n \sim_{n \rightarrow \infty} \mu n^\alpha \quad \mu \neq 1, \quad \alpha > 0$

the poles are asymptotically proportional to n^α , i.e. $s_n \sim_{n \rightarrow \infty} C \cdot n^\alpha$ where the constant C is a function only of μ .

This conjecture is supported by several facts.

1. It is known to be true for $\alpha = 1$. For $\alpha = 2$, it is also verified in all instances where the spectrum is known exactly: see the previous example and also the finite case of Roehner and Valent (1982).

2. (a) The series

$$\sum_{n=2}^{\infty} \frac{1}{s_n}$$

converges if $\alpha > 1$. (This has been shown, at least for particular forms of λ_n and μ_n , in Corollaries 2 and 3.)

(b) The series

$$\sum_{n=2}^{\infty} \frac{1}{s_n^2}$$

converges if $\alpha > \frac{1}{2}$ (Ledermann and Reuter (1954)).

3. Finally, the poles may be computed numerically as eigenvalues of the finite matrix obtained by cutting the infinite matrix at some sufficiently high rank. This computation supports the conjecture.

6. Application to non-asymptotically-proportional processes

The processes

$$\begin{aligned} \lambda_n &= \lambda n + l \\ \mu_n &= \mu n^2 \end{aligned}$$

correspond to the deterministic equation $dx/dt = \lambda x - \mu x^2 + l$ which is the logistic equation. They are therefore given the same name.

Corollary 4. For the process

$$\begin{aligned}\lambda_n &= \lambda(n+1)^\alpha & \lambda > 1 \\ \mu_n &= n^\beta & \beta > \alpha \geq 1 & n \geq 0\end{aligned}$$

the spectrum is discrete with the following lower bound:

$$\begin{aligned}s_1 &= 0, & s_2 > \frac{1}{m} & m = \frac{dF(\lambda; \alpha, \beta)}{d\lambda} + \frac{\tilde{n}}{\lambda} F(\lambda; \alpha, \beta) + \lambda \zeta(2\beta - \alpha - 1) + \zeta(\beta) \\ F(\lambda; \alpha, \beta) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^{\beta-\alpha}}.\end{aligned}$$

\tilde{n} is the position of the minimum of the sequence $((n-1)!)^{\beta-\alpha}/\lambda^n n^\alpha$. In particular:

1. If $\alpha = 1, \beta = 2$ (logistic process)

$$m = e^\lambda \left(1 + \frac{\tilde{n}}{\lambda}\right) + (1 + \lambda) \frac{\pi^2}{6} \quad \tilde{n} = \frac{1}{2} \left(\sqrt{1 + \frac{4}{\lambda}} - 1 \right).$$

2. If $\alpha = 0, \beta = 2$

$$m = \frac{I_{-1}(2\sqrt{\lambda}) + I_1(2\sqrt{\lambda})}{2\sqrt{\lambda}} + \frac{I_0(2\sqrt{\lambda})}{\sqrt{\lambda}} + \lambda \zeta(3) + \frac{\pi^2}{6},$$

where $I_\nu(x)$ is the Bessel function.

Proof. The corresponding a_n are

$$a_{2n+1} = \frac{\lambda^n}{(n!)^{\beta-\alpha}} \quad n \geq 0; \quad a_{2n} = \frac{[(n-1)!]^{\beta-\alpha}}{\lambda^n n^\alpha} \quad n \geq 1,$$

and in this case, C will converge. Since we shall not try to obtain the 'best' possible bound, let us use the less accurate (but simpler) result $s_2 \geq 1/C$. We denote by \tilde{n} the value of n for which the sequence a_{2n} is minimum. We decompose C into a sum and a series:

$$\begin{aligned}C &= C_1 + C_2 \\ C_1 &= \sum_{n=1}^{\tilde{n}} a_{2n+1} \sum_{k=1}^n a_{2k} & C_2 &= \sum_{n=\tilde{n}+1}^{\infty} a_{2n+1} \sum_{k=1}^n a_{2k} \\ C_1 &< \frac{1}{\lambda} \sum_{n=1}^{\tilde{n}} n a_{2n+1} & & < \frac{1}{\lambda} \sum_{n=1}^{\infty} n \cdot a_{2n+1} = \lambda \frac{d}{d\lambda} F(\lambda; \alpha, \beta) \\ C_2 &= \sum_{n=\tilde{n}+1}^{\infty} a_{2n+1} \sum_{k=1}^{\tilde{n}} a_{2k} + \sum_{n=\tilde{n}+1}^{\infty} a_{2n+1} \sum_{k=\tilde{n}+1}^n a_{2k}.\end{aligned}$$

The first term can be estimated as in C_1 and in the second we take the term a_{2n} apart. Thus

$$C_2 < \frac{\tilde{n}}{\lambda} \sum_{n=\tilde{n}+1}^{\infty} a_{2n+1} + \sum_{n=\tilde{n}+1}^{\infty} (n-\tilde{n})a_{2n+1}a_{2(n-1)} + \sum_{n=\tilde{n}+1}^{\infty} \frac{1}{\mu_n}.$$

The second and third terms give rise to the zeta functions appearing in the expression of Corollary 4.

Particular cases. In the first case, $F(\lambda; \alpha, \beta)$ is merely an exponential and in the second we have (Gradshteyn and Ryzhik (1965), p. 961):

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^2} = I_0(2\sqrt{\lambda})$$

and (Gradshteyn and Ryzhik (1965), p. 970):

$$\frac{d}{d\lambda} I_0(2\sqrt{\lambda}) = \frac{I_{-1}(2\sqrt{\lambda}) + I_1(2\sqrt{\lambda})}{2\sqrt{\lambda}}.$$

7. Conclusion

We have obtained bounds for the spectra of very different processes; let us, however, recall here the limitations of the method. These are:

1. The condition $\lambda_0 \neq 0$ (the condition $\mu_0 = 0$, on the other hand, seems a natural one).
2. The condition that one of the series A , B or C converges. This excludes in particular the processes where $\mu_n \sim_{n \rightarrow \infty} n^\alpha$ $\alpha \leq 1$.

On the other hand, the basic property that the poles are the roots of a polynomial (tending to a series) with converging and positive coefficients may probably be used to extract additional information (besides the bounds) about the spectrum.

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Appendix

Since Stieltjes's work is written in French, we have attempted to make the present paper almost self-contained by giving here a brief account of the results used. All the page numbers in this appendix refer to Stieltjes (1918).

1. Let us first of all recall the general notion of the even (or odd, respectively) part of a continued fraction

$$(A.1) \quad \underline{a_1}/b_1 + \underline{a_2}/b_2 + \dots$$

The quantity

$$\frac{P_n}{Q_n} = \underline{a_1}/b_1 + \underline{a_2}/b_2 + \dots + \underline{a_n}/b_n$$

is called the n th approximant of the continued fraction (A.1). The limit, if it exists, of the sequence

$$\frac{P_{2n}}{Q_{2n}} \quad \text{or} \quad \frac{P_{2n+1}}{Q_{2n+1}}$$

is called the even (or odd) part of the fraction (A.1).

2. Stieltjes developed a theory for the fraction

$$(3) \quad \underline{1}/a_1 s + \underline{1}/a_2 s + \underline{1}/a_3 s + \dots \quad a_k > 0.$$

(a) A first general result concerns the poles of the approximants. The zeros of $Q_{2n}(s)$ and $Q_{2n+1}(s)$ are real, simple and negative (p. 411).

(b) Let us now introduce the following notations for the numerator and denominator of the approximants:

$$(A.2) \quad Q_{2n}(s) = 1 + B_1^{(2n)} s + \dots + B_n^{(2n)} s^n = \prod_{k=1}^n \left(1 + \frac{s}{s_k^{(2n)}} \right)$$

$$P_{2n+1}(s) = 1 + C_1^{(2n+1)} s + \dots + C_n^{(2n+1)} s^n$$

(A.3)

$$Q_{2n+1}(s) = D_1^{(2n+1)} s + D_2^{(2n+1)} s^2 + \dots + D_{n+1}^{(2n+1)} s^{n+1} = D_1^{(2n+1)} s \cdot \prod_{k=2}^{n+1} \left(1 + \frac{s}{s_k^{(2n+1)}} \right).$$

The following expressions hold:

$$(A.4) \quad \begin{aligned} B_1^{(2n)} &= \sum_{k=1}^n (a_1 + \dots + a_{2k-1}) a_{2k} \\ C_1^{(2n+1)} &= \sum_{k=1}^n (a_2 + \dots + a_{2k}) a_{2k+1} \end{aligned}$$

$$(A.5) \quad D_1^{(2n+1)} = \sum_{k=1}^n a_{2k+1} \quad D_2^{(2n+1)} = \sum_{l=1}^n a_{2l+1} \sum_{k=1}^l (a_1 + \dots + a_{2k-1}) a_{2k}.$$

3. Two different cases occur in the theory of the S -fraction (3).

Case 1. If the series

$$S = \sum_{k=1}^{\infty} a_n$$

converges, the four polynomials $P_{2n}(s)$, $Q_{2n}(s)$, $P_{2n+1}(s)$, $Q_{2n+1}(s)$ admit finite

limits $p(s)$, $q(s)$, $p_1(s)$, $q_1(s)$ which are analytic functions in the whole complex plane.

Let s_k denote the limits of the roots $s_k^{(2n)}$. Then (p. 446):

$$\frac{p(s)}{q(s)} = \sum_{k=1}^{\infty} \frac{r_k}{s + s_k}.$$

The s_k are real, simple, positive and such that the series

$$\sum_{k=1}^{\infty} \frac{1}{s_k}$$

converges.

Case 2. If the series S diverges, the odd and the even parts of the fraction have the same limit which is an analytic function in the complex plane deprived of the negative real axis.

4. *Particular cases* (p. 525). Suppose that S diverges. Then, the polynomials $P_n(s)$ and $Q_n(s)$ do not in general admit a finite limit. There are however two cases (and only two) where the odd (or even) polynomials still admit a finite limit.

(a) If the series

$$B = \sum_{n=1}^{\infty} (a_1 + \dots + a_{2n-1}) a_{2n}$$

converges, $Q_{2n}(s)$ has a finite limit $Q(s)$. The zeros of $Q(s)$ have the same properties as in Case 1.

Remark. The convergence of B implies the convergence of

$$\sum_{n=1}^{\infty} a_{2n}.$$

Since S should diverge, so does the series $\sum_{n=1}^{\infty} a_{2n+1}$.

(b) If the series

$$C = \sum_{n=1}^{\infty} (a_2 + \dots + a_{2n}) a_{2n+1}$$

converges, $P_{2n+1}(s)$ has a finite limit. Hence, $Q_{2n+1}(s)$ has a finite limit too and the zeros of the limit $Q(s)$ have the same properties as in Case 1. Moreover, because of the form (A.3) of $Q_{2n+1}(s)$, the odd approximants all admit the simple pole $s_1 = 0$.

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