Partition functions, tau-functions, and wall-crossing

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The following is about

some aspects related to

integrable models and cluster algebras

of a project on

topological string partition functions

pursued in collaboration with

I. Coman, P. Longhi and E. Pomoni.

Isomonodromic deformations

Schlesinger system: Consider connections $\nabla_\lambda = \lambda\partial_x - A(x)$,

$$A(x) = \sum_{r=1}^{n} \frac{A_r}{x - z_r}, \qquad A_r \in \mathfrak{sl}_2, \qquad \sum_{r=1}^{n} A_r = 0.$$

Poisson-structure (Goldman-Atiyah-Bott)

$$\left\{ A(x) \overset{\otimes}{,} A(y) \right\}_{\mathrm{GAB}} = \frac{1}{x - y}\left[\mathsf{P}\, ,\, A(x) \otimes 1 + 1 \otimes A(y) \right].$$

Hamiltonians:

$$H_r = \sum_{s \neq r} \frac{\mathrm{tr}(A_r A_s)}{z_r - z_s}.$$

Schlesinger's equations:

$$\frac{\partial}{\partial z_r} A_s = \left\{ A_s \, , \, H_r \right\}_{\mathrm{GAB}} \quad \Leftrightarrow \quad \textbf{Monodromy of } \nabla_\lambda \textbf{ is constant.}$$

Integrability: $\{H_r, H_s\} = 0$.

Conserved quantities: Monodromy data

Let $\Psi(x)$: solution to,

$$(\lambda\partial_x - A(x))\Psi(x) = 0, \qquad \Psi(x_0) = 0.$$

Monodromies M_r, defined by

$$\Psi(\gamma_r.x) = \Psi(x)M_r, \qquad M_r \in G = \mathrm{SL}(2,\mathbb{C}),$$

$\Psi(\gamma_r.x)$: analytic continuation of $\Psi(x)$ along contour γ_r encircling only z_r,

generate representations $\rho : \pi_1(C) \to G$, $C = \mathbb{P}^1 \setminus \{z_1, \ldots, z_n\}$.

Change of x_0 changes matrices M_r by overall conjugation.

$\rightsquigarrow$ **Space of monodromy data: Character variety** $\mathcal{M}_{\mathrm{char}}(C)$,

the space of all representations $\rho : \pi_1(C) \to G$ modulo overall conjugation.

From connections to monodromy data and back

There is a locally biholomorphic map Hol from $\mathcal{M}^{\lambda}_{\mathrm{dR}}(C)$ to $\mathcal{M}^{\lambda}_{\mathrm{B}}(C)$, assigning

representations $\rho : \pi_1(C) \to \mathrm{SL}(2,\mathbb{C})$ to $(\mathcal{E}, \nabla_\lambda)$,

defined by computing the holonomy/monodromy of ∇_λ.

The inverse of this map: Riemann-Hilbert correspondence. Classical formulation:

Find a matrix function $\Psi(x)$ *satisfying the following conditions:*

i) $\Psi(x)$ *is multivalued, analytic and invertible on* $C_{0,n}$.

ii) The monodromy of $\Psi(x)$ *is represented as*

$$\Psi(\gamma.x) = \Psi(x)\rho(\gamma), \qquad \rho : \pi_1(C) \to \mathrm{SL}(2,\mathbb{C}).$$

The tau-function: Unification of integrable structures I

Isomonodromic tau-function, classical definition (Sato-Miwa-Jimbo):

$$\frac{\partial}{\partial z_r} \log \mathcal{T}(\mu, \mathbf{z}) = H_r$$

where μ: monodromy data, $\mathbf{z} = (z_1, \ldots, z_n)$, H_r: Schlesinger-Hamiltonians.

Longstanding problems:

A) Calculate series expansions of $\mathcal{T}(\mu, \mathbf{z})$ around singular points.

B) What are natural ways to fix dependence on monodromy parameters μ?

Replace conserved quantities by initial values

For some fixed $\mathbf{z} = (z_1, \ldots, z_n)$ one may use holonomy map Hol to express the monodromy data μ as function $\mu(A, \lambda)$ of the data $(A(x), \lambda)$, representing the initial values of the isomonodromic deformation problem.

The tau-function $\mathcal{T}(\mu, \mathbf{z})$ can be used to define a function $\widehat{\Theta}(A, \mathbf{z}; \lambda)$,

$$\widehat{\Theta}(A, \lambda; \mathbf{z}) := \mathcal{T}(\mu(A, \mathbf{z}; \lambda), \mathbf{z}).$$

The space of initial values is $\mathcal{M}_{\mathrm{flat}}(C) \times \mathbb{C}^{\times}$, with

- $\mathcal{M}_{\mathrm{flat}}(C)$: moduli space of flat connections on C,
- λ: coordinate for $\mathbb{C}^{\times}$.

Variant of B): Are there natural ways to fix the normalisation of $\widehat{\Theta}(A, \lambda; \mathbf{z})$? Or:

How to extend the (locally defined!) functions $\widehat{\Theta}(A, \lambda; \mathbf{z})$ to a natural globally defined geometric object on $\mathcal{M}_{\mathrm{flat}}(C) \times \mathbb{C}^{\times}$?

The tau-function: Unification of integrable structures II

Explicit formula: (conjectured by Gamayun-Iorgov-Lisovyy[1], proofs by Iorgov-Lisovyy-J.T.[2], Bershtein-Shchechkin[3], Gavrylenko-Lisovyy[4]

$$\mathcal{T}(\sigma,\eta\,;\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^{n-3}} e^{i(\mathbf{n},\eta)}\,\mathcal{G}(\,\sigma+\mathbf{n}\,;\,\mathbf{z}\,),$$

where $\mathcal{G}(\,\sigma\,;\,\mathbf{z}\,)$: **instanton partition functions** $\overset{\mathrm{AGT}}{\leftrightarrow}$ **conformal blocks**

have **explicit** power series expansions:

[1] Inspired by/using results of Sato-Miwa-Jimbo, Moore, Moore-Nekrasov-Shatashvili, Nekrasov, Alday-Gaiotto-Tachikawa

[2] CFT: Monodromy of $\mathfrak{Vir}$-degenerate fields $\rightsquigarrow$ construction of solution of Riemann-Hilbert problem

[3] VOA duality (Bershtein-Feigin-Litvinov) $\rightsquigarrow$ bilinear equations of Hirota type, related to Nakajima-Yoshioka blow-up

[4] Combinatorial expansion of Fredholm determinants; Cafasso-Gavrylenko-Lisovyy: Relation with Sato-Segal-Wilson

$\mathcal{G}(\,\sigma\,;\,\mathbf{z}\,)$ have explicit power series expansions **(here combinatorics!):**

Example $n=4$**:** $\mathcal{G}(\,\sigma\,,\,\underline{\theta}\,;\,z\,)\,\mathcal{G}(\,\sigma\,,\,\underline{\theta}\,;\,z\,) = M(\sigma,\theta_4,\theta_3)M(\sigma,\theta_2,\theta_1)\mathcal{F}(\,\sigma\,,\,\underline{\theta}\,;\,z\,)$, where

- the functions $M(\theta_3,\theta_2,\theta_1)$ are defined as

$$M(\theta_3,\theta_2,\theta_1) \;=\; \frac{\prod_{\epsilon=\pm} G(1+\theta_3+\epsilon(\theta_2+\theta_1))G(1+\theta_3+\epsilon(\theta_2-\theta_1))}{G(1+2\theta_3)G(1-2\theta_2)G(1-2\theta_1)}\,,$$

where $G(p)$ is the Barnes G-function that satisfies $G(p+1)=\Gamma(p)G(p)$,

- $\mathcal{F}(\,\sigma\,,\,\underline{\theta}\,;\,z\,)$ can be represented by the following power series

$$\mathcal{F}(\,\sigma\,,\,\underline{\theta}\,;\,z\,) = z^{\sigma^2-\theta_1^2-\theta_2^2}(1-z)^{2\theta_2\theta_3}\sum_{\xi,\zeta\in\mathbb{Y}} z^{|\xi|+|\zeta|}\mathcal{F}_{\xi,\zeta}(\sigma,\underline{\theta}),$$

with $\mathbb{Y}$: set of partitions, coefficients $\mathcal{F}_{\xi,\zeta}(\sigma,\underline{\theta})$ explicitly given in

$$\mathcal{F}_{\xi,\zeta}(\sigma,\underline{\theta}) = \prod_{(i,j)\in\xi}\frac{((\theta_2+\sigma+i-j)^2-\theta_1^2)((\theta_3+\sigma+i-j)^2-\theta_4^2)}{(\xi_j'-i+\xi_i-j+1)^2(\xi_j'-i+\zeta_i-j+1+2\sigma)^2}$$
$$\prod_{(i,j)\in\zeta}\frac{((\theta_2-\sigma+i-j)^2-\theta_1^2)((\theta_3-\sigma+i-j)^2-\theta_4^2)}{(\zeta_j'-i+\zeta_i-j+1)^2(\zeta_j'-i+\xi_i-j+1-2\sigma)^2}.$$

ζ_i / ζ_i' arm / leg length of $(i,j)\in\mathbb{Y}$.

The coordinates $\sigma = (\sigma_1, \ldots, \sigma_d)$**,** $\eta = (\eta_1, \ldots, \eta_d)$**,** $d = n - 3$

appearing in magic formula

$$\mathcal{T}\big(\sigma, \eta\,;\,\mathbf{z}\big) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{i(\mathbf{n},\eta)}\,\mathcal{G}(\,\sigma + \mathbf{n}\,;\,\mathbf{z}\,),$$

are **very** special:

a) reflect integrable structure of $\mathcal{M}_{\mathrm{char}}(C)$

b) reflect algebraic structure of $\mathcal{M}_{\mathrm{char}}(C)$**:**

Some coordinates are better than others....

a) (σ, η) **reflect secondary integrable structure (for** $G = SL(2)$**):**

- Pick pants decomposition $(\gamma_1, \ldots, \gamma_d)$.

- Write $\mathrm{tr}(\rho(\gamma_r)) = 2\cos(2\pi\sigma_r)$.

$\rightsquigarrow$ **Commuting flows**: If F is a function on $\mathcal{M}_{\mathrm{char}}(C)$, let

$$\frac{\partial}{\partial \eta_r} F = \{\, F \,,\, \sigma_r \,\}_{\mathrm{GAB}}. \tag{1}$$

(σ, η), $\sigma = (\sigma_1, \ldots, \sigma_d)$, $\eta = (\eta_1, \ldots, \eta_d)$: Darboux coordinates,

$$\Omega_{\mathrm{GAB}} = \sum_{r=1}^{d} d\sigma_r \wedge d\eta_r,$$

Remark:

(1): Fenchel-Nielsen twist flows on Teichmüller component of real slice in $\mathcal{M}_{\mathrm{B}}(C)$

We call such coordinates **FN-type coordinates**.

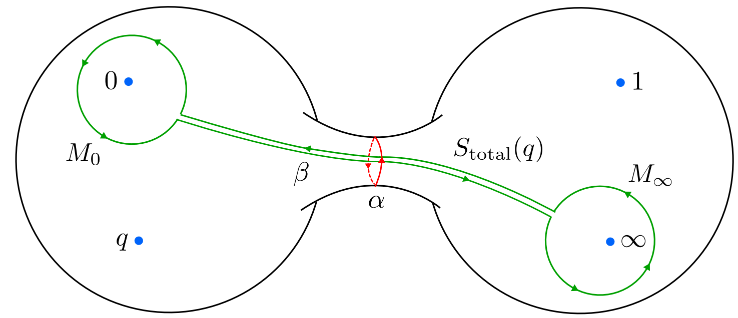
Some coordinates are better than others....

Explicit definition of η for $C = C_{0,4}$:

Coordinate ring ($k \in \{0, q, 1, \infty\}$):

Generators: $L_{0q} = \mathrm{Tr}(M_0 M_q)$, $L_{01} = \mathrm{Tr}(M_0 M_1)$,

$L_{0\infty} = \mathrm{Tr}(M_0 M_\infty)$, $L_k = \mathrm{Tr}\, M_k = 2\cos 2\pi\theta_k$,



$$L_0 L_q L_1 L_\infty + L_{0q} L_{01} L_{0\infty} + L_{0q}^2 + L_{01}^2 + L_{0\infty}^2 + L_0^2 + L_q^2 + L_1^2 + L_\infty^2 =$$
$$= (L_0 L_q + L_1 L_\infty)\, L_{0q} + (L_0 L_1 + L_q L_\infty)\, L_{01} + (L_q L_1 + L_0 L_\infty)\, L_{0\infty} + 4.$$

Pants decomposition $\rightsquigarrow$ Factorisation of holonomy:

$$L_{0\infty} = \mathrm{tr}(T^{-1} M_0 T M_\infty) = \mathrm{tr}\left(T^{-1} \begin{pmatrix} * & \mu_0^+ \\ \mu_0^- & * \end{pmatrix} T \begin{pmatrix} * & \mu_\infty^+ \\ \mu_\infty^- & * \end{pmatrix}\right) \qquad T = \begin{pmatrix} V & 0 \\ 0 & V^{-1} \end{pmatrix}$$
$$= \mu_0^- \mu_\infty^+ V^2 + N_0 + \mu_0^+ \mu_\infty^- V^{-2}, \qquad V = e^{\pi \mathrm{i}\, \eta},$$

and $\mu_0^\pm = \mu_0^\pm(\sigma)$, $\mu_\infty^\pm = \mu_\infty^\pm(\sigma)$ and $N_0 = N_0(\sigma)$ do not depend on η.

Cases $n > 4$ reduced to $n = 4$ by means of pants decomposition.

Coordinates (σ, η) related to work of `Nekrasov, Rosly, Shatashvili`.

Some coordinates are better than others....

a) Reflect integrable structure of $\mathcal{M}_{\mathrm{B}}(C)$!

b) Reflect algebraic structure of $\mathcal{M}_{\mathrm{B}}(C)$?

There is still a large freedom in the choice of η, $\eta \to \eta + f(\sigma)$.

However, there exists a small family of coordinates η of **rational FN-type** such that

$$L_{0\infty} = \frac{p_+(U)\,V^2 + p_0(U) + p_-(U)\,V^{-2}}{(U - U^{-1})^2}, \quad L_{01} = \frac{q_+(U)\,V^2 + q_0(U) + q_-(U)\,V^{-2}}{(U - U^{-1})^2},$$

$q_\pm(U) = -p_\pm(U)U^{\pm 1}$, and $p_\epsilon(U)$: **Laurent-polynomial** in $U = e^{2\pi \mathrm{i}\sigma}$.

Coordinate ring represented by **rational** functions of U and V $\leftrightarrow$ algebraic structure!

Residual, **finite** freedom: Note that

$$p_+(U)p_-(U) = \prod_{s,s'=\pm} 2\sin\pi(\sigma + s\sigma_0 + s'\sigma_q) \prod_{s,s'=\pm} 2\sin\pi(\sigma + s\sigma_1 + s'\sigma_\infty).$$

Choices for $p_\pm(U)$ from distributing factors $2\sin\pi(\dots)$ between $p_+(U)$ and $p_-(U)$.

Good coordinates

Good coordinates (p, q) satisfy a), b) and allow us to define

$$\mathcal{T}_Q(q + \delta_r, p; z) = \mathcal{T}_Q(q, p; z),$$
$$\mathcal{T}_Q(q, p + \delta_r; z) = e^{-2\pi \mathrm{i} q_r} \mathcal{T}_Q(q, p; z).$$

This is equivalent to existence of an expansion as generalised theta series

$$\mathcal{T}_Q(q, p; z) = \sum_{n \in \mathbb{Z}^d} e^{2\pi \mathrm{i}(n,q)} \mathcal{Z}_Q(p + n; z).$$

Note: There are preferred normalisations of $\mathcal{T}_Q$ caracterised by these conditions.

Example: Coordinates $(q, p) = (\eta, \sigma)$ with (η, σ) introduced above are good.

Questions:

a) Can we cover $\mathcal{M}_{\mathrm{char}}$ with good coordinates?

b) How much freedom is there in the choice of good coordinates?

Changes of coordinates induce change of normalisation

To a Poisson automorphism $(q_+, p_+) = (f(q_-, p_-), g(q_-, p_-))$ defined by an equation $p_+ = g(q_-, p_-)$ which can be partially inverted to define functions $q_- = q_-(p_+, p_-)$ we may assign a "difference generating function" $F(p_+, p_-)$ satisfying (here $d = 1$)

$$\begin{aligned} F(p_+ + 1, p_-) &= e^{-2\pi \mathrm{i}\, q_+(p_+,p_-)} F(p_+, p_-), \\ F(p_+, p_- + 1) &= e^{+2\pi \mathrm{i}\, q_-(p_+,p_-)} F(p_+, p_-), \end{aligned} \qquad q_+(p_+, p_-) = f(q_-(p_+, p_-), p_-).$$

The functions $\mathcal{T}_Q(q, p; z)$ associated to two different coordinate systems Q and Q' can differ by an overall μ-dependent factor $F_{QQ'}(p, p')$,

$$\mathcal{T}_Q(q, p; z) = F_{QQ'}(p, p')\mathcal{T}_{Q'}(q', p'; z).$$

Example:

If $(q, p) = (\eta, \sigma)$ with (η, σ) introduced above, and $(q', p') = (\eta', \sigma')$ with (η', σ') defined in the same way using another pants decomposition, $F_{QQ'}(p, p')$ has been found by Iorgov, Lisovyy, Tykhyy, and Its, Lisovyy, Prokhorov.

Perfect coordinates

Let us now switch attention to the coordinates $(\hat{q},\hat{p})$ on $\mathcal{M}_{\rm flat}(C)\times\mathbb{C}^{\times}$ defined by composing (q,p) with Hol. Note that $\hat{q}=q(A,\lambda)$, $\hat{p}=p(A,\lambda)$.

We call the coordinates $(\hat{q},\hat{p})$ **perfect** if they can be defined by Borel summation of the asymptotic expansion in powers of λ.

Key observation I (verified in Painlevé VI examples)

Coordinates (σ,η) can be perfect in subsets of $\mathcal{M}_{\rm flat}(C)\times\mathbb{C}^{\times}$.

Key observation II (work in progress by D. Allegretti, T. Bridgeland)

The Fock-Goncharov coordinates associated to the WKB-triangulation defined by (A,λ) are perfect.

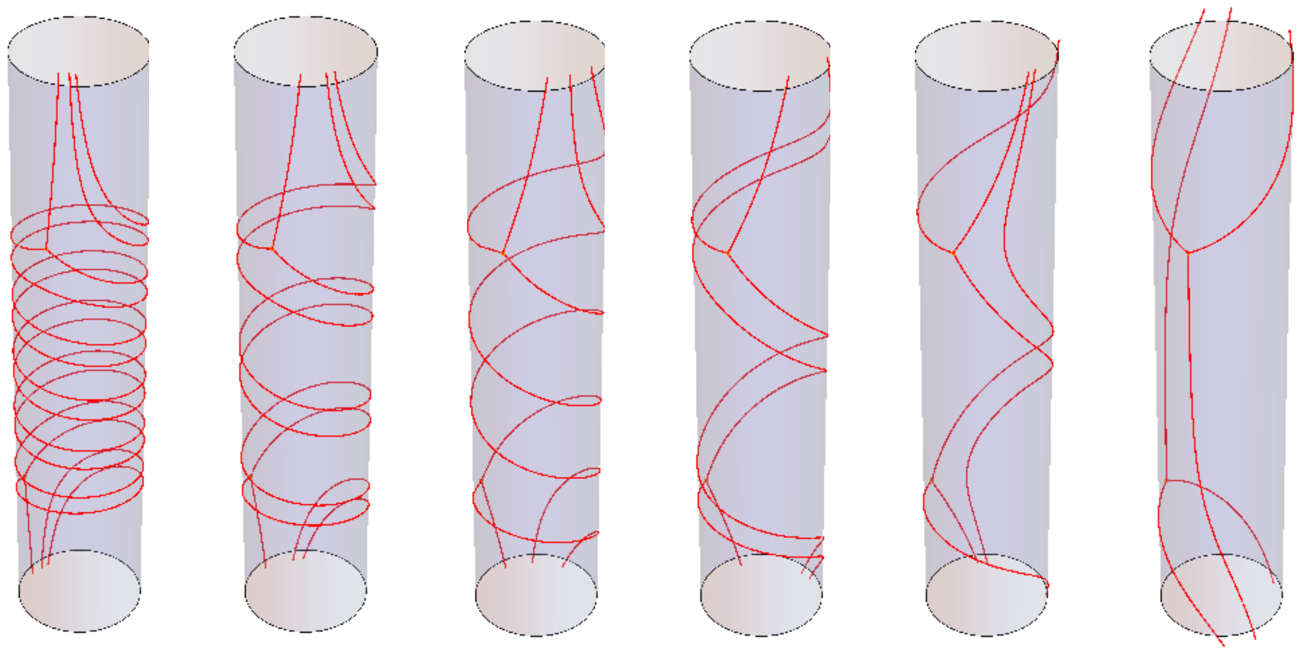
The analytic continuation of $\hat{Q}=(\hat{q},\hat{p})$ to the domain of $\hat{Q}'=(\hat{q}',\hat{p}')$ defines a difference generating function.

$\rightsquigarrow$ Can extend definition of function $\widehat{\Theta}(A,\lambda;\mathbf{z})$ from domain of $\hat{Q}$ to domain of $\hat{Q}'$!

Goal: Use this to define $\widehat{\Theta}(A,\lambda;\mathbf{z})$ globally.

An interesting discrete dynamics

It is interesting to consider dependence on λ for fixed A. Domains of FG-type coordinates: Wedges in λ-plane. Stokes graph and WKB triangulation change when crossing certain rays in λ-plane. Such rays are called "active".



Coordinates on two sides of a ray related by cluster trsf. $\rightsquigarrow$ Discrete evolution described by cluster mutations, "time" step: number of crossings of active rays.

Link to the programs initiated by Gaiotto, Moore and Neitzke, and the one of Bridgeland.

Simplify life by confluence Painlevé VI $\to$ Painlevé III

Isomonodromic deformations of $\partial_x - A(x)$,

$$A(x) = -\frac{\mathrm{i}r^2}{16}\sigma_3 - \frac{\mathrm{i}v}{4x}\sigma_1 + \frac{\mathrm{i}}{x^2}e^{-\frac{\mathrm{i}}{2}u\sigma_1}\sigma_3 e^{\frac{\mathrm{i}}{2}u\sigma_1}, \qquad \begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial H}{\partial v},\\ \frac{\partial v}{\partial r} &= -\frac{\partial H}{\partial u},\end{aligned} \qquad H = \frac{v^2}{2r} - r\cos(u).$$

Definition tau-function: $\dfrac{\partial}{\partial r}\ln\mathcal{T}\big(2^{-12}r^4\big) = -\dfrac{H}{8} + \dfrac{1}{4}\dfrac{\partial}{\partial r}\ln r\, e^{\mathrm{i}u}.$

$\exists$ pair of solutions $Y^{(0)}(x)$, $Y^{(\infty)}(x)$ of $(\partial_x - A(x))Y(x) = 0$ having monodromy

$$\begin{aligned} Y^{(0)}(e^{2\pi\mathrm{i}}x) &= Y^{(0)}(x)M_0,\\ Y^{(\infty)}(e^{2\pi\mathrm{i}}x) &= Y^{(\infty)}(x)M_\infty \end{aligned} \qquad M_0 = \sigma_x M_\infty \sigma_x = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & -2\cos 2\pi\sigma \end{pmatrix},$$

with $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The two solutions are related by

$$Y^{(\infty)}(x) = Y^{(0)}(x)E, \qquad E = \frac{1}{\sin 2\pi\sigma}\begin{pmatrix} \sin 2\pi\eta & -\mathrm{i}\sin 2\pi(\eta+\sigma) \\ \mathrm{i}\sin 2\pi(\eta-\sigma) & \sin 2\pi\eta \end{pmatrix}.$$

(η, σ): analogs of the coordinates used for Painlevé VI above.

Other good coordinates on $\mathcal{M}_{\rm char}$

Let us change coordinates from (σ, η) to (X, Y),

$$X = \frac{U - U^{-1}}{UV + (UV)^{-1}}, \qquad Y = \frac{V + V^{-1}}{U - U^{-1}}, \qquad \begin{aligned} U &= e^{2\pi \mathrm{i}\,\sigma}, \\ V &= e^{2\pi \mathrm{i}\,\eta}. \end{aligned}$$

It can be shown that

- $\log X$ and $\log Y$ are good coordinates (Its, Lisovyy, Tykhyy)
- U^2, V^2 are Fock-Goncharov (cluster) coordinates for $\mathcal{M}_{\rm flat}(C)$ for a certain triangulation of the annulus with one puncture on each boundary.

Question:

(σ, η) analogous to Gelfand-Zeitlin coordinates? ($\mathcal{A}$ represented by rational functions).

Discrete dynamics in FG versus FN coordinates

Poisson-automorphism of $\hat{\mathcal{A}}$

$$\tau_{\mathrm{FG}}(X) = Y^{-1}, \qquad \tau_{\mathrm{FG}}(Y) = X(1+Y^2).$$

Let us then perform the change of variables:

$$X = \frac{U - U^{-1}}{UV + (UV)^{-1}}, \qquad Y = \frac{V + V^{-1}}{U - U^{-1}}.$$

Defining $\tau_{\mathrm{FN}}(U) = U$, $\tau_{\mathrm{FN}}(V) = VU^{-1}$ we have

$$\begin{aligned}
\tau_{\mathrm{FN}}(X(U,V)) &= \tau_{\mathrm{FN}}\left(\frac{U - U^{-1}}{UV + (UV)^{-1}}\right) = \frac{U - U^{-1}}{V + V^{-1}} = \frac{1}{Y(U,V)} = (\tau_{\mathrm{FG}}(Y))(U,V), \\
\tau_{\mathrm{FN}}(Y(U,V)) &= \tau_{\mathrm{FN}}\left(\frac{V + V^{-1}}{U - U^{-1}}\right) = \frac{UV^{-1} + U^{-1}V}{U - U^{-1}} \\
&= \frac{U - U^{-1}}{UV + (UV)^{-1}}\left(1 + \frac{(V + V^{-1})^2}{(U - U^{-1})^2}\right) = (\tau_{\mathrm{FG}}(Y))(U,V).
\end{aligned}$$

The converse is also true $\rightsquigarrow$ dynamics becomes “free” in FN-type coordinates.

The resulting conjectural picture:

- FG type coordinates can be used to cover $\mathcal{M}_{\text{flat}}(C) \times \mathbb{C}^{\times}$.

- FN type coordinates not everywhere defined. When they are defined, they give equivalent descriptions of the dynamics generated by variation of $\arg(\lambda)$.

- Define a line bundle $\mathcal{L}_{\Theta}$ on $\mathcal{M}_{\text{flat}}(C) \times \mathbb{C}^{\times}$, transition functions: Difference generating functions of changes of variables between FG-type coordinates.

- There exist difference generating functions describing changes of coordinates from FN to FG-type.

- Choices of pants decomposition $\rightsquigarrow$ preferred sections of $\mathcal{L}_{\Theta}$: Partition functions $\widehat{\Theta}(A, \lambda; \mathbf{z})$.

The relevance of the resulting "beast" for topological string theory has been confirmed by explicit calculations using the topological vertex (Coman, Pomoni, J.T.).