Baxter's Q-operator
(for the open XXZ Heisenberg chain with diagonal boundaries)

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Based on joint work with Robert Weston (in progress, on arXiv soon)

Heisenberg XXZ spin chains

Let $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \cong \mathbb{C}^2$. Quantum-mechanical state space of spin-$\frac{1}{2}$ chain with N sites:

$$V^{\otimes N} = V \otimes \cdots \otimes V.$$

Quantum Hamiltonian of Heisenberg XXZ spin$-\frac{1}{2}$ chain (nearest-neighbour interaction):

$$H \propto \sum_{n=1}^{N-1} \left(\sigma_n^{\mathrm{x}} \sigma_{n+1}^{\mathrm{x}} + \sigma_n^{\mathrm{y}} \sigma_{n+1}^{\mathrm{y}} + \frac{q+q^{-1}}{2} \sigma_n^{\mathrm{z}} \sigma_{n+1}^{\mathrm{z}} \right) + \text{boundary terms}$$
$$\in \mathrm{End}(V^{\otimes N})$$

where

- $\sigma^{\mathrm{x}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^{\mathrm{y}} = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$, $\sigma^{\mathrm{z}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$;
- subscripts indicate in which tensor factor the operators $\sigma^{\mathrm{x,y,z}}$ act;
- $q \in \mathbb{C}^{\times}$ parametrizes the degree of isotropy - we assume $|q| < 1$.

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1. Construct $U^V(z) \in \mathrm{End}(V \otimes V^{\otimes N})$ (auxiliary copy of V, parameter $z \in \mathbb{C}$) by composing operators each of which acts nontrivially on tensor products of auxiliary V and at most one of the other Vs.

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2. Define transfer matrices

$$T(z) = \operatorname*{Tr}_V U^V(z) \in \mathrm{End}(V^{\otimes N}).$$

For "nice" and well-chosen constituent operators we have

$$[T(y), T(z)] = 0, \qquad H \propto \left(T(z)^{-1} \tfrac{\mathrm{d}}{\mathrm{d}z} T(z)\right)|_{z=1}.$$

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3. (*Algebraic Bethe ansatz*) Decompose $U^V(z)$ w.r.t. auxiliary V $U^V(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ with $A(z), B(z), C(z), D(z) \in \mathrm{End}(V^{\otimes N})$; derive commutation relations; *hope you have a joint eigenvector* v_0 *of* $A(z)$ *and* $D(z)$; show that $B(z_1)\cdots B(z_M)v_0$ is an eigenvector of $T(z) = A(z) + D(z)$ subject to cancellation of "unwanted terms"; show this cancellation is equivalent to a set of equations on $z_1, \ldots, z_M$: Bethe ansatz equations.

For step 1 and 2 also cf. "Keeler's Theorem" from [Futurama S6E10 (2010)]

Let $\widetilde{\beta}, \beta \in \mathbb{C}$. We will consider the open XXZ chain with diagonal boundaries:

$$H \propto \widetilde{\beta}\sigma_1^{\mathrm{z}} + \sum_{i=1}^{N-1} \left(\sigma_i^{\mathrm{x}} \sigma_{i+1}^{\mathrm{x}} + \sigma_i^{\mathrm{y}} \sigma_{i+1}^{\mathrm{y}} + \frac{q+q^{-1}}{2} \sigma_i^{\mathrm{z}} \sigma_{i+1}^{\mathrm{z}} \right) + \beta \sigma_N^{\mathrm{z}}.$$

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Sklyanin defined the *two-row transfer matrix*

$$T(z) = \operatorname*{Tr}_V \widetilde{K}_a^V(z) R_{1a}(z) \cdots R_{Na}(z) K_a^V(z) R_{aN}(z) \cdots R_{a1}(z)$$

(the auxiliary space has the label a). Then $[T(y), T(z)] = 0$ if

- $R(z) \in \mathrm{End}(V \otimes V)$ satisfies the Yang-Baxter equation
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Moreover, for the open XXZ chain with diagonal boundaries, choosing $R(z)$ to be the quantum affine $\mathfrak{sl}_2$ R-matrix and $K^V(z), \widetilde{K}^V(z)$ particular diagonal matrices, we also have $H = \frac{\mathrm{d}}{\mathrm{d}z} \log T(z)|_{z=1}$ and ABA [Sklyanin (1988)].

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Sklyanin's ABA cannot be done for the most general $K^V(z), \widetilde{K}^V(z)$.

An alternative method: Baxter's Q-operator

Suppose we have another family $\{Q(z) \in \mathrm{End}(V^{\otimes N})\}_{z \in \mathbb{C}}$ such that

- $$[T(y), T(z)] = [T(y), Q(z)] = [Q(y), Q(z)] = 0;$$
- $Q(z)$ and $T(z)$ are diagonalizable and entire functions of z (hence the eigenvalues of $Q(z)$ and $T(z)$ are entire functions of z);
- Baxter's TQ-relation holds
$$T(z)Q(z) = \alpha_+(z)Q(pz) + \alpha_-(z)Q(p^{-1}z)$$
for some $p \in \mathbb{C}^\times$ and $\alpha_+(z), \alpha_-(z) \in \mathbb{C}$ (meromorphic in z).

Then one can derive *equations for the zeroes of the eigenvalues of* $Q(z)$ in terms of their Weierstrass factorization and $\alpha_\pm(z)$.

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Q-operator according to Bazhanov, Lukyanov & Zamolodchikov (1996)

Mimic the construction of $T(z)$, i.e.

$$Q(z) = \operatorname*{Tr}_W \left(\text{factorized linear map on } W \otimes V^{\otimes N}\right)$$

with W an infinite-dimensional vector space.

Open problems

- Compare with [Frassek & Szécsényi (2015)] for the Q-operator for the open XXX chain (with diagonal boundaries).
- Extend to a pair $(Q(z), \tilde{Q}(z))$ and express $T(z)$ as a polynomial in these (Q-operators are fundamental objects).
- Connect with other representation-theoretic approaches to the Q-operator, in particular "asymptotic algebra" and "prefundamental representations" [Hernandez & Jimbo (2012); Frenkel & Hernandez (2015)].
- Generalize to other coideal subalgebras of $\mathcal{U}_q$; in particular the ones with nondiagonal $K(z)$, $\widetilde{K}(z)$.

Today: derivation of the TQ-relation

How do you derive something like

$$T(z)Q(z) = \alpha_+(z)Q(pz) + \alpha_-(z)Q(p^{-1}z) \quad ?$$

Note: $T(z) = \mathrm{Tr}_V(\text{operator})$, $Q(z) = \mathrm{Tr}_W(\text{operator})$.

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How do you derive something like

$$\operatorname*{Tr}_{V\otimes W}(\text{some operator}) = \operatorname*{Tr}_{W}(\text{another operator}) + \operatorname*{Tr}_{W}(\text{yet another operator}) \quad ?$$

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Proposition (Decomposition of a trace using a short exact sequence)

Consider a short exact sequence of vector spaces:

$$0 \longrightarrow L \stackrel{\iota}{\longrightarrow} M \stackrel{\tau}{\longrightarrow} L' \longrightarrow 0$$

Let $\phi \in \mathrm{End}(M)$. *If all traces are well-defined we have*

$$\operatorname*{Tr}_{M} \phi = \operatorname*{Tr}_{L} \iota^{-1} \circ \phi \circ \iota + \operatorname*{Tr}_{L'} \tau \circ \phi \circ \tau^{-1}$$

where ι^{-1} *is a left-inverse of* ι *and* τ^{-1} *is a right-inverse of* τ. *If also*

$\exists \psi \in \mathrm{End}(L): \quad \phi \circ \iota = \iota \circ \psi$ *and* $\exists \psi' \in \mathrm{End}(L'): \quad \tau \circ \phi = \psi' \circ \tau$

then $\mathrm{Tr}_M \phi = \mathrm{Tr}_L \psi + \mathrm{Tr}_{L'} \psi'$.

Definition (Quantum affine $\mathfrak{sl}_2$)

Let $\mathcal{U}_q$ be the algebra with generators E_i, F_i and invertible k_i ($i \in \{0,1\}$) and relations

$$k_i E_i = q^2 E_i k_i, \quad k_i F_i = q^{-2} F_i k_i, \quad E_i F_i - F_i E_i = \frac{k_i - k_i^{-1}}{q - q^{-1}}$$

$$\left.\begin{array}{cc} k_i k_j = k_j k_i & E_i F_j = F_j E_i \\ k_i E_j = q^{-2} E_j k_i & k_i F_j = q^2 F_j k_i \\ \multicolumn{2}{c}{\text{q-Serre relations}} \end{array}\right\} \text{if } j \neq i.$$

$\mathcal{U}_q$ is a Hopf algebra. In particular we have an algebra homomorphism $\Delta : \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$

$$\Delta(E_i) = E_i \otimes 1 + k_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes k_i^{-1} + 1 \otimes F_i, \quad \Delta(k_i) = k_i \otimes k_i.$$

Let $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \cong \mathbb{C}^2$. For $z \in \mathbb{C}^\times$ define the *evaluation representation* w.r.t. "principal grading"

$$\pi_z : \mathcal{U}_q \to \mathrm{End}(V)$$

by

$$E_0 \mapsto \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \quad F_0 \mapsto \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix} \quad k_0 \mapsto q^{\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$
$$E_1 \mapsto \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \quad F_1 \mapsto \begin{pmatrix} 0 & 0 \\ z^{-1} & 0 \end{pmatrix} \quad k_1 \mapsto q^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

Note: $\pi_z(E_0) = \pi_{z^{-1}}(F_1)$ and $\pi_z(E_1) = \pi_{z^{-1}}(F_0)$.

More standard choice ("homogeneous grading")

$$E_0 \mapsto \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \quad F_0 \mapsto \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix} \quad k_0 \mapsto q^{\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$
$$E_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F_1 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad k_1 \mapsto q^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

is less suitable here.

The quantum Borel subalgebras

$$\mathcal{U}_q^+ = \langle E_0, E_1, k_0^{\pm 1}, k_1^{\pm 1} \rangle, \qquad \mathcal{U}_q^- := \langle F_0, F_1, k_0^{\pm 1}, k_1^{\pm 1} \rangle$$

are Hopf subalgebras of $\mathcal{U}_q$.

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Consider

$$W = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathbb{C} w_j = \mathbb{C} w_0 \oplus \mathbb{C} w_1 \oplus \cdots$$

Define linear maps on W as follows:

$$a^\dagger(w_j) = (1 - q^{2(j+1)}) w_{j+1}, \qquad a(w_j) = w_{j-1}, \qquad f(D)(w_j) = f(j) w_j$$

for any function $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ (we have set $w_{-1} = 0$). For $r, z \in \mathbb{C}^\times$, define the representation $\rho^+_{z,r} : \mathcal{U}_q^+ \to \mathrm{End}(W)$ by

$$E_0 \mapsto \frac{z}{q^2 - 1} a^\dagger, \quad E_1 \mapsto \frac{z}{1 - q^{-2}} a, \quad k_0 \mapsto r q^{2D}, \quad k_1 \mapsto r^{-1} q^{-2D}.$$

Recall that $\pi_z(E_0) = \pi_{z^{-1}}(F_1)$ and $\pi_z(E_1) = \pi_{z^{-1}}(F_0)$. It would be nice to have an algebra automorphism ψ_q of $\mathcal{U}_q$ such that

$$\pi_z = \pi_{z^{-1}} \circ \psi_q, \qquad \psi_q(\mathcal{U}_q^{\pm}) = \mathcal{U}_q^{\mp}.$$

Then we could define a "compatible" representation of $\mathcal{U}_q^-$ on W via

$$\rho^-_{z,r} := \rho^+_{z^{-1},r^{-1}} \circ \psi_q.$$

There are actually many such ψ_q. How do we decide what is the best $\rho^-_{z,r}$? We'll want to have some nice intertwiners

Intertwiners for $\mathcal{U}_q^+$

Lemma

There are $\mathcal{U}_q^+$-intertwiners (unique up to a scalar)

$$\begin{aligned} \iota^+(r) : \quad & (W, \rho^+_{qz,qr}) \to (W \otimes V, \rho^+_{z,r} \otimes \pi_z) \\ \tau^+(r) : \quad & (W \otimes V, \rho^+_{z,r} \otimes \pi_z) \to (W, \rho^+_{q^{-1}z,q^{-1}r}). \end{aligned}$$

W.r.t. the basis (v_0, v_1) of V they are given by

$$\iota^+(r) = \begin{pmatrix} q^{-D} a^\dagger \\ q^{D+1} r \end{pmatrix} \qquad \tau^+(r) = \begin{pmatrix} q^D & -q^{-D} r^{-1} a^\dagger \end{pmatrix}.$$

Moreover, we have the short exact sequence

$$(W, \rho^+_{qz,qr}) \overset{\iota^+(r)}{\hookrightarrow} (W \otimes V, \rho^+_{z,r} \otimes \pi_z) \overset{\tau^+(r)}{\twoheadrightarrow} (W, \rho^+_{q^{-1}z,q^{-1}r})$$

for all $r, z \in \mathbb{C}^\times$.

Note that $\mathcal{U}_q$ is quasitriangular w.r.t. category of level-0 representations. Universal R-matrix $\mathcal{R}$ lying in (completion of) $\mathcal{U}_q^+ \otimes \mathcal{U}_q^-$ [Khoroshkin & Tolstoy, 1991].

In particular, for $z = z_1/z_2$, define

$$R(z) := \text{convenient scalar} \times (\pi_{z_1} \otimes \pi_{z_2})(\mathcal{R})$$
$$= \begin{pmatrix} 1-q^2z^2 & 0 & 0 & 0 \\ 0 & q(1-z^2) & (1-q^2)z & 0 \\ 0 & (1-q^2)z & q(1-z^2) & 0 \\ 0 & 0 & 0 & 1-q^2z^2 \end{pmatrix} \in \mathrm{End}(V \otimes V)$$

$$L^+(z,r) := \text{convenient scalar} \times (\rho^+_{z_1,r} \otimes \pi_{z_2})(\mathcal{R})$$
$$= \begin{pmatrix} 1 & -q^{-1}za^\dagger \\ -qza & 1-q^{2(D+1)}z^2 \end{pmatrix} \begin{pmatrix} q^D r & 0 \\ 0 & q^{-D} \end{pmatrix} \in \mathrm{End}(W \otimes V).$$

For any two vector spaces V_1, V_2 define $P : V_1 \otimes V_2 \to V_2 \otimes V_1$ by $P(v_1 \otimes v_2) = v_2 \otimes v_1$ for $v_i \in V_i$. We have a $\mathcal{U}_q$-intertwiner $\check{R}(z)$ and a $\mathcal{U}_q^+$-intertwiner $\check{L}^+(z, r)$:

$$\begin{aligned} \check{R}(\tfrac{z_1}{z_2}) &:= PR(\tfrac{z_1}{z_2}) && : (V \otimes V, \pi_{z_1} \otimes \pi_{z_2}) \to (V \otimes V, \pi_{z_2} \otimes \pi_{z_1}) \\ \check{L}^+(\tfrac{z_1}{z_2}, r) &:= PL^+(\tfrac{z_1}{z_2}, r) && : (W \otimes V, \rho^+_{z_1,r} \otimes \pi_{z_2}) \to (V \otimes W, \pi_{z_2} \otimes \rho^+_{z_1,r}) \end{aligned}$$

Also recall the $\mathcal{U}_q^+$-intertwiner

$$\iota^+(r) : (W, \rho^+_{qz,qr}) \to (W \otimes V, \rho^+_{z,r} \otimes \pi_z).$$

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$$\check{R}(\tfrac{z_1}{z_2}) := PR(\tfrac{z_1}{z_2}) \quad : (V \otimes V, \pi_{z_1} \otimes \pi_{z_2}) \to (V \otimes V, \pi_{z_2} \otimes \pi_{z_1})$$
$$\check{L}^+(\tfrac{z_1}{z_2}, r) := PL^+(\tfrac{z_1}{z_2}, r) \quad : (W \otimes V, \rho^+_{z_1,r} \otimes \pi_{z_2}) \to (V \otimes W, \pi_{z_2} \otimes \rho^+_{z_1,r})$$

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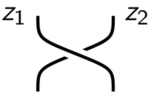
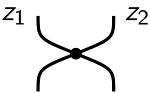
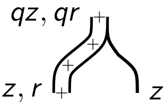
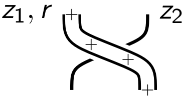
Pictures

$\check{R}(\frac{z_1}{z_2}) =$ 

(since $\check{R}(\frac{z_2}{z_1})^{-1} \propto \check{R}(\frac{z_1}{z_2})$, you *may* think of it as 

)

$\iota^+(r) =$ 

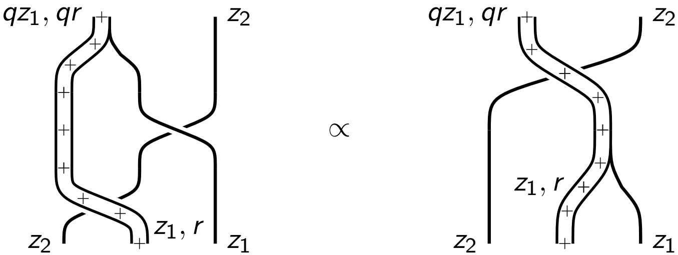
$\check{L}^+(\frac{z_1}{z_2}, r) =$ 

From the coproduct property of the universal R-matrix we obtain

$$(\check{L}^+(z,r)\otimes \mathrm{Id})(\mathrm{Id}\otimes\check{R}(z))(\iota^+(r)\otimes\mathrm{Id}) = (z^2-1)(\mathrm{Id}\otimes\iota^+(r))\check{L}^+(qz,qr)$$

$$(\mathrm{Id}\otimes\tau^+(r))(\check{L}^+(z,r)\otimes\mathrm{Id})(\mathrm{Id}\otimes\check{R}(z)) = q(q^2z^2-1)\check{L}^+(qz,qr)(\tau^+(r)\otimes\mathrm{Id})$$

where $z=z_1/z_2$. The former in pictures:



Intertwiners for $\mathcal{U}_q^-$.

We define an algebra automorphism ψ_q of $\mathcal{U}_q$ via the assignments

$$E_0 \mapsto q^{-1}k_1^{-1}F_1 \qquad F_0 \mapsto qE_1k_1 \qquad k_0 \mapsto k_1^{-1}$$
$$E_1 \mapsto q^{-1}k_0^{-1}F_0 \qquad F_1 \mapsto qE_0k_0 \qquad k_1 \mapsto k_0^{-1}.$$

It satisfies $\pi_{z^{-1}} = \pi_z \circ \psi_q$. Note that $\psi_q(\mathcal{U}_q^\pm) = \mathcal{U}_q^\mp$ so can construct a representation of $\mathcal{U}_q^-$ on W by $\rho^-_{z^{-1},r^{-1}} := \rho^+_{z,r} \circ \psi_q$. We have:

Lemma

There are $\mathcal{U}_q^-$-intertwiners (unique up to a scalar)

$$\iota^- : \quad (W, \rho^-_{q^{-1}z, q^{-1}r}) \to (V \otimes W, \pi_z \otimes \rho^-_{z,r})$$
$$\tau^- : \quad (V \otimes W, \pi_z \otimes \rho^-_{z,r}) \to (W, \rho^-_{qz,qr}).$$

They are given by

$$\iota^- = \begin{pmatrix} a^\dagger \\ q \end{pmatrix} \qquad \tau^- = \begin{pmatrix} 1 & -q^{-1}a^\dagger \end{pmatrix}.$$

Recall that $\mathcal{R} \in$ completion of $\mathcal{U}_q^+ \otimes \mathcal{U}_q^-$. Define, for $z = z_1/z_2$,

$$L^-(z,r) := \text{convenient scalar} \times (\pi_{z_1} \otimes \rho^-_{z_2,r^{-1}})(\mathcal{R})$$
$$= \begin{pmatrix} q^D r & 0 \\ 0 & q^{-D} \end{pmatrix} \begin{pmatrix} 1 & -q^{-1} z a^\dagger \\ -qza & 1 - q^{2(D+1)} z^2 \end{pmatrix} \in \mathrm{End}(V \otimes W).$$

Note $L^-(z,r) \neq PL^+(z',r')P$. We define the $\mathcal{U}_q^-$-intertwiner

$$\check{L}^-(z,r) = PL(z,r) : (V \otimes W, \pi_{z_1} \otimes \rho^-_{z_2,r^{-1}}) \to (W \otimes V, \rho^-_{z_2,r^{-1}} \otimes \pi_{z_1}).$$

We have, for $z = z_1/z_2$,

$$(\mathrm{Id} \otimes \check{L}^-(z,r))(\check{R}(z) \otimes \mathrm{Id})(\mathrm{Id} \otimes \iota^-) = (z^2 - 1)(\iota^- \otimes \mathrm{Id})\check{L}^-(qz,qr)$$
$$(\tau^- \otimes \mathrm{Id})(\mathrm{Id} \otimes \check{L}^-(z,r))(\check{R}(z) \otimes \mathrm{Id}) = q(q^2 z^2 - 1)\check{L}^-(qz,qr)(\tau^- \otimes \mathrm{Id})$$

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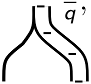
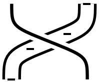
More pictures

$\iota^- =$ 

$\check{L}^-(\frac{z_1}{z_2}, r) =$ 

Reflection equations for the right boundary

Let $\xi \in \mathbb{C}$. Consider

$$K^V(z) = \begin{pmatrix} \xi z^2 - 1 & 0 \\ 0 & \xi - z^2 \end{pmatrix} \in \mathrm{End}(V)$$

$$K^W(z) = \prod_{i=1}^{D} (q^{2i} z^2 - \xi) \in \mathrm{End}(W).$$

$K^V(z)$ is the K-matrix used in [Sklyanin (1988)].

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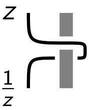
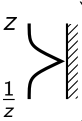
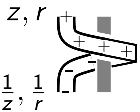
$K^V(z) =$ 

(since $K^V(z^{-1})^{-1} \propto K^V(z)$, you *may* think of it as 

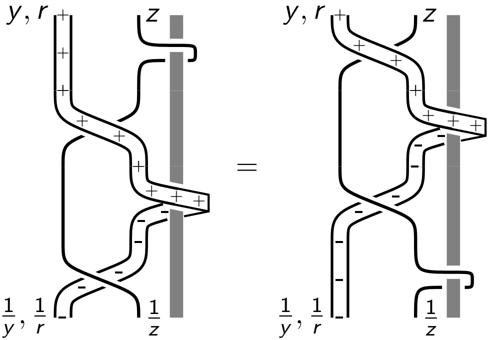
)

$K^W(z) =$ 

They satisfy

$$\check{R}(\tfrac{y}{z})(\mathrm{Id} \otimes K^V(y))\check{R}(yz)(\mathrm{Id} \otimes K^V(z)) =$$
$$= (\mathrm{Id} \otimes K^V(z))\check{R}(yz)(\mathrm{Id} \otimes K^V(y))\check{R}(\tfrac{y}{z}) \in \mathrm{End}(V \otimes V)$$

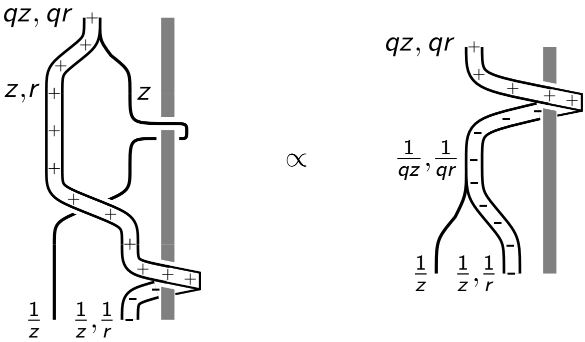
$$\check{L}^-(\tfrac{y}{z}, r)(\mathrm{Id} \otimes K^W(y))\check{L}^+(yz, r)(\mathrm{Id} \otimes K^V(z)) =$$
$$= (\mathrm{Id} \otimes K^V(z))\check{L}^-(yz, r)(\mathrm{Id} \otimes K^W(y))\check{L}^+(\tfrac{y}{z}, r) \in \mathrm{End}(W \otimes V)$$



See [Cherednik, 1992] for the first appearance of such "generalized reflection equations".

You should think of $K^W(z)$ as turning $\mathcal{U}_q^+$-modules into $\mathcal{U}_q^-$-modules in the following sense:

$$(\mathrm{Id} \otimes K^W(z))\check{L}^+(z^2, r)(\mathrm{Id} \otimes K^V(z))\iota^+(r) = r(z^4 - 1)(q^2z^2 - \xi)\iota^- K^W(qz)$$
$$\tau^-(\mathrm{Id} \otimes K^W(z))\check{L}^+(z^2, r)(\mathrm{Id} \otimes K^V(z)) = r(\xi z^2 - 1)K^W(q^{-1}z)\tau^+(r),$$



The left boundary

There are also solutions $\widetilde{K}^V(z) \in \mathrm{End}(V)$ and $\widetilde{K}^W(z) \in \mathrm{End}(W)$ of "dual" reflection equations turning $\mathcal{U}_q^-$-reps back into $\mathcal{U}_q^+$-reps. They satisfy

$$(\mathrm{Id} \otimes \widetilde{K}^V(z))\widetilde{L}^+(z^2, r)P(\mathrm{Id} \otimes \widetilde{K}^W(z))\iota^- = r^{-1}\frac{1-\widetilde{\xi}q^2z^2}{1-q^2z^4}\iota^+(r)\widetilde{K}^W(qz)$$

$$\tau^+(\mathrm{Id} \otimes \widetilde{K}^V(z))\widetilde{L}^+(z^2, r)P(\mathrm{Id} \otimes \widetilde{K}^W(z)) = r^{-1}\frac{1-q^4z^2}{1-q^2z^4}(z^2-\widetilde{\xi})\widetilde{K}^W(q^{-1}z)\tau^-$$

where $\widetilde{L}^+(z^2, r) = ((L^+(z, r)^{\mathrm{t}_V})^{-1})^{\mathrm{t}_V}$.

More on $K^V(z)$ and $K^W(z)$

Consider derived Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ and its involutive automorphism

$$\theta = (\text{Chevalley involution}) \circ (\text{nontrivial diag. automorphism}).$$

Let $c_0, c_1 \in \mathbb{C}^\times$. The subalgebra

$$\mathcal{B}_{c_0,c_1} = \langle E_0 - c_0 F_1 k_0, \quad E_1 - c_1 F_0 k_1, \quad k_0^{\pm 2} \rangle \subset \mathcal{U}_q$$

is a left coideal, i.e. $\Delta(\mathcal{B}_{c_0,c_1}) \subset \mathcal{U}_q \otimes \mathcal{B}_{c_0,c_1}$, and satisfies
$\mathcal{B}_{c_0,c_1} \xrightarrow{q \to 1} \mathcal{U}(\widehat{\mathfrak{sl}}_2^\theta)$ if $c_0, c_1 \in q^{\mathbb{Z}}$, see [Kolb, 2014].
Note: $\lim_{q \to 1} \psi_q = \mathrm{Ad}(\chi)\theta$ where $\chi(\alpha_0) = \chi(\alpha_1) = -1$.

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Note: $\lim_{q\to 1} \psi_q = \mathrm{Ad}(\chi)\theta$ where $\chi(\alpha_0) = \chi(\alpha_1) = -1$.

If $q/c_0 = c_1/q =: \xi$ then $K^V(z)$ is the unique-up-to-a-scalar intertwiner for the $\mathcal{B}_{c_0,c_1}$-modules (V, π_z) and $(V, \pi_{1/z})$:

$$K^V(z)\pi_z(X) = \pi_{1/z}(X) K^V(z) \qquad \text{for all } X \in \mathcal{B}_{c_0,c_1}.$$

More on $K^V(z)$ and $K^W(z)$

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$$K^V(z)\pi_z(X) = \pi_{1/z}(X) K^V(z) \qquad \text{for all } X \in \mathcal{B}_{c_0,c_1}.$$

Since $\mathcal{B}_{c_0,c_1} \not\subseteq \mathcal{U}_q^{\pm}$, we cannot evaluate $\rho^{\pm}_{z,r}(X)$ for all $X \in \mathcal{B}_{c_0,c_1}$ so the following does not make sense:

$$K^W(z)\rho^{\pm}_{z,r}(X) = \rho^{\pm}_{1/z,1/r}(X) K^W(z) \qquad \text{for all } X \in \mathcal{B}_{c_0,c_1}.$$

Transfer matrix and Q-operator

Let $\boldsymbol{t} = (t_1, \ldots, t_N) \in (\mathbb{C}^\times)^N$. The Q-operator and transfer matrix are given by

$$Q(z;\boldsymbol{t}) := \left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{\otimes N} \operatorname*{Tr}_W \widetilde{K}^W_a(z)\mathcal{M}^W_a(z;\boldsymbol{t}),$$
$$T(z;\boldsymbol{t}) := \operatorname*{Tr}_V \widetilde{K}^V_b(z)\mathcal{M}^V_b(z;\boldsymbol{t})$$

where

$$\mathcal{M}^W_a(z;\boldsymbol{t}) := L^-_{1a}(t_1 z,1)L^-_{2a}(t_2 z,1)\cdots L^-_{Na}(t_N z,1)\cdot$$
$$\cdot K^W_a(z)L^+_{aN}(\tfrac{z}{t_N},1)\cdots L^+_{a2}(\tfrac{z}{t_2},1)L^+_{a1}(\tfrac{z}{t_1},1)$$
$$\mathcal{M}^V_b(z;\boldsymbol{t}) := R_{1b}(t_1 z)R_{2b}(t_2 z)\cdots R_{Nb}(t_N z)K^V_b(z)R_{bN}(\tfrac{z}{t_N})\cdots R_{b2}(\tfrac{z}{t_2})R_{b1}(\tfrac{z}{t_1}).$$

- Here a labels the auxiliary space W, b labels the auxiliary space V.
- If $|\xi/\widetilde{\xi}| < |q|^{2N}$ then $Q(z;\boldsymbol{t})$ is well-defined for generic z.

Generalizing arguments from [Sklyanin (1988)], we have

$$Q(z;\boldsymbol{t})T(z;\boldsymbol{t}) = \left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{\otimes N} \operatorname*{Tr}_{W\otimes V} \widetilde{K}^V_b(z)\widetilde{L}^+_{ab}(z^2,1)\widetilde{K}^W_a(z)\times$$
$$\times \mathcal{M}^W_a(z,1;\boldsymbol{t})L^+_{ab}(z^2,1)\mathcal{M}^V_b(z;\boldsymbol{t}).$$

Generalizing arguments from [Sklyanin (1988)], we have

$$Q(z;\boldsymbol{t})T(z;\boldsymbol{t}) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes N} \operatorname*{Tr}_{W\otimes V} \widetilde{K}_b^V(z)\widetilde{L}_{ab}^+(z^2,1)\widetilde{K}_a^W(z)\times$$
$$\times \mathcal{M}_a^W(z,1;\boldsymbol{t})L_{ab}^+(z^2,1)\mathcal{M}_b^V(z;\boldsymbol{t}).$$

Combining the results in "Intertwiners and suchlike" we have

$$\widetilde{K}_b^V(z)\widetilde{L}_{ab}^+(z^2,r)\widetilde{K}_a^W(z)\mathcal{M}_a^W(z,r;\boldsymbol{t})L_{ab}^+(z^2,r)\mathcal{M}_b^V(z;\boldsymbol{t})(\iota^+(r)\otimes \mathrm{Id}) =$$
$$= \alpha_+(z;\boldsymbol{t})(\iota^+(r)\otimes \mathrm{Id})\widetilde{K}_a^W(qz)\mathcal{M}_a^W(qz,qr;\boldsymbol{t})$$
$$(\tau^+(r)\otimes \mathrm{Id})\widetilde{K}_b^V(z)\widetilde{L}_{ab}^+(z^2,r)\widetilde{K}_a^W(z)\mathcal{M}_a^W(z,r;\boldsymbol{t})L_{ab}^+(z^2,r)\mathcal{M}_b^V(z;\boldsymbol{t}) =$$
$$= \alpha_-(z;\boldsymbol{t})\widetilde{K}_a^W(q^{-1}z)\mathcal{M}_a^W(q^{-1}z,q^{-1}r;\boldsymbol{t})(\tau^+(r)\otimes \mathrm{Id})$$

where

$$\alpha_+(z;\boldsymbol{t}) = \frac{z^4-1}{q^2z^4-1}(q^2\widetilde{\xi}z^2-1)(q^2z^2-\xi)\prod_{n=1}^N((zt_n)^2-1)((\tfrac{z}{t_n})^2-1)$$

$$\alpha_-(z;\boldsymbol{t}) = q^{2N}\frac{q^4z^4-1}{q^2z^4-1}(z^2-\widetilde{\xi})(\xi z^2-1)\prod_{n=1}^N((qzt_n)^2-1)((\tfrac{qz}{t_n})^2-1).$$

We invoke the Proposition to decompose the trace over $W \otimes V$:

$$Q(z;\boldsymbol{t})T(z;\boldsymbol{t}) = \alpha_+(z;\boldsymbol{t}) \left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{\otimes N} \operatorname*{Tr}_W \widetilde{K}_0^W(qz)\mathcal{M}_0^W(qz,q;\boldsymbol{t}) + \\ + \alpha_-(z;\boldsymbol{t}) \left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{\otimes N} \operatorname*{Tr}_W \widetilde{K}_0^W(q^{-1}z)\mathcal{M}_0^W(q^{-1}z,q^{-1};\boldsymbol{t}).$$

Since the r-dependence factors out of $L^+(z,r)$ and $L^-(z,r)$ and $Q(z;\boldsymbol{t})$ commutes with $\begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes N}$, we obtain

$$\begin{aligned} Q(z;\boldsymbol{t})T(z;\boldsymbol{t}) &= \alpha_+(z;\boldsymbol{t}) \begin{pmatrix} (qz)^2 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes N} \operatorname*{Tr}_W \widetilde{K}_0^W(qz)\mathcal{M}_0^W(qz,1;\boldsymbol{t}) + \\ &\quad + \alpha_-(z;\boldsymbol{t}) \begin{pmatrix} (q^{-1}z)^2 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes N} \operatorname*{Tr}_W \widetilde{K}_0^W(q^{-1}z)\mathcal{M}_0^W(q^{-1}z,1;\boldsymbol{t}) \\ &= \alpha_+(z;\boldsymbol{t})Q(qz;\boldsymbol{t}) + \alpha_-(z;\boldsymbol{t})Q(q^{-1}z;\boldsymbol{t}). \end{aligned}$$

One can now proceed to re-obtain the Bethe ansatz equations found by [Sklyanin (1988)].