Exactly solvable models of tilings and Littlewood–Richardson coefficients

P. Zinn-Justin

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Outline of the talk

1. Introduction
2. Lozenge tilings and Schur functions
   - Plane partitions, lozenge tilings
   - NILPs and Fermionic Fock space
   - Schur functions and skew-Schur functions
3. Square-triangle-rhombus tilings and LR coefficients
   - Interacting fermions
   - Puzzles and square-triangle tilings
   - A new “integrable” proof
4. Inhomogeneities and equivariance
   - Cohomology of Grassmannians and Schur functions
   - MS-alt puzzles, Equivariant puzzles
   - Another “integrable” proof
5. Conclusion and prospects
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5. Conclusion and prospects
Random tilings

- Random tilings are simple models whose main purpose is to describe quasi-crystals.
- They typically correspond to a high-temperature limit where entropy considerations dominate.
- All (known) random tiling models can be thought of as fluctuating surfaces (i.e. bosonic fields) in a higher-dimensional space.
- Typical configurations may have “forbidden” symmetries. For example, the square/triangle model has 12-fold symmetry!
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Schur functions are the most important family (basis) of symmetric functions in algebraic combinatorics.

- They are also characters of $GL(N)$.
- They form bases of the cohomology ring of Grassmannians. (related to Schubert varieties)
- Littlewood–Richardson coefficients are structure constants of the algebra of Schur functions.
- Geometrically, they correspond to intersection theory on Grassmannians.
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Plane partitions
Plane partitions
Lozenge tilings
Non-Intersecting Lattice Paths
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Define a *partition* to be a weakly decreasing finite sequence of non-negative integers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We usually represent partitions as *Young diagrams*: for example $\lambda = (5, 2, 1, 1)$ is depicted as

\[
\lambda = \begin{array}{cccc}
\square & \square & \square & \\
\square & \square & \\
\square & \\
\end{array}
\]
To each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ one associates a fermionic state $|\lambda\rangle$ so that the black (resp. red) sites correspond to vertical (resp. horizontal) edges:

$$F = \bigoplus_\lambda \mathbb{C} |\lambda\rangle$$

is the fermionic Fock space (with charge 0).
Definition of Schur polynomials

To a pair of Young diagrams $\lambda$, $\mu$ one associates the skew Schur polynomial $s_{\lambda/\mu}(x_1, \ldots, x_n)$:

The (usual) Schur polynomial is $s_\lambda = s_{\lambda/\emptyset}$.

Remark: the number of plane partitions in $a \times b \times c$ is $s_{[a \times c]}(x_1 = \cdots = x_{a+b} = 1)$. 
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\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array}$

$\mu = \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}$

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Solvable tilings and Littlewood–Richardson coefficients
Example

\[ s_{\square}(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2 \]
Consider the operator $T(x)$ on $\mathcal{F}$ with matrix elements

$$\langle \mu | T(x) | \lambda \rangle = s_{\lambda/\mu}(x)$$

It corresponds to the addition of one row of the tiling. In particular

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = \langle \mu | T(x_1) \ldots T(x_n) | \lambda \rangle$$
Properties

• “Integrability” property:

\[ [T(x), T(x')] = 0 \Rightarrow s_{\lambda/\mu} \text{ symmetric polynomial} \]

• Stability property:

\[ T(0) = 1 \Rightarrow s_{\lambda/\mu}(x_1, \ldots, x_n, x_{n+1} = 0) = s_{\lambda/\mu}(x_1, \ldots, x_n) \]

Thus, the \( s_{\lambda/\mu} \) are symmetric functions (symmetric polynomials in an infinite number of variables).

In fact, the \( s_{\lambda} \) are known to be a basis of the space of symmetric functions (which is thus isomorphic to \( \mathcal{F} \)).
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Thus, the \( s_{\lambda/\mu} \) are symmetric functions (symmetric polynomials in an infinite number of variables). In fact, the \( s_{\lambda} \) are known to be a basis of the space of symmetric functions (which is thus isomorphic to \( \mathcal{F} \)).
Some identities

- An identity that can be derived using the formalism above:

\[
\sum_{\mu} s_{\lambda/\mu}(x_1, \ldots, x_n)s_{\mu/\rho}(y_1, \ldots, y_m) = s_{\lambda/\rho}(x_1, \ldots, x_n, y_1, \ldots, y_m)
\]

- Identities which remain mysterious:

\[
s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{\nu} c_{\lambda,\mu}^{\nu}s_{\nu}(x_1, \ldots, x_n)
\]

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\]
Two species of fermions

Pilings of (hyper)cubes in \textit{four} dimensions!
Two species of fermions

Pilings of (hyper)cubes in four dimensions!
Two species of fermions

Pilings of (hyper)cubes in *four* dimensions!
The interaction
Theorem

If \( x + y + z = 0 \), then

\[
\begin{array}{c}
\text{z} \\
\text{x} \\
\text{y}
\end{array}
\begin{array}{c}
\text{y}
\end{array}
\begin{array}{c}
\text{z}
\end{array} =
\begin{array}{c}
\text{x}
\end{array}
\begin{array}{c}
\text{y}
\end{array}
\begin{array}{c}
\text{z}
\end{array}
\]

for any fixed boundaries and where tile \( x \) (resp. \( y, z \)) is only allowed where marked.
**Theorem**

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\begin{array}{c}
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**Example:**

\[
\begin{array}{c}
\text{+} \\
\text{+} \\
\text{=} \text{0}
\end{array}
\]
**Yang–Baxter equation**

**Theorem**

If \( x + y + z = 0 \), then

\[
\begin{array}{c}
\begin{array}{ccc}
    z & y & x \\
    x & y & z \\
\end{array}
\end{array}
\]

= \[
\begin{array}{c}
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    x & y & z \\
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\begin{array}{c}
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\text{ } & \text{ } & \text{ } \\
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+ \[
\begin{array}{c}
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\end{array}
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= 0
Puzzles

Remove all tiles $x$, $y$, $z$:
Some history... 

- 1993: M. Widom introduces the square-triangle model, deforms it into a regular triangular lattice (∼ puzzles) and proves integrability.
- 1994: P. Kalugin (partially) solves the Coordinate Bethe Ansatz equations (size→ ∞).
- 2008: K. Purbhoo reformulates puzzles as mosaics (∼ square-triangle tilings).
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- **2003–2004:** A. Knutson, T. Tao and C. Woodward reexpress it in terms of puzzles.
- **2008:** K. Purbhoo reformulates puzzles as mosaics (≈ square-triangle tilings).
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Interacting fermions
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A new “integrable” proof

\[ \sum_{\lambda, \mu, \nu} c_{\mu, \nu}^\lambda s_{\mu}(\tilde{x}^{-1}) s_{\nu}(y^{-1}) \]
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$s_{\lambda}(\tilde{x}^{-1}, y^{-1})$

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Solvable tilings and Littlewood–Richardson coefficients
Cohomology of Grassmannians

The cohomology ring of $Gr(n,k) = \{ V \subset \mathbb{C}^n, \dim V = k \}$ is the quotient of the ring of symmetric functions by the span of the $s_\lambda$, $\lambda \not\subset [k \times (n-k)]$.

Given a fixed flag, one can build Schubert varieties indexed by $\lambda \subset [k \times (n-k)]$ such that the $s_\lambda$ are their cohomology classes. There is a torus $T = (\mathbb{C}^\times)^n$ acting on $Gr(n,k)$ and a corresponding equivariant cohomology ring. It is a module over $\mathbb{Z}[y_1, \ldots, y_n]$, with basis the $\tilde{s}_\lambda$, $\lambda \subset [k \times (n-k)]$.

If flag and torus are compatible (so that the Schubert varieties are $T$-invariant), the $\tilde{s}_\lambda$ are the equivariant cohomology classes of the Schubert varieties.
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There is a torus \( T = (\mathbb{C}^\times)^n \) acting on \( Gr(n, k) \) and a corresponding equivariant cohomology ring. It is a module over \( \mathbb{Z}[y_1, \ldots, y_n] \), with basis the \( \tilde{s}_\lambda, \lambda \subset [k \times (n - k)] \).

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Double Schur functions

The $\tilde{s}_\lambda$ can be represented as polynomials $s_\lambda(x_1, \ldots, x_n|y_1, \ldots, y_n)$.
(such that $s_\lambda(x_1, \ldots, x_n|0, \ldots, 0)_k = s_\lambda(x_1, \ldots, x_n)$).
Product formulae

- **Knutson–Tao problem:**

\[
s_\lambda(x_1, \ldots, x_k|z_1, \ldots, z_n)s_\mu(x_1, \ldots, x_k|z_1, \ldots, z_n) = \sum_{\nu} c_{\nu,\lambda}^{\mu}(z_1, \ldots, z_n)s_\nu(x_1, \ldots, x_k|z_1, \ldots, z_n)
\]

- **Molev–Sagan problem:**

\[
s_\lambda(x_1, \ldots, x_k|z_1, \ldots, z_n)s_\mu(x_1, \ldots, x_k|y_1, \ldots, y_n) = \sum_{\nu} e_{\nu,\lambda}^{\mu}(y_1, \ldots, y_n; z_1, \ldots, z_n)s_\nu(x_1, \ldots, x_k|y_1, \ldots, y_n)
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Unifying solution of these two problems!
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Solvable tilings and Littlewood–Richardson coefficients

P. Zinn-Justin
\[ s_\lambda(x|z)s_\mu(x|y) = \sum e^\nu_{\lambda,\mu}(y;z)s_\nu(x|y) \]
“Integrable” proofs of combinatorial identities?

- Coproduct formula for double Schur functions?
- Use of Bethe Ansatz?
- Generalization to other families of symmetric polynomials? (Jack, Hall–Littlewood, Macdonald)
- Generalization to other families of polynomials of geometric origin? (Schubert, Grothendieck)
- Application to FPLs / Razumov–Stroganov conjecture? (cf Nadeau’s talk)
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