Universal formula for Hilbert series of minimal nilpotent orbits

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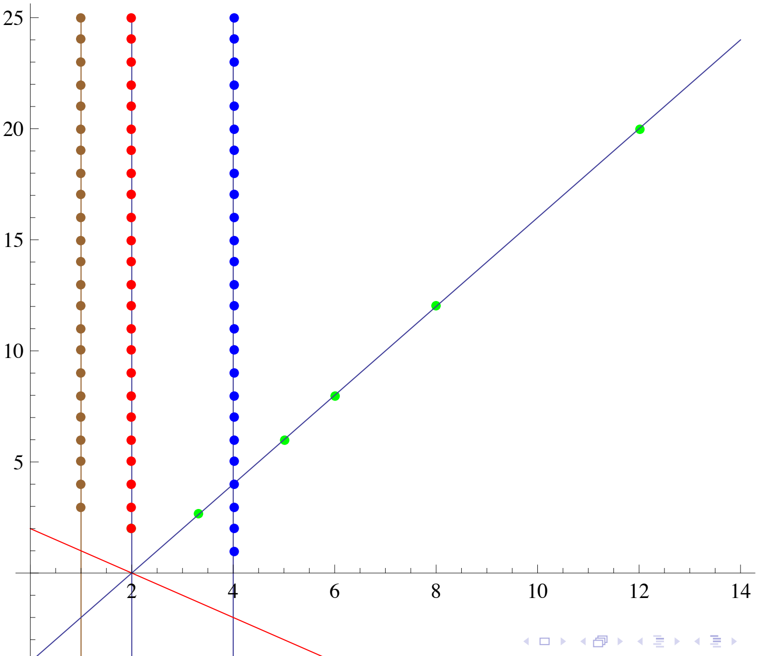
Table: Vogel's parameters for simple Lie algebras

Type	Lie algebra	α	β	γ	$t = h^\vee$
A_n	$\mathfrak{sl}_{n+1}$	-2	2	$n+1$	$n+1$
B_n	$\mathfrak{so}_{2n+1}$	-2	4	$2n-3$	$2n-1$
C_n	$\mathfrak{sp}_{2n}$	-2	1	$n+2$	$n+1$
D_n	$\mathfrak{so}_{2n}$	-2	4	$2n-4$	$2n-2$
G_2	$\mathfrak{g}_2$	-2	$10/3$	$8/3$	4
F_4	$\mathfrak{f}_4$	-2	5	6	9
E_6	$\mathfrak{e}_6$	-2	6	8	12
E_7	$\mathfrak{e}_7$	-2	8	12	18
E_8	$\mathfrak{e}_8$	-2	12	20	30

Vogel's map



Vogel's parameters and universal formulae

Consider the decomposition

$$S^2\mathfrak{g} = \mathbb{C} \oplus Y_2(\alpha) \oplus Y_2(\beta) \oplus Y_2(\gamma)$$

and choose an invariant bilinear form (Casimir).
In Vogel's parametrisation the Casimir eigenvalues of the 3 components are $4t - 2\alpha, 4t - 2\beta, 4t - 2\gamma$, where

$$t = \alpha + \beta + \gamma,$$

which defines the parameters uniquely up to a common multiple. If we normalise $\alpha = -2$, then $t = h^\vee$ is *dual Coxeter number* and we have Table 1.

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Vogel, 1999: universal formulae for the dimensions

$$\dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma},$$

$$dim\, Y_2(\alpha) = -\frac{(3\,\alpha - 2\,t)\,(\beta - 2\,t)\,(\gamma - 2\,t)\,t\,(\beta + t)\,(\gamma + t)}{\alpha^2\,(\alpha - \beta)\,\beta\,(\alpha - \gamma)\,\gamma}.$$

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Exceptional (Deligne) line:

$$dim\ Y_2(\gamma) = 0: \quad 3\gamma - 2t = 0, \quad \gamma = 2\beta - 4,$$

containing

$$\mathfrak{sl}_3,\ \mathfrak{g}_2,\ \mathfrak{so}_8,\ \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_{7+\frac{1}{2}},\ \mathfrak{e}_8.$$

Minimal nilpotent orbits

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Example. In sl_{n+1}-case $X = P(\mathcal{O}_{min})$ is the hyperplane section of the Segre variety

$$\Sigma_{n,n} = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{(n+1)^2-1}.$$

Indeed, $\mathcal{O}_{min}$ consists of the nilpotent rank one matrices, which can be written as $p \otimes q$ with $p, q \in \mathbb{C}^{n+1}$ satisfying

$$(p,q) = p_1 q_1 + \cdots + p_{n+1} q_{n+1} = 0.$$

Hilbert series and polynomial

For a projective variety $X \subset \mathbb{P}^n$ the Hilbert series $H_X(z)$ is defined as the generating function

$$H_X(z) = \sum_{k=0}^{\infty} \dim(S(X)_k) z^k,$$

where $S(X) = \mathbb{C}[x_0, \ldots, x_n]/I(X)$ is the homogeneous coordinate ring of X and $S(X)_k$ is the component of degree k.

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It is known that

$$h_X(x) = \deg X \, \frac{x^d}{d!} + \ldots,$$

where $d = \dim X$, so $h_X(x)$ determines both dimension and degree of X.

Universal formula

Let α, β, γ be Vogel's parameters and introduce

$$a_1 = 2b_1 + 2b_2 - 3, \ a_2 = b_1 + 2b_2 - 2, \ a_3 = 2b_1 + b_2 - 2, \ a_4 = b_3 + 1,$$

$$b_1 = -\frac{\beta}{\alpha}, \ b_2 = -\frac{\gamma}{\alpha}, \ b_3 = -\frac{2t + \alpha}{2\alpha}.$$

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Consider the generalized hypergeometric function

$${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n (a_4)_n}{(b_1)_n (b_2)_n (b_3)_n} \frac{z^n}{n!},$$

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Matsuo, APV 2017: *The Hilbert series of* $X = \mathbb{P}(\mathcal{O}_{min})$ *has the following universal form*

$$H_X(z) = {}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z) = \left(1 + \frac{2}{a_1} z \frac{d}{dz}\right) {}_3F_2(a_1, a_2, a_3; b_1, b_2; z).$$

The Hilbert polynomial of $X = \mathbb{P}(\mathcal{O}_{min})$ *is*

$$h_X(x) = \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \left(1 + \frac{2x}{a_1}\right) \frac{\Gamma(a_1+x)\Gamma(a_2+x)\Gamma(a_3+x)}{\Gamma(b_1+x)\Gamma(b_2+x)\Gamma(1+x)},$$

with $h_X(k) = \dim(S(X)_k)$ *for* **all** $k \geq 0$.

Degree and dimension

Proof follows from **Borel-Hirzebruch-Kostant** formula

$$S(X) = \bigoplus_{k=0}^{\infty} V(k\theta),$$

where θ is the maximal root of $\mathfrak{g}$ and $V(\lambda)$ is the irreducible representation with the highest weight λ, and from the universal formula for $\dim V(k\theta)$ found by **Landsberg and Manivel 2006**.

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Corollary *The dimension of $X = P(\mathcal{O}_{min})$ is*

$$\dim X = 2a_1 - 1 = 2h^\vee - 3,$$

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The degree of X *is*

$$\deg(X) = \frac{2\,\Gamma(2a_1)\,\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1+1)\,\Gamma(a_2)\Gamma(a_3)}.$$

Table

Type	a_1	a_2	a_3	b_1	b_2	$\dim X$	$\deg X$
A_n	n	n	$\frac{n+1}{2}$	1	$\frac{n+1}{2}$	$2n-1$	$\binom{2n}{n}$
B_n	$2n-2$	$2n-3$	$n+\frac{1}{2}$	2	$n-\frac{3}{2}$	$4n-5$	$\frac{4}{2n-1}\binom{4n-4}{2n-2}$
C_n	n	$n+\frac{1}{2}$	$\frac{n}{2}$	$\frac{1}{2}$	$\frac{n}{2}+1$	$2n-1$	2^{2n-1}
D_n	$2n-3$	$2n-4$	n	2	$n-2$	$4n-7$	$\frac{4}{2n-2}\binom{4n-6}{2n-3}$
E_6	11	9	8	3	4	21	151164
E_7	17	14	12	4	6	33	141430680
E_8	29	24	20	6	10	57	126937516885200
F_4	8	$\frac{13}{2}$	6	$\frac{5}{2}$	3	15	4992
G_2	3	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{5}{3}$	$\frac{4}{3}$	5	18

Table: Parameters, dimension and degree of $X = P(\mathcal{O}_{min})$.

Example: sl_{n+1}-case

In sl_{n+1}-case $X = P(\mathcal{O}_{min})$ is the **hyperplane section of the Segre variety**

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Our universal formula for the degree gives in this case

$$\deg X = \frac{2\,\Gamma(2n)\Gamma(1)\Gamma(\frac{n+1}{2})}{\Gamma(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})} = \frac{2 \cdot (2n-1)!}{n!(n-1)!} = \binom{2n}{n},$$

which agrees with the well-known result:

$$< (\alpha + \beta)^n, [\mathbb{P}^n \times \mathbb{P}^n] >= \binom{2n}{n},$$

since $\alpha^{n+1} = \beta^{n+1} = 0$.

Comparison with Gross and Wallach

Gross and Wallach 2011 used Weyl's dimension formula to show that

$$h_X(q) = \prod_{\alpha \in R_+} \left(1 + \frac{(\theta, \alpha^\vee)}{(\rho, \alpha^\vee)} q \right),$$

where ρ is the half-sum of the positive roots of $\mathfrak{g}$.

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The Hilbert series of X can be written then as

$$H_X(z) = h_X\left(z\frac{d}{dz}\right)\frac{1}{1-z},$$

which implies **Borel-Hirzebruch 1959** formula

$$\deg(X) = d! \prod_{\alpha} \frac{(\theta, \alpha^\vee)}{(\rho, \alpha^\vee)},$$

where the product is taken over positive roots such that $(\theta, \alpha^\vee) \neq 0$.

It would be interesting to deduce from here our universal formula for $\deg(X)$.

Numerology

Our formulae are symmetric in β and γ, but not in α. It is natural to ask for possible meaning of the corresponding Hilbert series when we permute α with β or γ (cf. **Landsberg, Manivel 2006**).

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Our formulae predict that the corresponding "virtual varieties" Y and Z must have degree 0 and negative dimensions:

$$\dim Y = -\frac{4t}{\beta} - 3, \quad \dim Z = -\frac{4t}{\gamma} - 3.$$

In particular, for A_n type

$$\dim Y = -2n - 5, \ \dim Z = -7,$$

and for E_8

$$\dim Y = -13, \ \dim Z = -9.$$

Is there any geometry behind this?

Vogel's parameters for Lie superalgebras

Vogel's approach for the basic classical classical Lie superalgebras and leads to the following table:

Table: Vogel's parameters for basic classical Lie superalgebras

Lie superalgebra	α	β	γ	t
$\mathfrak{sl}_{m,n}$	-2	2	$m-n$	$m-n$
$\mathfrak{osp}_{p,q}$	-2	4	$p-q-4$	$p-q-2$
$\mathfrak{f}_4$	-2	2	3	3
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Note that in Vogel's approach exceptional Lie superalgebras $\mathfrak{f}_4$ and $\mathfrak{g}_3$ are equivalent to $\mathfrak{sl}_3$ and $\mathfrak{sl}_2$ respectively and in the (potentially most interesting) case of $\mathfrak{D}_{2,1,\lambda}$ the parameter $t = \lambda_1 + \lambda_2 + \lambda_3 = 0$ (red line on Vogel's map).

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Is there a superanalogue of our results?

Vogel's map